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Complex analysis and geometry / *Analyse et géométrie complexe*

Bohr radius and its asymptotic value for holomorphic functions in higher dimensions

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Abstract. We establish sharp Bohr phenomena for holomorphic functions defined on a bounded balanced domain G in a complex Banach space X, which map into a simply connected domain or a convex domain Ω in the complex plane $\mathbb{C}$. Taking X as the n-dimensional complex plane and G as the open unit polydisk, we consider a version of the Bohr inequality stronger than the above mentioned one and study the exact asymptotic behaviour of the Bohr radius. Explicit lower bounds on the Bohr radii of this type are also provided. Extending a recent result of Liu and Ponnusamy, we further record a refined form of the Bohr inequality for the particular case $\Omega = \mathbb{D}$, i.e. the open unit disk in $\mathbb{C}$.

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1. Introduction and the main results

Let X be a complex Banach space and $G \subset X$, $\Omega \subset \mathbb{C}$ be two domains. For any holomorphic mapping $f : G \to \Omega$, let $D^k f(x)$ denote the k^{th} Fréchet derivative ($k \in \mathbb{N}$) of f at $x \in G$, which is a bounded symmetric k-linear mapping from $\prod_{i=1}^{k} X$ to $\mathbb{C}$. Any such f can be expanded into the series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(x_0)\left((x - x_0)^k\right) \tag{1}$$

in a neighborhood of any given $x_0 \in G$. It is understood that $D^0 f(x_0)(x^0) = f(x_0)$ and

$$D^k f(x_0)\left(x^k\right) = D^k f(x_0)(\underbrace{x, x, \cdots, x}_{k\text{-times}})$$

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for $k \geq 1$. The reader is referred to [14] for a more general and detailed discussion in this area. Now, let us denote by $K_X^G(\Omega)$ the supremum of all $r \in [0,1]$ such that the inequality

$$\sum_{k=1}^{\infty} \left| \frac{1}{k!} D^k f(0)\left(x^k\right) \right| \leq d\left(f(0), \partial\Omega\right) \tag{2}$$

holds for all $x \in rG$ and for all holomorphic mappings f from a bounded balanced domain $G \subset X$ to $\Omega \subset \mathbb{C}$ with an expansion (1) in a neighborhood of $x_0 = 0$. We clarify that G is balanced if $uG \subset G$ for all $u \in \overline{\mathbb{D}}$, and $d(f(0), \partial\Omega)$ is the Euclidean distance between $f(0)$ and the boundary $\partial\Omega$ of the domain Ω. The sharp version of the famous theorem of Harald Bohr [8] states that $K_{\mathbb{C}}^{\mathbb{D}}(\mathbb{D}) = 1/3$. After this theorem found an application to the characterization problem of Banach algebras satisfying the von Neumann inequality (cf. [12]), problems of similar type started being studied extensively in different settings (see for example [1–7, 10, 11, 15–21] and the references therein), and gained popularity by the name *Bohr phenomenon*. It is worth mentioning here that Bohr inequalities of type (2) have been considered in [2, 15, 18]. Of particular interest to us is [1, Theorem 8], which shows that for any balanced domain G centered at 0 in $\mathbb{C}^n$, $K_{\mathbb{C}^n}^G(\mathbb{D}) \geq 1/3$, and assuming G convex it was shown that $K_{\mathbb{C}^n}^G(\mathbb{D}) = 1/3$. As a consequence of a more general theorem, it was further proved in [15, Corollary 3.2] that $K_X^G(\mathbb{D}) = 1/3$ for any bounded balanced domain $G \subset X$, X being a complex Banach space. In the following theorem, we replace $\mathbb{D}$ with more general domains Ω and establish sharp Bohr phenomena. To this end, we define the following two quantities for any given complex Banach space X and a bounded balanced domain $G \subset X$:

$$\widetilde{K}_X^G = \inf\left\{K_X^G(\Omega) : \Omega \subset \mathbb{C} \text{ is simply connected}\right\},$$

and

$$\widetilde{\widetilde{K}}_X^G = \inf\left\{K_X^G(\Omega) : \Omega \subset \mathbb{C} \text{ is convex}\right\}.$$

Theorem 1. *$\widetilde{K}_X^G = 3 - 2\sqrt{2}$ and $\widetilde{\widetilde{K}}_X^G = 1/3$.*

For $X = \mathbb{C}$ and $G = \mathbb{D}$, the above theorem gives [18, Theorem 1 and Remark 1] back. Also, in some sense, the second part of our Theorem 1 generalizes [15, Corollary 3.2].

Before we proceed further, we need to introduce some concepts. Let $\mathbb{D}^n = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : \|z\|_\infty := \max_{1 \leq k \leq n} |z_k| < 1\}$ be the open unit polydisk in the n-dimensional complex plane $\mathbb{C}^n$. Any holomorphic $f : \mathbb{D}^n \to \mathbb{C}$ can be expanded in the power series

$$f(z) = c_0 + \sum_{|\alpha| \in \mathbb{N}} c_\alpha z^\alpha, z \in \mathbb{D}^n. \tag{3}$$

Here and hereafter, we use the standard multi-index notation: α means an n-tuple $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ of nonnegative integers, $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$, z denotes an n-tuple $(z_1, z_2, \cdots, z_n)$ of complex numbers, and z^α is the product $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$. It is evident that in our previous discussion, if the Banach space X is chosen to be $\mathbb{C}^n$ and $G = \mathbb{D}^n$, then for any $k \in \mathbb{N}$ and for any f as in (3) we have

$$\frac{1}{k!} D^k f(0)\left(z^k\right) = \sum_{|\alpha| = k} c_\alpha z^\alpha, z \in \mathbb{D}^n.$$

Hence, we are motivated to consider a "stronger" Bohr phenomenon in this case. To be more specific, we denote by $K_n(\Omega)$ the supremum of all $r \in [0,1)$ such that

$$\sum_{k=1}^{\infty} \left(\sum_{|\alpha| = k} \left| c_\alpha z^\alpha \right| \right) \leq d\left(f(0), \partial\Omega\right) \tag{4}$$

for all $z \in \mathbb{D}^n$ satisfying $\|z\|_\infty \leq r$ and for all holomorphic $f : \mathbb{D}^n \to \Omega$ with an expansion (3). Lower and upper bounds for $K_n(\mathbb{D})$ were obtained in [7, 10] and the recent article [4] has improved over previously known lower bounds. Although for any $n > 1$ the exact value for $K_n(\mathbb{D})$ is yet unknown, it is known from [3] that $K_n(\mathbb{D})$ behaves asymptotically as $\sqrt{\log n}/\sqrt{n}$. Let us define

$$\widetilde{K}_n := \inf\left\{K_n(\Omega) : \Omega \subset \mathbb{C} \text{ is simply connected}\right\}.$$

In the next theorem, we show that $\widetilde{K}_n$ has the same asymptotic behaviour as $K_n(\mathbb{D})$.

Theorem 2. $\lim_{n\to\infty} \widetilde{K}_n \sqrt{n}/(\sqrt{\log n}) = 1$.

The aim of the penultimate theorem of this article is to give lower bounds on $K_n(\Omega)$ for simply connected and convex domains Ω in $\mathbb{C}$. For this purpose, we need to be familiar with the quantity $S(k,n)$, known as the Sidon constant for the index set $\{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) : |\alpha| = k\}$. $S(k,n)$ is defined as the smallest constant C such that

$$\sum_{|\alpha|=k} |a_\alpha| \le C \sup_{z\in\mathbb{D}^n} \left| \sum_{|\alpha|=k} a_\alpha z^\alpha \right|$$

for any k-homogeneous polynomial $P(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$ in n-complex variables (see f.i. [11]).

Theorem 3. *Suppose $\Omega \subset \mathbb{C}$ is a simply connected domain and $f : \mathbb{D}^n \to \Omega$ is holomorphic with an expansion* (3)*. Then $K_n(\Omega) \ge r_0$, r_0 being the only root in* $(0,1)$ *of the equation*

$$y + \sum_{k=2}^{\infty} kS(k,n) y^k = \frac{1}{4}. \tag{5}$$

In addition, if Ω is assumed to be convex, then $K_n(\Omega) \ge r_1$, r_1 being the only root in $(0,1)$ *of the equation*

$$y + \sum_{k=2}^{\infty} S(k,n) y^k = \frac{1}{2}. \tag{6}$$

We remark here that the lower bound r_1 of $K_n(\Omega)$ in the second part of the above Theorem 3 was obtained for $\Omega = \mathbb{D}$ in [4, Theorem 3.3]. Finally, we concentrate on the functions mapping into $\mathbb{D}$. In particular, very recently a refined version of [1, Theorem 8] has appeared in [17, Theorem 2.1] along with several other Bohr or Bohr-type inequalities (see also [20, Theorem 2] for one variable versions of [17, Theorem 2.1]). Like [1, Theorem 8], [17, Theorem 2.1] also deals with the holomorphic functions defined on a balanced domain $G \subset \mathbb{C}^n$, and the result was shown to be sharp for G convex. Motivated by [15, Corollary 3.2], we show in the last theorem of this article that [17, Theorem 2.1] extends for the holomorphic functions defined on a bounded balanced domain G in any complex Banach space X.

Theorem 4. *Suppose X is a complex Banach space, $G \subset X$ is a bounded balanced domain and $f : G \to \mathbb{D}$ is a holomorphic function with an expansion* (1) *in a neighborhood of $x_0 = 0$. Then*

$$a^p + \sum_{k=1}^{\infty} \left| \frac{1}{k!} D^k f(0)\left(x^k\right) \right| + \left(\frac{1}{1+a} + \frac{r}{1-r} \right) \sum_{k=1}^{\infty} \left| \frac{1}{k!} D^k f(0)\left(x^k\right) \right|^2 \le 1 \tag{7}$$

for $x \in r_p(a)G$ and $r \le r_p(a)$, where $r_p(a) = (1-a^p)/(2-a^2-a^p)$, $a = |f(0)|$ and $p > 0$. The number $r_p(a)$ and the factor $1/(1+a)$ in (7) *cannot be improved.*

It may be noted that the other Bohr-like inequalities, i.e. [17, the Theorems 2.3, 2.6, 2.7 and the Corollary 2.8] can also be proved in sharp form for the holomorphic functions defined on a bounded balanced domain of a complex Banach space X in a similar manner as in Theorem 4.

2. Proofs of the theorems

Proof of Theorem 1. We start with any arbitrary simply connected domain $\Omega \subset \mathbb{C}$. For any holomorphic $f : G \to \Omega$ with an expansion (1) around $x_0 = 0$, we observe that the holomorphic function

$$f_1(u) := f(u\beta) = f(0) + \sum_{k=1}^{\infty} \left(\frac{1}{k!} D^k f(0) \left(\beta^k\right) \right) u^k, u \in \mathbb{D} \tag{8}$$

maps $\mathbb{D}$ into the same domain Ω for any fixed $\beta \in G$. Therefore, f_1 is subordinate to g in $\mathbb{D}$, where g is the univalent Riemann mapping from $\mathbb{D}$ onto Ω, satisfying $g(0) = f_1(0) = f(0)$. Now, using the well-known theorem of de Branges (cf. [9] and [13, p. 197]), we have

$$\left| \frac{1}{k!} D^k f(0) \left(\beta^k\right) \right| \le k \left| f_1'(0) \right|$$

for all $k \ge 1$. Since β is arbitrary, [18, Lemma 1] gives

$$\left| \frac{1}{k!} D^k f(0) \left(x^k\right) \right| \le 4kd\left(f(0), \partial\Omega\right) \tag{9}$$

for any $x \in G$. Hence, given any $r \in [0,1)$,

$$\sum_{k=1}^{\infty} \left| \frac{1}{k!} D^k f(0) \left(y^k\right) \right| \le \frac{4r}{(1-r)^2} d\left(f(0), \partial\Omega\right)$$

for all $y \in rG$. Thus, inequality (2) is satisfied whenever $4r/(1-r)^2 \le 1$, i.e. if $r \le 3 - 2\sqrt{2}$, which implies that $K_X^G(\Omega) \ge 3 - 2\sqrt{2}$, and therefore $\widetilde{K}_X^G \ge 3 - 2\sqrt{2}$. To show that $\widetilde{K}_X^G$ is actually equal to $3 - 2\sqrt{2}$, we adopt the approach of [15]. For any $\tilde{r} \in (3 - 2\sqrt{2}, 1)$, there exists $c \in (0,1)$ and $V \in \partial G$ such that $c\tilde{r} > 3 - 2\sqrt{2}$ and $c \sup_{x \in \partial G} \|x\| < \|V\|$. Now, we consider the Koebe function $K(u) = u/(1-u)^2, u \in \mathbb{D}$ and define the holomorphic function f on G by $f(x) = K(c\phi_V(x)/\|V\|)$, where ϕ_V is a bounded linear functional on X with $\phi_V(V) = \|V\|$ and $\|\phi_V\| = 1$. It is easy to see that f maps inside a simply connected domain Ω, which is, in this case, the range of K, i.e. the whole plane $\mathbb{C}$ minus the part of the negative real axis from $-1/4$ to infinity. Thus, for $x = \tilde{r}V$,

$$\sum_{k=1}^{\infty} \left| \frac{1}{k!} D^k f(0) \left(x^k\right) \right| = \frac{c\tilde{r}}{(1-c\tilde{r})^2} > \frac{1}{4} = d\left(f(0), \partial\Omega\right),$$

showing that $\widetilde{K}_X^G$ cannot be bigger than $3 - 2\sqrt{2}$. Similar argument can be used for completing the proof of the case $\widetilde{\widetilde{K}}_X^G = 1/3$. We only need to note that the right hand side of the inequality (9) will be replaced by $2d(f0), \partial\Omega)$ (see [18, Lemmas 2, 3]), and for the proof of the sharpness of the constant $1/3$, we have to use $L(u) = u/(1-u), u \in \mathbb{D}$ instead of $K(u)$, and observe that L maps $\mathbb{D}$ onto the half-plane $c(w) > -1/2$. □

Proof of Theorem 2. We follow the ideas of [3] in this proof. Given any k-homogeneous ($k \ge 1$) complex polynomial $P(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$ defined in $\mathbb{C}^n$ and for any pre-assigned $\epsilon > 0$, there exists $\mu > 0$ such that

$$\left(\sum_{|\alpha|=k} |a_\alpha|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \le \mu(1+\epsilon)^k \sup_{\|z\|_\infty = 1} \left| \sum_{|\alpha|=k} a_\alpha z^\alpha \right| \tag{10}$$

(see [3, Theorem 1.1]). Now, for any holomorphic function f which maps $\mathbb{D}^n$ into a simply connected domain $\Omega \subset \mathbb{C}$ and has an expansion (3), it is immediate from (9) that

$$\left(\sum_{|\alpha|=k} |c_\alpha|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \le 4\mu k(1+\epsilon)^k d\left(f(0), \partial\Omega\right).$$

Hence, using the Hölder's inequality and the estimate

$$\binom{n+k-1}{k} \le \frac{(n+k-1)^k}{k!} < \left(\frac{e}{k}\right)^k (n+k-1)^k < e^k \left(1 + \frac{n}{k}\right)^k,$$

we get, by setting $r=(1-2\epsilon)\sqrt{(\log n)/n}$

$$\begin{aligned}\sum_{k=1}^{\infty} r^k \sum_{|\alpha|=k}|c_\alpha| &\le \sum_{k=1}^{\infty} r^k \left(\sum_{|\alpha|=k}|c_\alpha|^{\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}} \binom{n+k-1}{k}^{\frac{k-1}{2k}}\\ &< 4\mu d\left(f(0),\partial\Omega\right)\sum_{k=1}^{\infty} k\left(r\sqrt{e}(1+\epsilon)\right)^k\left(1+\frac{n}{k}\right)^{\frac{k-1}{2}}\\ &= 4\mu d\left(f(0),\partial\Omega\right)\sum_{k=1}^{\infty} k\left(\sqrt{\frac{\log n}{n}}\sqrt{e}(1-2\epsilon)(1+\epsilon)\right)^k\left(1+\frac{n}{k}\right)^{\frac{k-1}{2}}.\end{aligned}$$

For n large enough,

$$t_n := \frac{\sqrt{\log n}}{n^{1/4}}\sqrt{2e}(1-2\epsilon)(1+\epsilon)<1,$$

and for $k>\sqrt{n}$, observe that

$$\left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} < \left(2\sqrt{n}\right)^{\frac{k}{2}}.$$

Using both the above facts,

$$\sum_{k>\sqrt{n}} k\left(\sqrt{\frac{\log n}{n}}\sqrt{e}(1-2\epsilon)(1+\epsilon)\right)^k\left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} \le \sum_{k>\sqrt{n}} k\left(\frac{\sqrt{\log n}}{n^{1/4}}\sqrt{2e}(1-2\epsilon)(1+\epsilon)\right)^k \le \frac{t_n}{(1-t_n)^2}$$

which goes to 0 as $n\to\infty$. For $k\le\sqrt{n}$, we start by making n sufficiently large such that $2<k_0\le\log n$ can be chosen for which the inequalities

$$k_0^{\frac{1}{k_0-1}} \le 1+\frac{\epsilon}{2},\quad \sum_{k_0\le k\le\sqrt{n}} k\left((1-2\epsilon)(1+\epsilon)^{3/2}\right)^k \le \frac{1}{8\mu} \text{ and } \left(\frac{1}{n}\right)^{\frac{k_0-2}{2(k_0-1)}} \le \frac{\epsilon}{2}$$

are satisfied. Observing that $x^{1/(x-1)}$ is decreasing and $(x-2)/2(x-1)$ is increasing in $(1,\infty)$, we obtain, for $k\ge k_0$:

$$\begin{aligned}\left(k^{\frac{k}{k-1}}\left(\frac{1}{n}+\frac{1}{k}\right)\right)^{\frac{k-1}{k}} &\le \left(\left(\frac{1}{n}\right)^{\frac{k-2}{2(k-1)}} + k^{\frac{1}{k-1}}\right)^{\frac{k-1}{k}}\\ &\le \left(\left(\frac{1}{n}\right)^{\frac{k_0-2}{2(k_0-1)}} + k_0^{\frac{1}{k_0-1}}\right)^{\frac{k-1}{k}} \le (1+\epsilon)^{\frac{k-1}{k}} \le 1+\epsilon,\end{aligned}$$

which, after a little simplification, gives

$$\left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} \le (1+\epsilon)^{\frac{k}{2}}\frac{n^{\frac{k}{2}}}{n^{\frac{1}{2}}k^{\frac{k}{2}}}.$$

Therefore, observing that $x\mapsto n^{1/x}x$ is decreasing upto $x=\log n$ and increasing thereafter, we get

$$\begin{aligned}&\sum_{k_0\le k\le\sqrt{n}} k\left(\sqrt{\frac{\log n}{n}}\sqrt{e}(1-2\epsilon)(1+\epsilon)\right)^k\left(1+\frac{n}{k}\right)^{\frac{k-1}{2}}\\ &\quad\le \sum_{k_0\le k\le\sqrt{n}} k\left(\sqrt{e\log n}(1-2\epsilon)(1+\epsilon)^{3/2}\sqrt{\frac{1}{n^{1/k}k}}\right)^k\\ &\quad\le \sum_{k_0\le k\le\sqrt{n}} k\left((1-2\epsilon)(1+\epsilon)^{3/2}\right)^k \le \frac{1}{8\mu}.\end{aligned}$$

It remains to analyze the case $1 \le k \le k_0$. In this case, we observe that for n large enough,

$$\frac{k}{n} + 1 \le \frac{k_0}{n} + 1 \le \epsilon + 1,$$

and hence

$$\left(1 + \frac{n}{k}\right)^{\frac{k-1}{2}} \le (1+\epsilon)^{\frac{k}{2}} \left(\frac{n}{k}\right)^{\frac{k-1}{2}}.$$

Making use of the above inequality and the fact that $x \mapsto n^{1/x} x$ is decreasing in $[1, k_0]$, it is easily seen that

$$\sum_{k=1}^{k_0} k \left(\sqrt{\frac{\log n}{n}} \sqrt{e} (1-2\epsilon)(1+\epsilon) \right)^k \left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} \le \sum_{k=1}^{k_0} k \left(\sqrt{e \log n} (1-2\epsilon)(1+\epsilon)^{3/2} \frac{k^{1/(2k)}}{k_0^{1/2} n^{1/(2k_0)}} \right)^k,$$

which tends to 0 as $n \to \infty$. Combining all the above three estimates, we have

$$\sum_{k=1}^{\infty} r^k \sum_{|\alpha|=k} |c_\alpha| \le 4\mu d\left(f(0), \partial\Omega\right) \left(\frac{1}{8\mu} + o(1)\right) \le d\left(f(0), \partial\Omega\right)$$

for n large enough. Therefore, $K_n(\Omega) \ge (1-2\epsilon)\sqrt{(\log n)/n}$, provided n is sufficiently large. On the other hand, it is known from [7] that $\overline{\lim}_{n\to\infty} K_n(\mathbb{D})\sqrt{n/(\log n)} \le 1$. Since $\widetilde{K}_n \le K_n(\mathbb{D})$, our proof is complete. □

Proof of Theorem 3. First suppose that Ω is simply connected. Hence, using the definition of Sidon constant, it follows from (9) that

$$\sum_{|\alpha|=k} |c_\alpha| \le 4kS(k,n)d\left(f(0), \partial\Omega\right)$$

for all $k \ge 1$. Assume that $\|z\|_\infty \le r$. Then applying the above inequality and noting that $S(1,n) = 1$, it is clear that (4) is satisfied if

$$4\sum_{k=1}^{\infty} kS(k,n)r^k = 4\left(r + \sum_{k=2}^{\infty} kS(k,n)r^k\right) \le 1. \tag{11}$$

It is easily seen that

$$R(y) := y + \sum_{k=2}^{\infty} kS(k,n)y^k - 1/4$$

is increasing in $(0,1)$, $R(0) = -1/4 < 0$, $R(1/2) = 1/4 + \sum_{k=2}^{\infty} kS(k,n)(1/2)^k > 0$, and therefore R has exactly one root r_0 in $(0,1)$. As a consequence, the inequality (11) holds if $\|z\|_\infty \le r_0$, where r_0 is the only root in $(0,1)$ of the equation (5), i.e. $K_n(\Omega) \ge r_0$. For convex Ω, we only have to start from the inequality

$$\sum_{|\alpha|=k} |c_\alpha| \le 2S(k,n)d\left(f(0), \partial\Omega\right)$$

for all $k \ge 1$, and argue exactly as above. □

Proof of Theorem 4. We construct f_1 as in the proof of Theorem 1, which then becomes a holomorphic self mapping of $\mathbb{D}$ with an expansion (8). Since $\beta \in G$ is arbitrary, [17, Theorem A (b)] asserts the validity of (7) under the conditions $x \in r_p(a)G$ and $r \le r_p(a)$, $r_p(a)$ as defined in the statement of Theorem 4. To prove the sharpness part, we again need to use arguments similar to that of the article [15]. For the sake of completeness, it is included here. Given any $a \in [0,1)$, we begin by considering the function $F(u) = (a-u)/(1-au)$, $u \in \mathbb{D}$. For any $\widetilde{r} \in (r_p(a), 1)$, there exists $c \in (0,1)$ and $V \in \partial G$ such that $c\widetilde{r} > r_p(a)$ and $c \sup_{x\in\partial G} \|x\| < \|V\|$. Now we define the holomorphic function f on G by $f(x) = F(c\phi_V(x)/\|V\|)$, where ϕ_V is a bounded linear functional on X with $\phi_V(V) = \|V\|$ and $\|\phi_V\| = 1$. Hence, for $x = \tilde{r}V$ and $r = \tilde{r}$, the left hand side of the inequality (7) reduces to

$$\eta(a,\tilde{r}) := a^p + \frac{c\tilde{r}\left(1-a^2\right)}{1-ac\tilde{r}} + \frac{1+a\tilde{r}}{(1+a)(1-\tilde{r})}\,\frac{\left(1-a^2\right)^2 (c\tilde{r})^2}{1-(ac\tilde{r})^2} > a^p + \left(1-a^2\right)\frac{c\tilde{r}}{1-c\tilde{r}} > 1.$$

On the other hand, let us assume that the quantity $1/(1+a)$ in (7) can be replaced by a bigger number A and the resulting inequality is still valid for all $x \in r_p(a)G$ and $r \le r_p(a)$. We use the same f and F as already defined, but instead of fixing some $\tilde{r}$, we will work with $r_p(a)$ itself; and for any $c \in (0,1)$, we get a $V \in \partial G$ as above. Now for $x = cr_p(a)V$ and $r = r_p(a)$, the left hand side of the modified inequality (7) is bigger than $\eta(a, cr_p(a))$, which is again bigger than $a^p + (1-a^2)(c^2 r_p(a))/(1-c^2 r_p(a))$. It is evident that the last quantity approaches to 1 as $c \to 1-$, and at the same time the modified inequality (7) is satisfied for this particular x and r as well. Therefore,

$$\begin{aligned}
\lim_{c\to 1-}&\left(a^p + \frac{c^2 r_p(a)\left(1-a^2\right)}{1-ac^2 r_p(a)} + \left(A + \frac{r_p(a)}{1-r_p(a)}\right)\frac{\left(1-a^2\right)^2\left(c^2 r_p(a)\right)^2}{1-\left(ac^2 r_p(a)\right)^2}\right)\\
&= a^p + \frac{r_p(a)\left(1-a^2\right)}{1-ar_p(a)} + \left(A + \frac{r_p(a)}{1-r_p(a)}\right)\frac{\left(1-a^2\right)^2\left(r_p(a)\right)^2}{1-\left(ar_p(a)\right)^2}\\
&= 1 = a^p + \left(1-a^2\right)\frac{r_p(a)}{1-r_p(a)},
\end{aligned}$$

i.e. $A = 1/(1+a)$. Summarizing the above discussion, we conclude that neither the number $r_p(a)$ nor the factor $1/(1+a)$ could be improved. □

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