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On the continued fraction expansions of $(1+\sqrt{pq})/2$ and $\sqrt{pq}$

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Abstract. The evenness and the values modulo 4 of the lengths of the periods of the continued fraction expansions of $\sqrt{p}$ and $\sqrt{2p}$ for $p \equiv 3 \pmod 4$ a prime are known. Here we prove similar results for the continued fraction expansion of $\sqrt{pq}$, where $p, q \equiv 3 \pmod 4$ are distinct primes.

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1. Introduction

Let α be a real quadratic irrational number. Its continued fraction expansion $\alpha = [a_0, a_1, a_2, \dots]$ is periodic, i.e. there exists $k \geq 0$ and $l \geq 1$ such that $a_{i+l} = a_i$ for $i \geq k$. In that case we write $\alpha = [a_0, \dots, a_{k-1}, \overline{a_k, \dots, a_{k+l-1}}]$. The least such l is called the length of the period of the periodic continued fraction expansion of α. The evenness of the length of the period of the continued fraction expansion of $\sqrt{p}$ for $p \equiv 3 \pmod 4$ a prime is well known. In [8] we determined its value modulo 4 and gave a similar result for $\sqrt{2p}$:

Theorem 1. *Take $d = p$ or $d = 2p$, where $p \equiv 3 \pmod 4$ is a prime integer. Let $l \geq 1$ be the length of the period of the periodic continued fraction expansion $\sqrt{d} = [a_0, \overline{a_1, \dots, a_l}]$. Then,*

(i) $a_0 = \lfloor \sqrt{d} \rfloor$, $a_l = 2a_0$ *and* $a_k = a_{l-k}$ *for* $1 \leq k \leq l-1$,
(ii) $l = 2L$ *is even and* L *is even if and only if* $p \equiv 7 \pmod 8$,
(iii) $a_{l/2} = a_L$ *is the integer in* $\{a_0 - 1, a_0\}$ *of the same parity as* d.

This behavior in the case of $d = p$ had already been proved in [3, Corollary 2 p. 2071]. Our proof was different and applied both to $d = p$ and $d = 2p$. It was based on the arithmetic of quadratic number fields and their ideal class groups in the narrow sense (as in [6] and [7]). Let $\mathscr{I}$ be an integral ideal of the ring of algebraic integers $\mathbb{Z}_K$ of a real quadratic number field K. Recall that $\mathscr{I}$ is principal if and only if there exists $\alpha \in \mathbb{Z}_K$ such that $\mathscr{I} = \alpha\mathbb{Z}_K$, whereas $\mathscr{I}$ is principal in the narrow sense if there exists a totally positive element $\alpha \in \mathbb{Z}_K$ such that $\mathscr{I} = \alpha\mathbb{Z}_K$. Here, bearing on a similar approach, we prove:

Theorem 2. *Let p, q be two prime integers equal to* $3 \pmod 4$*, with* $3 \le p < q$*. Let* $l \ge 1$ *be the length of the period of the periodic continued fraction expansion* $(1+\sqrt{pq})/2 = [a_0, \overline{a_1, \dots, a_l}]$*. Then,*

(i) $a_l = 2a_0 - 1$ *and* $a_k = a_{l-k}$ *for* $1 \le k \le l-1$,
(ii) $l = 2L$ *is even and* $(-1)^L = \left(\frac{p}{q}\right)$ *(Legendre's symbol),*
(iii) $a_{l/2} = a_L$ *is the unique odd integer in* $\{\lfloor\sqrt{q/p}\rfloor - 1, \lfloor\sqrt{q/p}\rfloor\}$.

Theorem 3. *Let p, q be two prime integers equal to* $3 \pmod 4$*, with* $3 \le p < q$*. Let* $l \ge 1$ *be the length of the period of the periodic continued fraction expansion* $\sqrt{pq} = [a_0, \overline{a_1, \dots, a_l}]$*. Then,*

(i) $a_0 = \lfloor\sqrt{pq}\rfloor$, $a_l = 2a_0$ *and* $a_k = a_{l-k}$ *for* $1 \le k \le l-1$,
(ii) $l = 2L$ *is even and* $(-1)^L = \left(\frac{p}{q}\right)$ *(Legendre's symbol),*
(iii) $a_{l/2} = a_L = 2\lfloor\sqrt{q/p}\rfloor$ *is even.*

Part of Theorem 3 was proved in [10, Corollary 1], [2, Theorem 2] and [1], but notice that point (iii) of Theorem 3 is much more precise than [1, Theorem 1.2].

2. On the continued fraction expansions of some real quadratic irrational numbers

(i). Let ω be a real quadratic irrational number. Hence $\omega = (P+\sqrt{d})/Q$ for some non-square integer $d > 1$, some $P \in \mathbb{Z}$ and some $Q \in \mathbb{Z} \setminus \{0\}$ dividing $d - P^2$. Then ω is called *reduced* if $\omega > 1$ and $-1/\omega' > 1$, where $\omega' = (P-\sqrt{d})/Q$ is the conjugate of ω in $\mathbb{Q}(\sqrt{d})$. Hence, ω is reduced if and only if $P+\sqrt{d} > Q > \sqrt{d} - P > 0$, which implies $0 < Q < 2\sqrt{d}$, $|P| < \sqrt{d}$, $2\sqrt{d}/Q - 1 < \omega < 2\sqrt{d}/Q$ and $\lfloor\omega\rfloor \in \{\lfloor 2\sqrt{d}/Q\rfloor - 1, \lfloor 2\sqrt{d}/Q\rfloor\}$.

(ii). The *continued fraction expansion* $\omega_0 = [a_0, a_1, \dots]$ of $\omega_0 = (P_0+\sqrt{d})/Q_0$ with $P_0, Q_0 \in \mathbb{Z}$, and $Q_0 \ne 0$ dividing $d - P_0^2$, can be computed inductively by writing $\omega_k = [a_k, \dots]$ as $\omega_k = (P_k+\sqrt{d})/Q_k$, where the $P_k, Q_k \in \mathbb{Z}$ with $Q_k \ne 0$ dividing $d - P_k^2$ are inductively computed, using $a_k = \lfloor\omega_k\rfloor$ and $\omega_k = a_k + 1/\omega_{k+1}$, by $P_{k+1} = a_k Q_k - P_k$ and $Q_{k+1} = (d - P_{k+1}^2)/Q_k = (d - P_k^2)/Q_k + 2a_k P_k - a_k^2 Q_k$. (Hence Q_1 is a non-zero rational integer, $Q_{k+1} = Q_{k-1} + 2a_k P_k - a_k^2 Q_k$ for $k \ge 1$ and the Q_k's are non-zero rational integers, by induction on k.)

(iii). Assume that $\omega_0 = (P_0+\sqrt{d})/Q_0$ is reduced. Using $\omega_k = a_k + 1/\omega_{k+1}$, we obtain that all the ω_k's are reduced, by induction. Hence $0 < Q_k < 2\sqrt{d}$ and $|P_k| < \sqrt{d}$ for $k \ge 0$ and there are only finitely many pairwise distinct ω_k's. It follows that $\omega_m = \omega_n$ for some $m > n \ge 0$, which implies $\omega_{k+l} = \omega_k$ and $a_{k+l} = a_k$ for $k \ge b$, where $l := m - n \ge 1$. Hence, the continued fraction expansion of ω_0 is l-periodic. In fact is purely periodic, which we write $\omega_0 = [\overline{a_0, \dots, a_{l-1}}]$, i.e. $\omega_{k+l} = \omega_k$ and $a_{k+l} = a_k$ for $k \ge 0$, where $l := m - n \ge 1$. (Notice that $\omega_{k+l} = \omega_k$ and $k \ge 1$ imply $\omega_{k-1} - a_{k-1} = 1/\omega_k = 1/\omega_{k+l} = \omega_{k+l-1} - a_{k+l-1}$, hence imply $\omega_{k+l-1} - \omega_{k-1} = a_{k+l-1} - a_{k-1} \in \mathbb{Z}$ and $\omega_{k+l-1} - \omega_{k-1} = \omega'_{k+l-1} - \omega'_{k-1} \in (-1,1) \cap \mathbb{Z}$, hence imply $\omega_{k+l-1} = \omega_{k-1}$.) The least such $l \ge 1$ is called *the length* of the purely periodic continued fraction expansion of the reduced quadratic irrational number ω_0.

In that case $-1/\omega'_0 = [\overline{a_{l-1}, \dots, a_0}]$ (e.g. see [4, XV page 311]).

(iv). If $\omega_0 = [\overline{a_0, a_1, \dots, a_{l-1}}] \in \mathbb{Q}(\sqrt{d})$ is reduced, using $\omega_k = a_k + 1/\omega_{k+1}$ we obtain $\mathbb{M}_k := \mathbb{Z} + \mathbb{Z}\omega_k = \mathbb{Z} + \mathbb{Z}\omega_{k+1}^{-1} = \omega_{k+1}^{-1}\mathbb{M}_{k+1}$ and $\mathbb{M}_0 = \omega_1^{-1}\mathbb{M}_1 = \omega_1^{-1}\omega_2^{-1}\mathbb{M}_2 = \cdots = \varepsilon^{-1}\mathbb{M}_l = \varepsilon^{-1}\mathbb{M}_0$, where $\varepsilon = \omega_1\omega_2\dots\omega_l = \omega_0\omega_1\dots\omega_{l-1}$. Therefore, ε is a unit of norm $N(\varepsilon) = \prod_{k=0}^{l-1}(\omega_k\omega'_k) = (-1)^l$ of the $\mathbb{Z}$-module $\mathbb{M}_0 = \mathbb{Z} + \mathbb{Z}\omega_0 \subseteq \mathbb{Q}(\sqrt{d})$ (as $\omega_k > 1$ and $-1/\omega'_k > 1$).

(v). See [4, p. 305–322], [5, Chapter 10] and [9] for more information on continued fractions.

3. Proof of Theorem 2

Let $d \equiv 1 \pmod 4$ be a non-square integer, with $d \geq 5$. Let $g' \geq 1$ be the unique odd integer in $[\sqrt{d}-2, \sqrt{d})$. Then $\omega_0 = (P_0 + \sqrt{d})/Q_0 = (g' + \sqrt{d})/2$ is reduced. Its continued fraction expansion $\omega_0 = [\overline{g', a_1, \dots, a_{l-1}}]$ is purely periodic and $\omega_1 = [\overline{a_1, \dots, a_{l-1}, g'}] = 1/(\omega_0 - g') = 2/(\sqrt{d} - g') = -1/\omega_0' = [\overline{a_{l-1}, \dots, a_1, g'}]$. Hence, $a_k = a_{l-k}$ for $1 \leq k \leq l-1$. Using $Q_0 = 2$, the oddness of $P_0 = g'$, the evenness of $Q_1 = (d - P_0^2)//Q_0 = (d - g'^2)/2$ and the identities $Q_{k+1} = Q_{k-1} + 2a_k P_k - a_k^2 Q_k$ for $k \geq 1$ and $P_{k+1} = a_k Q_k - P_k$ for $k \geq 0$, we obtain that the Q_k's are even and the P_k's are odd for $k \geq 0$. Consequently, if d is square-free then $\mathbb{M}_0$ is equal to the ring of algebraic integers $\mathbb{Z}_{\mathbb{K}}$ of the real quadratic number field $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ and the $\mathbb{Z}$-modules

$$\mathscr{I}_k := (Q_k/2)\mathbb{M}_k = (Q_k/2)\mathbb{Z} + \frac{P_k + \sqrt{d}}{2}\mathbb{Z} = (Q_k/2)\omega_1 \dots \omega_k \mathbb{M}_0 = \alpha_k \mathbb{Z}_{\mathbb{K}}$$

are primitive, principal, integral ideals of norms $Q_k/2$ of the real quadratic number field $\mathbb{Q}(\sqrt{d})$, where $\alpha_k = (Q_k/2)\omega_1 \dots \omega_k \in \mathscr{I}_k \subseteq \mathbb{Z}_{\mathbb{K}}$ is an algebraic integer of norm $(-1)^k (Q_k/2)$ (recall that $\omega_k > 1$ and $-1/\omega_k' > 1$ for $k \geq 0$). Hence, $\mathscr{I}_k$ is principal in the narrow sense if and only if k is even.

Now, assume that d is divisible by a prime $p \equiv 3 \pmod 4$. Since the congruence $x^2 - dy^2 \equiv -4 \pmod p$ has no solution in rational integers, any algebraic unit of $\mathbb{Q}(\sqrt{d})$ has norm $+1$. The algebraic unit $\varepsilon = \omega_0\omega_1 \dots \omega_{l-1} := \overline{\mathbb{Z} + \mathbb{Z}\omega_0}$ being of norm $(-1)^l$, $l = 2L$ is even, $\omega_0 = [\overline{g', a_1, \dots, a_{L-1}, a_L, a_{L-1}, \dots, a_1}]$, $\omega_L = [\overline{a_L, \dots, a_1, g', a_1, \dots, a_{L-1}}]$ and $\omega_{L+1} = [\overline{a_{L-1}, \dots, a_1, g', a_1, \dots, a_L}] = -1/\omega_L'$. Hence, $-1 = \omega_{L+1}\omega_L' = \frac{P_{L+1}+\sqrt{d}}{Q_{L+1}} \frac{P_L - \sqrt{d}}{Q_L}$, which implies $P_{L+1} = P_L$. Since $P_{L+1} = a_L Q_L - P_L$, we have $P_{L+1} = a_L(Q_L/2)$ and a_L is odd. Moreover,

$$d - P_{L+1}^2 = d - a_L^2 (Q_L/2)^2 = 4(Q_L/2)(Q_{L+1}/2).$$

Hence, $Q_L/2$ divides d. Finally, $\omega_L = (P_L + \sqrt{d})/Q_L$ being reduced, we have $1 < Q_L < 2\sqrt{d}$ and $a_L = \lfloor \omega_L \rfloor \in \{\lfloor 2\sqrt{d}/Q_L \rfloor, \lfloor 2\sqrt{d}/Q_L \rfloor - 1\}$, and we obtain the following Proposition and Corollary from which Theorem 2 follows:

Proposition 4. *Let $d \equiv 1 \pmod 4$ be a square-free integer, with $d \geq 5$ such that at least one prime $p \equiv 3 \pmod 4$ divides d. Let $g' \geq 1$ be the unique odd integer in the interval $[\sqrt{d}-2, \sqrt{d})$. Set $\omega_0 = (g' + \sqrt{d})/2$. $l \geq 1$ be the length of the period of the purely periodic continued fraction expansion $\omega_0 = [\overline{g', a_1, \dots, a_{l-1}}]$. Then*

- (i) $a_k = a_{l-k}$ *for* $1 \leq k \leq l-1$;
- (ii) $l = 2L$ *is even;*
- (iii) $Q_L/2$ *divides d and* $1 < Q_L/2 < \sqrt{d}$;
- (iv) a_L *is odd and* $a_L = \lfloor \omega_L \rfloor \in \{\lfloor 2\sqrt{d}/Q_L \rfloor, \lfloor 2\sqrt{d}/Q_L \rfloor - 1\}$;
- (v) *The integral ideal* $\mathscr{I}_L = (Q_L/2)\mathbb{Z} + \frac{P_L+\sqrt{d}}{2}\mathbb{Z}$ *of norm $Q_L/2$ is principal and L is even if and only $\mathscr{I}_L$ is principal in the narrow sense.*

Corollary 5. *Let p, q be two prime integers equal to $3 \pmod 4$, with $3 \leq p < q$. Take $d = pq \equiv 1 \pmod 4$. Then $Q_L/2 = p$. Hence, $\mathscr{I}_L$ is the prime ramified ideal $\mathscr{P}$ of norm p of the ring of algebraic integers of the real quadratic field $\mathbb{Q}(\sqrt{d})$ and a_L is the unique odd integer in $\{\lfloor \sqrt{q/p} \rfloor, \lfloor \sqrt{q/p} \rfloor - 1\}$. Moreover, $\mathscr{P}$ is principal in the narrow sense if and only if $\left(\frac{p}{q}\right) = +1$.*

Proof. Let $\mathscr{P}$ and $\mathscr{Q}$ be the prime ideals above p and q, respectively. Hence $\mathscr{P} = \mathscr{I} = (\alpha)$ is principal and $\mathscr{P}\mathscr{Q} = (\sqrt{d})$ is also clearly principal. Since $\sqrt{d} \in \mathscr{P}\mathscr{Q} \subseteq \mathscr{P} = (\alpha)$, we have $\sqrt{d} = \alpha\beta$ for some algebraic integer β. Hence $\mathscr{Q} = (\beta)$ is also principal. Since $N(\alpha)N(\beta) = N(\alpha\beta) = N(\sqrt{d}) = -d < 0$, only one of the two principal ideals $\mathscr{P}$ or $\mathscr{Q}$ is principal in the narrow sense. If $\mathscr{P} = (\alpha)$ is principal in the narrow sense, with $\alpha = (x + y\sqrt{d})/2$ such that $p = N(\alpha) = (x^2 - pqy^2)/4$, then p divides $x = pX$, $4 = pX^2 - qy^2$ and $\left(\frac{p}{q}\right) = +1$. If $\mathscr{P}$ is not principal in the narrow sense, then $\mathscr{Q} = (\beta)$ is principal in the narrow sense, with $\beta = (x + y\sqrt{d})/2$ such that $q = N(\beta) = (x^2 - pqy^2)/4$. Hence, q divides $x = qX$, $4 = qX^2 - py^2$ and $\left(\frac{-p}{q}\right) = -\left(\frac{p}{q}\right) = +1$. □

4. Proof of Theorem 3

Let $d \equiv 1 \pmod 4$ be a non-square integer, with $d \geq 5$. Set $g = \lfloor\sqrt{d}\rfloor$. Then $\omega_0 = (P_0+\sqrt{d})/Q_0 = g+\sqrt{d}$ is reduced. Since $\mathbb{M}_0 = \mathbb{Z}[\omega_0] = \mathbb{Z}[\sqrt{d}]$ is not the ring of algebraic integers of $\mathbb{Q}(\sqrt{d})$, the proof of Theorem 3 is a little more tricky than the one of Theorem 2. Here again, the continued fraction expansion $\omega_0 = [\overline{2g, a_1,\dots,a_{l-1}}]$ is purely periodic and $\omega_1 = [\overline{a_1,\dots,a_{l-1},2g}] = 1/(\omega_0 - 2g) = 1/(\sqrt{d}-g) = -1/\omega_0' = [\overline{a_{l-1},\dots,a_1,2g}]$. Hence, $a_k = a_{l-k}$ for $1 \leq k \leq l-1$. Suppose that we had $Q_n \equiv 2 \pmod 4$ for some $n \geq 0$. Then P_{n+1} would be odd and Q_{n+1} would be even, as $Q_nQ_{n+1} = d - P_{n+1}^2$. Therefore, all the Q_k's would be even for $k \geq n$, as $Q_{k+1} = Q_{k-1} + 2a_kP_k - a_k^2Q_k$ for $k \geq 1$, hence for $k \geq 0$, by pure periodicity of the continued fraction expansion of ω_0. Since Q_0 is odd, we deduce that $Q_k \not\equiv 2 \pmod 4$ for $k \geq 0$.

Now, assume that d is divisible by a prime $p \equiv 3 \pmod 4$. As above, $l = 2L$ is even and

$$2P_{L+1} = a_LQ_L, \quad \text{and} \quad 4d - a_L^2Q_L^2 = 4Q_LQ_{L+1}.$$

Hence, Q_L divides $4d$ and 4 does not divide Q_L and we obtain the following Proposition from which Theorem 3 follows, by Corollary 5:

Proposition 6. *Let $d \equiv 1 \pmod 4$ be a square-free integer, with $d \geq 5$ such that at least one prime $p \equiv 3 \pmod 4$ divides d.*

Set $\omega_0 = g+\sqrt{d}$, where $g = \lfloor\sqrt{d}\rfloor$. Let $l \geq 1$ be the length of the period of the purely periodic continued fraction expansion $\omega_0 = [\overline{2g, a_1,\dots,a_{l-1}}]$. Then

(i) *$a_k = a_{l-k}$ for $1 \leq k \leq l-1$;*
(ii) *$l = 2L$ is even;*
(iii) *Q_L divides $2d$ and $1 < Q_L < 2\sqrt{d}$;*
(iv) *$a_L = \lfloor\omega_L\rfloor \in \{\lfloor 2\sqrt{d}/Q_L\rfloor, \lfloor 2\sqrt{d}/Q_L\rfloor - 1\}$;*
(v) *if $d = pq$, where p,q are prime numbers equal to 3 modulo 4 with $p < q$, then $a_L = 2\lfloor\sqrt{q/p}\rfloor$, $Q_L = p$, the prime ideal $\mathscr{P}$ of norm p of the ring of algebraic integers of the real quadratic field $\mathbb{Q}(\sqrt{d})$ is principal and L is even if and only if $\mathscr{P}$ is principal in the narrow sense.*

Proof. It remains to prove point (v). Since Q_L divides $2pq$, $Q_L \not\equiv 2 \pmod 4$ and $Q_L < 2\sqrt{pq}$, we have $Q_L \in \{p,q\}$. Since $Q_L = q$ would yield the contradiction $4qQ_{L+1} = 4d - a_L^2Q_L^2 \leq 4pq - 4q^2 < 0$, we have $Q_L = p$. Hence, a_L is even, as $2P_L = a_LQ_L$, and $a_L \in \{\lfloor 2x\rfloor, \lfloor 2x\rfloor - 1\}$, where $x = \sqrt{d}/Q_L$. Since $\lfloor 2x\rfloor \in \{2\lfloor x\rfloor, 2\lfloor x\rfloor + 1\}$ for x real, we have that a_L is even and $a_L \in \{2\lfloor x\rfloor - 1, 2\lfloor x\rfloor, 2\lfloor x\rfloor + 1\}$. Therefore, $a_L = 2\lfloor x\rfloor = 2\lfloor\sqrt{d}/Q_L\rfloor = 2\lfloor\sqrt{q/p}\rfloor$.

Finally, set $\beta_L = Q_L\omega_1\dots\omega_L$. Then

$$\mathscr{I}_L := \beta_L\mathbb{Z}[\sqrt{d}] = \beta_L\mathbb{M}_0 = Q_L\omega_1\dots\omega_L\mathbb{M}_0 = Q_L\mathbb{M}_L = Q_L\mathbb{Z} + (P_L+\sqrt{d})\mathbb{Z} \subseteq \mathbb{Z}[\sqrt{d}].$$

Hence, $\beta_L = x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ and $\{Q_L, P_L+\sqrt{d}\}$ and $\{\beta_L, \beta_L\sqrt{d}\} = \{x+y\sqrt{d}, dy+x\sqrt{d}\}$ are two $\mathbb{Z}$-bases of $\mathscr{I}_L$ and the change of basis matrix

$$A = \begin{pmatrix} \frac{x-yP_L}{Q_L} & \frac{dy-xP_L}{Q_L} \\ y & x \end{pmatrix}$$

is in $M_2(\mathbb{Z})$ and of determinant ± 1, i.e. $\pm 1 = (x^2-dy^2)/Q_L = N(\beta_L)/p$. Therefore, $N(\beta_L) = (-1)^Lp$, with $\beta_L \in \mathbb{Z}[\sqrt{d}]$. It follows that the prime ideal $\mathscr{P}$ of the ring of algebraic integers $\mathbb{Z}_{\mathbb{K}} = \mathbb{Z}[(1+\sqrt{d})/2]$ of $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ lying above p is principal and equal to (β_L) and that $\mathscr{P}$ it is principal in the narrow sense if and only if L is even. □

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