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The Monotonicity of the Principal Frequency of the Anisotropic p-Laplacian

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Abstract. Let $D > 1$ be a fixed integer. Given a smooth bounded, convex domain $\Omega \subset \mathbb{R}^D$ and $H : \mathbb{R}^D \to [0,\infty)$ a convex, even, and 1-homogeneous function of class $C^{3,\alpha}(\mathbb{R}^D \setminus \{0\})$ for which the Hessian matrix $D^2(H^p)$ is positive definite in $\mathbb{R}^D \setminus \{0\}$ for any $p \in (1,\infty)$, we study the monotonicity of the principal frequency of the anisotropic p-Laplacian (constructed using the function H) on Ω with respect to $p \in (1,\infty)$. As an application, we find a new variational characterization for the principal frequency on domains Ω having a sufficiently small inradius. In the particular case where H is the Euclidean norm in $\mathbb{R}^D$, we recover some recent results obtained by the first two authors in [3, 4].

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1. Introduction and main results

For each positive integer D let $\mathscr{E}^D$ be the Euclidean norm in $\mathbb{R}^D$. We define the set $\mathscr{H}^D$ as follows: if $D = 1$, $\mathscr{H}^1 := \{\mathscr{E}^1\}$; if $D \geq 2$ we let $\mathscr{H}^D$ be the family of all maps $H : \mathbb{R}^D \to [0,\infty)$ which are convex, even, 1-homogeneous, and of class $C^{3,\alpha}(\mathbb{R}^D \setminus \{0\})$ such that the Hessian matrix $D^2(H^p)$ is

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positive definite in $\mathbb{R}^D \setminus \{0\}$ for all $p \in (1,\infty)$. For $D \geq 2$ and $H \in \mathscr{H}^D$, let $H^\circ : \mathbb{R}^D \to [0,\infty)$ be the polar function of H, defined by

$$H^\circ(\eta) := \sup_{\xi \in \mathbb{R}^D \setminus \{0\}} \frac{\langle \xi, \eta \rangle}{H(\xi)}, \quad \eta \in \mathbb{R}^D.$$

Next, for each positive integer D and $H \in \mathscr{H}^D$, define

$$\mathscr{P}^{D,H} := \left\{ \Omega \subset \mathbb{R}^D \,\middle|\, \Omega \text{ is a } C^2, \text{bounded, convex domain with nonnegative anisotropic mean curvature} \right\}.$$

For $\Omega \in \mathscr{P}^{D,H}$ let $\delta_{H,\Omega} : \Omega \to [0,\infty)$ be the anisotropic distance function to the boundary of Ω, given by

$$\delta_{H,\Omega}(x) := \inf_{y \in \partial\Omega} H^\circ(x-y), \quad x \in \Omega.$$

Further, for $\Omega \in \mathscr{P}^{D,H}$ and $s > 0$, define

$$\mathscr{P}^{D,H}(s) := \left\{ \Omega \in \mathscr{P}^{D,H} : \|\delta_{H,\Omega}\|_{L^\infty(\Omega)} = s \right\}.$$

Finally, for $\Omega \in \mathscr{P}^{D,H}$ and $p \in (1,\infty)$, we define the principal Dirichlet frequency of the anisotropic p-Laplacian by

$$\lambda_H\big(p;\Omega\big) := \min_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega H(\nabla u)^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}. \tag{1}$$

It is known (see e.g. Belloni, Ferone & Kawohl [1] or Belloni, Kawohl & Juutinen [2]) that $\lambda_H(p;\Omega)$ is the lowest eigenvalue λ of the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{D} \frac{\partial}{\partial x_i} \left[H(\nabla u)^{p-2} \mathscr{K}_i(\nabla u) \right] = \lambda |u|^{p-2} u & \text{if } x \in \Omega \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{2}$$

where $\mathscr{K}_i(\xi) := \frac{\partial}{\partial \xi_i}(\frac{1}{2} H(\xi)^2)$, for all $\xi \in \mathbb{R}^D$ and $i \in \{1, \cdots, D\}$. In the particular case when $H = \mathscr{E}^D$, the differential operator involved in the eigenvalue problem (2) reduces to the classical p-Laplace operator Δ_p. For this reason, (2) is called the *eigenvalue problem for the anisotropic p-Laplacian.* The main goal of this paper is to analyze the monotonicity of the function $p \mapsto \lambda_H(p;\Omega)$ with respect to $p \in (1,\infty)$ for given $H \in \mathscr{H}^D$ and $\Omega \in \mathscr{P}^{D,H}$.

When $D = 1$ and $\Omega = (a,b)$ with $a, b \in \mathbb{R}$, it is well known (see [11]) that the principal frequency of the p-Laplacian ($H = \mathscr{E}^1$) is given by the explicit formula

$$\lambda_{\mathscr{E}^1}\big(p;(a,b)\big) = (p-1)\left(\frac{2}{b-a}\right)^p \left(\frac{\pi/p}{\sin\big(\pi/p\big)}\right)^p.$$

It can be shown that when $\frac{b-a}{2} \in (1,\infty)$ there exists $p^\star = p^\star(\frac{b-a}{2}) \in (1,\infty)$ such that $p \mapsto \lambda_{\mathscr{E}^1}(p;(a,b))$ is increasing on $(1,p^\star)$ and decreasing on $(p^\star,\infty)$ (see, Kajikiya, Tanaka & Tanaka [10, Theorem 1.1 (ii)]). On the other hand, it is easy to check that if $\frac{b-a}{2} \leq 1$ then the map $p \mapsto \lambda_{\mathscr{E}^1}(p;(a,b))$ is increasing on $(1,\infty)$ (see, Kajikiya, Tanaka & Tanaka [10, Theorem 1.1 (i)]). A similar result was established in the case when $D \geq 2$ and $H = \mathscr{E}^D$ by the first two authors of this paper in [4, Theorem 1]. Our main goal here is to show that the results obtained in [4] continue to hold in the anisotropic case where $H \in \mathscr{H}^D$ is a general function as described above. Our main result is stated in the following theorem.

Theorem 1. *Let $D \geq 2$ and $H \in \mathscr{H}^D$ be fixed, and let M_H be defined by*

$$M_H := \sup\left\{ s > 0 \,\middle|\, \lambda_H(p;\Omega) < \lambda_H(q;\Omega) \ \forall\ 1 < p < q < \infty \quad \text{and} \quad \Omega \in \mathscr{P}^{D,H}(s) \right\}. \tag{3}$$

Then $M_H \in [e^{-1},1]$, and $\lambda_H(p;\Omega) \leq \lambda_H(q;\Omega)$ whenever $1 < p < q < \infty$ and $\Omega \in \mathscr{P}^{D,H}(M_H)$. Moreover, for any $s > M_H$ there exists a domain $\Omega \in \mathscr{P}^{D,H}(s)$ for which the map $(1,+\infty) \ni p \mapsto \lambda_H(p;\Omega)$ is not monotone.

Next, using Theorem 1, we obtain a new variational characterization of $\lambda_H(p;\Omega)$ for domains $\Omega \in \mathscr{P}^{D,H}(s)$ with $s \in (0, M_H]$, where M_H is defined by (3). Precisely, we prove

Theorem 2. *Let $D \geq 2$ and $H \in \mathscr{H}^D$ be fixed. For each $\Omega \in \mathscr{P}^{D,H}$ and $p \in (1,\infty)$, define*

$$\Lambda_H(p;\Omega) := \inf_{u \in X_0 \setminus \{0\}} \frac{\displaystyle\int_\Omega \left[\exp\left(H(\nabla u)^p\right) - 1\right]\, dx}{\displaystyle\int_\Omega \left[\exp\left(|u|^p\right) - 1\right]\, dx},$$

where

$$X_0 := W^{1,\infty}(\Omega) \cap \left(\cap_{q>1} W_0^{1,q}(\Omega)\right).$$

If $\|\delta_{H,\Omega}\|_{L^\infty(\Omega)} > 1$ then $\Lambda_H(p;\Omega) = 0$, while if $\|\delta_{H,\Omega}\|_{L^\infty(\Omega)} \leq 1$ we have $\Lambda_H(p;\Omega) > 0$. Moreover, if $\|\delta_{H,\Omega}\|_{L^\infty(\Omega)} \leq M_H$, with M_H defined by (3), then $\Lambda_H(p;\Omega) = \lambda_H(p;\Omega)$.

Note that a similar result was proved in [3, Theorem 2] in the particular case $H = \mathscr{E}^D$.

2. The principal frequency of the anisotropic p-Laplacian

In this section we recall a number of known properties of $\lambda_H(p;\Omega)$ (with $H \in \mathscr{H}^D$, $p \in (1,\infty)$ and $D \geq 2$) that will be useful in the sequel. We begin with a result of Belloni, Kawohl & Juutinen [2] (see also Juutinen, Lindqvist & Manfredi [9] or Fukagai, Ito & Narukawa [7] for the case where $H = \mathscr{E}^D$). Let $\lambda_H(\infty;\Omega)$ be defined by

$$\lambda_H(\infty;\Omega) := \min\left\{ \frac{\left\|H(\nabla\varphi)\right\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^\infty(\Omega)}} \;\middle|\; \varphi \in X_0 \setminus \{0\} \right\}, \tag{4}$$

where $X_0 := W^{1,\infty}(\Omega) \cap (\cap_{q>1} W_0^{1,q}(\Omega))$. Then, the minimum in (4) is always achieved at $\delta_{H,\Omega}$, the anisotropic distance function to the boundary of Ω, and

$$\lambda_H(\infty;\Omega) = \left\|\delta_{H,\Omega}\right\|_{L^\infty(\Omega)}^{-1}. \tag{5}$$

Moreover, by [2, Lemma 3.1], we have

$$\lim_{p\to\infty} \sqrt[p]{\lambda_H(p;\Omega)} = \lambda_H(\infty;\Omega) = \left\|\delta_{H,\Omega}\right\|_{L^\infty(\Omega)}^{-1}. \tag{6}$$

It is also well known that $\lambda_H(\infty;\Omega) = R_H(\Omega)^{-1}$, where $R_H(\Omega)$ stands for the anisotropic inradius of Ω with respect to $H \in \mathscr{H}^D$. The following lower bound for $\lambda_H(p;\Omega)$ is due to Della Pietra, di Blasio, and Gavitone (see [5, Theorem 5.1]).

$$\frac{p-1}{R_H(\Omega)^p}\left(\frac{\pi/p}{\sin(\pi/p)}\right)^p \leq \lambda_H(p;\Omega) \quad \forall\ p > 1. \tag{7}$$

Note that the left-hand side in (7) is exactly $\lambda_{\mathscr{E}^1}(p;(-R_H(\Omega), R_H(\Omega)))$, i.e. the principal frequency of the p-Laplacian on the interval $(-R_H(\Omega), R_H(\Omega))$ when $D = 1$. Moreover, by [5, Theorem 5.8], equality in (7) is achieved when Ω approaches a suitable infinite slab. More precisely, if we define, for $k > 0$, $\Omega(k) := (-R_H(\Omega)(H^\circ(e_1))^{-1}, R_H(\Omega)(H^\circ(e_1))^{-1}) \times (-k,k)^{D-1}$, then

$$\lim_{k\to\infty} \lambda_H\left(p;\Omega(k)\right) = \frac{p-1}{R_H(\Omega)^p}\left(\frac{\pi/p}{\sin\left(\pi/p\right)}\right)^p \quad \forall\ p > 1. \tag{8}$$

By [5, Proposition 2.2 (iii)] (see also [6, Proposition 3.3 (iii)] or [12, Theorem 3.2] for the case $H = \mathscr{E}^D$)

$$p\sqrt[p]{\lambda_H\big(p;\Omega\big)} \le q\sqrt[q]{\lambda_H\big(q;\Omega\big)}, \quad \forall\ 1 < p < q < \infty. \tag{9}$$

Finally, note that for each $R > 0$, considering the rescaled domain $\Omega_R := R\Omega = \{Rx \mid x \in \Omega\}$, we have (see, e.g., [5, Proposition 2.2 (i)] or [6, Proposition 3.3 (v)])

$$\lambda_H\big(p;\Omega_R\big) = R^{-p}\lambda_H\big(p;\Omega\big) \quad \forall\ p > 1. \tag{10}$$

Moreover, it is easy to check that in this case $\|\delta_{H,\Omega_R}\|_{L^\infty(\Omega_R)} = R\|\delta_{H,\Omega}\|_{L^\infty(\Omega)}$.

3. Proof of the main results

3.1. *Proof of Theorem 1*

First, in view of [10, Theorem 1.1 (ii)] and [5, Theorem 5.8] we have the following result.

Proposition 3. *Let $D \ge 2$ and $H \in \mathscr{H}^D$. For any $s \in (1,\infty)$, there exists a domain $\Omega \in \mathscr{P}^{D,H}(s)$ for which the function $p \mapsto \lambda_H(p;\Omega)$ is not monotone on $(1,\infty)$.*

Proof. We start by observing that since $s > 1$, (6) yields

$$\lim_{p\to\infty} \lambda_H\big(p;\Omega\big) = 0, \quad \forall\ \Omega \in \mathscr{P}^{D,H}(s). \tag{11}$$

For each $k > 0$, define $\Omega_s(k) := (-s(H^\circ(e_1))^{-1}, s(H^\circ(e_1))^{-1}) \times (-k,k)^{D-1}$. It is clear that $\Omega_s(k) \in \mathscr{P}^{D,H}(s)$ whenever $k \ge s$ is sufficiently large. By (8), we know that

$$\lambda_{\mathscr{E}^1}\big(p;(-s,s)\big) = \lim_{k\to\infty} \lambda_H\big(p;\Omega_s(k)\big) \quad \forall\ p > 1. \tag{12}$$

Moreover, [10, Theorem 1.1 (ii)] guarantees that the function $\lambda_{\mathscr{E}^1}(\cdot;(-s,s))$ is not monotone.

We claim that there exists $k \ge s$ for which the function $\lambda_H(\cdot;\Omega_s(k))$ is not monotone and prove this by contradiction. Thus, let us assume that $\lambda_H(\cdot;\Omega_s(k))$ is monotone for every $k \ge s$ sufficiently large. In view of (11), $\lambda_H(\cdot;\Omega_s(k))$ must be non-increasing, so that

$$\lambda_H\big(p;\Omega_s(k)\big) \ge \lambda_H\big(q;\Omega_s(k)\big) \quad \forall\ 1 < p < q < \infty \quad \text{and} \quad k > s \text{ sufficiently large.}$$

Letting $k \to \infty$ and taking (12) into account we obtain

$$\lambda_{\mathscr{E}^1}\big(p;(-s,s)\big) \ge \lambda_{\mathscr{E}^1}\big(q;(-s,s)\big) \quad \forall\ 1 < p < q < \infty,$$

which contradicts the fact that $\lambda_{\mathscr{E}^1}(\cdot;(-s,s))$ is not monotone. This concludes the proof of Proposition 3. □

Proposition 4. *Let $\Omega \in \mathscr{P}^{D,H}(s)$ with $s \in (0,e^{-1}]$. Then $\lambda_H(p;\Omega)$ is strictly increasing as a function of p on $(1,\infty)$.*

The proof of Proposition 4 follows from the next two lemmas.

Lemma 5. *Let $\Omega \in \mathscr{P}^{D,H}$ and suppose that $\lambda_H(q;\Omega) \le \lambda_H(p;\Omega)$ for some $1 < p < q < \infty$. Then*

$$\lambda_H\big(p;\Omega\big) < e^q.$$

Proof. Combining the hypothesis with (9), we have

$$p\sqrt[p]{\lambda_H\big(p;\Omega\big)} \le q\sqrt[q]{\lambda_H\big(q;\Omega)\big)} \le q\sqrt[q]{\lambda_H\big(p;\Omega)\big)}.$$

Consequently,

$$\lambda_H\big(p;\Omega\big) \le \left(\frac{q}{p}\right)^{\frac{pq}{q-p}} = x^{\frac{q}{x-1}}, \quad \text{with} \quad x := \frac{q}{p} > 1.$$

Since $x^{1/(x-1)} < e$, the result follows (note that the function $t \mapsto t^{1/(t-1)}$ is strictly decreasing on $(1,\infty)$ and $\lim_{t\to 1^+} t^{1/(t-1)} = e$). □

Lemma 6. *Let* $\Omega \in \mathscr{P}^{D,H}(s)$ *with* $s > 0$. *Then*

$$\frac{1}{s^p} < \lambda_H\big(p;\Omega\big) \quad \forall\ p > 1.$$

Proof. Observing that

$$(p-1)\left(\frac{\pi/p}{\sin(\pi/p)}\right)^p = \lambda_{\mathscr{E}^1}\big(p;(-1,1)\big),$$

(7) can be rewritten as

$$\frac{\lambda_{\mathscr{E}^1}\big(p;(-1,1)\big)}{s^p} \le \lambda_H\big(p;\Omega\big) \quad \forall\ p > 1.$$

Hence, using the fact that $\lambda_{\mathscr{E}^1}(\cdot;(-1,1))$ is strictly increasing and $\lim_{p\to 1^+}\lambda_{\mathscr{E}^1}(p;(-1,1)) = 1$ (see [10, Theorem 1.1 (i)] and [8, Theorem 3.3], respectively), we obtain

$$\frac{1}{s^p} < \frac{\lambda_{\mathscr{E}^1}\big(p;(-1,1)\big)}{s^p} \le \lambda_H\big(p;\Omega\big) \quad \forall\ p > 1.$$

□

Having proven Lemmas 5 and 6, the conclusion of Proposition 4 is now immediate. Indeed, assume by contradiction that there exists $1 < p < q < \infty$ such that $\lambda_H(q;\Omega) \le \lambda_H(p;\Omega)$. Then, combining the inequalities in Lemmas 5 and 6, we have

$$\frac{1}{s^q} < \lambda_H\big(q;\Omega\big) \le \lambda_H\big(p;\Omega\big) < e^q,$$

which leads to the contradiction $s > e^{-1}$. This concludes the proof.

Lemma 7. *If for some* $r \in (0,1]$ *and any domain* $\Omega \in \mathscr{P}^{D,H}(r)$ *we have*

$$\lambda_H\big(p;\Omega\big) \le \lambda_H\big(q;\Omega\big) \quad \forall\ 1 < p < q < \infty,$$

then for any $s \in (0,r)$ *and any* $\Omega \in \mathscr{P}^{D,H}(s)$ *we also have*

$$\lambda_H\big(p;\Omega\big) < \lambda_H\big(q;\Omega\big) \quad \forall\ 1 < p < q < \infty.$$

Proof. Indeed, if $\Omega \in \mathscr{P}^{D,H}(r)$ then for each $R \in (0,1)$ we have $\|\delta_{H,\Omega_R}\|_{L^\infty(\Omega_R)} = Rr < r$ and, in view of (10), we get that

$$\lambda_H\big(p;\Omega_R\big) = \frac{1}{R^p}\lambda_H\big(p;\Omega\big) \quad \forall\ 1 < p < \infty.$$

Now, fix $s \in (0,r)$ and take $R := s/r \in (0,1)$. If $\Omega \in \mathscr{P}^{D,H}(s)$ then $\Omega_{R^{-1}} \in \mathscr{P}^{D,H}(r)$ and, consequently,

$$\lambda_H\big(p;\Omega\big) = \frac{1}{R^p}\lambda_H\big(p;\Omega_{R^{-1}}\big) \quad \forall\ 1 < p < \infty.$$

But

$$\lambda_H\big(p;\Omega_{R^{-1}}\big) \le \lambda_H\big(q;\Omega_{R^{-1}}\big) \quad \forall\ 1 < p < q < \infty,$$

and since $R \in (0,1)$, we deduce that

$$\frac{1}{R^p}\lambda_H\big(p;\Omega_{R^{-1}}\big) < \frac{1}{R^q}\lambda_H\big(q;\Omega_{R^{-1}}\big) \quad \forall\ 1 < p < q < \infty.$$

Equivalently, $\lambda_H(p;\Omega) < \lambda_H(q;\Omega) \quad \forall\ 1 < p < q < \infty$. □

We are now ready to complete the proof of Theorem 1. Let $D \ge 2$ be a fixed integer, $H \in \mathscr{H}^D$, and M_H be defined by (3). In view of Proposition 4, we have that $M_H \in [e^{-1},1]$. If $M_H = 1$, then by Proposition 3 and the definition of M_H it follows that for any $s > M_H = 1$ there exists a domain $\Omega \in \mathscr{P}^{D,H}(s)$ for which the function $\lambda_H(p;\Omega)$ is not monotone in p on the interval $(1,\infty)$. This conclusion is still valid in the case where $M_H \in [e^{-1},1)$. Indeed, if $M_H \in [e^{-1},1)$ and $s \in (M_H,1]$ then, if $\lambda_H(\cdot;\Omega)$ were monotone for every $\Omega \in \mathscr{P}^{D,H}(s)$, one would have (noting that $\lambda_H(p;\Omega) \to \infty$ as $p \to \infty$, since $s \le 1$) that $\lambda_H(\cdot;\Omega)$ must be nondecreasing (but not necessarily increasing). Hence, by fixing $r \in (M_H, s)$ and applying Lemma 7, one can show that $\lambda_H(\cdot;\Omega)$ is strictly increasing for every $\Omega \in \mathscr{P}^{D,H}(r)$. This contradicts the definition of M_H.

Up to this point we have shown that for any $r \in (0, M_H)$ and any domain $\Omega \in \mathscr{P}^{D,H}(r)$ we have $\lambda_H(p;\Omega) < \lambda_H(q;\Omega) \quad \forall\ 1 < p < q < \infty$. To finish the proof of Theorem 1 it remains to prove that we still have $\lambda_H(p;\Omega) \le \lambda_H(q;\Omega) \ \ \forall\ 1 < p < q < \infty$ when $\Omega \in \mathscr{P}^{D,H}(M_H)$. To justify this, note that for any $R \in (0,1)$, $\|\delta_{H,\Omega_R}\|_{L^\infty(\Omega_R)} = RM_H \in (0, M_H)$ and hence $\lambda_H(p;\Omega_R) < \lambda_H(q;\Omega_R) \quad \forall\ 1 < p < q < \infty$, or, by virtue of (10),

$$\frac{1}{R^p}\lambda_H\left(p;\Omega\right) < \frac{1}{R^q}\lambda_H\left(q;\Omega\right) \quad \forall\ 1 < p < q < \infty.$$

Letting $R \nearrow 1$ we are led to $\lambda_H(p;\Omega) \le \lambda_H(q;\Omega) \quad \forall\ 1 < p < q < \infty$, as claimed.

3.2. *Proof of Theorem 2*

The proof follows from the following lemmas.

Lemma 8. *If* $\|\delta_{H,\Omega}\|_{L^\infty(\Omega)} > 1$ *then* $\Lambda_H(p;\Omega) = 0$ *for all* $p \in (1,\infty)$.

Proof. Let $\epsilon_0 > 0$ and ω be an open subset of Ω having positive Lebesgue measure $|\omega| > 0$, such that $\delta_{H,\Omega}(x) \ge 1 + \epsilon_0$ for any $x \in \omega$. Since $\delta_{H,\Omega} \in X_0 \setminus \{0\}$,

$$\Lambda_H(p;\Omega) \le \frac{\displaystyle\int_\Omega \left[\exp\left(H\left(\nabla\left(n\delta_{H,\Omega}\right)\right)^p\right) - 1\right]\, dx}{\displaystyle\int_\Omega \left[\exp\left(\left(n\delta_{H,\Omega}\right)^p\right) - 1\right]\, dx} \quad \forall\ n \ge 1. \tag{13}$$

Taking into account the fact that $H(\nabla\delta_{H,\Omega}(x)) = 1$ for a.e. $x \in \Omega$, we have

$$\begin{aligned}
\frac{\displaystyle\int_\Omega \left[\exp\left(H\left(\nabla\left(n\delta_{H,\Omega}\right)\right)^p\right) - 1\right]\, dx}{\displaystyle\int_\Omega \left[\exp\left(\left(n\delta_{H,\Omega}\right)^p\right) - 1\right]\, dx} &= \frac{|\Omega|\left[\exp\left(n^p\right) - 1\right]}{\displaystyle\int_\Omega \left[\exp\left(n^p\delta_{H,\Omega}(x)^p\right) - 1\right]\, dx} \\
&\le \frac{|\Omega|\left[\exp\left(n^p\right) - 1\right]}{\displaystyle\int_\omega \left[\exp\left(n^p\delta_{H,\Omega}(x)^p\right) - 1\right]\, dx} \\
&\le \frac{|\Omega|\left[\exp\left(n^p\right) - 1\right]}{|\omega|\left[\exp\left(n^p\left(1+\epsilon_0\right)^p\right)\right) - 1\right]}
\end{aligned}$$

for every integer $n \ge 1$. Letting $n \to \infty$ in (13) gives $\Lambda_H(p;\Omega) = 0$. □

Lemma 9. *If* $\|\delta_{H,\Omega}\|_{L^\infty(\Omega)} \in (0,1]$ *then* $\Lambda_H(p;\Omega) > 0$ *for all* $p \in (1,\infty)$.

Proof. First, we claim that

$$\Lambda_H(p;\Omega) \ge \inf_{k\in\mathbb{N}\setminus\{0\}} \lambda_H(kp;\Omega). \tag{14}$$

Indeed, recall that the definition of $\lambda_H(kp;\Omega)$ implies that if $u \in X_0 \setminus \{0\}$ (which, in particular, means that $u \in W_0^{1,q}(\Omega) \setminus \{0\}$ for any $q > 1$) then

$$\begin{aligned}
\frac{\displaystyle\sum_{k=1}^\infty \frac{1}{k!}\int_\Omega H(\nabla u)^{kp}\, dx}{\displaystyle\sum_{k=1}^\infty \frac{1}{k!}\int_\Omega |u|^{kp}\, dx} &\ge \frac{\displaystyle\sum_{k=1}^\infty \frac{\lambda_H\left(kp;\Omega\right)}{k!}\int_\Omega |u|^{kp}\, dx}{\displaystyle\sum_{k=1}^\infty \frac{1}{k!}\int_\Omega |u|^{kp}\, dx} \\
&\ge \inf_{k\in\mathbb{N}\setminus\{0\}} \lambda_H\left(kp;\Omega\right).
\end{aligned}$$

Passing to the infimum over all $u \in X_0 \setminus \{0\}$ gives (14).

Next, in view of (7), and since $R_H(\Omega) = \|\delta_{H,\Omega}\|_{L^\infty(\Omega)}$, we arrive at

$$\frac{q-1}{\|\delta_{H,\Omega}\|^q_{L^\infty(\Omega)}}\left(\frac{\pi/q}{\sin(\pi/q)}\right)^q \leq \lambda_H(q;\Omega) \quad \forall\ q\in(1,\infty). \tag{15}$$

Further, recall the fact that the function

$$(1,\infty)\ni p\mapsto (p-1)\left(\frac{\pi/p}{\sin(\pi/p)}\right)^p,$$

is increasing (see, e.g. [10, Theorem 1.1 (i)] for the proof). Taking into account (15) and the fact that $\|\delta_{H,\Omega}\|_{L^\infty(\Omega)}\in(0,1]$, we obtain

$$0<(p-1)\left(\frac{\pi/p}{\sin(\pi/p)}\right)^p \leq \lambda_H(kp;\Omega) \quad \forall\ k\in\mathbb{N}\setminus\{0\} \text{ and } p\in(1,\infty).$$

Thus, by (14), $\Lambda_H(p;\Omega)>0$ for any $p\in(1,\infty)$. This concludes the proof of Lemma 9. □

Lemma 10. *If M_H is defined by* (3) *and* $\|\delta_{H,\Omega}\|_{L^\infty(\Omega)}\in(0,M_H]$, *then* $\Lambda_H(p;\Omega)=\lambda_H(p;\Omega)$ *for all* $p\in(1,\infty)$.

Proof. First, we show that $\Lambda_H(p;\Omega)\leq\lambda_H(p;\Omega)$ for all $p\in(1,\infty)$. To this aim, first note that

$$\Lambda_H(p;\Omega)\leq\frac{\int_\Omega\left[\exp\left(H(\nabla(tu))^p\right)-1\right]\,dx}{\int_\Omega\left[\exp\left(|tu|^p\right)-1\right]\,dx} \quad \forall\ u\in C_0^\infty(\Omega)\setminus\{0\}\subset X_0\setminus\{0\} \quad\text{and}\quad t\in(0,1).$$

Hence,

$$\Lambda_H(p;\Omega)\leq\frac{\sum_{k=1}^\infty\int_\Omega\frac{H(\nabla(tu))^{kp}}{k!}\,dx}{\sum_{k=1}^\infty\int_\Omega\frac{|tu|^{kp}}{k!}\,dx}=\frac{\int_\Omega H(\nabla u)^p\,dx+\sum_{k=2}^\infty t^{(k-1)p}\int_\Omega\frac{H(\nabla u)^{kp}}{k!}\,dx}{\int_\Omega|u|^p\,dx+\sum_{k=2}^\infty t^{(k-1)p}\int_\Omega\frac{|u|^{kp}}{k!}\,dx}$$

for any $u\in C_0^\infty(\Omega)\setminus\{0\}$ and $t\in(0,1)$. Letting $t\to0^+$ in the above inequality we get

$$\Lambda_H(p;\Omega)\leq\frac{\int_\Omega H(\nabla u)^p\,dx}{\int_\Omega|u|^p\,dx} \quad \forall\ u\in C_0^\infty(\Omega)\setminus\{0\}.$$

We obtain

$$\Lambda_H(p;\Omega)\leq\lambda_H(p;\Omega) \quad \forall\ p\in(1,\infty), \tag{16}$$

as claimed.

Next, taking into account that $\|\delta_{H,\Omega}\|_{L^\infty(\Omega)}\in(0,M_H]$ by Theorem 1 we deduce that $\lambda_H(p;\Omega)\leq\lambda_H(q;\Omega)$ whenever $1<p<q<\infty$. Combining this with (14) we are led to

$$\lambda_H(p;\Omega)\leq\Lambda_H(p;\Omega) \quad \forall\ p\in(1,\infty). \tag{17}$$

The conclusion of Lemma 10 now follows from (16) and (17). □

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