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On a Pólya's inequality for planar convex sets

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Abstract. In this short note, we prove that for every bounded, planar and convex set Ω, one has

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \leq \frac{\pi^2}{12}\cdot\left(1+\sqrt{\pi}\frac{r(\Omega)}{\sqrt{|\Omega|}}\right)^2,$$

where λ_1, T, r and $|\cdot|$ are the first Dirichlet eigenvalue, the torsion, the inradius and the volume. The inequality is sharp as the equality asymptotically holds for any family of thin collapsing rectangles.

As a byproduct, we obtain the following bound for planar convex sets

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \leq \frac{\pi^2}{12}\left(1+\frac{2\sqrt{2(6+\pi^2)}-\pi^2}{4+\pi^2}\right)^2 \approx 0.996613\ldots$$

which improves Polyá's inequality $\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} < 1$ and is slightly better than the one provided in [3].

The novel ingredient of the proof is the sharp inequality

$$\lambda_1(\Omega) \leq \frac{\pi^2}{4}\cdot\left(\frac{1}{r(\Omega)}+\sqrt{\frac{\pi}{|\Omega|}}\right)^2,$$

recently proved in [8].

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1. Introduction

Let Ω be an open set in $\mathbb{R}^n$ with finite Lebesgue measure $|\Omega|$. We denote

$$T(\Omega) := \sup_{w\in H_0^1(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega w\,\mathrm{d}x\right)^2}{\int_\Omega |\nabla w|^2\mathrm{d}x}$$

the torsion of the set Ω, and

$$\lambda_1(\Omega) := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\mathrm{d}x}{\int_\Omega u^2\mathrm{d}x}$$

the fundamental frequency of Ω which corresponds to the first eigenvalue of the Laplace operator with Dirichlet boundary condition of the set Ω.

In 1951, G. Pólya [13] proved the following inequality:

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \le 1,$$

which was recently improved by M. van den Berg et al. in [3], where the authors prove the inequality

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \le 1 - \frac{2n\omega_n^{2/n}}{n+2}\frac{T(\Omega)}{|\Omega|^{1+\frac{2}{n}}}, \tag{1}$$

where ω_n is the measure of the ball with radius 1 in $\mathbb{R}^n$. They also show, by using homogenization arguments, that the upper bound 1 is optimal in the class of open sets with finite measure, see [3, Theorem 1.2], but is not optimal for the class of bounded, *planar* and convex sets. Indeed, in this case they obtain the following improved estimate

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \le 1 - \frac{1}{11560} \approx 0.999913\ldots \tag{2}$$

see [3, Theorem 1.5]. The optimal upper bound is conjectured to be given by $\frac{\pi^2}{12}$, which is asymptotically attained by any sequence of thin collapsing rectangles.

Let us now state the main results of the present note.

Theorem 1. *We have for every bounded planar convex set* Ω*:*

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \le \frac{\pi^2}{12}\cdot\left(1+\sqrt{\pi}\frac{r(\Omega)}{\sqrt{|\Omega|}}\right)^2. \tag{3}$$

The inequality is sharp as it is asymptotically attained by any family of thin collapsing rectangles. Moreover, for every sequence (Ω_k) *of planar convex sets such that* $|\Omega_k| = 1$ *and* $\lim_{k\to+\infty} d(\Omega_k) = +\infty$*, we have*

$$\limsup_{k\to+\infty} \frac{\lambda_1(\Omega_k)T(\Omega_k)}{|\Omega_k|} \le \frac{\pi^2}{12}.$$

As a byproduct, we obtain the following slight improvement of inequality (2):

Corollary 2. *We have for every bounded planar convex set* Ω*:*

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \le \frac{\pi^2}{12}\left(1+\frac{2\sqrt{2(6+\pi^2)}-\pi^2}{4+\pi^2}\right)^2 \approx 0.996613\ldots \tag{4}$$

We provide the complete proofs of Theorem 1 and Corollary 2 in Section 2 and state some comments in Section 3.

2. Proofs of The main results

2.1. *Proof of Theorem 1*

Let Ω be a bounded, planar and convex set. We combine the following inequality proved in [8]:

$$\lambda_1(\Omega) < \frac{\pi^2}{4}\cdot\left(\frac{1}{r(\Omega)}+\sqrt{\frac{\pi}{|\Omega|}}\right)^2, \tag{5}$$

with the following Makai's inequality [11]:

$$T(\Omega) < \frac{1}{3}|\Omega| r(\Omega)^2.$$

We have

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} < \frac{1}{|\Omega|}\times\frac{\pi^2}{4}\cdot\left(\frac{1}{r(\Omega)}+\sqrt{\frac{\pi}{|\Omega|}}\right)^2\times\frac{1}{3}|\Omega| r(\Omega)^2 = \frac{\pi^2}{12}\cdot\left(1+\sqrt{\pi}\frac{r(\Omega)}{\sqrt{|\Omega|}}\right)^2.$$

The inequality is sharp as it is an asymptotically an equality for any family of thin vanishing rectangles, see [2, Proposition 3.2].

Let us now consider (Ω_k) a sequence of planar convex sets such that $|\Omega_k| = 1$ and $\lim_{k\to+\infty} d(\Omega_k) = +\infty$. We have for every $k \in \mathbb{N}$:

$$0 < \frac{r(\Omega_k)}{\sqrt{|\Omega_k|}} \le \frac{1}{\sqrt{|\Omega_k|}} \frac{2|\Omega_k|}{P(\Omega_k)} \le \frac{2\sqrt{|\Omega_k|}}{2d(\Omega_k)} = \frac{1}{d(\Omega_k)} \underset{k\to+\infty}{\longrightarrow} 0,$$

where we used the classical inequalities $r(\Omega_k) \le \frac{2|\Omega_k|}{P(\Omega_k)}$ (see [5]) and $2d(\Omega_k) \le P(\Omega_k)$.

We then conclude by using inequality (3) that

$$\limsup_{k\to+\infty} \frac{\lambda_1(\Omega_k)T(\Omega_k)}{|\Omega_k|} \le \frac{\pi^2}{12}.$$

2.2. *Proof of Corollary 2*

Let us prove the inequality

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} < 1 - \frac{\pi}{\frac{\pi^2}{4}\left(\frac{\sqrt{|\Omega|}}{r(\Omega)} + \sqrt{\pi}\right)^2 + \pi}. \tag{6}$$

We have by inequality (1) (see [3, Theorem 1.1]):

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \le 1 - \pi\frac{T(\Omega)}{|\Omega|^2},$$

which is equivalent to the inequality:

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \le 1 - \frac{\pi}{|\Omega|\lambda_1(\Omega) + \pi}.$$

Indeed:

$$\begin{aligned}\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \le 1 - \pi\frac{T(\Omega)}{|\Omega|^2} &\iff \frac{T(\Omega)}{|\Omega|}\left(\lambda_1(\Omega) + \frac{\pi}{|\Omega|}\right) < 1\\ &\iff \frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} < \frac{\lambda_1(\Omega)}{\lambda_1(\Omega) + \frac{\pi}{|\Omega|}} = 1 - \frac{\pi}{|\Omega|\lambda_1(\Omega) + \pi}.\end{aligned}$$

Thus, by using the inequality (5), we obtain the inequality (6).

We then combine the inequalities (3) and (6) to conclude:

$$\begin{aligned}\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} &\le \min\left(\frac{\pi^2}{12}\cdot\left(1 + \sqrt{\pi}\frac{r(\Omega)}{\sqrt{|\Omega|}}\right)^2, 1 - \frac{\pi}{\frac{\pi^2}{4}\left(\frac{\sqrt{|\Omega|}}{r(\Omega)} + \sqrt{\pi}\right)^2 + \pi}\right)\\ &\le \max_{x\in\left(0,\frac{1}{\sqrt{\pi}}\right]} \min\left(\frac{\pi^2}{12}\cdot\left(1+\sqrt{\pi}x\right)^2, 1 - \frac{\pi}{\frac{\pi^2}{4}\left(\frac{1}{x} + \sqrt{\pi}\right)^2 + \pi}\right)\\ &= \frac{\pi^2}{12}\left(1 + \frac{2\sqrt{2(6+\pi^2)} - \pi^2}{4+\pi^2}\right)^2.\end{aligned}$$

The second inequality is a consequence of the estimate $0 < \frac{r(\Omega)}{\sqrt{|\Omega|}} \le \frac{r(B)}{\sqrt{|B|}} = \frac{1}{\sqrt{\pi}}$, where B is any ball of $\mathbb{R}^2$. Let us now detail the computations leading to the last equality. We introduce the functions $f : x \longmapsto \frac{\pi^2}{12}\cdot\left(1+\sqrt{\pi}x\right)^2$ and $g : x \longmapsto 1 - \frac{\pi}{\frac{\pi^2}{4}\left(\frac{1}{x}+\sqrt{\pi}\right)^2+\pi}$. The function f is increasing on $(0, \frac{1}{\sqrt{\pi}}]$ and g is decreasing on $(0, \frac{1}{\sqrt{\pi}}]$, moreover, $\lim_{x\to0^+} f(x) = \frac{\pi^2}{12} < 1 = \lim_{x\to0^+} g(x)$ and $f(\frac{1}{\sqrt{\pi}}) = \frac{\pi^2}{3} > \frac{1}{1+\pi^2} = g(\frac{1}{\sqrt{\pi}})$. Thus the function $x \longmapsto \min(f(x), g(x))$ attains its maximum on $(0, \frac{1}{\sqrt{\pi}}]$ at the point x_0 such that $f(x_0) = g(x_0)$ (see Figure 1). It remains then to solve the equation $f(x) = g(x)$ on $(0, \frac{1}{\sqrt{\pi}}]$.

The equation $f(x) = g(x)$ can be written in the algebraic form $ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are explicit constants. We then notice that $-\frac{1}{\sqrt{\pi}}$ is a solution, which means that the equation is equivalent to $a(x + \frac{1}{\sqrt{\pi}})(x^2 + b'x + c') = 0$, where b' and c' are explicit constants. It remains then to solve the second order equation $x^2 + b'x + c' = 0$ that admits two solutions

$$x_1 = -\frac{\pi^{3/2}}{4+\pi^2} - \frac{2\sqrt{\frac{2(6+\pi^2)}{\pi}}}{4+\pi^2} < 0 \quad \text{and} \quad x_0 = \frac{2\sqrt{\frac{2(6+\pi^2)}{\pi}}}{4+\pi^2} - \frac{\pi^{3/2}}{4+\pi^2} \in (0, \frac{1}{\sqrt{\pi}}].$$

Thus

$$\max_{x\in(0,\frac{1}{\sqrt{\pi}}]} \min(f(x), g(x)) = f(x_0) = g(x_0) = \frac{\pi^2}{12}\left(1 + \frac{2\sqrt{2(6+\pi^2)} - \pi^2}{4+\pi^2}\right)^2.$$

This ends the proof.

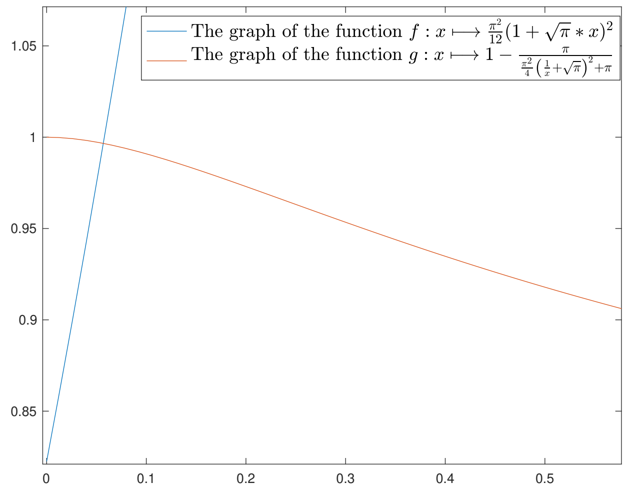

Figure 1. Graphs of the functions f and g.

3. Some comments

We close the paper by the following comments:

- The result of the paper is limited to the planar case because it mainly relies on the inequality

$$\lambda_1(\Omega) < \frac{\pi^2}{4}\left(\frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}\right)^2,$$

  which is obtained by combining the inequalities $\lambda_1(\Omega) < \frac{\pi^2}{4} h(\Omega)^2$ (proved in [12]), where h is the Cheeger constant, and $h(\Omega) \le \frac{1}{r(\Omega)} + \sqrt{\frac{\pi}{|\Omega|}}$ (proved in [8]). Up to our knowledge, the last inequality is only known for the planar case and a generalization to higher dimensions seems to be difficult as the proof relies on the explicit characterization of the Cheeger constant of planar convex sets (see [9]).

- Let us denote $B_{r(\Omega)}$ a ball of radius $r(\Omega)$ inscribed in Ω. The inclusion $B_{r(\Omega)} \subset \Omega$ combined with the monotonicity of λ_1 for the inclusion yield the inequality

$$\lambda_1(\Omega) \le \frac{j_{01}^2}{r(\Omega)^2}, \tag{7}$$

  where j_{01} denotes the first zero of the first Bessel function. We note that even if inequality (7) is better than (5) for sets that are close to the ball (we refer to [8, Section 5.1.1] for more details), it is not sufficient to improve the upper bound given in Theorem 2.
- We recall that the infimum of the functional $\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|}$ on the class of open sets is zero (see [2, Remark 2.4]. As for the class of bounded convex subsets of $\mathbb{R}^n$ (with $n \ge 2$), it is conjectured that the infimum is given by the constant $\frac{\pi^2}{12}\frac{6}{(n+1)(n+2)}$, see [1, Conjecture 4.2]. In [3], the authors prove the following estimates

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \ge \left(\frac{\pi}{2}\right)^2 \frac{1}{n^{n+2}(n+2)}, \quad \text{for } n \ge 3,$$

  and

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \ge \frac{\pi^2}{48}, \quad \text{for } n = 2,$$

  that have been improved in [6, Remark 4.1], where the authors provide the lower bound

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \ge \left(\frac{\pi}{2}\right)^2 \frac{1}{n(n+2)}.$$

  We also refer to [4, Theorems 1.4 & 1.5] for finer lower bounds in restricted classes of convex planar sets.
- An interesting tool to visualize the inequalities relating three functionals is the Blaschke–Santaló diagram. In our case, we are interested in the functionals: torsion T, fundamental frequency λ_1 and measure $|\cdot|$ in the case of planar convex sets, which leads to consider set of points

$$\mathcal{D} = \{(\lambda_1(\Omega), T^{-1}(\Omega)) \mid \Omega \subset \mathbb{R}^2 \text{ is convex and } |\Omega| = 1\}.$$

  We note that this diagram has been theoretically studied in [10] and we refer to [1, 7] for results in the case of open sets. In Figure 2, we plot an approximation of the diagram $\mathcal{D}$ obtained by randomly generating 10^5 convex polygons (we used the algorithm presented in [14]) for which we compute the involved functionals, we also plot the curves corresponding to the best known inequalities relating the 3 functionals, namely:
  - the Kohler-Jobin inequality

$$T(\Omega)\lambda_1(\Omega)^2 \ge T(B)\lambda_1(B)^2,$$

  - the Faber–Krahn inequality

$$|\Omega|\lambda_1(\Omega) \ge |B|\lambda_1(B),$$

  - the inequality (4)

$$\frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} \le \frac{\pi^2}{12}\left(1 + \frac{2\sqrt{2(6+\pi^2)} - \pi^2}{4+\pi^2}\right)^2 \approx 0.996613\ldots$$

  We also plot in dashed line the curves corresponding to the conjectures

$$\frac{\pi^2}{24} < \frac{\lambda_1(\Omega)T(\Omega)}{|\Omega|} < \frac{\pi^2}{12}.$$

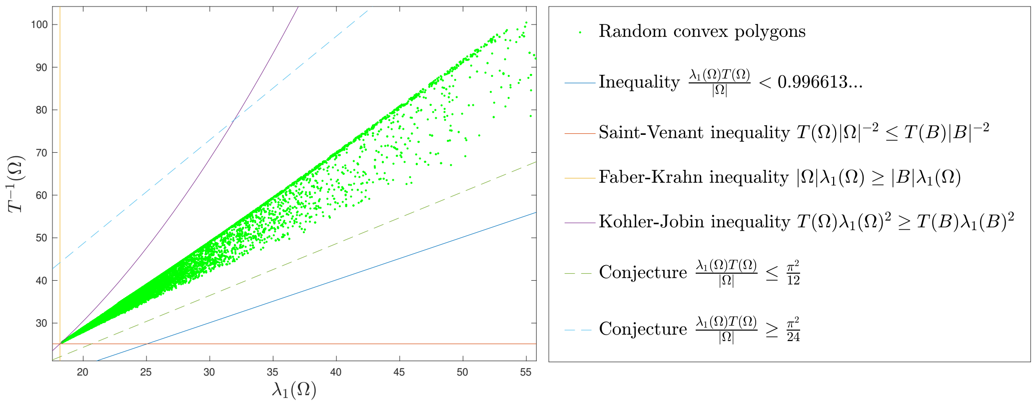

Figure 2. Blaschke–Santaló diagram $\mathscr{D}$ and inequalities.

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