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Periodic Fourier integral operators in L^p-spaces

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Abstract. In this note we give sufficient conditions for the L^p boundedness of periodic Fourier integral operators. We also refer to them as Fourier series operators (FSOs). The main tool will be the notion of full symbol and the periodic analysis on the torus introduced by Ruzhansky and Turunen [34].

Résumé. Dans cette note nous présentons les conditions suffisantes pour la continuité des opérateurs intégraux de Fourier périodique qui sont appelés aussi séries des opérateurs de Fourier. Le principal outil est la notion des opérateurs intégraux de Fourier et l'analyse discrête notamment l'analyse périodique dans le tore introduite par Ruzhansky et Turunen [34].

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1. Introduction

In this note we investigate the L^p-boundedness of periodic Fourier integral operators which, by following Ruzhansky and Turunen [34], will be called Fourier series operators. If $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$, is a (symbol) function defined on the phase space $\mathbb{T}^n \times \mathbb{Z}^n$, where $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus, the Fourier series operator (FSO) associated with a is defined by:

$$Af(x) := \sum_{\xi \in \mathbb{Z}^n} e^{i\phi(x,\xi)} a(x,\xi)(\mathscr{F}_{\mathbb{T}^n} f)(\xi), \quad f \in C^\infty(\mathbb{T}^n), \tag{1}$$

where $\mathscr{F}_{\mathbb{T}^n} f$ is the toroidal Fourier transform of f. These operators appear as solutions of several periodic hyperbolic problems, and their microlocal properties, in particular, an associated symbolic calculus for them, as well as, their boundedness properties on L^2, were studied by M. Ruzhansky and V. Turunen in [34, Chapter 4]. Applications of the boundedness of Fourier series operators to hyperbolic differential equations with periodic conditions can be found in [34, p. 410].

Fourier series operators are periodic versions of Fourier integral operators which, in the sense of Hörmander, are defined by:

$$Tf(x) := \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)(\mathscr{F}_{\mathbb{R}^n} f)(\xi)\, d\xi, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{2}$$

where $\mathscr{F}_{\mathbb{R}^n} f$ is the Euclidean Fourier transform of f. The general theory of Fourier integral operators (FIOs) was developed, in a very satisfactory way, by Hörmander [21], and Duistermaat and Hörmander [18]. For analysts, the investigation of the mapping properties of FIOs is a very fundamental task due to its applications in PDE, for instance, in estimating the solutions of a long variety of hyperbolic problems (see e.g. Stein [39]). The subject has been studied by several authors. We refer the reader to Fujiwara [20], Asada and Fujiwara [2], Miyachi [25], Peral [28], Seeger, Sogge, and Stein [38], Ruzhansky [29], Coriasco and Ruzhansky [14,15], and Tao [40], were several of the fundamental theorems on the subject have been proved.

In (1), when $\phi(x,\xi) := 2\pi\langle x,\xi\rangle = 2\pi\sum_{i=1}^n x_i\xi_i$, we obtain the global definition of a periodic pseudo-differential operator. These operators appear in the early works of Volevich and Agranovich [1], and subsequent developments are due to McLean [24], Turunen and Vainikko [41], and Ruzhansky and Turunen [35]. Nevertheless, the mapping properties for periodic pseudo-differential operators have been treated by Ruzhansky and Turunen [35], Cardona [3–8], Cardona and Kumar [9–11], Delgado [16] and Molahajloo and Wong [26,27]. The fundamental goal of this paper is to announce our results in [12] about the L^p-boundedness of Fourier series operators, where we cover a suitable class of non-degenerate phase functions ϕ on $\mathbb{T}^n \times \mathbb{R}^n$, and a family of symbols a on $\mathbb{T}^n \times \mathbb{Z}^n$ belonging to the toroidal Hörmander classes (see [34, Chapter 4] for details).

As one of the reviewers of this note remarked, it is important to mention that several results about the mapping properties of continuous linear operators on the torus, can be obtained from ones obtained in the context of compact Lie groups, we refer the reader to [13, 17, 37] and references therein just to mention a few.

This note is organised as follows. In Section 2 we present some basics on the analysis of periodic operators. To motivate our main results, in Section 3 we record several of the fundamental L^p-estimates for Fourier integral operators on $\mathbb{R}^n$. Finally, in Section 4 we present our main results.

2. Preliminaries

In this section we present some basics on the theory of Fourier integral operators on the torus. For a complete background about the periodic analysis of pseudo-differential operators (and Fourier series operators) on the torus, we refer the reader to, Ruzhansky and Turunen [34], for instance.

The torus is the quotient space defined as $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n = \mathbb{R}^n/\mathbb{Z}^n$, where $\mathbb{Z}^n$ denotes the additive group of integral coordinates (the addition being, of course, the one derived from the vector structure of $\mathbb{R}^n$). Let $C^\infty(\mathbb{T}^n)$ be the family of smooth functions on $\mathbb{T}^n$ with its usual Fréchet structure. Let $\mathscr{S}(\mathbb{Z}^n)$ be the discrete Schwartz space which consists of all complex functions $\psi : \mathbb{Z}^n \to \mathbb{C}$, such that, for every $N \in \mathbb{Z}$, there exists $C_N > 0$ satisfying that $|\psi(m)| \leq C_N(1+|m|)^{-N}$, for every $m \in \mathbb{Z}^n$.

In order to define FSOs on the torus we will use, by following [34], the Fourier analysis on $\mathbb{T}^n$. So, let us recall that the toroidal/periodic Fourier transform $\mathscr{F}_{\mathbb{T}^n} : C^\infty(\mathbb{T}^n) \to \mathscr{S}(\mathbb{Z}^n)$ is defined by:

$$(\mathscr{F}_{\mathbb{T}^n} f)(\xi) \equiv \widehat{f}(\xi) := \int_{\mathbb{T}^n} e^{-i2\pi\langle x,\xi\rangle} f(x)\, dx, \quad f \in C^\infty(\mathbb{T}^n), \quad \xi \in \mathbb{Z}^n. \tag{3}$$

In this case, the Fourier transform is a bijection from $C^{\infty}(\mathbb{T}^n)$ into $\mathscr{S}(\mathbb{Z}^n)$ and the Fourier inversion formula gives:

$$(\mathscr{F}_{\mathbb{T}^n}^{-1}\psi)(x) = \sum_{\xi\in\mathbb{Z}^n} e^{i2\pi\langle x,\xi\rangle}\psi(\xi), \quad \psi\in\mathscr{S}(\mathbb{Z}^n), \quad x\in\mathbb{T}^n.$$

We are interested in the action of FIOs on periodic Lebesgue spaces. As usually, we denote by $L^p(\mathbb{T}^n)$ the space of Borel-measurable complex functions u on the torus $\mathbb{T}^n$, satisfying

$$\|u\|_{L^p(\mathbb{T}^n)} := \left(\int_{\mathbb{T}^n} |u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty,$$

for $0 < p < \infty$, and for $p = \infty$, let $L^{\infty}(\mathbb{T}^n)$ be the space of all complex functions u such that:

$$\|u\|_{L^{\infty}(\mathbb{T}^n)} := \operatorname{esssup}_{x\in\mathbb{T}^n}|u(x)| < \infty.$$

The instrumental tool in the theory of FSOs is the notion of periodic symbol. In order to classify the regularity of symbols associated to FIOs we need the notion of discrete derivatives (difference operators). These operators will be introduced in the next definition.

Definition 1 (Difference operators). *Let $\sigma:\mathbb{Z}^n\to\mathbb{C}$ and $1\le i,j\le n$. Let $\delta_j\in\mathbb{N}_0^n$ be defined by:*

$$(\delta_j)_i := \begin{cases} 1, & \text{if } i=j\\ 0, & \text{if } i\neq j.\end{cases}$$

We define the forward and backward partial difference operators Δ_{ξ_j} and $\overline{\Delta_{\xi_j}}$ respectively, by:

$$\Delta_{\xi_j}\sigma(\xi) = \sigma(\xi+\delta_j) - \sigma(\xi), \quad \overline{\Delta}_{\xi_j}\sigma(\xi) = \sigma(\xi) - \sigma(\xi-\delta_j),$$

and for $\alpha\in\mathbb{N}_0^n$ define:

$$\Delta_\xi^\alpha := \Delta_{\xi_1}^{\alpha_1}\dots\Delta_{\xi_n}^{\alpha_n}, \quad \overline{\Delta}_\xi^\alpha := \overline{\Delta}_{\xi_1}^{\alpha_1}\dots\overline{\Delta}_{\xi_n}^{\alpha_n}.$$

Now, we consider symbols in $C^{\infty}(\mathbb{T}^n\times\mathbb{Z}^n)$. These classes are motivated by the treatment of (periodic) elliptic and hypoelliptic problems (see [34, Chapter 4] for details). Here $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$ is the Japanese bracket.

Definition 2 (Toroidal Hörmander class $S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{Z}^n)$). *Let $m\in\mathbb{R}$, $0\le\delta,\rho\le 1$. Then the toroidal symbol class $S^m_{\rho,\delta}(\mathbb{Z}^n\times\mathbb{T}^n)$ consists of those functions $a(x,\xi)\in C^{\infty}(\mathbb{Z}^n\times\mathbb{T}^n)$ and satisfy the toroidal symbol inequalities:*

$$|\Delta_\xi^\alpha\partial_x^\beta a(x,\xi)| \le C_{\alpha\alpha\beta m}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|},$$

for every $x\in\mathbb{T}^n$, for every α, $\beta\in\mathbb{N}_0^n$ and for all $\xi\in\mathbb{Z}^n$.

As it is well known, difference and differential operators satisfy the Leibniz rule.

Important examples of FSOs are periodic pseudo-differential operators. We recall the following definition from Ruzhansky and Turunen [34].

Definition 3 (Toroidal pseudo-differential operators). *If $a\in S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{Z}^n)$, we denote by $a(X,D)$ the corresponding toroidal pseudo-differential operator defined by*

$$a(X,D)f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{i2\pi\langle x,\xi\rangle} a(x,\xi)\widehat{f}(\xi). \tag{4}$$

The series (4) converges, e.g., if $f\in C^{\infty}(\mathbb{T}^n)$. The set of operators of the form (4) with $a\in S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{Z}^n)$ is denoted by $\mathrm{Op}(S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{Z}^n))$. If an operator A satisfies $A\in\mathrm{Op}(S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{Z}^n))$ we denote its toroidal symbol by $\sigma_A = \sigma_A(x,\xi)$, $\quad x\in\mathbb{T}^n$, $\quad \xi\in\mathbb{Z}^n$.

To be precise, we will define the class of Fourier series operators that we will investigate.

Definition 4 (Fourier series operators). *A continuous linear operator $A: C^\infty(\mathbb{T}^n)\to\mathscr{D}'(\mathbb{T}^n)$ is a Fourier series operator, if there exist a real-valued phase function $\phi:\mathbb{T}^n\times\mathbb{Z}^n\to\mathbb{R}$, homogeneous of order 1 in $\xi\neq 0$, and a symbol $a\in S^m_{1,0}(\mathbb{T}^n\times\mathbb{Z}^n)$ such that*

$$Af(x)\equiv A_{\phi,a}f(x):=\sum_{\xi\in\mathbb{Z}^n} e^{i2\pi\phi(x,\xi)}a(x,\xi)(\mathscr{F}_{\mathbb{T}^n}f)(\xi),\quad f\in C^\infty(\mathbb{T}^n). \tag{5}$$

In our further analysis we will use the close relation between toroidal symbols (in the sense of Ruzhansky and Turunen [34]) and periodic Hörmander classes. We introduce such classes as follows.

Definition 5 (Periodic symbol class $S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{R}^n)$). *Symbols in $S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{R}^n)$ are symbols in $S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ (see [22, 30]) of order m which are 1-periodic in x. If $a(x,\xi)\in S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{R}^n)$, the corresponding pseudo-differential operator is defined by*

$$a(x,D_x)u(x):=\int_{\mathbb{T}^n}\int_{\mathbb{R}^n} e^{i2\pi\langle x-y,\xi\rangle}a(x,\xi)u(y)\,\mathrm{d}\xi\,\mathrm{d}y,\quad u\in C^\infty(\mathbb{T}^n). \tag{6}$$

We refer the reader to Cardona and Kumar [9], and references therein, for the multilinear aspects of the periodic pseudo-differential theory.

3. L^p-boundedness of Fourier integral operators on $\mathbb{R}^n$

In order to motivate our main results, let us summarise several of the fundamental boundedness properties for Fourier integral operators on $\mathbb{R}^n$. Indeed, in the case of general phases, according to the theory of FIOs developed by Hörmander [21], the phase functions ϕ are positively homogeneous of order 1 and smooth at $\xi\neq 0$, and the symbols satisfy estimates of the form

$$\sup_{(x,y)\in K}|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)|\leq C_{\alpha,\beta,K}(1+|\xi|)^{\kappa-|\alpha|} \tag{7}$$

for every compact subset K of $\mathbb{R}^{2n}$. So, as it was pointed out in Ruzhansky and Sugimoto [33], the L^p-properties of the FIO defined in (2) can be summarized as follows.

- If $\kappa\leq 0$, then T is $(L^2_{\text{comp}},L^2_{\text{loc}})$-bounded (Hörmander [21] and Eskin [19]).
- If $\kappa\leq\kappa_p:=-(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$, then T is $(L^p_{\text{comp}},L^p_{\text{loc}})$-bounded (Seeger, Sogge and Stein [38]).
- If $\kappa\leq-\frac{1}{2}(n-1)$, then T is $(H^1_{\text{comp}},L^1_{\text{loc}})$-bounded (Seeger, Sogge and Stein [38]).
- If $\kappa\leq-\frac{1}{2}(n-1)$, then T is locally weak $(1,1)$ type (Tao [40]).
- Other conditions can be found in Miyachi [25], Peral [28], Asada and Fujiwara [2], Fujiwara [20], Kumano-go [23], Coriasco and Ruzhansky [14, 15], Ruzhansky and Sugimoto [30–33], and Ruzhansky [29].
- A periodic version of the L^2-result by Hörmander and Eskin mentioned above, was proved by Ruzhansky and Turunen (see Theorem 8). See also dispersive estimates for FIOs in Ruzhansky and Wirth [36].

4. L^p-boundedess of Fourier series operators on $\mathbb{T}^n$

The main results of this paper are Theorems 6 and 7 below. In Theorem 6 we investigate how the L^p-boundedness of FIOs implies the L^p-boundedness of FSOs. Since the phase functions ϕ are usually considered homogeneous of order 1 in $\xi\neq 0$, i.e., $\phi(x,\lambda\xi)=\lambda\phi(x,\xi)$ we will work with phase functions defined on $\mathbb{T}^n\times\mathbb{R}^n$ instead of phase functions on $\mathbb{T}^n\times\mathbb{Z}^n$.

Theorem 6. *Let $1<p<\infty$. Let us assume that ϕ is a real valued continuous function defined on $\mathbb{T}^n\times\mathbb{R}^n$. If $a:\mathbb{T}^n\times\mathbb{R}^n\to\mathbb{C}$ is a continuous bounded function and the Fourier integral operator*

$$T_{\phi,a}f(x):=\int_{\mathbb{R}^n} e^{i2\pi\phi(x,\xi)}a(x,\xi)(\mathscr{F}_{\mathbb{R}^n}f)(\xi)\,\mathrm{d}\xi \tag{8}$$

extends to a bounded operator $T_{\phi,a} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, *then the Fourier series operator*

$$A_{\phi,a}f(x) := \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi\phi(x,\xi)} a(x,\xi)(\mathscr{F}_{\mathbb{T}^n} f)(\xi) \tag{9}$$

also extends to a bounded operator $A_{\phi,a} : L^p(\mathbb{T}^n) \to L^p(\mathbb{T}^n)$. *Moreover, for some* $C_p > 0$, *the estimate* $\|A_{\phi,a}\|_{\mathscr{B}(L^p(\mathbb{T}^n))} \leq C_p \|T_{\phi,a}\|_{\mathscr{B}(L^p(\mathbb{R}^n))}$ *holds true.*

In the next result we establish the L^p-boundedness of FSOs, which in this context, is an analogy for FSOs of the celebrated (local) L^p-estimate proved by Seeger, Sogge and Stein in [38] (and also, of the global L^p-estimate proved by Coriasco and Ruzhansky [14, 15]).

Theorem 7. *Let us assume that* $\phi : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ *is a real-valued phase function positively homogeneous of order* 1 *in* $\xi \neq 0$. *Let us assume that* $\partial_x^{\gamma'} \partial_\xi^{\gamma} \phi \in S^0_{0,0}(\mathbb{T}^n \times (\mathbb{R}^n \setminus \{0\}))$ *when* $|\gamma| = |\gamma'| = 1$, *that*

$$|\det(\partial_y \partial_\xi \phi(y,\xi))| \geq C > 0, \quad |\partial_y^\alpha \phi(y,\xi)| \leq C_\alpha |\xi|, \quad \xi \neq 0, \tag{10}$$

$$\langle \nabla_\xi \phi(y,\xi) \rangle \asymp 1, \quad \langle \nabla_y \phi(y,\xi) \rangle \asymp \langle \xi \rangle, \tag{11}$$

and the symbol inequalities

$$|\partial_x^\beta \Delta_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{\kappa - |\alpha|}, \quad \kappa \leq \kappa_p := -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|, \quad |\beta| \leq \left[\frac{n}{p}\right] + 1, \tag{12}$$

hold true for every $(x,\xi) \in \mathbb{T}^n \times \mathbb{Z}^n$. *Then, the Fourier series operator in* (9) *extends to a bounded operator* $A_{\phi,a} : L^p(\mathbb{T}^n) \to L^p(\mathbb{T}^n)$ *for all* $1 < p < \infty$.

Theorem 7 gives the L^2-boundedness of FSOs provided that $\kappa \leq 0$. Our conditions however are different from the following sharp L^2-result due to Ruzhansky and Turunen [34, p. 407].

Theorem 8 (Ruzhansky–Turunen). *Let us assume that* $\phi : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{R}$ *is a real-valued phase function, homogeneous of order* 1 *in* $\xi \neq 0$. *Let us assume that* $\Delta_\xi^\gamma \phi \in S^0_{0,0,2n+1,0}(\mathbb{T}^n \times \mathbb{R}^n \setminus \{0\})$ *when* $|\gamma| = 1$ *and the symbol inequalities*

$$|\partial_x^\beta a(x,\xi)| \leq C, \quad |\beta| \leq 2n+1, \tag{13}$$

hold true. Assume also that

$$|\nabla_x \phi(x,\xi) - \nabla_x \phi(x,\xi')| \geq C|\xi - \xi'|, \quad \xi, \xi' \in \mathbb{Z}^n. \tag{14}$$

Then, the Fourier series operator in (9) *extends to a bounded operator* $A_{\phi,a} : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$.

In relation with the L^2-results mentioned above, we present the following dispersive estimate for a family of Fourier series operators of the form (see Ruzhansky and Wirth [36] for the case of $\mathbb{R}^n$),

$$A_t f(x) := \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi x\xi + i2\pi t\phi(t,x,\xi)} a(t,x,\xi)(\mathscr{F}_{\mathbb{T}^n} f)(\xi), \quad 0 < t_0 \leq t < \infty. \tag{15}$$

The corresponding assertion is the following.

Theorem 9. *Let us consider the parametrized family of Fourier series operators in* (15). *Let us assume that* $\phi : [t_0,\infty) \times \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ *is a real-valued phase function, homogeneous of order* 1 *in* $\xi \neq 0$, *and satisfies*

$$|\det(I + t\partial_x \partial_\xi \phi(t,x,\xi))| \geq C_0 > 0, \quad |\partial_x^\beta \partial_\xi^\alpha \phi(t,x,\xi)| \leq C_{\alpha,\beta} t^{-|\beta|}, \quad t \geq t_0 > 0, \tag{16}$$

for all $x \in \mathbb{T}^n$, $\xi \neq 0$, *and for* $1 \leq |\beta|, |\alpha| \leq 2n+2$. *Let us assume that* $a : [t_0,\infty) \times \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ *is supported in* $t|\xi| \geq C$ *for some constant* $C > 0$ *and that*

$$|\partial_x^\beta \Delta_\xi^\alpha a(t,x,\xi)| \leq C_{\alpha,\beta} t^{-|\beta|}, \quad t \geq t_0 > 0, \quad |\alpha|, |\beta| \leq 2n+2. \tag{17}$$

Then, the family A_t, $0 < t_0 \leq t < \infty$, *is uniformly bounded on* $L^2(\mathbb{T}^n)$. *Moreover*

$$\|A_t f\|_{L^2(\mathbb{T}^n)} \leq C \sup_{|\alpha|,|\beta| \leq 2n+2} C_{\alpha,\beta} \cdot \|f\|_{L^2(\mathbb{T}^n)}. \tag{18}$$

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