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Control theory / *Théorie du contrôle*

Spectral stabilization of linear transport equations with boundary and in-domain couplings

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Abstract. In this work, the problem of stabilization of general systems of linear transport equations with in-domain and boundary couplings is investigated. It is proved that the unstable part of the spectrum is of finite cardinal. Then, using the pole placement theorem, a linear full state feedback controller is synthesized to stabilize the unstable finite-dimensional part of the system. Finally, by a careful study of semigroups, we prove the exponential stability of the closed-loop system. As a by product, the linear control constructed before is saturated and a fine estimate of the basin of attraction is given.

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1. Introduction

1.1. *Literature review*

In the work, we investigate boundary stabilization of a class of linear first-order hyperbolic systems of Partial Differential Equations (PDEs) on a finite space domain $x \in [0,1]$. Such systems are predominant in modeling of traffic flow [1], heat exchangers [32], open channel flow [3, Chapter 1.4] or multiphase flow [10, 12, 14]. The couplings between states traveling in opposite directions, both in-domain and at the boundaries, may induce instability leading to undesirable behaviors. For example, oscillatory two-phase flow regimes occurring on oil and gas production systems directly result, in some cases, from these mechanisms [12]. The dynamics of most of these industrial systems are described by nonlinear transport equations. If we linearize systems presented before, one obtains a system of equations of the form:

$$\begin{cases} R_t + \Lambda R_x = MR \\ R_1(t,0) = u(t) \\ R_2(t,1) = HR_1(t,1) \end{cases} \tag{1}$$

where $R = (R_1, R_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and:

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & -\Lambda_2 \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

Velocity matrices have dimensions $\Lambda_1 \in D^+_{d_1}(\mathbb{R})$, $\Lambda_2 \in D^+_{d_2}(\mathbb{R})$ where for all $m \in \mathbb{N}$, $D^+_m(\mathbb{R})$ stands for the set of definite positive diagonal matrices. For couplings, dimensions are $M_{12} \in M_{d_1 \times d_2}(\mathbb{R})$, $M_{21} \in M_{d_2 \times d_1}(\mathbb{R})$, $M_{11} \in M_{d_1 \times d_1}(\mathbb{R})$ and $M_{22} \in M_{d_2 \times d_2}(\mathbb{R})$. It should be noticed that in most of the cases presented in the first paragraph, the linearized system is not homogeneous in the sense that matrices depend on the space variable.

To exponentially stabilize system (1), feedback controls $u(t)$ depending on the boundary values $R(t,0), R(t,1)$ were designed in the literature. Lyapunov techniques allows to establish exponential stabilization in Sobolev or C^p spaces when term M is supposed to be small. Applications to linearized Saint Venant systems are given in [4, 13, 18–20].

However when the in-domain coupling term M is too large, simple quadratic Lyapunov function may not be found [2, 19]. Moreover, spectral analysis shows that when the entries of M exceed a certain amplitude, the system is unstable for any control of the form $u(t) = FR(t,1)$ ($F \in M_{d_1 \times d_2}(\mathbb{R})$) [3, Proposition 5.2]. Note that in [3, Proposition 5.2], this was proven only for $d_1 = d_2 = 1$ and $M = \left(\begin{smallmatrix} 0 & c \\ c & 0 \end{smallmatrix}\right)$.

To overcome this problem, one can relax the assumption of a control depending only on the value of the state at the boundary. Doing so, it is possible to construct a full-state feedback depending on the value of R on all the domain $[0,1]$. In this work, we consider an integral feedback of the form $u(t) = \int_0^1 k(x)R(x)\mathrm{d}x$ where k is a kernel to be defined. As a consequence, to use the proposed method, one needs to measure the state R on all the domain, which is sometimes impossible in industrial applications (see [10] for example). Some works [5, 11, 15, 17, 31] solve this difficulty designing a boundary observer of the state R. Here for simplicity, it is assumed that the observation of the state is complete in order to focus only on the effect of the control.

The full-state feedback strategy has already succeeded in stabilizing hyperbolic systems like (1). One can cite [9, 21] where authors use the backstepping method to locally stabilize quasi-linear hyperbolic systems in H^2. Additionally, backstepping can be used to stabilize system (1) in finite time [8]. For an introduction to this method, the book [23] gives a wide overview of the topic.

In this work, the method used differs from backstepping and is based on spectral theory applied to a well-behaved open-loop operator for which the spectrum is reduced to its point spectrum. Our analysis is greatly inspired from works [24, 25, 28] where the authors study a vast class of linear hyperbolic system with in domain and boundary couplings. In order to explain the great lines of the proof of exponential stabilization, it is needed to define the problem in a semigroup form. This is the object of the next two sections.

1.2. *Preliminaries*

Let $\mathcal{H}$ be an Hilbert space. The scalar product on $\mathcal{H}$ is denoted $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the associated norm is given by $\| \cdot \|_{\mathcal{H}}$.

The following formalism is taken from the book [30, Chapter 2]. Here, concepts are introduced without proof to ease the presentation. For more details, we refer to [30, Chapter 2].

1.2.1. *Generator of a strongly continuous semigroup*

The notion of semigroup is fundamental in this article. Its definition is given here:

Definition 1. *A family $(T_t)_{t\geq 0}$ of operators in $\mathscr{L}(\mathscr{H})$ is a strongly continuous semigroup on $\mathscr{H}$ if:*

- $T_0 = I$.
- $\forall\ t,\tau \geq 0,\ T_{t+\tau} = T_t T_\tau$ *(the semigroup property)*
- $\forall\ z \in \mathscr{H},\ \lim_{t\to 0} T_t z = z$ *(the strong continuity property)*

The generator of the semigroup $(T_t)_{t\geq 0}$ is defined as follows:

Definition 2. *Let D be the subset of $\mathscr{H}$ such that:*

$$D := \left\{ R \in \mathscr{H} \,\middle|\, \lim_{t\to 0^+} \frac{T_t R - R}{t} \text{ exists in } \mathscr{H} \right\}.$$

Then, the operator $\mathscr{A} : D \to \mathscr{H}$ is defined such that:

$$\mathscr{A} R = \lim_{t\to 0^+} \frac{T_t R - R}{t}, \quad \forall\ R \in D.$$

This operator is called the generator of $(T_t)_{t\geq 0}$ and for the rest of the paper, we use the very classic notation $T_t \leftarrow e^{\mathscr{A} t}, D \leftarrow D(\mathscr{A})$. It is also said that $\mathscr{A}$ generates the semigroup $e^{\mathscr{A} t}$.

The Lumer–Phillips Theorem states necessary and sufficient conditions on an unbounded operator $(\mathscr{A}, D(\mathscr{A}))$ to generate a strongly continuous semigroup.

Theorem 3 (Lumer–Phillips). *Let $\mathscr{A}$ be an unbounded operator defined on $D(\mathscr{A}) \subset \mathscr{H}$. The operator $\mathscr{A}$ generates a strongly continuous semigroup $e^{\mathscr{A} t}$ if and only if:*

- *$D(\mathscr{A})$ dense in $\mathscr{H}$*
- *$\mathscr{A}$ is closed*
- *$\forall\ R \in D(\mathscr{A}),\ \langle R, \mathscr{A} R\rangle_{\mathscr{H}} \leq \zeta \|R\|^2_{\mathscr{H}}$ where $\zeta \in \mathbb{R}$. This property is called the ζ dissipativity of $\mathscr{A}$.*
- *The resolvent set of $\mathscr{A}$*

$$\rho(\mathscr{A}) := \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda I - \mathscr{A} \text{ is invertible and } (\lambda I - \mathscr{A})^{-1} \in \mathscr{L}(\mathscr{H}) \right\}$$

  is not empty.

Remark 4. The operator $(\lambda I - \mathscr{A})^{-1}$ appearing in the definition of $\rho(\mathscr{A})$ is denoted by $R(\lambda, \mathscr{A})$ in this paper. Moreover, one can easily generalize the notion of spectrum to unbounded operators; the spectrum of $\mathscr{A}$ is given by:

$$\sigma(\mathscr{A}) := \mathbb{C} \setminus \rho(\mathscr{A}).$$

The point spectrum of $\mathscr{A}$ is included in $\sigma(\mathscr{A})$ and is given as:

$$\sigma_p(\mathscr{A}) := \left\{ \lambda \in \sigma(\mathscr{A}) \,\middle|\, \lambda I - \mathscr{A} \text{ is not injective} \right\}.$$

If $R_0 \in D(\mathscr{A})$, it is not difficult to prove that $T(t) R_0 \in D(\mathscr{A})$ for all time $t \geq 0$ and:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\mathscr{A} t} R_0 = \mathscr{A} e^{\mathscr{A} t} R_0, \quad \forall\ t \geq 0.$$

In other words, if we note $R(t) := e^{\mathscr{A} t} R_0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} R(t) = \mathscr{A} R(t), \quad \forall\ t \geq 0. \tag{2}$$

Hence, the semigroup representation can be very useful to treat PDEs when the initial data is in the domain of the operator considered.

1.2.2. *The adjoint semigroup*

When R_0 is not in $D(\mathscr{A})$, things become more difficult since we are not allowed to differentiate $e^{\mathscr{A}t}R_0$. In order to consider less regular solutions, we need to introduce duality. The adjoint semigroup is a key tool to understand this aspect.

Let us define the space:

$$D(\mathscr{A}^\star) := \left\{ \varphi \in \mathscr{H} \;\middle|\; \sup_{R \in D(\mathscr{A}), R \neq 0} \frac{\langle \mathscr{A} R, \varphi \rangle_{\mathscr{H}}}{\|R\|_{\mathscr{H}}} < \infty \right\}$$

and the adjoint operator $\mathscr{A}^\star$ is defined by the Riesz representation theorem as:

$$\langle \mathscr{A} R, \varphi \rangle_{\mathscr{H}} = \langle R, \mathscr{A}^\star \varphi \rangle_{\mathscr{H}}, \quad \forall\, R \in D(\mathscr{A}), \varphi \in D(\mathscr{A}^\star).$$

Hence when R is supposed to be in $\mathscr{H}$ only, we can still define $\mathscr{A} R$ weakly, writing $\langle \mathscr{A} R, \varphi \rangle = \langle R, \mathscr{A}^\star \varphi \rangle$ when $\varphi \in D(\mathscr{A}^\star)$ so that the problem (2) is written in the duality form:

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} R(t), \varphi \right\rangle_{\mathscr{H}} = \langle R(t), \mathscr{A}^\star \varphi \rangle_{\mathscr{H}}, \quad \forall\, \varphi \in D(\mathscr{A}^\star).$$

Here φ corresponds to the test functions of our problem. Moreover and by density, one can prove that $e^{\mathscr{A}t}R_0$ is a solution to previous equation and we just found a solution to problem (2) when R_0 is in $\mathscr{H}$ only.

1.2.3. *The embedding $\mathscr{H}_1^d \subset \mathscr{H} \subset \mathscr{H}_{-1}$*

In this paper, the control operator is singular in the sense that it does not belong to $\mathscr{L}(\mathbb{U}, \mathscr{H})$ ($\mathbb{U}$ is the control space). Hence, we cannot use the bracket $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ when the control operator is involved. Fortunately, one can generalize the notion of duality to a larger space than $\mathscr{H}$.

Let $\mathscr{H}_1^d$ be a normed dense subset of $\mathscr{H}$. For all $R \in \mathscr{H}$, the $\mathscr{H}_{-1}$ norm of R writes:

$$\|R\|_{\mathscr{H}_{-1}} := \sup_{\varphi \in \mathscr{H}_1^d,\ \|\varphi\|_{\mathscr{H}_1^d} \leq 1} \langle R, \varphi \rangle_{\mathscr{H}}.$$

The space $\mathscr{H}_{-1}$ is then defined as follows:

Definition 5. *The space $\mathscr{H}_{-1}$ is the completion of $\mathscr{H}$ with respect to the norm $\|\cdot\|_{\mathscr{H}_{-1}}$ and hence $\mathscr{H}_1^d \subset \mathscr{H} \subset \mathscr{H}_{-1}$. Moreover, for all $R \in \mathscr{H}_{-1}$ and all sequence $(R_n)_n$ of elements of $\mathscr{H}$ such that $\lim_{n\to\infty} \|R - R_n\|_{\mathscr{H}_{-1}} = 0$, we define the duality bracket as:*

$$\forall\, \varphi \in \mathscr{H}_d^1, \quad \langle R, \varphi \rangle_{\mathscr{H}_{-1}, \mathscr{H}_1^d} := \lim_{n \to \infty} \langle R_n, \varphi \rangle_{\mathscr{H}}.$$

The space $\mathscr{H}$ which defines the scalar product is called the pivot space. The space $\mathscr{H}_{-1}$ is named as the dual of $\mathscr{H}_1^d$ with respect to the pivot space $\mathscr{H}$.

One can extend the operator $\mathscr{A}$ and its associated semigroup using the following theorem.

Theorem 6. *Let $\lambda \in \rho(\mathscr{A})$ and define:*

$$\mathscr{H}_1^d := (D(\mathscr{A}^\star), \|(\bar{\lambda} I - \mathscr{A}^\star) \cdot \|_{\mathscr{H}})$$

and $\mathscr{H}_{-1}$ be the completion of $\mathscr{H}$ with respect to the norm $\|(\lambda I - \mathscr{A})^{-1} \cdot \|_{\mathscr{H}}$. Spaces $\mathscr{H}_1^d$ and $\mathscr{H}_{-1}$ are independent on the choice of λ.

Additionally, the space $\mathscr{H}_{-1}$ is the dual of $\mathscr{H}_1^d$ with respect to the pivot space $\mathscr{H}$. Moreover, one can extend the operator $\mathscr{A}$ so that (keeping the same notation for the extension) $\mathscr{A} \in \mathscr{L}(\mathscr{H}, \mathscr{H}_{-1})$ and the associated semigroup $(e^{\mathscr{A}t})$ can be extended in $\mathscr{L}(\mathscr{H}_{-1})$.

Remark 7. In the proof of last theorem, the extension of $\mathscr{A}$ is built from $\mathscr{A}^{\star\star}$.

As a consequence, if the operator of control $\mathscr{B}$ is in $\mathscr{L}(\mathbb{U},\mathscr{H}_{-1})$, it is easy to define a solution to:

$$\frac{\mathrm{d}}{\mathrm{d}t}R = \mathscr{A}R + \mathscr{B}u$$

with $u \in L^2([0,T],\mathbb{U})$ $(T>0)$. Using the duality bracket, one can write the last equation as:

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}R,\varphi \right\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} = \langle R,\mathscr{A}^\star\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} + \langle \mathscr{B}u,\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d}, \quad \forall\, \varphi \in D(\mathscr{A}^\star)$$

and the closed-loop problem is well-defined.

1.3. *The abstract problem*

Now that the semigroup framework has been recalled, we can apply it to our problem. Let $\mathscr{H} = L^2([0,1];\mathbb{C}^d)$ where $d := d_1 + d_2$, be the base space embedded with the usual scalar product:

$$\langle f,g\rangle_{\mathscr{H}} := \sum_{i=1}^{d}\int_0^1 \overline{f_i(x)}g_i(x)\mathrm{d}x.$$

The corresponding norm on $\mathscr{H}$ is denoted $\|\cdot\|_{\mathscr{H}}$. The open-loop operator $\mathscr{A}$ is given by:

$$\begin{cases} D(\mathscr{A}) = \left\{R \in \mathscr{H} \;\middle|\; R' \in \mathscr{H}, R_1(0) = 0, R_2(1) = HR_1(1)\right\} \\ \quad \mathscr{A}R = -\Lambda R' + MR. \end{cases}$$

It is easy to check (left for the reader) that $\mathscr{A}$ is a closed densely defined operator that satisfies the hypothesis of Lumer–Phillips Theorem. Hence, it generates a strongly continuous semigroup denoted $(e^{\mathscr{A}t})_{t\geq 0}$ on $\mathscr{H}$. For its adjoint $\mathscr{A}^\star$, it can be shown that:

$$\begin{cases} D(\mathscr{A}^\star) = \left\{R \in \mathscr{H} \;\middle|\; R' \in \mathscr{H}, R_2(0) = 0, \Lambda_1 R_1(1) = H^T\Lambda_2 R_2(1)\right\} \\ \quad \mathscr{A}^\star R = \Lambda R' + M^T R. \end{cases}$$

The control space $L^2(\mathbb{R}_+,\mathbb{U}) := L^2(\mathbb{R}_+,\mathbb{R}^{d_1})$ $(\mathbb{U} := \mathbb{R}^{d_1})$ is embedded with the canonical norm of $L^2(\mathbb{R}_+,\mathbb{R}^{d_1})$. To define the control operator, we introduce the space $\mathscr{H}_{-1}$ which is the dual of $D(\mathscr{A}^\star)$ when we take $\mathscr{H}$ as pivot space, namely:

$$\mathscr{H}_{-1} := \overline{(D(\mathscr{A}), \|R(\lambda,\mathscr{A})\cdot\|_{\mathscr{H}})}.$$

where $\lambda \in \rho(\mathscr{A})$ is taken arbitrarily in the resolvant set of $\mathscr{A}$. Moreover, the primal of $\mathscr{H}_{-1}$ is denoted:

$$\mathscr{H}_1^d := (D(\mathscr{A}^\star), \|(\lambda I - \mathscr{A}^\star)\cdot\|_{\mathscr{H}}).$$

The control operator $\mathscr{B} \in \mathscr{L}(\mathbb{U},\mathscr{H}_{-1})$ writes:

$$\mathscr{B}u := (\sqrt{\Lambda_1}u, 0_{d_2})\delta(x) \in \mathscr{H}_{-1}$$

where δ is the usual Dirac delta distribution. Its dual $\mathscr{B}^\star \in \mathscr{L}(\mathscr{H}_d^1,\mathbb{U})$ then writes:

$$\forall\, \varphi \in \mathscr{H}_d^1,\ \mathscr{B}^\star\varphi = \sqrt{\Lambda_1}\varphi_1(x=0) \tag{3}$$

where φ_1 is the function gathering the first d_1 components of φ.

Note that in this paper, we will not make the difference between $\mathscr{A},\mathscr{A}^\star$ and their canonical extension $(\mathscr{A})^{\star\star}, (\mathscr{A}^\star)^{\star\star}$ in $\mathscr{L}(\mathscr{H},\mathscr{H}_{-1})$.

With this notation, the abstract evolution problem on $\mathscr{H}_{-1}$ is defined here (see Definition 9 for a rigorous definition) :

$$\begin{cases} \dfrac{\mathrm{d}R}{\mathrm{d}t} = \mathscr{A}R + \mathscr{B}u(t) \\ R(0) = R_0. \end{cases} \tag{4}$$

The notation being introduced, one can present the main theorem of this work and the rest of the paper is dedicated to its proof.

Theorem 8. *There exists a linear feedback control of the form* $u := \mathscr{K} R$ *with* $\mathscr{K} \in \mathscr{L}(\mathscr{H}_{-1}, \mathbb{U})$ *for which there exists* $C, \delta > 0$ *such that for all initial condition* $R_0 \in \mathscr{H}_{-1}$*, the system* (4) *is well-posed and the corresponding unique solution* $R \in C(\mathbb{R}^+, \mathscr{H}_{-1})$ *to* (4) *verifies:*

$$\|R(t)\|_{\mathscr{H}_{-1}} \le Ce^{-\delta t}\|R_0\|_{\mathscr{H}_{-1}}, \ \forall \ t \ge 0.$$

A sketch of the proof of this exponential stabilizability result is given below.

- First, we prove that $\mathscr{A}$ has only a discrete spectrum with a finite number of unstable eigenvalues.
- Then, the focus is on the unstable finite-dimensional part $\mathscr{M}$ of the system using a projection on the unstable eigenspace. Proving a controlability result for the operator $(\mathscr{A}_{|\mathscr{M}}, \mathscr{B}_{|\mathscr{M}})$, it is possible to use the pole placement theorem to find a state feedback control stabilizing the unstable part.
- Finally, we prove that the whole closed-loop system is well-posed and exponentially stable. This is not immediate since the control synthesized from the finite-dimensional unstable part can destabilize the remaining one.

Outline

The article is organized as follows. In Section 2, we give the definition of solution, state and prove an admissibility condition for well-posedness. Section 3 is dedicated to the spectral study of the open-loop operator $\mathscr{A}$. Then, Section 4 gives a rigorous proof of Theorem 8. Next, the control constructed in Theorem 8 is saturated in a certain sense and we give a local stability result in Section 5. In the same part, numerical illustrations of the results previously presented, are exposed. Finally, as a conclusion, perspectives and open problems are stated.

2. Solution definition and admissibility condition

Let $u \in L^2(0, T, \mathbb{U})$ be given. The definition of a solution to (4) is proposed below:

Definition 9. *The function* $R \in C^1([0,T], \mathscr{H}_{-1}) \cap C([0,T]; \mathscr{H})$ *is a solution to* (4) *if for all* $\varphi \in \mathscr{H}_1^d$

$$\langle R(t), \varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} = \langle R_0, e^{\mathscr{A}^\star t}\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} + \int_0^t \langle u(s), \mathscr{B}^\star e^{\mathscr{A}^\star(t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}} \mathrm{d}s, \quad \forall\, 0 \le t \le T. \tag{5}$$

where $\mathbb{U}$ *is identified with its dual.*

The following lemma is an admissibility result allowing to prove that the solution has regularity in $\mathscr{H}$.

Lemma 10. *The admissibility property holds for all time* $T \ge 0$*:*

$$\int_0^T \|\mathscr{B}^\star e^{\mathscr{A}^\star(T-t)}\varphi\|_{\mathbb{U}}^2 \mathrm{d}t \le C^2\|\varphi\|_{\mathscr{H}}^2, \quad \forall\, \varphi \in \mathscr{H}. \tag{6}$$

where C *is a constant depending on the parameters of the problem.*

Proof. Let $\varphi \in \mathscr{H}_1^d$; the case $\varphi \in \mathscr{H}$ is easily deduced by a density argument. Let $z(t) := e^{t\mathscr{A}^\star}\varphi$ the solution to:

$$\begin{cases} z_t - \Lambda z_x = M^T z \\ z_2(0) = 0, \quad \Lambda_1 z_1(1) - H^T\Lambda_2 z_2(1) = 0 \\ z(t=0) = \varphi. \end{cases}$$

We define the functional V by:

$$V(z) := \int_0^1 z_1^T z_1 e^{-\gamma x} + z_2^T z_2 e^{-\gamma(1-x)} \mathrm{d}x = \int_0^1 z^T \Gamma z \mathrm{d}x$$

where $\Gamma(x) := \operatorname{diag}(e^{-\gamma x} I_{d_1}, e^{-\gamma(1-x)} I_{d_2})$ and $\gamma > 0$ will be chosen later. Using integration by parts, we get:

$$\begin{aligned}\frac{\mathrm{d}V(z)}{\mathrm{d}t} &= \int_0^1 z_t^T \Gamma z + z^T \Gamma z_t \mathrm{d}x \\ &= \int_0^1 \left(\Lambda z_x + M^T z\right)^T \Gamma z + z^T \Gamma \left(\Lambda z_x + M^T z\right) \mathrm{d}x \\ &= 2\int_0^1 z^T \Lambda \Gamma z_x \mathrm{d}x + \int_0^1 z^T \left(\Gamma M^T + M\Gamma\right) z \mathrm{d}x \\ &= \left[z^T \Lambda \Gamma z\right]_0^1 - \int_0^1 z^T \Lambda \Gamma_x z \mathrm{d}x + \int_0^1 z^T \left(\Gamma M^T + M\Gamma\right) z \mathrm{d}x.\end{aligned}$$

Using the fact that $z \in \mathscr{H}_1^d$ for all time, the boundary terms are estimated below:

$$\begin{aligned}\left[z^T \Lambda \Gamma z\right]_0^1 &= -z_1(0)^T \Lambda_1 z_1(0) + z_2(0)^T \Lambda_2 z_2(0) e^{-\gamma} + z_1(1)^T \Lambda_1 z_1(1) e^{-\gamma} - z_2(1)^T \Lambda_2 z_2(1) \\ &= -z_1(0)^T \Lambda_1 z_1(0) + z_2(1)^T \Lambda_2 H^T \Lambda_1^{-1} H^T \Lambda_2 z_2(1) e^{-\gamma} - z_2(1)^T \Lambda_2 z_2(1).\end{aligned}$$

Thus, for $\gamma > 0$ large enough:

$$\left[z^T \Lambda \Gamma z\right]_0^1 \leq -\frac{1}{2} z_1(0)^T \Lambda_1 z_1(0) = -\frac{1}{2} \|\mathscr{B}^\star e^{t\mathscr{A}^\star} \varphi\|_{\mathbb{U}}^2$$

where the last equality comes from the definition (3) of $\mathscr{B}^\star$. Hence:

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -\frac{1}{2} \|\mathscr{B}^\star e^{t\mathscr{A}^\star} \varphi\|_{\mathbb{U}}^2 - \int_0^1 z^T \Lambda \Gamma_x z \mathrm{d}x + \int_0^1 z^T (\Gamma M^T + M\Gamma) z \mathrm{d}x.$$

Using the fact that $\Gamma_x \Lambda = -\gamma \Gamma |\Lambda|$, the following estimate on V holds:

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -\frac{1}{2} \|\mathscr{B}^\star e^{\mathscr{A}^\star t} \varphi\|_{\mathbb{U}}^2 + C_{m,\gamma} V$$

where $C_{m,\gamma}$ depends on M, γ. Integrating, one obtains:

$$V(z(t)) - V(z(0)) e^{C_{m,\gamma} t} \leq -\frac{1}{2} \int_0^t \|\mathscr{B}^\star e^{\mathscr{A}^\star s} \varphi\|_{\mathbb{U}}^2 e^{C_{m,\gamma}(t-s)} ds$$

which immediately gives the existence of a constant C depending on t and the parameters of the problem such that:

$$C^2 V(z(0) = \varphi) \geq \int_0^T \|\mathscr{B}^\star e^{\mathscr{A}^\star (T-t)} \varphi\|_{\mathbb{U}}^2 \mathrm{d}t$$

which is the required result since V is equivalent to the square norm on $\mathscr{H}$. The proof of the general case of $\varphi \in \mathscr{H}$ follows by density of $\mathscr{H}_1^d$ in $\mathscr{H}$. □

By [7, Theorem 2.37], Lemma 10 gives the following results.

Lemma 11. *If $u \in L^2(0, T, \mathbb{U})$ then there exists a unique solution to* (4) *in the sense of Definition 9.*

The aim of this paper is to stabilize (4) showing the existence of an admissible operator (the notion of admissible operator will be defined latter) $\mathscr{K} : \mathscr{H}_{-1} \to \mathbb{U}$ such that:

$$\begin{cases} \dfrac{\mathrm{d}R}{\mathrm{d}t} = (\mathscr{A} + \mathscr{B}\mathscr{K}) R \\ R(0) = R_0 \end{cases} \tag{7}$$

is well-posed and its solution verifies the bound:

$$\|R(t, \cdot)\|_{\mathscr{H}_{-1}} \leq C e^{-\delta t} \|R_0\|_{\mathscr{H}_{-1}}, \forall\ t \geq 0$$

where $\delta, C > 0$ do not depend on R_0.

3. Spectral analysis of the open-loop problem

Before going into the proof of Theorem 8, we need some information the open-loop operator $\mathscr{A}$. More precisely, it is proved that $\mathscr{A}$ has a spectrum reduced to its point spectrum with a finite number of unstable eigenvalues.

Proposition 12. *The spectrum of the open-loop operator verifies:*

- $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A})$.
- *There exists* $r > 0$ *such that* $\sigma_p(\mathscr{A}) \subset \{z \in \mathbb{C} \mid \Re z < r\}$.
- *The unstable part of the spectrum;* $\sigma_p(\mathscr{A}) \cap \{\lambda \in \mathbb{C} \mid \Re\lambda \geq 0\}$ *has a finite cardinal.*

The proof will be the object of this section.

3.1. *Structure of the spectrum*

The first lemma states that the spectrum of $\mathscr{A}$ is in fact its point spectrum. Its proof is an adaptation of the proof of [24, Lemma 2.2].

Lemma 13. *The spectra of* $\mathscr{A}$ *consists of isolated eigenvalues of finite geometric multiplicity i.e.* $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A})$.

Proof. Let $\lambda \in \mathbb{C}$ and consider the unique solution to:

$$\begin{cases} -\Lambda R' + MR = \lambda R \\ R_1(0) = 0 \\ R_2(1) = HR_1(1). \end{cases} \tag{8}$$

In particular, by defining:

$$T(x, y, \lambda) = e^{-\Lambda^{-1}(\lambda I_d - M)(x-y)}.$$

The solution to (8) is given by:

$$R(x) = T(x, 0, \lambda)(0, I_{d_2})^T v(0)$$

where $v(0) \in \mathbb{C}^{d_2}$. In order to satisfy the right border boundary condition, we need to impose:

$$(H, -I_{d_2})T(1, 0, \lambda)(0, I_{d_2})^T v(0) = 0_{d_1}.$$

Hence, denoting:

$$U(\lambda) := (H, -I_{d_2})T(1, 0, \lambda)(0, I_{d_2})^T,$$

the point spectrum of the system is given by the zeros of the following characteristic equation:

$$\det(U(\lambda)) = 0. \tag{9}$$

Moreover, for all $\lambda \in \sigma_p(\mathscr{A})$, the corresponding eigenspace is given by:

$$\mathrm{Eig}(\mathscr{A}, \lambda) = \left\{T(x, 0, \lambda)(0, I_{d_2})^T v(0) \;\middle|\; v(0) \in \ker(U(\lambda))\right\}.$$

The geometric multiplicity is less than d_2 (the dimension of $v(0)$) and for $\lambda \in \mathbb{C} \setminus \sigma_p(\mathscr{A})$:

$$R(\lambda, \mathscr{A})G = T(x, 0, \lambda)(0, I_{d_2})v(0) - \int_0^x T(x, y, \lambda)\Lambda^{-1}G\,\mathrm{d}y, \quad \forall\, G \in \mathscr{H} \tag{10}$$

with:

$$v(0) = U(\lambda)^{-1}(H, -I_{d_2})\int_0^1 T(x, y, \lambda)\Lambda^{-1}G\,\mathrm{d}y.$$

With (10), it is easy to see that $R(\lambda, \mathscr{A})$ is bounded in $\mathscr{L}(\mathscr{H})$ when $\lambda \notin \sigma_p(\mathscr{A})$ and hence $\sigma(\mathscr{A}) = \sigma_p(\mathscr{A})$. □

Remark 14. In the proof of the previous lemma, we have shown that:

$$R(\lambda,\mathscr{A}) = U(\lambda)^{-1}E(\lambda),\ \forall\ \lambda \in \rho(\mathscr{A}) \tag{11}$$

where $E(\lambda)$ is a $\mathscr{H}$ valued entire function of λ.

The expression (11) gives immediately that:

Lemma 15. *The algebraic multiplicity of $\lambda \in \sigma(\mathscr{A})$ is given by the multiplicity of the zeros of $\kappa(\lambda) := \det(U(\lambda))$.*

3.2. *Analysis of the unstable part of the spectrum*

In this section, we end the proof of Proposition 12. The result given next ensures that the characteristic equation of the spectrum (9) can be approximated by the same equation removing the effect of non-diagonal 0th order term coming from the matrix M for which the spectrum is known. To be clear, the velocity matrix can always be decomposed as follows:

$$\Lambda = \operatorname{blockdiag}(\overline{\lambda}_i I_{\delta_i})_{1\le i\le n_1+n_2}$$

where $\delta_i > 0, \delta_i \in \mathbb{N}$ such that $\sum_{i=1}^{n_1}\delta_{i=1} = d_1, \sum_{i=n_1+1}^{n_2}\delta_i = d_2$. Moreover, for all $1 \le i \le n_1$, $\overline{\lambda}_i > 0$ and for $n_1+1 \le i \le n_2$, $\overline{\lambda}_i < 0$. Concerning the matrix M, one can also use the bloc decomposition $M = (\overline{M}_{ij})_{1\le i,j\le n_1+n_2}$ where $\overline{M}_{ij} \in M_{\delta_i\times\delta_j}(\mathbb{R})$. The matrix M_0 is then defined by:

$$M_0 := \operatorname{blockdiag}(\overline{M}_{ii})_{1\le i\le n_1+n_2}.$$

and $\mathscr{A}_0$ the operator in $\mathscr{H}$:

$$\begin{cases} D(\mathscr{A}_0) = \left\{R \in \mathscr{H} \mid R' \in \mathscr{H}, R_1(0) = 0, R_2(1) = HR_1(1)\right\} \\ \mathscr{A}_0 R = -\Lambda R' + M_0 R. \end{cases}$$

Obviously, we can prove Lemma 13 for operator $\mathscr{A}_0$ and exhibit a characteristic equation for $\mathscr{A}_0$:

$$\kappa_0(\lambda) := \det(U_0(\lambda)) = 0. \tag{12}$$

where:

$$U_0(\lambda) := (H, -I_{d_2})e^{-\Lambda^{-1}(\lambda I_d - M_0)}(0, I_{d_2})^T. \tag{13}$$

By simple computations:

$$\det(U_0(\lambda)) = \det(-e^{-\Lambda_2^{-1}(\lambda - M_{0,22})})$$

where $M_{0,22} := (M_{0,ij})_{d_1<i,j\le d_1+d_2} \in M_{d_2\times d_2}(\mathbb{R})$ and this functional does not have zeros. This means that the operator $\mathscr{A}_0$ generates a semigroup and $\rho(\mathscr{A}_0) = \mathbb{C}$. We present the following result taken from [24] which states that spectrum of $\mathscr{A}_0$ and $\mathscr{A}$ are closed for large imaginary part:

Lemma 16. *Let $r > 0$ be such that $\mathbb{C}_r := \{z \in \mathbb{C} \mid |\Re z| \le r\}$. We have the following:*

$$\lim_{|\Im\lambda|\to\infty} |U(\lambda) - U_0(\lambda)| = 0 \tag{14}$$

and the convergence is uniform on $\mathbb{C}_r$ for all $r > 0$.

Moreover, we have the following property which will be exploited for the complex functional κ_0 defined in (12):

Lemma 17 ([24]). *Let f be an exponential polynomial of the form $f(\lambda) = \sum_{i=1}^{r} a_j e^{b_k\lambda}$ ($\lambda, a_j \in \mathbb{C}, b_j \in \mathbb{R}$). Let $Z := \left\{\lambda \in \mathbb{C} \mid f(\lambda) = 0\right\}$ denotes the zero set of f. For all $\delta > 0, \alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ there exists a constant $m(\delta,\alpha,\beta) > 0$ such that for all $\lambda \in \mathbb{C}$ satisfying $\operatorname{dist}(\lambda, Z) > \delta$, $\alpha < \Re\lambda < \beta$, we have $|f(\lambda)| > m(\delta,\alpha,\beta)$.*

A direct consequence of previous lemma is that on all strips of the form $\alpha < \Re\lambda < \beta$, $\inf|\kappa_0(\lambda)| > 0$ (κ_0 does not admit zeros and is an exponential polynomial).

Corollary 18. *For all $\alpha < \beta$, we have*

$$\inf_{\lambda\in\mathbb{C},\ \alpha<\Re\lambda<\beta} |\kappa_0(\lambda)| > 0.$$

To conclude, Rouché's Theorem is recalled here:

Theorem 19 (Rouché). *Let $U \subset \mathbb{C}$ be an open connected set and f, g two meromorphic functions on U with finite number of zeros and poles. Let γ be a closed smooth curve in U that does not intersect the set of zeros of f or g and that forms the border ∂K of a compact set K. If*

$$|f(z) - g(z)| < |g(z)|,\ \forall\ z \in \gamma,$$

then:

$$Z_f - P_f = Z_g - P_g$$

where Z_f, Z_g designate the number of zeros of f and g in K and P_f, P_g designate the number of poles of f and g in K.

By applying Rouché's Theorem to $f = \kappa$ and $g = \kappa_0$ and using Lemma 16 and Corollary 18, one has that $\kappa(\lambda)$ has zeros located near the real axis. Owing this and the fact that $\kappa(\lambda)$ is an entire function, it has a finite number of zeros in the right-half plane. Combining this with Lemma 13, we easily conclude on the proof of Proposition 12.

4. Proof of Theorem 8

Let us denote $\mathscr{M}$ the finite-dimensional unstable generalized eigenspace (Jordan blocks) of $\mathscr{A}$ and $\mathscr{M}'$ its topological complement. We denote by $\alpha := \dim(\mathscr{M})$. Let $P : \mathscr{H} \to \mathscr{M}$ be the projection onto $\mathscr{M}$ defined as [22, Theorem 6.17]:

$$P = -\frac{1}{2i\pi}\oint_\Gamma R(\lambda, \mathscr{A})\mathrm{d}\lambda$$

where Γ is any contour enclosing the unstable eigenvalues of $\mathscr{A}$ (this is possible because of the separation of unstable eigenvalues). The following technical lemma allows to extend P as an element of $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H})$:

Lemma 20. *The projector $P \in \mathscr{L}(\mathscr{H})$ can be extended as an element $\widetilde{P}$ of $\mathscr{L}(\mathscr{H}_{-1}, \mathscr{H})$. Moreover,*

$$\operatorname{Ran}\widetilde{P} = \operatorname{Ran}P = \mathscr{M}. \tag{15}$$

Proof. First, we prove that P has an extension in $\mathscr{H}_{-1}$. We define $\widetilde{P}$ by duality, as follows:

$$\forall\ R \in \mathscr{H}_{-1}, \varphi \in \mathscr{H}^1_d, \quad \langle \widetilde{P}R, \varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}^1_d} := \langle R, P^\star\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}^1_d}.$$

The operator $\widetilde{P}$ is obviously an extension of P in $\mathscr{H}_{-1}$. Now we will prove that $\widetilde{P} \in \mathscr{L}(\mathscr{H}_{-1}, \mathscr{H})$ and to do so let us take $R \in \mathscr{H}_{-1}$, $\varphi \in \mathscr{H}^1_d$ and $\lambda \in \rho(\mathscr{A})$.

$$\begin{aligned}\langle PR, \varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}^1_d} &= \langle R, P^\star\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}^1_d}\\ &\le \|R\|_{\mathscr{H}_{-1}}\|P^\star\varphi\|_{\mathscr{H}^1_d}\\ &\le \|R\|_{\mathscr{H}_{-1}}\|(\bar{\lambda}I - \mathscr{A}^\star)P^\star\varphi\|_{\mathscr{H}}.\end{aligned}$$

The operator $(\bar{\lambda}I - \mathscr{A}^\star)P^\star$ writes:

$$\begin{aligned}(\bar{\lambda}I - \mathscr{A}^\star)P^\star &= (\bar{\lambda}I - \mathscr{A}^\star)\frac{1}{2\pi i}\int_\Gamma (\bar{\xi}I - \mathscr{A}^\star)^{-1}\mathrm{d}\xi\\ &= \frac{1}{2\pi i}\int_\Gamma (\bar{\xi} - \bar{\lambda})(\bar{\xi}I - \mathscr{A}^\star)^{-1}\mathrm{d}\xi \in \mathscr{L}(\mathscr{H})\end{aligned}$$

where we used the resolvant formula from [6, Proposition 3.18].

Hence,

$$\langle PR, \varphi\rangle_{\mathcal{H}_{-1},\mathcal{H}_d^1} \leq \|R\|_{\mathcal{H}_{-1}} \|(\bar{\lambda} I - \mathcal{A}^\star)P^\star\|_{\mathcal{L}(\mathcal{H})}\|\varphi\|_{\mathcal{H}}$$

and $P \in \mathcal{L}(\mathcal{H}_{-1}, \mathcal{H})$ with $\|P\|_{\mathcal{L}(\mathcal{H}_{-1},\mathcal{H})} \leq \|(\bar{\lambda} I - \mathcal{A}^\star)P^\star\|_{\mathcal{L}(\mathcal{H})}$.

Finally, we have to prove (15). Let $R \in \mathcal{H}_{-1}$ and let $(R_n)_n$ be a sequence of elements of $\mathcal{H}$ converging to R in $\mathcal{H}_{-1}$. As $\mathcal{M}$ is of finite dimension α, there exists a family $(e_i)_i$ which is an orthogonal basis of $\mathcal{M}$.

Let $\lambda \in \rho(\mathcal{A})$. In the finite-dimensional space $\mathcal{M}$, $\lambda I - \mathcal{A}$ as an operator from $\mathcal{M}$ to $\mathcal{M}$ is an automorphism (it can be identified to a matrix with Jordan blocks). We define:

$$\begin{cases} f_i := (\lambda I - \mathcal{A})e_i \in \mathcal{M} \\ \tilde{f}_i := R(\bar{\lambda}, \mathcal{A}^\star)R(\lambda, \mathcal{A}) f_i \in \mathcal{H}_d^1. \end{cases}$$

and $(f_i)_i, (\tilde{f}_i)_i$ are still a basis of $\mathcal{M}$ since they are the image of a basis by an automorphism.

On can write PR_n in the basis $(f_i)_i$:

$$PR_n = \sum_{i=1}^{\alpha} \beta_n^i f_i$$

where the β_n^i are the coefficient of PR_n in this basis. For $j \leq \alpha$:

$$\begin{aligned} \langle PR_n, \tilde{f}_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d} &= \sum_{i=1}^{\alpha} \beta_n^i \langle f_i, \tilde{f}_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d} \\ &= \sum_{i=1}^{\alpha} \beta_n^i \langle f_i, R(\bar{\lambda}, \mathcal{A}^\star)R(\lambda,\mathcal{A}) f_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d} \\ &= \sum_{i=1}^{\alpha} \beta_n^i \langle R(\lambda,\mathcal{A}) f_i, R(\lambda,\mathcal{A}) f_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d} \\ &= \sum_{i=1}^{\alpha} \beta_n^i \langle e_i, e_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d} \\ &= \sum_{i=1}^{\alpha} \beta_n^i \langle e_i, e_j\rangle_{\mathcal{H}} \\ &= \beta_n^j. \end{aligned}$$

Aside from that, $\langle PR_n, \tilde{f}_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d} = \langle R_n, P^\star \tilde{f}_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d} \to_{n\to+\infty} \langle R, P^\star \tilde{f}_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d}$ since $\lim_{+\infty} R_n = R$ in $\mathcal{H}_{-1}$. Consequently, for all $j \leq \alpha$ the sequence $(\beta^j)_n$ is convergent and its limit is $\langle R, P^\star \tilde{f}_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d}$. To conclude, we have that:

$$\widetilde{P}R = \sum_{i=1}^{\alpha} \langle R, P^\star \tilde{f}_j\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d} f_i$$

which is an element of $\mathcal{M}$. This concludes the proof of the lemma. □

From now on, we will not make a difference between P and its extension $\widetilde{P}$ keeping the notation without tilda.

Corollary 21. *The operator P is an element of $\mathcal{L}(\mathcal{H}_{-1})$.*

Proof. The proof is immediate owing Lemma 20 and the fact that $\mathcal{H}$ is continuously embedded in $\mathcal{H}_{-1}$. □

As P and $\mathcal{A}$ commute, it is possible to decompose $\mathcal{A}$ on the topological sum $\mathcal{M} \oplus \mathcal{M}' = \mathcal{H}$ and the abstract stabilization problem becomes:

$$\begin{cases} P\dot{R} = P\mathcal{A}PR + P\mathcal{B}u(t) \\ (I-P)\dot{R} = (I-P)\mathcal{A}(I-P)R + (I-P)\mathcal{B}u(t). \end{cases} \tag{16}$$

4.1. *Stabilization of the finite-dimensional part*

First, we stabilize the finite-dimensional part without considering the infinite-dimensional part taking $u(t)$ of the form $u(t) = KPR(t)$ where K is a matrix of dimension $d_1 \times \alpha$. Hence, we have to solve a finite-dimensional stabilization problem where the open-loop matrix is the restriction denoted $\mathscr{A}_{\mathscr{M}}$ of $\mathscr{A}$ on $\mathscr{M}$ and the control matrix is $\mathscr{B}_{\mathscr{M}} = P\mathscr{B} \in M_{\alpha\times d_1}(\mathbb{R})$.

Proposition 22. *The system $(\mathscr{A}_{\mathscr{M}}, \mathscr{B}_{\mathscr{M}})$ is controllable. Hence, there exists $K \in M_{d_1\times\alpha}(\mathbb{R})$ such that $\mathscr{A}_{\mathscr{M}} + \mathscr{B}_{\mathscr{M}} K$ is Hurwitz.*

Proof. We show the result by using the Fattorini–Hautus test (also known as the Popov–Belevitch–Hautus test). It is necessary to prove that:

$$\ker(\lambda I - \mathscr{A}^{\star}_{\mathscr{M}}) \cap \ker \mathscr{B}_{\mathscr{M}}{}^{\star} = \{0\}, \quad \forall\, \lambda \in \mathbb{C} \tag{17}$$

which reduces the analysis to eigenspaces only (and not generalized eigenspaces). In order to prove (17), the eigenvalues of $\mathscr{A}^{\star}$ are calculated. This is equivalent to solve:

$$\begin{cases} \Lambda R' + M^T R = \lambda R \\ \qquad R_1(1) = \Lambda_1^{-1} H^T \Lambda_2 R_2(1) \\ \qquad R_2(0) = 0. \end{cases} \tag{18}$$

To do so, we introduce the operator:

$$\widetilde{T}(x,y,\lambda) := e^{\Lambda^{-1}(\lambda I_d - M^T)(x-y)}.$$

The solution to (18) is given by:

$$R(x) = \widetilde{T}(x,0,\lambda)(I_{d_1},0)^T v^{\star}(0)$$

where $v^{\star}(0) \in \mathbb{C}^{d_1}$. In order to satisfy the right-border boundary condition, it is needed to impose:

$$(-I_{d_1}, \Lambda_1^{-1} H^T \Lambda_2)\widetilde{T}(1,0,\lambda)(I_{d_1},0)^T v^{\star}(0) = 0_{d_1}.$$

Hence the spectrum of $\mathscr{A}^{\star}$ is given by the equation:

$$\det(\widetilde{U}(\lambda)) := \det((-I_{d_1}, \Lambda_1^{-1} H^T \Lambda_2)\widetilde{T}(1,0,\lambda)(0, I_{d_2})^T) = 0, \tag{19}$$

where:

$$\widetilde{U}(\lambda) = (-I_{d_1}, \Lambda_1^{-1} H^T \Lambda_2)\widetilde{T}(1,0,\lambda)(0, I_{d_2})^T.$$

Similarly to $\mathscr{A}$, the spectrum of $\mathscr{A}^{\star}$ corresponds to its point spectrum and for all $\lambda \in \sigma_p(\mathscr{A}^{\star})$:

$$\mathrm{Eig}(\mathscr{A}^{\star}, \lambda) = \left\{ \widetilde{T}(x,0,\lambda)(I_{d_1},0)^T v(0) \;\middle|\; v(0) \in \ker(\widetilde{U}(\lambda)) \right\}.$$

To conclude, it suffices to remark that for all $\lambda \in \sigma(\mathscr{A}^{\star})$ and all $v^{\star}(0) \in \ker(\widetilde{U}(\lambda))$:

$$\mathscr{B}^{\star}\widetilde{T}(x,0,\lambda)(I_{d_1},0)^T v^{\star}(0) = 0 \iff v^{\star}(0) = 0.$$

This proves Fattorini's condition:

$$\ker(\lambda I - \mathscr{A}^{\star}) \cap \ker \mathscr{B}^{\star} = \{0\}, \quad \forall\, \lambda \in \mathbb{C}$$

which immediately implies (17). The finite-dimensional system $(\mathscr{A}_{\mathscr{M}}, \mathscr{B}_{\mathscr{M}})$ is controllable and we can apply a pole placement theorem to find a matrix gain K such that $\mathscr{A}_{\mathscr{M}} + \mathscr{B}_{\mathscr{M}} K$ is Hurwitz. □

4.2. *Well-posedness of the closed-loop system*

Now we take a gain matrix K stabilizing the finite-dimensional part of (16), define $\mathscr{K} := KP$ and system (7) split as follows:

$$\begin{cases} P\dot{R} = (P\mathscr{A}P + P\mathscr{B}KP)R \\ (I-P)\dot{R} = ((I-P)\mathscr{A}(I-P) + (I-P)\mathscr{B}KP)R. \end{cases} \tag{20}$$

Our notion of solution is given in the following definition:

Definition 23. *If $R_0 \in \mathscr{H}$ is the initial data considered and $T > 0$. The element $R \in C^1([0,T],\mathscr{H}_{-1}) \cap C([0,T];\mathscr{H})$ is a solution to* (7) *if for all $\varphi \in \mathscr{H}_1^d$*

$$\langle R(t),\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} = \langle R_0, e^{\mathscr{A}^\star t}\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} + \int_0^t \langle KPR, \mathscr{B}^\star e^{\mathscr{A}^\star(t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s, \quad \forall\, 0 \le t \le T. \tag{21}$$

where $\mathbb{U}$ is identified with its dual.

Proposition 24. *There exists a unique solution to* (7) *in the sense of Definition 23.*

Proof. Let $T > 0$. We use a Banach–Picard fixed-point theorem proving existence and uniqueness at the same time. Let us define $\mathscr{T} : C([0,T],\mathscr{H}) \to C([0,T],\mathscr{H})$ the application such that for all $\varphi \in \mathscr{H}$:

$$\langle(\mathscr{T}R)(t),\varphi\rangle_{\mathscr{H}} = \langle R_0, e^{\mathscr{A}^\star t}\varphi\rangle_{\mathscr{H}} + \int_0^t \langle KPR(s), \mathscr{B}^\star e^{\mathscr{A}^\star(t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s, \quad \forall\, R \in C([0,T],\mathscr{H}), 0 \le t \le T.$$

Let $R, Q \in C([0,T],\mathscr{H})$, we have:

$$\langle(\mathscr{T}(R-Q))(t),\varphi\rangle_{\mathscr{H}} = \int_0^t \langle KP(R(s)-Q(s)), \mathscr{B}^\star e^{\mathscr{A}^\star(t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s, \quad \forall\, 0 \le t \le T.$$

By Cauchy–Schwartz inequality and Lemma 10:

$$\begin{aligned}\langle(\mathscr{T}(R-Q))(t),\varphi\rangle_{\mathscr{H}} &\le \sqrt{\int_0^t \|\mathscr{B}^\star e^{(t-s)\mathscr{A}^\star}\varphi\|_{\mathbb{U}}^2\mathrm{d}s} \times \sqrt{\int_0^t \|KP(R(s)-Q(s))\|_{\mathbb{U}}^2\mathrm{d}s} \\ &\le \sqrt{C\|\varphi\|_{\mathscr{H}}^2} \times \sqrt{t\|KP\|_{\mathscr{L}(\mathscr{H},\mathbb{U})}^2 \|R-Q\|_{C([0,T],\mathscr{H})}^2}\end{aligned}$$

where we have used the fact that KP is in $\mathscr{L}(\mathscr{H},\mathbb{U})$. Indeed, P is a projection, hence $P \in \mathscr{L}(\mathscr{H})$ and K is a matrix. As a consequence,

$$\langle(\mathscr{T}(R-Q))(t),\varphi\rangle_{\mathscr{H}} \le C\sqrt{T}\|KP\|_{\mathscr{L}(\mathscr{H},\mathbb{U})}\|R-Q\|_{C([0,T],\mathscr{H})}\|\varphi\|_{\mathscr{H}}, \quad \forall\, 0 \le t \le T.$$

Taking T sufficiently small (uniformly with respect to R_0), it holds:

$$\langle(\mathscr{T}(R-Q))(t),\varphi\rangle_{\mathscr{H}} \le \frac{\|R-Q\|_{C([0,T],\mathscr{H})}}{2}\|\varphi\|_{\mathscr{H}}, \quad \forall\, 0 \le t \le T.$$

As a consequence,

$$\|\mathscr{T}(R-Q))(t)\|_{\mathscr{H}} \le \frac{\|R-Q\|_{C([0,T],\mathscr{H})}}{2}, \quad \forall\, 0 \le t \le T.$$

We can apply Banach–Picard theorem to assert the existence of a unique fixed point of $\mathscr{T}$ in $C([0,T],\mathscr{H})$ for T sufficiently small. By a bootstrap argument, we conclude on the existence and uniqueness in $C([0,T],\mathscr{H})$ for all $T > 0$. This unique solution is denoted by R and for $\varphi \in \mathscr{H}_1^d$:

$$\langle R(t),\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} = \langle R_0, e^{\mathscr{A}^\star t}\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} + \int_0^t \langle KPR(s), \mathscr{B}^\star e^{\mathscr{A}^\star(t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}}ds, \quad \forall\, 0 \le t \le T. \tag{22}$$

The equation (22) is equivalent to:

$$\langle R(t),\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} = \langle R_0, e^{\mathscr{A}^\star t}\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} + \int_0^t \langle \mathscr{B}KPR(s), e^{\mathscr{A}^\star(t-s)}\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d}ds, \quad \forall\, 0 \le t \le T.$$

Owing the fact that $R \in C([0,T],\mathscr{H})$, $\mathscr{B} \in \mathscr{L}(\mathbb{U},\mathscr{H}_{-1})$ and $s \mapsto e^{\mathscr{A}^\star s}\varphi \in C^1([0,T],\mathscr{H}_1^d)$, one deduces that $R \in C^1([0,T],\mathscr{H}_{-1})$. Moreover:

$$\left\langle \frac{dR}{dt}(t),\varphi \right\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} = \langle R_0, \mathscr{A}^\star e^{\mathscr{A}^\star t}\varphi\rangle_{\mathscr{H}} + \langle KPR(t), \mathscr{B}^\star \varphi\rangle_{\mathbb{U},\mathbb{U}} - \int_0^t \langle KPR(s), \mathscr{B}^\star e^{\mathscr{A}^\star (t-s)}\mathscr{A}^\star\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s, \qquad \forall\, 0 \le t \le T.$$

All terms in last equation are convergent because of Lemma 10. Indeed,

$$\begin{aligned}\int_0^t \langle KPR(s), \mathscr{B}^\star \mathscr{A}^\star e^{\mathscr{A}^\star (t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s &\le C\|\mathscr{A}^\star\varphi\|_{\mathscr{H}} \times \|KP\|_{\mathscr{L}(\mathscr{H})}\|R\|_{C([0,T],\mathscr{H})}\\ &\le C\|\varphi\|_{\mathscr{H}_1^d} \times \|KP\|_{\mathscr{L}(\mathscr{H})}\|R\|_{C([0,T],\mathscr{H}_{-1})}.\end{aligned}$$

This concludes the proof of Proposition 24. □

When the initial data is only known to be in $\mathscr{H}_{-1}$, we have the following well-posedness result:

Proposition 25. *Let $T > 0$. If $R_0 \in \mathscr{H}_{-1}$, there exists a unique solution in $C([0,T],\mathscr{H}_{-1})$ to* (7) *in the sense that for all $\varphi \in \mathscr{H}_d^1$,* (21) *holds.*

Proof. The proof is very similar to the one of Proposition 24 and this is why, we only give a sketch of the proof here.

Let us define $\mathscr{T} : C([0,T],\mathscr{H}_{-1}) \to C([0,T],\mathscr{H}_{-1})$ the application such that for all $\varphi \in \mathscr{H}_d^1$, $0 \le t \le T$:

$$\langle(\mathscr{T}R)(t),\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_d^1} = \langle R_0, e^{\mathscr{A}^\star t}\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_d^1} + \int_0^t \langle KPR(s), \mathscr{B}^\star e^{\mathscr{A}^\star (t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s,\ \ \forall\, R \in C([0,T],\mathscr{H}_{-1}).$$

Let $R, Q \in C([0,T],\mathscr{H}_{-1})$. To prove the fixed-point argument, we need to estimate the following term:

$$\begin{aligned}&\int_0^t \langle KP(R(s)-Q(s)), \mathscr{B}^\star e^{\mathscr{A}^\star (t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s\\ &\quad= \int_0^t \langle \mathscr{B}KP(R(s)-Q(s)), e^{\mathscr{A}^\star (t-s)}\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_d^1}\mathrm{d}s\\ &\quad\le \|\mathscr{B}\|_{\mathscr{L}(\mathbb{U},\mathscr{H}_{-1})}\|K\|_{\mathscr{L}(\mathscr{H}_{-1},\mathbb{U})}\|P\|_{\mathscr{L}(\mathscr{H}_{-1})}\int_0^t \|e^{\mathscr{A}^\star (t-s)}\varphi\|_{\mathscr{H}_d^1}\mathrm{d}s \times \|R-Q\|_{C([0,T],\mathscr{H}_{-1})}\end{aligned}$$

With the following estimate ($\lambda \in \rho(\mathscr{A})$):

$$\begin{aligned}\|e^{\mathscr{A}^\star (t-s)}\varphi\|_{\mathscr{H}_d^1} &= \|(\bar{\lambda}I - \mathscr{A}^\star)e^{\mathscr{A}^\star (t-s)}\varphi\|_{\mathscr{H}}\|e^{\mathscr{A}^\star (t-s)}(\bar{\lambda}I - \mathscr{A}^\star)\varphi\|_{\mathscr{H}}\\ &\le \|e^{\mathscr{A}^\star (t-s)}\|_{\mathscr{L}(\mathscr{H})}\|\varphi\|_{\mathscr{H}_d^1}\\ &\le Ce^{\omega(t-s)}\|\varphi\|_{\mathscr{H}_d^1},\end{aligned}$$

where ω is the growth bound of the semigroup generated by $\mathscr{A}^\star$, one gets:

$$\begin{aligned}&\int_0^t \langle KP(R(s)-Q(s)), \mathscr{B}^\star e^{\mathscr{A}^\star (t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s\\ &\quad\le \|\mathscr{B}\|_{\mathscr{L}(\mathbb{U},\mathscr{H}_{-1})}\|K\|_{\mathscr{L}(\mathscr{H}_{-1},\mathbb{U})}\|P\|_{\mathscr{L}(\mathscr{H}_{-1})}\frac{e^{\omega t}-1}{\omega}\|\varphi\|_{\mathscr{H}_d^1} \times \|R-S\|_{C([0,T],\mathscr{H}_{-1})}.\end{aligned}$$

For t small enough, we get the contractive property of $\mathscr{T}$ and the rest of the proof follows the same steps as the one of Proposition 24. □

4.3. *Conclusion on the stability of the whole system*

We choose a vector gain $K \in M_{d_1 \times \alpha}(\mathbb{R})$ (recall that α is the dimension of $\mathscr{M}$) such that the finite-dimensional system exponentially converges to zero with rate $\tau > 0$ fixed:

$$\|PR(t)\|_{\mathscr{H}} \leq Ce^{-\tau t}\|PR_0\|_{\mathscr{H}}, \quad \forall\, t \geq 0. \tag{23}$$

In order to conclude on the whole stability, we have to prove that the infinite-dimensional part in (20) is not destabilized by the control. To prove the stability of the infinite-dimensional part, it suffices to see the second equation of (20) as an inhomogeneous Cauchy problem on $\mathscr{M}'$:

$$(I-\dot{P})R = \mathscr{A}_{\mathscr{M}'}(I-P)R + (I-P)\mathscr{B}u(t)$$

where $u(t)$ verifies the following estimate:

$$|u|_{\mathbb{U}} \leq Ce^{-\tau t}, \quad \forall\, t \geq 0$$

and $\mathscr{A}_{\mathscr{M}'}$ is the restriction of $\mathscr{A}$ on $\mathscr{M}'$. The way we separated the spectrum gives that $\mathscr{A}_{\mathscr{M}'}$ is stable in the sense that:

$$\sigma(\mathscr{A}_{\mathscr{M}'}) \subset \{z \in \mathbb{C} \mid \Re z < -\widetilde{\tau}\}$$

with $\widetilde{\tau} > 0$. To conclude, we need the following spectral mapping Theorem from [24]:

Theorem 26. *The spectral mapping theorem holds true:*

$$\sigma(e^{\mathscr{A}t}) \setminus \{0\} = \overline{e^{\sigma(\mathscr{A})t}} \setminus \{0\}, \quad \forall\, t \geq 0. \tag{24}$$

Corollary 27. *Property* (24) *is also true for* $\mathscr{A}_{\mathscr{M}'}$.

Proof. To characterize the spectrum of a semigroup when comparing it with its generator, we will need the following theorem:

Theorem 28 (Gearhart-Prüss Spectral Mapping Theorem [16, 27]). *Let* $(e^{At})_{t\geq 0}$ *be a* C_0 *semi-group generated by* A *in a Hilbert space. Then* $e^{\lambda t} \in \rho(e^{At})$ *iff*

$$\left\{\lambda + i2\pi t^{-1}z \mid z \subset \mathbb{Z}\right\} \in \rho(A) \quad \text{and} \quad \sup_{z\in\mathbb{Z}} \|R(\lambda + i2\pi t^{-1}z, A)\| < \infty.$$

We first show that $\sigma(e^{\mathscr{A}_{\mathscr{M}'}t}) \setminus \{0\} \subset \overline{e^{\sigma(\mathscr{A}_{\mathscr{M}'})t}} \setminus \{0\}$. Let $\lambda \in \mathbb{C}$ such that $e^{\lambda t} \notin \overline{e^{\sigma(\mathscr{A}_{\mathscr{M}'})t}}$. We have to show that $e^{\lambda t} \in \rho(e^{\mathscr{A}_{\mathscr{M}'}t})$. By the definition of λ, there exists $\delta > 0$ such that:

$$\bigcup_{z\in\mathbb{Z}} B(\lambda + 2i\pi z/t, \delta) \subset \rho(\mathscr{A}_{\mathscr{M}'}). \tag{25}$$

We need to prove that $R(\lambda + 2i\pi z/t, \mathscr{A}_{\mathscr{M}'}) = R(\lambda + 2i\pi z/t, \mathscr{A})_{|\mathscr{M}'}$ [22, Theorem 6.17] is uniformly bounded with respect to $z \in \mathbb{Z}$. Note that for z large enough, $\lambda + 2i\pi z/t \in \rho(\mathscr{A})$ because of Lemma 16 and Rouché theorem. As a consequence, it suffices to prove that $R(\lambda + 2i\pi z/t, \mathscr{A})$ is bounded when z goes to infinity. The following lemma will help us to conclude:

Lemma 29 ([24]). *Let* $U \subset \rho(\mathscr{A})$ *so that* $\sup_{\lambda\in U} |\Re\lambda| < \infty$ *and* $\inf_{\lambda\in U} |\kappa_0(\lambda)| > 0$. *Then, there exists* $d > 0$ *such that for* $\lambda \in U$ *and* $|\Im\lambda| \geq d$*:*

$$R(\lambda, \mathscr{A}) = R(\lambda, \mathscr{A}_0) + \mathscr{E}(\lambda, \mathscr{A})/\lambda$$

with $R(\lambda, \mathscr{A}_0)$ *and* $\mathscr{E}(\lambda, \mathscr{A})$ *bounded on* U. *In particular,* $R(\lambda, \mathscr{A})$ *is bounded on* U.

Because of Corollary 18, one has:

$$\inf_{\bigcup_{z\in\mathbb{Z}} B(\lambda+2i\pi z/t,\delta)} |\kappa_0(\lambda)| \neq 0. \tag{26}$$

We apply previous lemma to $U = \bigcup_{z\in\mathbb{Z}} B(\lambda + 2i\pi z/t, \delta)$ which gives immediately that $R(\lambda + 2i\pi z/t, \mathscr{A})$ is bounded when z goes to infinity. Hence,

$$\sigma(e^{\mathscr{A}_{\mathscr{M}'}t}) \setminus \{0\} \subset \overline{e^{\sigma(\mathscr{A}_{\mathscr{M}'})t}} \setminus \{0\}.$$

The reverse inclusion is classic for all closed densely defined operators [26, Theorem 2.3] . This finishes the proof of Corollary 27. □

Property (24) holds true for $\mathscr{A}_{\mathscr{M}'}$. Hence, the growth bound of $\mathscr{A}_{\mathscr{M}'}$ verifies:

$$\omega(\mathscr{A}_{\mathscr{M}'}) = \sup\left\{\Re\lambda \;\middle|\; \lambda \in \sigma(\mathscr{A}_{\mathscr{M}'})\right\} =: -\widetilde{\tau} < 0.$$

This allows to prove the following stability result:

Lemma 30. *There exists $C > 0$ such that for all $t \geq 0$:*

$$\|e^{\mathscr{A}_{\mathscr{M}'}t}(I-P)\|_{\mathscr{L}(\mathscr{H})} \leq Ce^{-\tilde{\tau}t} \tag{27}$$

and

$$\|e^{\mathscr{A}_{\mathscr{M}'}t}(I-P)\|_{\mathscr{L}(\mathscr{H}_{-1})} \leq Ce^{-\tilde{\tau}t}. \tag{28}$$

Proof. The first inequality is proven by Gelfand formula [30, Remark 2.2.16] and Corollary 27. For the second inequality, take $z \in \mathscr{H}_{-1}, \varphi \in \mathscr{H}_1^d$.

$$\begin{aligned}\langle e^{\mathscr{A}_{\mathscr{M}'}t}(I-P)z,\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} &= \langle z, e^{\mathscr{A}^\star t}(I-P^\star)\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d}\\ &\leq \|z\|_{\mathscr{H}_{-1}}\|e^{\mathscr{A}^\star t}(I-P^\star)\varphi\|_{\mathscr{H}_1^d}\\ &= \|z\|_{\mathscr{H}_{-1}}\|(\lambda I-\mathscr{A}^\star)e^{\mathscr{A}^\star t}(I-P^\star)\varphi\|_{\mathscr{H}}\end{aligned}$$

As $e^{\mathscr{A}^\star t}$ and $\lambda I - \mathscr{A}^\star$ commutes and as $\mathscr{A}^\star$ commutes with $P^\star$, one gets:

$$\begin{aligned}\langle e^{\mathscr{A}_{\mathscr{M}'}t}(I-P)z,\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} &\leq \|z\|_{\mathscr{H}_{-1}}\|e^{\mathscr{A}^\star t}(I-P^\star)(\lambda I-\mathscr{A}^\star)\varphi\|_{\mathscr{H}}\\ &\leq \|z\|_{\mathscr{H}_{-1}}\|e^{\mathscr{A}^\star_{\mathscr{M}'}t}(I-P^\star)\|_{\mathscr{L}(\mathscr{H})}\|(\lambda I-\mathscr{A}^\star)\varphi\|_{\mathscr{H}}\\ &= \|z\|_{\mathscr{H}_{-1}}\|e^{\mathscr{A}^\star_{\mathscr{M}'}t}(I-P^\star)\|_{\mathscr{L}(\mathscr{H})}\|\varphi\|_{\mathscr{H}_1^d}\end{aligned}$$

To conclude, we use the fact that the norm of the adjoint is also the norm of the anti-adjoint and by (27):

$$\langle e^{\mathscr{A}_{\mathscr{M}'}t}z,\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} \leq \|z\|_{\mathscr{H}_{-1}}Ce^{-\tau t}\|\varphi\|_{\mathscr{H}_1^d}$$

which finishes the proof of the lemma. □

By the definition of solution (Definition 23), we have for all $\varphi \in \mathscr{H}_1^d$:

$$\langle R(t),\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} = \langle R_0, e^{\mathscr{A}^\star t}\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} + \int_0^t \langle KPR, \mathscr{B}^\star e^{\mathscr{A}^\star(t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s.$$

The following decomposition is used to conclude on the exponential stability:

$$\langle R(t),\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} = \langle R(t), P^\star\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} + \langle R(t),(I-P^\star)\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d}$$

and we estimate both terms on the right hand side of last equation.

For the simplest one:

$$\begin{aligned}\langle R(t),P^\star\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} &= \langle PR(t),\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d}\\ &\leq Ce^{-\tau t}\|PR_0\|_{\mathscr{H}}\|\varphi\|_{\mathscr{H}}\\ &\leq Ce^{-\tau t}\|PR_0\|_{\mathscr{H}}\|\varphi\|_{\mathscr{H}_d^1}\end{aligned} \tag{29}$$

where we used (23) and the fact that $\mathscr{H}_1^d$ is continuously embedded in $\mathscr{H}$.

For the other estimate:

$$\begin{aligned}
&\langle R(t),(I-P^\star)\varphi\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d}\\
&=\langle R_0, e^{\mathcal{A}^\star t}(I-P^\star)\varphi\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d}+\int_0^t\langle KPR,\mathcal{B}^\star e^{\mathcal{A}^\star(t-s)}(I-P^\star)\varphi\rangle_{\mathbb{U},\mathbb{U}}\mathrm{d}s\\
&=\langle e^{\mathcal{A}t}(I-P)R_0,\varphi\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d}+\int_0^t\langle e^{\mathcal{A}(t-s)}(I-P)\mathcal{B}KPR,\varphi\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d}\mathrm{d}s\\
&=\langle e^{\mathcal{A}_{\mathcal{M}'}t}(I-P)R_0,\varphi\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d}+\int_0^t\langle e^{\mathcal{A}_{\mathcal{M}'}(t-s)}(I-P)\mathcal{B}KPR,\varphi\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d}\mathrm{d}s\\
&\le\Big(\|e^{\mathcal{A}_{\mathcal{M}'}t}(I-P)\|_{\mathcal{L}(\mathcal{H}_{-1})}\|R_0\|_{\mathcal{H}_{-1}}\\
&\qquad+\int_0^t\|\mathcal{B}\|_{\mathcal{L}(\mathbb{U},\mathcal{H}_{-1})}\|K\|_{\mathcal{L}(\mathcal{H}_{-1},\mathbb{U})}\|PR(s)\|_{\mathcal{H}_{-1}}\|e^{\mathcal{A}_{\mathcal{M}'}(t-s)}(I-P)\|_{\mathcal{L}(\mathcal{H}_{-1})}\mathrm{d}t\Big)\|\varphi\|_{\mathcal{H}_1^d}
\end{aligned}\tag{30}$$

where we used the fact that P and $e^{s\mathcal{A}}$ commute.

The inequality (23) combined with Lemma 20 and the fact that $\mathcal{H}$ is continuously embedded in $\mathcal{H}_{-1}$ gives:

$$\|PR(s)\|_{\mathcal{H}_{-1}}\le Ce^{-\tau t}\|R_0\|_{\mathcal{H}_{-1}}$$

Combining last equation with (28) to bound the left hand side of (30), one gets:

$$\|(I-P)R(t)\|_{\mathcal{H}_{-1}}\le e^{-\tilde{\tau}t}\|(I-P)R_0\|_{\mathcal{H}_{-1}}+C\int_0^t e^{-(t-s)\tilde{\tau}}e^{-\tau s}\mathrm{d}s\|R_0\|_{\mathcal{H}_{-1}}\tag{31}$$

where C depends on K and the parameters of the problem. Hence, the exponential stability in $\mathcal{H}_{-1}$ holds with rate (at least) $\min(\tilde{\tau},\tau)$. This finishes the proof of Theorem 8.

Remark 31. If instead of (1), we focus on the stabilization of:

$$\begin{cases}R_t+\Lambda R_x=MR\\ R_1(t,0)=FR_2(t,0)+u(t)\\ R_2(t,1)=HR_1(t,1)\end{cases}\tag{32}$$

where F is a real matrix of suitable dimension. Then, the characteristic equation of the system will be modified giving eigenvalues at high frequency. Indeed, in this case:

$$\kappa_0(\lambda)=\det((H,-I_{d_2})e^{-\Lambda^{-1}(\lambda I_d-M_0)}(F,I_{d_2})^T)$$

which may have zeros at $|\Im\lambda|\gg 1$. To illustrate this, we give an example with $d_1=d_2=1$:

$$\begin{cases}R_t+\Lambda R_x=MR\\ R_1(t,0)=2R_2(t,0)\\ R_2(t,1)=2R_1(t,1)\end{cases}$$

with:

$$\Lambda=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad M=\begin{pmatrix}1&-1\\1&1\end{pmatrix}.$$

Then, the characteristic equation $\kappa_0(\lambda)=0$ becomes:

$$4e^{-(\lambda-1)}-e^{\lambda-1}=0$$

for which solutions are explicit:

$$\forall\, k\in\mathbb{Z},\quad \lambda_k=\log(2)+1+2ik\pi$$

which immediately implies that the number of unstable poles is infinite.

5. Towards a nonlinear control

5.1. *The saturation of the control*

In this section, the following abstract problem is considered:

$$\begin{cases} \dfrac{\mathrm{d}R}{\mathrm{d}t} = \mathscr{A}R + \mathscr{B}\sigma(\mathscr{K}R) \\ R(0) = R_0 \end{cases} \tag{33}$$

where $\sigma : \mathbb{U} \to \mathbb{U}$ is the usual saturation by components with $\sigma_s > 0$ the saturation level. More precisely, for all $R \in \mathbb{U}(= \mathbb{R}^{d_1}), 1 \le i \le d_1, \sigma(R)_i = \sigma_i(R_i)$ with $\sigma_i : \mathbb{R} \to \mathbb{R}$ defined as:

$$\sigma(x) = \begin{cases} x & \text{if } |x| \le \sigma_s \\ \sigma_s \operatorname{sign} x & \text{otherwise.} \end{cases}$$

If $\mathscr{K} := KP$ where P is the projection on the unstable space of $\mathscr{A}$ and K is a matrix, we can easily give the notion of solution:

Definition 32. *If $R_0 \in \mathscr{H}_{-1}$ is the initial data considered. We say that $R \in C([0,T], \mathscr{H}_{-1})$ is a solution to* (7) *if for all $\varphi \in \mathscr{H}_1^d$*

$$\langle R(t), \varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} = \langle R_0, e^{\mathscr{A}^\star t}\varphi\rangle_{\mathscr{H}_{-1},\mathscr{H}_1^d} + \int_0^t \langle \sigma(KPR), \mathscr{B}^\star e^{\mathscr{A}^\star (t-s)}\varphi\rangle_{\mathbb{U},\mathbb{U}} \mathrm{d}s, \quad \forall\, t \ge 0. \tag{34}$$

where $\mathbb{U}$ is identified with its dual.

Using a fixed point argument as in previous section, it is easy to prove that such a solution exists and is unique. As we have seen in the previous section, the stability of the finite-dimensional part implies the stability of the entire system. As a consequence, to estimate a basin of attraction, techniques from finite-dimension literature will be used. More precisely, the following theorem from [29, Theorem 3.1, p. 125] is a key tool:

Theorem 33. *If there exist a symmetric definite positive matrix $W \in M_{\alpha\times\alpha}(\mathbb{R})$, $S > 0$, a matrix $Z \in M_{\alpha\times 1}(\mathbb{R})$ such that:*

$$\begin{pmatrix} W(\mathscr{A}_{\mathscr{M}} + (P\mathscr{B})K)^T + (\mathscr{A}_{\mathscr{M}} + (P\mathscr{B})K)W & (P\mathscr{B})S - Z^T \\ S(P\mathscr{B})^T - Z & -2S \end{pmatrix} < 0 \tag{35}$$

and

$$\begin{pmatrix} W & WK^T - Z^T \\ KW - Z & \sigma_s^2 \end{pmatrix} \ge 0 \tag{36}$$

then, the cylinder $\mathscr{E}(W^{-1}, 1) := \left\{R \in \mathscr{H}_{-1} \mid (PR)^T W^{-1}(PR) < 1\right\}$ is a region of stability for system (33).

This theorem has a surprising consequence. The basin of attraction is expressed in the unstable finite-dimensional subspace of $\mathscr{A}$. As a consequence, if we suppose that we can find a solution W, S, Z for the matrix inequalities (35)–(36). Then, taking an initial data $R_0 \in \mathscr{H}_{-1}$ in the cylinder $\mathscr{E}(W^{-1}, 1)$, one can take an arbitrarily large stable part $(I - P)R_0$ such that the corresponding solution is stable. Hence, the basin of attraction is not bounded in $\mathscr{H}_{-1}$.

5.2. *Numerical simulations*

5.2.1. *Linear feedback*

In this section, an example illustrating the problem of stabilization, is discretized. The space and time steps are denoted respectively $\mathrm{d}x$ and $\mathrm{d}t$. We denote $N := E(1/\mathrm{d}x) \in \mathbb{N}$. For this example, the dimensions are $d_1 = d_2 = 1$ and:

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad H = 1.$$

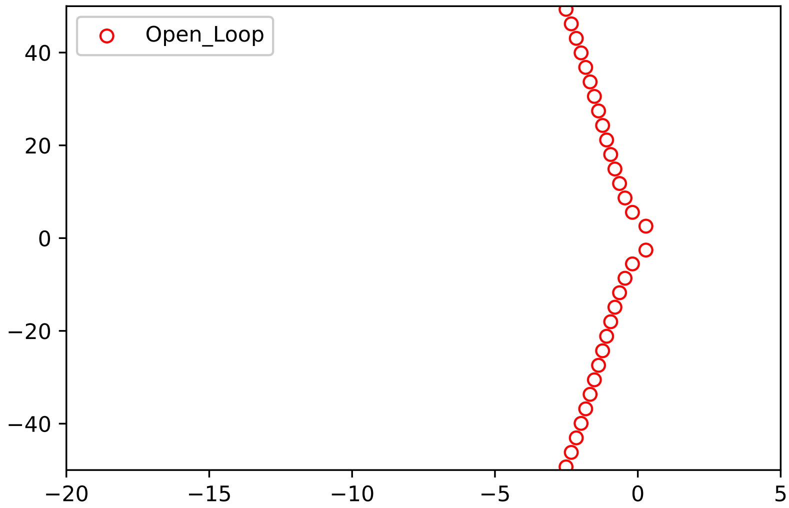

Figure 1. The spectrum of the open-loop operator A for $\mathrm{d}x = 10^{-3}$

The state vector $R \in M_{2N\times 1}(\mathbb{R})$ is now a vector such that $R(1\cdots N), R(N+1\cdots 2N)$ is the numerical version of R_1, R_2 respectively. The discretized open-loop operator $A \in M_{2N\times 2N}(\mathbb{R})$ is expressed using a classical upwind scheme:

$$A = \left[\begin{array}{cccc|cccc} -\frac{1}{\mathrm{d}x}+M_{11} & 0 & \cdots & 0 & M_{12} & 0 & \dots & 0 \\ \frac{1}{\mathrm{d}x} & -\frac{1}{\mathrm{d}x}+M_{11} & \ddots & 0 & 0 & M_{12} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\mathrm{d}x} & -\frac{1}{\mathrm{d}x}+M_{11} & 0 & \cdots & \cdots & M_{12} \\ \hline M_{21} & 0 & \cdots & 0 & -\frac{1}{\mathrm{d}x}+M_{22} & \frac{1}{\mathrm{d}x} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & M_{21} & 0 & 0 & \ddots & -\frac{1}{\mathrm{d}x}+M_{22} & \frac{1}{\mathrm{d}x} \\ 0 & \cdots & 0 & M_{21}+\frac{H}{\mathrm{d}x} & 0 & \cdots & 0 & -\frac{1}{\mathrm{d}x}+M_{22} \end{array}\right].$$

Using the function "eig" from python, we plot the spectrum of A:

One gets $2N$ simple eigenvalues with two unstable modes. The corresponding eigenvectors are stored in a matrix $V_{ec} \in M_{2N\times 2N}(\mathbb{C})$ whose eigenvectors are its columns. For complex conjugate eigenvalues, the corresponding couple of complex conjugate eigenvectors (denoted $F, \overline{F}$) is replaced by $\Re(F), \Im(F)$ to work with real matrices. Thus, $V_{ec} \in M_{2N\times 2N}(\mathbb{R})$ is now a real matrix. For the control operator $B \in M_{2N\times 1}(\mathbb{R})$, it can be written as:

$$B = \begin{bmatrix} 1/\mathrm{d}x \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

To compute the control, matrices A, B are projected on the unstable eigenspace. To do so, V_{ec}^{-1} is computed and we denote the first two columns of V_{ec} by $P := (V_{ec,i,j})_{1\le i\le 2N, 1\le j\le 2} \in M_{2N\times 2}(\mathbb{R})$.

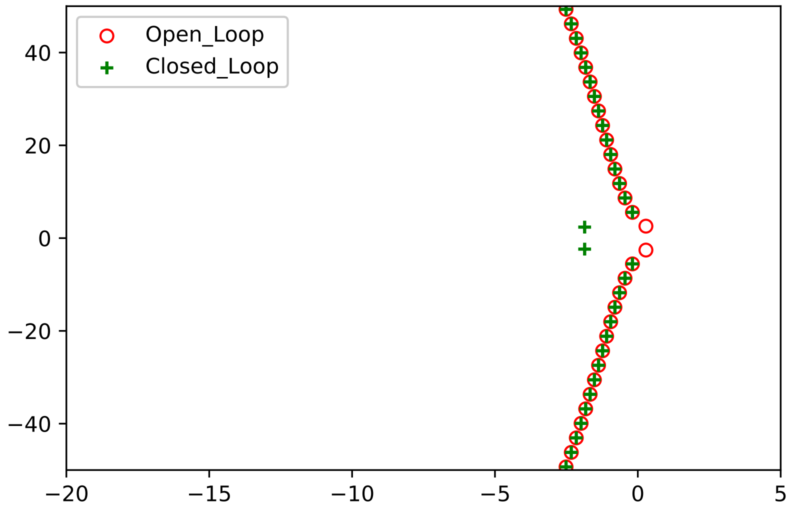

Figure 2. The spectrum of the closed-loop operator $A+BK$ for $\mathrm{d}x = 10^{-3}$

Similarly, $P^\star := (V^{-1}_{ec,i,j})_{1\le i\le 2, 1\le j\le 2N} \in M_{2\times 2N}(\mathbb{R})$ is the first two lines of V^{-1}_{ec}. The projection on the unstable eigenspace writes:

$$A_{\text{proj}} = P^\star AP,\ B_{\text{proj}} = P^\star B.$$

Then, we apply a classic pole placement algorithm to matrices $A_{\text{proj}}, B_{\text{proj}}$ to move unstable poles to the desired locus. It gives a feedback matrix $K_{\text{proj}} \in M_{1\times 2}(\mathbb{R})$ stabilizing the open-loop. To obtain the non-projected feedback matrix, it suffices to return to the state space:

$$K := K_{\text{proj}} P^\star \in M_{1\times 2N}(\mathbb{R}).$$

We get the following spectrum for the following closed-loop operator $A+BK$:

One can observe that poles are now all stable which implies that the closed loop system is stable.

5.2.2. *Saturated feedback*

Now, the focus is on the case where we saturate the control with a saturation level $\sigma_s := 1$. For computations, we take $\mathrm{d}x = 5\times 10^{-3}$. We then evaluate the relevance of the estimation of the basin of attraction given in Theorem 33. It has been observed that the unstable space is of dimension 2 and is generated by vectors $P_1 := (P_{i,1})_i, P_2 := (P_{i,2})_i$. Hence, to study the basin of attraction, it is sufficient to study initial conditions of the form:

$$R_0 \in xP_1 + yP_2 \in M_{2N\times 1}(\mathbb{R}) \tag{37}$$

where $(x,y)\in\mathbb{R}^2$.

In Figure 3, the x-axis corresponds to the variable x in (37) and the y-axis corresponds to the variable y in (37). To estimate the real basin of attraction, at each initial data given in (37) we compute the solution using a time explicit Euler scheme (the time step being $\mathrm{d}t = 0.9\times \mathrm{d}x$) and observe the exponential rate of its L^2 norm. This corresponds to the colormap in Figure 3. Divergence is associated to the red color whereas convergence corresponds to the blue color. The green curve represents the contour related to a convergence rate equal to zero. Finally, solving

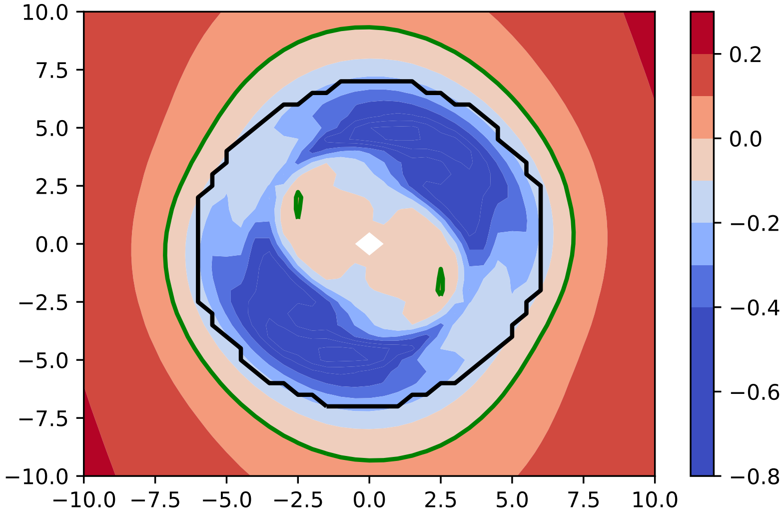

Figure 3. The basin of attraction and its estimation ($\mathrm{d}x = 5 \times 10^{-3}$)

matrix inequalities (35)-(36) from Theorem 33, we give an estimation of the basin of attraction $\mathscr{E}(W^{-1}, 1)$ represented by the closed black curve in Figure 3. Note that we have used the package cxvpy for Python to solve the matrix inequalities problem. We remark that the estimation is very close to the real basin of attraction.

6. Conclusion

In this work, a general system of linear balance laws coupled by the domain and the boundary is studied. More precisely, the problem of stabilizability is treated using a spectral method. The main idea is to use a pole placement theorem applied on the unstable finite-dimensional part of the system. As a by-product, the linear control has been saturated to give a more realistic model of controller and a result is given to estimate the basin of attraction. In the final part, numerical simulation are given to illustrate results from the linear and the saturated theory.

Some questions remain open. A priori, it is not guarantied that the control proposed is robust with respect to perturbation of the control. Proving this can be difficult since we do not use Lyapunov methods. Another important question is that the full state feedback imposes the full observation which may be problematic in real applications. In order to have a realistic model of controller, one should analyze the coupling between the controller and an observer. It is far from being obvious that the stability will be preserved. Finally, the problem of local stability of nonlinear balance laws using pole placement methods was discarded.

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