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"Elementary" Number Theory / *Théorie "élémentaire" des nombres*

On a congruence involving q-Catalan numbers

Sur une congruence impliquant des q-nombres de Catalan

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Abstract. Based on a q-congruence of the author and Petrov, we set up a q-analogue of Sun–Tauraso's congruence for sums of Catalan numbers, which extends a q-congruence due to Tauraso.

Résumé. À partir d'une q-congruence de l'auteur et Petrov, nous établissons un q-analogue de la congruence de Sun–Tauraso pour des sommes de nombres de Catalan, qui étend la q-congruence due à Tauraso.

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1. Introduction

In combinatorics, the Catalan numbers are a sequence of natural numbers, which play an important role in various counting problems. The nth Catalan number is given by the following binomial coefficient:

$$C_n = \binom{2n}{n}\frac{1}{n+1} = \binom{2n}{n} - \binom{2n}{2n+1}.$$

Closely related numbers are the central binomial coefficients $\binom{2n}{n}$ for $n \geq 0$.

Both Catalan numbers and central binomial coefficients satisfy many interesting congruences (see, for instance, [7, 9–11]). In 2011, Sun and Tauraso [11] proved that for primes $p \geq 5$,

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \tag{1}$$

$$\sum_{k=0}^{p-1} C_k \equiv \frac{3}{2}\left(\frac{p}{3}\right) - \frac{1}{2} \pmod{p^2}, \tag{2}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

In the past few years, q-analogues of congruences (q-congruences) for indefinite sums of binomial coefficients as well as hypergeometric series attracted many experts' attention (see, for example, [2–6, 8, 12, 13]). It is worth mentioning that Guo and Zudilin [6] developed an interesting microscoping method to prove many q-congruences.

In order to discuss q-congruences, we first recall some q-series notation. The q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

where the q-shifted factorial is given by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geq 1$ and $(a;q)_0 = 1$. Moreover, the q-integers are defined by $[n]_q = (1-q^n)/(1-q)$, and the nth cyclotomic polynomial is given by

$$\Phi_n(q) = \prod_{\substack{1\leq k\leq n \\ (n,k)=1}} (q - e^{2k\pi i/n}).$$

Recently, the author and Petrov [8] established a q-analogue for (1) as follows:

$$\sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix} \equiv \left(\frac{n}{3}\right) q^{\frac{n^2-1}{3}} \pmod{\Phi_n(q)^2}, \tag{3}$$

which was originally conjectured by Guo [2] and generalises a q-congruence of Tauraso [12]. There are several natural q-analogues of Catalan numbers (see [1]). Here and throughout the paper, we consider the following q-analogue of Catalan numbers:

$$C_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix} = \begin{bmatrix} 2n \\ n \end{bmatrix} - q \begin{bmatrix} 2n \\ n+1 \end{bmatrix}. \tag{4}$$

In 2012, Tauraso [12] obtained a weak q-version of (2) as follows:

$$\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} q^{\lfloor n/3 \rfloor} & \text{if } n \equiv 0,1 \pmod 3 \\ -1 - q^{(2n-1)/3} & \text{if } n \equiv 2 \pmod 3 \end{cases} \pmod{\Phi_n(q)},$$

where $\lfloor x \rfloor$ denotes the integral part of real x. In this note, we aim to set up a q-analogue of (2) as well as another related q-congruence for sums of binomial coefficients.

Theorem 1. *For any positive integer n, the following holds modulo $\Phi_n(q)^2$:*

$$\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} -q^{\frac{n^2-1}{3}} - q^{\frac{n(2n-1)}{3}} & \text{if } n \equiv 2 \pmod 3 \\ q^{\frac{n^2-1}{3}} - \frac{n-1}{3}(q^n - 1) & \text{if } n \equiv 1 \pmod 3. \end{cases} \tag{5}$$

In order to prove (5), we shall establish the following q-congruence.

Theorem 2. *For any positive integer n, the following holds modulo $\Phi_n(q)^2$:*

$$\sum_{k=0}^{n-1} q^{k+1} \begin{bmatrix} 2k \\ k+1 \end{bmatrix} \equiv \begin{cases} q^{\frac{n(2n-1)}{3}} & \text{if } n \equiv 2 \pmod 3, \\ \frac{n-1}{3}(q^n - 1) & \text{if } n \equiv 1 \pmod 3. \end{cases} \tag{6}$$

It is clear that (5) can be directly deduced from (3), (4) and (6). The remainder of the paper is organized as follows. We first set up a preliminary result in the next section, and prove Theorem 2 in Section 3.

2. An auxiliary result

Lemma 3. *For any positive integer n, the following holds modulo $\Phi_n(q)$:*

$$\sum_{k=1}^{n-1}\left(\frac{k-1}{3}\right)\frac{(-1)^k q^{\frac{1}{3}\left(2k^2-k\left(\frac{k-1}{3}\right)\right)-\frac{k(k-1)}{2}}}{1-q^k}\equiv\begin{cases}0 & \text{if } n\equiv 2 \pmod 3,\\ \frac{n-1}{6} & \text{if } n\equiv 1 \pmod 3.\end{cases}\tag{7}$$

Proof. Note that

$$\sum_{k=1}^{n-1}(-1)^k\left(\frac{k-1}{3}\right)\frac{q^{\frac{1}{3}\left(2k^2-k\left(\frac{k-1}{3}\right)\right)-\frac{k(k-1)}{2}}}{1-q^k}=\sum_{k=0}^{\lfloor\frac{n-3}{3}\rfloor}\frac{(-1)^k q^{\frac{(k+1)(3k+2)}{2}}}{1-q^{3k+2}}-\sum_{k=1}^{\lfloor\frac{n-1}{3}\rfloor}\frac{(-1)^k q^{\frac{k(3k+5)}{2}}}{1-q^{3k}}.$$

We shall distinguish two cases to prove (7).

Case 1. $n\equiv 2 \pmod 3$**.** This case is equivalent to

$$\sum_{k=0}^{n-1}\frac{(-1)^k q^{\frac{(k+1)(3k+2)}{2}}}{1-q^{3k+2}}-\sum_{k=1}^{n}\frac{(-1)^k q^{\frac{k(3k+5)}{2}}}{1-q^{3k}}\equiv 0 \pmod{\Phi_{3n+2}(q)}.\tag{8}$$

Let ω be a primitive $(3n+2)$th root of unity. Letting $k\to n-k$ in the following sum gives

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{(-1)^k\omega^{\frac{(k+1)(3k+2)}{2}}}{1-\omega^{3k+2}}&=\sum_{k=1}^{n}\frac{(-1)^{n-k}\omega^{\frac{(n-k+1)(3n-3k+2)}{2}}}{1-\omega^{3n-3k+2}}\\&=\sum_{k=1}^{n}\frac{(-1)^{n-k}\omega^{\frac{k(3k-1)}{2}+\frac{(3n+2)(n+1)}{2}-(3n+2)k}}{1-\omega^{3n-3k+2}}\\&=\sum_{k=1}^{n}\frac{(-1)^k\omega^{\frac{k(3k+5)}{2}}}{1-\omega^{3k}},\end{aligned}$$

where we have used the fact that $\omega^{\frac{(3n+2)(n+1)}{2}}=(-1)^{n+1}$. Thus,

$$\sum_{k=0}^{n-1}\frac{(-1)^k\omega^{\frac{(k+1)(3k+2)}{2}}}{1-\omega^{3k+2}}-\sum_{k=1}^{n}\frac{(-1)^k\omega^{\frac{k(3k+5)}{2}}}{1-\omega^{3k}}=0,$$

which is equivalent to (8).

Case 2. $n\equiv 1 \pmod 3$**.** Let ζ be a primitive $(3n+1)$th root of unity. It suffices to show that

$$\sum_{k=0}^{n-1}\frac{(-1)^k\zeta^{\frac{(k+1)(3k+2)}{2}}}{1-\zeta^{3k+2}}-\sum_{k=1}^{n}\frac{(-1)^k\zeta^{\frac{k(3k+5)}{2}}}{1-\zeta^{3k}}=\frac{n}{2}.\tag{9}$$

Note that

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{(-1)^k\zeta^{\frac{(k+1)(3k+2)}{2}}}{1-\zeta^{3k+2}}&=\sum_{k=n+1}^{2n}\frac{(-1)^{2n-k}\zeta^{\frac{(2n-k+1)(6n-3k+2)}{2}}}{1-\zeta^{6n-3k+2}}\\&=\sum_{k=n+1}^{2n}\frac{(-1)^k\zeta^{\frac{k(3k-1)}{2}+(3n+1)(2n-2k+1)}}{1-\zeta^{-3k}}\\&=-\sum_{k=n+1}^{2n}\frac{(-1)^k\zeta^{\frac{k(3k+5)}{2}}}{1-\zeta^{3k}},\end{aligned}$$

where we replace k by $2n-k$ in the first step. Thus,

$$\sum_{k=0}^{n-1}\frac{(-1)^k\zeta^{\frac{(k+1)(3k+2)}{2}}}{1-\zeta^{3k+2}}-\sum_{k=1}^{n}\frac{(-1)^k\zeta^{\frac{k(3k+5)}{2}}}{1-\zeta^{3k}}=-\sum_{k=1}^{2n}\frac{(-1)^k\zeta^{\frac{k(3k+5)}{2}}}{1-\zeta^{3k}}.\tag{10}$$

Furthermore, letting $k\to 2n+1-k$ on the right-hand side of (10) gives

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{(-1)^k\zeta^{\frac{(k+1)(3k+2)}{2}}}{1-\zeta^{3k+2}}-\sum_{k=1}^{n}\frac{(-1)^k\zeta^{\frac{k(3k+5)}{2}}}{1-\zeta^{3k}}&=-\sum_{k=1}^{2n}\frac{(-1)^{2n+1-k}\zeta^{\frac{(2n+1-k)(6n-3k+8)}{2}}}{1-\zeta^{3(2n+1-k)}}\\&=-\sum_{k=1}^{2n}\frac{(-1)^{1-k}\zeta^{\frac{(3k-1)(k-2)}{2}+(3n+1)(2n+3-2k)}}{1-\zeta^{1-3k}}\\&=-\sum_{k=1}^{2n}\frac{(-1)^k\zeta^{\frac{k(3k-1)}{2}}}{1-\zeta^{3k-1}}.\end{aligned}\tag{11}$$

An identity due to the author and Petrov [8, (2.4)] says

$$\sum_{k=1}^{2n}\frac{(-1)^k\zeta^{\frac{k(3k-1)}{2}}}{1-\zeta^{3k-1}}=-\frac{n}{2}.\tag{12}$$

Then the proof of (9) follows from (11) and (12). □

3. Proof of Theorem 2

Now we are in a position to prove Theorem 2. We recall the following identity:

$$\sum_{k=0}^{n-1}q^k\begin{bmatrix}2k\\k+1\end{bmatrix}=\sum_{k=0}^{n-1}\left(\frac{n-k-1}{3}\right)q^{\frac{1}{3}\left(2(n-k)^2-(n-k)\left(\frac{n-k-1}{3}\right)-3\right)}\begin{bmatrix}2n\\k\end{bmatrix},\tag{13}$$

which was proved by Tauraso in a more general form (see [12, Theorem 4.2]). Since $1-q^n\equiv 0 \pmod{\Phi_n(q)}$, we have

$$1-q^{2n}=(1+q^n)(1-q^n)\equiv 2(1-q^n)\pmod{\Phi_n(q)^2}.$$

It follows that for $1\le k\le n-1$,

$$\begin{aligned}\begin{bmatrix}2n\\k\end{bmatrix}&=\frac{(1-q^{2n})(1-q^{2n-1})\cdots(1-q^{2n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}\\&\equiv 2(1-q^n)\frac{(1-q^{-1})\cdots(1-q^{-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}\pmod{\Phi_n(q)^2}\\&=2(q^n-1)\frac{(-1)^kq^{-\frac{k(k-1)}{2}}}{1-q^k}.\end{aligned}\tag{14}$$

Multiplying both sides of (13) by q and substituting (14) into the right-hand side of (13), we arrive at

$$\begin{aligned}&\sum_{k=0}^{n-1}q^{k+1}\begin{bmatrix}2k\\k+1\end{bmatrix}\\&=\left(\frac{n-1}{3}\right)q^{\frac{1}{3}\left(2n^2-n\left(\frac{n-1}{3}\right)\right)}+\sum_{k=1}^{n-1}\left(\frac{n-k-1}{3}\right)q^{\frac{1}{3}\left(2(n-k)^2-(n-k)\left(\frac{n-k-1}{3}\right)\right)}\begin{bmatrix}2n\\k\end{bmatrix}\\&\equiv\left(\frac{n-1}{3}\right)q^{\frac{1}{3}\left(2n^2-n\left(\frac{n-1}{3}\right)\right)}\\&\quad+2(q^n-1)\sum_{k=1}^{n-1}\left(\frac{n-k-1}{3}\right)\frac{(-1)^kq^{\frac{1}{3}\left(2(n-k)^2-(n-k)\left(\frac{n-k-1}{3}\right)\right)-\frac{k(k-1)}{2}}}{1-q^k}\pmod{\Phi_n(q)^2}.\end{aligned}\tag{15}$$

Furthermore,

$$\sum_{k=1}^{n-1}\left(\frac{n-k-1}{3}\right)\frac{(-1)^k q^{\frac{1}{3}\left(2(n-k)^2-(n-k)\left(\frac{n-k-1}{3}\right)\right)-\frac{k(k-1)}{2}}}{1-q^k}$$
$$=\sum_{k=1}^{n-1}\left(\frac{k-1}{3}\right)\frac{(-1)^{n-k} q^{\frac{1}{3}\left(2k^2-k\left(\frac{k-1}{3}\right)\right)-\frac{(n-k)(n-k-1)}{2}}}{1-q^{n-k}}$$
$$=\sum_{k=1}^{n-1}\left(\frac{k-1}{3}\right)\frac{(-1)^{n-k} q^{\frac{1}{3}\left(2k^2-k\left(\frac{k-1}{3}\right)\right)-\frac{n(n-1)}{2}-\frac{k(k+1)}{2}+nk}}{1-q^{n-k}}$$
$$\equiv\sum_{k=1}^{n-1}\left(\frac{k-1}{3}\right)\frac{(-1)^{k} q^{\frac{1}{3}\left(2k^2-k\left(\frac{k-1}{3}\right)\right)-\frac{k(k-1)}{2}}}{1-q^{k}} \pmod{\Phi_n(q)},$$

where we set $k \to n-k$ in the first step. Thus,

$$\sum_{k=0}^{n-1} q^{k+1}\begin{bmatrix}2k\\k+1\end{bmatrix} \equiv \left(\frac{n-1}{3}\right) q^{\frac{1}{3}\left(2n^2-n\left(\frac{n-1}{3}\right)\right)} + 2(q^n-1)\sum_{k=1}^{n-1}\left(\frac{k-1}{3}\right)\frac{(-1)^{k} q^{\frac{1}{3}\left(2k^2-k\left(\frac{k-1}{3}\right)\right)-\frac{k(k-1)}{2}}}{1-q^{k}} \pmod{\Phi_n(q)^2}. \quad (16)$$

We complete the proof of (6) by combining (7) and (16).

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