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Nir Lazarovich and Arielle Leitner

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Local Limits of Connected Subgroups of $\mathrm{SL}_3(\mathbb{R})$

Limites Locales des Sous-groupes Connectés de $\mathrm{SL}_3(\mathbb{R})$

Nir Lazarovich [a] and Arielle Leitner [*, b]

[a] Department of Mathematics, Technion, Haifa 32000 Israel
URL: https://lazarovich.net.technion.ac.il/

[b] Faculty of Math and C.S., The Weizmann Institute of Science, 234 Herzl Street, Rehovot 7610001 Israel

E-mails: lazarovich@technion.ac.il (N. Lazarovitch), aleitner.math@gmail.com (A. Leitner)

Abstract. In this paper we describe the local limits under conjugation of all closed connected subgroups of $\mathrm{SL}_3(\mathbb{R})$ in the Chabauty topology.

Résumé. Dans cet article, nous décrivons les limites locaux sous conjugaison de tous les sous-groupes fermés connectés de $\mathrm{SL}_3(\mathbb{R})$ dans la topologie de Chabauty.

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1. Introduction

Let G be a locally compact second countable group. The set $\mathrm{Sub}(G)$ of closed subgroups of G may be endowed with the *Chabauty topology*, with which it is a compact space. The Chabauty topology was introduced in several places, among them [3, 8]. For an overview we recommend [13]. We are interested in *convergence* in the Chabauty topology, which for a locally compact second countable group can be defined as follows:

Definition 1 (See [2]). *A sequence of closed subgroups* $\{H_n\} \leq G$ converges *to* $H \leq G$ *if the following two conditions hold*

[*] Corresponding author.

(1) *For every $h \in H$, there exists a sequence of elements $h_n \in H_n$ so that $h_n \to h$.*
(2) *Given a sequence of elements $h_n \in H_n$, for every convergent subsequence $(h_{n_k}) \to h$ then $h \in H$.*

A group $H \le G$ converges to a group $L \le G$ under conjugacy *if there exists a sequence $p_n \in G$ such that $p_n H p_n^{-1}$ converges to L in the sense of the definition above.*

A connected subgroup H locally converges *to a connected subgroup L under conjugacy if there is a sequence $p_n \in G$ such that $p_n H p_n^{-1} \to L'$ and L is the identity component of L'.*

Limits of connected subgroups are not necessarily connected. For instance, [5, Example 2 in Section 3.2] shows that there is a sequence of conjugates of the rotation subgroup $\mathrm{SO}(2) \le \mathrm{SL}_2(\mathbb{R})$ that converges to the subgroup $\{\left(\begin{smallmatrix} \pm 1 & x \\ 0 & \pm 1 \end{smallmatrix}\right) : x \in \mathbb{R}\}$ which has two connected components. It is therefore necessary to pass to the identity component of the limit in the definition of local convergence.

Let $G = \mathrm{SL}_3(\mathbb{R})$. Our main result is a description of the local convergence of connected subgroups of G. Note that we consider subgroups up to conjugacy, and use the classification of subalgebras of $\mathfrak{g}$ up to conjugacy by Winternitz [18], their images under the exponential map are connected subgroups of G. In Section 3, we have written a subsection for each dimension of subgroups.

Theorem 2. *The local convergence of the connected subgroups of* $\mathrm{SL}_3(\mathbb{R})$ *of each dimension is described by the chart of limits in the corresponding section in the paper.*

The connected subgroups of $\mathrm{SL}_3(\mathbb{R})$ were classified by Winternitz [18], who provided a full list of subalgebras of $\mathfrak{sl}_3(\mathbb{R})$ up to conjugacy. In each section, we first list the subgroups together with their normalizers and properties, and then we provide a chart which shows which groups locally limit to others by conjugation. Following each chart, we prove that this is indeed the complete chart of limits.

Theorem 2 gives a partial understanding of the closure of the connected subgroups in $\mathrm{Sub}(G)$ in the following sense. The conjugacy class of each connected subgroup $H \in \mathrm{Sub}(G)$ is a subspace homeomorphic to $G/N_G(H)$. The closure of the conjugacy class of H in $\mathrm{Sub}(G)$ consists of conjugacy classes of subgroups whose identity components are the local limits of H described by our main result.

Using work of [5, 7] we prove the following proposition which is a component of the proof of Theorem 2.

Proposition 3. *Let $G = \mathrm{SL}_3(\mathbb{R})$ and let $H \le G$ be a connected subgroup and L a local limit of H. Then* $\dim H = \dim L$.

Note that this fails in $\mathrm{SL}_4(\mathbb{R})$ as shown in [5]. This also implies that the local limit can be seen as a limit of the corresponding Lie subalgebras under the $\mathrm{Ad}(G)$ action.

The proof of Proposition 3 relies on the following Theorem 4, which we believe might be of independent interest.

Theorem 4. *Let $X \in \mathfrak{gl}_d(\mathbb{C})$ be a matrix with an eigenvalue which is not purely imaginary. Then the local conjugacy limits of the one-dimensional closed subgroup $H = \{\exp(tX) \mid t \in \mathbb{R}\} \le \mathrm{GL}_d(\mathbb{C})$ are one-dimensional.*

In all but a few cases the homeomorphism type of all of $\mathrm{Sub}(G)$ is still unknown. However, progress towards understanding the topology has been made on the Heisenberg group by [1], on $\mathbb{R} \times \mathbb{Z}$ by [11], on the set of Cartan subgroups of $\mathrm{SL}_n(\mathbb{R})$ by [12, 16, 17], on $\mathbb{R}^2$ by [14], on $\mathbb{R}^n$ by [15], and [5] make progress on limits of symmetric subgroups in $\mathrm{PGL}_n(\mathbb{R})$.

The Chabauty compactification $\mathrm{Sub}(G)$ may be used to compactify Bruhat–Tits buildings or symmetric spaces, by identifying points in those spaces with their stabilizers in G viewed as points in $\mathrm{Sub}(G)$. See [4, 9, 10].

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2. Notation and Methods

As a convention, throughout the paper: a, b, z are parameters for infinite families, $s, t, *$ are variables. We think of $\mathbb{C}$ as $\left(\begin{smallmatrix} * \\ * \end{smallmatrix}\right)$, and $\mathbb{C}^*$ as $\left(\begin{smallmatrix} s & t \\ -t & s \end{smallmatrix}\right)$. We avoid using set builder notation as it is clunky.

Throughout this paper, we will follow the notation given by Winternitz [18] where $W_{d,k}$ denotes the k^{th} group in the list of subgroups of $\mathrm{SL}_3(\mathbb{R})$ with Lie algebra of dimension d. Sometimes a group depends on some parameters, which we denote in the superscript; sometimes a group is not conjugate to its transpose, and so we might list them together using the / notation. See e.g. the 3-dimensional groups on p. 10.

We begin by proving Proposition 3. To do so, we first review some theorems which we apply throughout the paper.

Theorem 5 ([5, Theorem 3.1]). *Let G be an algebraic group (defined over $\mathbb{C}$ or $\mathbb{R}$). Suppose that H is an algebraic subgroup and L a conjugacy limit of H. Then L is algebraic and* $\dim L = \dim H$.

A more general notion than an algebraic group is a *definable* group, in the sense of an O-minimal structure, see [6]. Many of the properties of algebraic sets carry over to this more general setting. [7, Proposition 3.1] implies that the limit of a definable group is definable, and the dimension stays constant under taking a limit.

There are three flavors of non-algebraic groups among connected subgroups of $\mathrm{SL}_3(\mathbb{R})$:

$$\begin{pmatrix} e^{at} & \cdot & \cdot \\ 0 & e^{bt} & \cdot \\ 0 & 0 & e^{-(a+b)t} \end{pmatrix} : a, b \in \mathbb{R} \text{ fixed} \qquad \begin{pmatrix} e^{t} & te^{t} & \cdot \\ 0 & e^{t} & \cdot \\ 0 & 0 & e^{-2t} \end{pmatrix} \qquad \begin{pmatrix} e^{zt} & \vdots \\ 0 & e^{-2\Re(z)t} \end{pmatrix} : z \in \mathbb{C} \text{ fixed}$$

here the $\cdot$ can be zero or any element of $\mathbb{R}$. The first and last are infinite families of groups, since we can choose any fixed $a, b \in \mathbb{R}$ or $z \in \mathbb{C}$. The second item is only one group. The first two families are definable where the O-minimal structure defined including real exponential functions, so by [7] limits cannot increase in dimension. The last family of group is not definable, since e^{zt} is not definable in any O-minimal structure. There are (up to taking transpose) two families of groups of this sort:

$$W_{1,2}^{z} := \begin{pmatrix} e^{zt} & 0 \\ 0 & e^{-2\Re(z)t} \end{pmatrix}, \qquad \text{and} \qquad W_{3,8/9}^{z} := \begin{pmatrix} e^{zt} & * \\ 0 & e^{-2\Re(z)t} \end{pmatrix}.$$

We will show both these families of groups have limits which stay constant in dimension. Lemma 7 treats the one parameter group $W_{1,2}^{z}$ and Lemma 8 completes the proof for $W_{3,8/9}^{z}$. Both lemmas are corollaries of Theorem 4.

The inspiration for Theorem 4 is [5, Section 3.2] which gives the following example of a limit $p_n H p_n^{-1} \to L$:

$$H = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} \qquad p_n = \begin{pmatrix} n^{-1} & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix}.$$

This example satisfies $\dim(L) = 2 > 1 = \dim(H)$. The group H is a one-parameter subgroup of $L \simeq \mathbb{R} \times \mathbb{S}^1$ that looks like a helix. Conjugating by p_n coils the helix more tightly so that the orbits of H accumulate. One obstruction for such a phenomenon is given by Theorem 4, which we prove next.

Proof of Theorem 4. Let us denote by B the Borel subgroup of $\mathrm{GL}_d(\mathbb{C})$ of upper triangular matrices, and let $\mathfrak{b}$ be its corresponding Lie subalgebra. Since we work over $\mathbb{C}$ we can replace X by its conjugate Jordan form, which is an upper triangular matrix. We therefore assume without loss of generality that $X \in \mathfrak{b}$. Let us denote by α the non purely imaginary eigenvalue of X, and by $v \in \mathbb{C}^d$ its eigenvector.

The map $t \mapsto \|\exp(tX)v\| = \|e^{t\alpha}v\| = e^{t\Re(\alpha)}\|v\|$ is a homeomorphism since $\Re(\alpha) \neq 0$. It follows that the map $\mathbb{R} \to \mathrm{GL}_d(\mathbb{C})$ given by $t \mapsto \exp(tX)$ is proper, and H is closed.

Now let $g_n \in \mathrm{GL}_d(\mathbb{C})$ be a sequence such that $g_n H g_n^{-1} \to L$ in $\mathrm{Sub}(\mathrm{GL}_d(\mathbb{C}))$, and suppose for contradiction that $\dim L > \dim H = 1$. Denote by $H_n = g_n H g_n^{-1}$ and $X_n = g_n X g_n^{-1}$. So $g_n H g_n^{-1} = \{\exp(tX_n) \mid t \in \mathbb{R}\}$. By the Iwasawa decomposition $\mathrm{GL}_d(\mathbb{C}) = U_d \cdot B$ where U_d is the unitary group. Write $g_n = u_n b_n$ for $u_n \in U_d$ and $b_n \in B$. Since the unitary group is compact, we may assume, up to passing to a subsequence, that $u_n \to u$. Thus $b_n H b_n^{-1} \to u^{-1} L u$. Therefore, we may assume without loss of generality that $g_n \in B$, and hence also $X_n = g_n X g_n^{-1} \in \mathfrak{b}$.

It follows from $\dim L > \dim H$ that for every small enough identity neighborhood $I \in V \subset \mathrm{GL}_d(\mathbb{C})$, the number of components of $V \cap H_n$ goes to infinity. In particular, $V \cap H_n$ has more than one component for all large enough n. So H_n leaves V and returns to it, and we have

$$\text{there exist } 0 < s_n < t_n \text{ such that } \exp(s_n X_n) \notin V \text{ but } \exp(t_n X_n) \in V. \tag{1}$$

Fix some norm $\|\cdot\|$ on $\mathfrak{gl}_d(\mathbb{C})$ and let B_ε be the ball of radius ε in $\mathfrak{gl}_d(\mathbb{C})$ around 0. Denote by $V_\varepsilon = \exp(B_\varepsilon)$. The exponential map $\exp : \mathfrak{gl}_d(\mathbb{C}) \to \mathrm{GL}_d(\mathbb{C})$ is a local homeomorphism at 0. Let ε_0 be small enough so that $\exp : B_{\varepsilon_0} \to V_{\varepsilon_0}$ is a homeomorphism and V_{ε_0} is open. These neighborhoods have the following *trapping property:* for every $0 < \delta < \varepsilon_0$, and $0 < t$ and $Y \in \mathfrak{gl}_d(\mathbb{C})$, if $\exp(sY) \in V_{\varepsilon_0}$ for all $s \in [0,t]$ and $\exp(tY) \in V_\delta$ then $\exp(sY) \in V_\delta$ for all $s \in [0,t]$. The following upgraded form of (1) follows: for every $\delta > 0$ and large enough n,

$$\text{there exist } 0 < s_n < t_n \text{ such that } \exp(s_n X_n) \notin V_{\varepsilon_0} \text{ but } \exp(t_n X_n) \in V_\delta. \tag{2}$$

Let $\varepsilon = \frac{\varepsilon_0}{2}$. For every $\delta \in (0,\varepsilon)$, let $t_n > 0$ be the first return of the curve $t \mapsto \exp(tX_n)$ to $\overline{V_\delta}$ after leaving V_{ε_0}.

Claim 6. $\exp(\frac{t_n}{2} X_n) \notin V_\varepsilon$.

Assume for contradiction that $\exp(\frac{t_n}{2} X_n) \in V_\varepsilon$. We first show that $\exp(\frac{t_n}{2} X_n) \in \overline{V_\delta}$. Indeed, let $Y \in B_\varepsilon$ be such that $\exp(Y) = \exp(\frac{t_n}{2} X_n)$. For all $s \in [0,2]$, $\exp(sY) \in V_{\epsilon_0}$ (since $\epsilon = \epsilon_0/2$), and

$$\exp(2Y) = \exp(Y)^2 = \exp\left(\frac{t_n}{2} X_n\right)^2 = \exp(t_n X_n) \in \overline{V_\delta}.$$

It follows by the trapping property that $\exp(Y) = \exp(\frac{t_n}{2} X_n) \in \overline{V_\delta}$.

Recall that t_n is defined as the first return of $t \mapsto \exp(tX_n)$ to $\overline{V_\delta}$ after leaving V_{ε_0}. Therefore $\exp(sX_n) \in V_{\varepsilon_0}$ for all $s \in [0, \frac{t_0}{2}]$. By the trapping property, it follows that $\exp(sX_n) \in \overline{V_\delta}$ for all $s \in [0, \frac{t_0}{2}]$. Hence,

$$\exp(sX_n) = \exp\left(\frac{s}{2} X_n\right)^2 \in \overline{V_\delta}^2 \subseteq V_{\varepsilon_0} \text{ for all } \in [0, t_0].$$

This contradicts the assumption that the curve leaves V_{ε_0}, and proves the claim.

Thus we may further upgrade (2). For every $\delta \in (0,\varepsilon)$, and large enough n,

$$\text{there exist } 0 < t_n \text{ such that } \exp\left(\frac{t_n}{2} X_n\right) \notin V_\varepsilon \text{ but } \exp(t_n X_n) \in V_\delta. \tag{3}$$

By choosing an appropriate subsequence, assume $h_n = \exp(\frac{t_n}{2} X_n)$ satisfies (3) for $\delta = \frac{1}{n}$. In particular, $h_n \not\to 1$ but $h_n^2 \to 1$. Let $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ be the eigenvalues of X and thus also of $X_n = g_n X g_n^{-1}$. By definition of h_n it follows that $e^{t_n\alpha}$ is an eigenvalue of $h_n^2 = \exp(t_n X_n)$. Since $h_n^2 \to 1$, we have $e^{t_n\alpha} \to 1$. From the assumption $\Re(\alpha) \neq 0$ it follows that $t_n \to 0$. It follows that the eigenvalues of h_n, namely $e^{\frac{t_n}{2}\alpha_1}, \dots, e^{\frac{t_n}{2}\alpha_d}$ tend to 1 as well.

We get a sequence of matrices $h_n \in B$ satisfying

$$h_n^2 \to 1, h_n \not\to 1, \text{ and all eigenvalues of } h_n \text{ tend to } 1. \tag{4}$$

Let us suppress indices and write $h = h_n$. The matrix h is upper triangular so we may write

$$h = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1d} \\ 0 & x_{22} & \cdots & x_{2d} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & x_{dd} \end{pmatrix} \qquad h^2 = \begin{pmatrix} x_{11}^2 & x_{12}(x_{11}+x_{22}) & \cdots & \sum_{k=1}^{d} x_{1k}x_{kd} \\ 0 & x_{22}^2 & \cdots & \sum_{k=1}^{d} x_{2k}x_{kd} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & x_{dd}^2 \end{pmatrix}.$$

By assumption, as $n \to \infty$, the eigenvalues of h tend to 1, i.e $x_{ii} \to 1$ for all $1 \le i \le d$. We also have $h^2 \to 1$ as $n \to \infty$. In particular, looking at the super-diagonal entries of h^2 we have $x_{i,i+1}(x_{ii} + x_{i+1,i+1}) \to 0$. But since $x_{ii} + x_{i+1,i+1} \to 2$ we can deduce that $x_{i,i+1} \to 0$ for all $1 \le i \le d-1$. Proceeding by induction to next diagonal, we get $x_{i,j} \to 0$ for all $1 \le i < j \le d$. We conclude that $h \to 1$ as $n \to \infty$. However this contradicts (4), and completes the proof of Theorem 4. □

Lemma 7. *Limits of $W_{1,2}^z$ are 1 dimensional.*

Proof. The group $W_{1,2}^z$ is given by $\{\exp(tX) \mid t \in \mathbb{R}\}$ for

$$X = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{R})$$

with $a \neq 0, b \in \mathbb{R}$. This matrix has a non-zero real eigenvalue. As a subgroup of $\mathrm{GL}_3(\mathbb{C})$ this satisfies the assumptions of Theorem 4, and therefore every local limit of this subgroup under conjugation in $\mathrm{GL}_3(\mathbb{C})$ is one-dimensional. In particular, the same conclusion also holds for conjugates of the closed subgroup $W_{1,2}^z$ in $\mathrm{SL}_3(\mathbb{R})$, since $\mathrm{SL}_3(\mathbb{R})$ is a closed subgroup of $\mathrm{GL}_3(\mathbb{C})$. □

Lemma 8. *Limits of $W_{3,8/9}^z$ are 3 dimensional.*

Proof. Set $H = W_{3,8}^z$ for some $z \notin i\mathbb{R}$, and assume that $p_n H p_n^{-1} \to L$ we wish to show that $\dim L = 3$. We first claim we can reduce to the case that the p_n are upper triangular. Use the Iwasawa decomposition to write $G = KB$ where $K = \mathrm{SO}(3)$ and B is the subgroup of upper triangular matrices in $\mathrm{SL}_3(\mathbb{R})$. Hence we can write $p_n = k_n b_n$ for $k_n \in K$ and $b \in B$. By compactness of K we may assume, up to passing to a subsequence, that $k_n \to k$. Thus $b_n H b_n^{-1} \to k^{-1} L k$. Since $\dim k^{-1} L k = \dim L$, the claim follows.

Now, both H and B are contained in the parabolic subgroup

$$Q = W_{6,1} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix},$$

and hence it suffices to look at limits in $\mathrm{Sub}(Q)$. Let $p : Q \to \mathrm{GL}_2(\mathbb{R})$ be the homomorphism sending a matrix in Q to its upper left 2×2 block. The homomorphism p induces a pullback map $p^* : \mathrm{Sub}(\mathrm{GL}_2(\mathbb{R})) \to \mathrm{Sub}(Q)$, by $p^*(\widetilde{A}) = p^{-1}(\widetilde{A})$. The map p^* is a homeomorphism between

$\mathrm{Sub}(\mathrm{GL}_2(\mathbb{R})))$ and $\{A \in \mathrm{Sub}(Q) | A \geq \ker p\}$, as it has an inverse $p_* : \{A \in \mathrm{Sub}(Q) | A \geq \ker p\} \to \mathrm{Sub}(\mathrm{GL}_2(\mathbb{R}))$ defined by $p_*(A) = p(A)$. The group H contains

$$\ker p = \left(\begin{smallmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix}\right).$$

The image of H under this homeomorphism is the 1-dimensional subgroup $\widetilde{H} = (e^{zt}) \in \mathrm{Sub}(GL_2(\mathbb{R}))$. By Theorem 4 limits of $\widetilde{H}$ in $\mathrm{Sub}(\mathrm{GL}_2(\mathbb{R})$ are 1 dimensional since $z \notin i\mathbb{R}$ by assumption. Let L be a limit of conjugates of H, then $L \leq p^*(\widetilde{L})$ for some limit $\widetilde{L}$ of conjugates of $\widetilde{H}$. Since $\ker p$ is 2-dimensional, and $\bar{L}$ is 1-dimension, then $p^*(\widetilde{L})$ is 3-dimensional. Hence L is 3-dimensional. □

We will also extensively use the following propositions from [4, 5] to identify which subgroups cannot limit to other subgroups.

Denote the normalizer of a subgroup $H \leq G$ by $N_G(H)$. Denote the connected component of the identity by H_0. The next theorem says that the dimension of the normalizer increases under taking a limit.

Proposition 9 ([5, Proposition 3.2]). *Let G be an algebraic Lie group (defined over $\mathbb{C}$ or $\mathbb{R}$), let H be an algebraic subgroup and let L be any limit of H. Then* $\dim N_G(H_0) \leq \dim N_G(L_0)$ *with equality if and only if L and H are conjugate.*

The same statement works for non-algebraic groups as well as long as the dimension does not increase when taking a limit. In this case, taking a local limit of groups by conjugation is equivalent to taking a limit of their Lie algebras by $\mathrm{Ad}(G)$ action, and the same proof idea works.

Any element $A \in \mathfrak{gl}(n)$ has a well defined characteristic polynomial, denoted $char(A)$. Given a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{gl}(n)$, we denote by $Char(\mathfrak{h})$ the closure of the subset $\{char(A) : A \in \mathfrak{h}\} \subset \mathbb{R}[x]$. Thus $Char(\mathfrak{h})$ is closed and invariant under conjugation of $\mathfrak{h}$. The next Proposition 10 implies that limits have smaller sets of characteristic polynomials.

Proposition 10 ([5, Proposition 3.4]). *Suppose H is a closed algebraic subgroup of* $\mathrm{GL}_n(\mathbb{R})$, *and L is a conjugacy limit of H. Then $Char(\mathfrak{l}) \subseteq Char(\mathfrak{h})$, where $\mathfrak{h}, \mathfrak{l} \subset \mathfrak{gl}(n)$ denote the Lie algebras of H and L respectively.*

The next Proposition 11 implies that limits of abelian groups are abelian. A group H satisfies a universal relation if there is a finitely generated free group F and a word $w \in F$ such that for all homomorphisms $\theta : F \to H$ we have $\theta(w) = 1$.

Proposition 11 ([4, Proposition 2.2] idea due to Daryl Cooper). *If $H \leq G$ satisfies a universal relation, w, then so does every G-conjugacy limit L of H.*

To organize the local limit charts of Theorem 2, we note that by Proposition 3 we can treat each dimension separately. In view of Proposition 9, normalizers of (non-conjugate) limits must increase in dimension, it is therefore convenient to arrange the columns (or rows) of the chart by the dimension of the normalizer. Arrows thus can only go to the right (or down if arranged by rows). To complete the proof in each section, we need to provide a conjugating sequence of matrices for each arrow in the chart, and prove nonexistence of any arrows from left to right (up to down), which we do using the remainder of the theorems and propositions from this section.

3. The classification of local limits in $\mathrm{SL}_3(\mathbb{R})$

Dimension 1

We begin with a table of the one-parameter subgroups of $\mathrm{SL}_3(\mathbb{R})$ and their properties. After the table, the terminology *singular* and *non-singular* and other conventions used in the table are explained.

Name	Group	Normalizer	Properties
$W_{1,1}^{(a,b)}$	$\begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{-(a+b)t} \end{pmatrix}$ $(a,b) \in \mathbb{R} \setminus \{0\}$	$W_{2,2}$ if nonsingular $W_{4,1}$ if singular	definable algebraic
$W_{1,2}^{z}$	$\begin{pmatrix} e^{zt} & 0 \\ 0 & e^{-2\Re(z)t} \end{pmatrix}$ $z \in \mathbb{C}$	$W_{2,1}$	limits 1 dimensional algebraic if $z \in i\mathbb{R}$
$W_{1,3}$	$\begin{pmatrix} e^{t} & te^{t} & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$	$W_{2,3}$	definable
$W_{1,4}$	$\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	conjugate to $W_{5,3}$	algebraic
$W_{1,5}$	$\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$	$W_{3,7}$	algebraic

The group $W_{1,1}^{(a,b)}$ depends on a choice of two real numbers $a, b \in \mathbb{R}$. If none of the weights a, b and $-a-b$ match, the group is *nonsingular*. If any of the weights match, i.e. if $a = b$, $a = -a-b$ or $b = -a-b$, it is called *singular*. We abuse notation and denote the nonsingular case by $W_{1,1}^{a\neq b}$ and the singular case $W_{1,1}^{a=b}$, and use this notation throughout the rest of the paper. Note that if $z \in \mathbb{R}$ then $W_{1,2}^{z} = W_{1,1}^{(a,a)}$. Therefore we abuse notation in writing $W_{1,2}^{z}$ to assume also $z \notin \mathbb{R}$.

The possible local limits of each group are represented in the following transitive chart.

$$\begin{tikzcd}
W_{1,1}^{a\neq b} \arrow[dr] & & W_{1,1}^{a=b} \arrow[dr] & \\
W_{1,2}^{z} \arrow[r] & W_{1,5} \arrow[rr] & & W_{1,4} \\
W_{1,3} \arrow[ur] \arrow[uurr, bend right] & & &
\end{tikzcd}$$

Another way to see the local limits of one parameter groups is to take a sequence of elements in the groups that converges to the limit. However, we wanted all the sections of the paper to be consistent. So, we give a sequence of conjugating matrices, p_n in the sense of Definition 1, for each arrow that appears in the chart:

$$W_{1,5} \to W_{1,4} \quad \begin{pmatrix} \frac{1}{n} & 0 & 0 \\ 0 & \frac{1}{n} & 0 \\ 0 & 0 & n^2 \end{pmatrix} \qquad W_{1,3} \to W_{1,5}: \quad \begin{pmatrix} n & 0 & \frac{n}{9} \\ 0 & 1 & \frac{-1}{3} \\ 0 & 0 & \frac{1}{n} \end{pmatrix} \qquad W_{1,1}^{a\neq b} \to W_{1,5}: \quad \begin{pmatrix} 1 & n & \frac{(a-b)^2 n^2}{2a^2+5ab+2b^2} \\ 0 & 1 & \frac{(a-b)y}{a+2b} \\ 0 & 0 & 1 \end{pmatrix}$$

$$W_{1,1}^{a=b} \to W_{1,4}: \quad \begin{pmatrix} 1 & 0 & n \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad W_{1,2}^{z} \to W_{1,5}: \quad \begin{pmatrix} n & \frac{3an}{b} & \frac{(9a^2+b^2)n}{b^2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{n} \end{pmatrix} \qquad W_{1,3} \to W_{1,1}^{a=b}: \quad \begin{pmatrix} \frac{1}{n} & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Recall we have assumed $z \notin \mathbb{R}$. Next we explain nonexistence of the missing arrows. The first subscript of the group in the normalizer column is the dimension of the normalizer. Proposition 9 explains the missing arrows except for $W_{1,5} \not\to W_{1,1}^{a=b}$, $W_{1,1}^{a\neq b} \not\to W_{1,1}^{a=b}$, and $W_{1,2}^{z} \not\to W_{1,1}^{a=b}$ which follow from Proposition 10.

Dimension 2

Name	Group	Normalizer	Properties
$W_{2,1}$	$\begin{pmatrix} \mathbb{C}^* & 0 \\ 0 & \det^{-1} \end{pmatrix}$	$W_{2,1}$	$\cong \mathbb{C}^*$
$W_{2,2}$	$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$	$W_{2,2}$	$\cong (\mathbb{R}^2,+)$
$W_{2,3}$	$\begin{pmatrix} e^t & * & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$	conjugate to $W_{3,1}$	$\cong (\mathbb{R}^2,+)$
$W_{2,4}$	$\begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$W_{5,2}$	$\cong (\mathbb{R}^2,+)$
$W_{2,5}$	$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$	$W_{5,1}$	$\cong (\mathbb{R}^2,+)$
$W_{2,6}$	$\begin{pmatrix} 1 & t & * \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$	$W_{4,6}^{(1,0)}$	$\cong (\mathbb{R}^2,+)$
$W_{2,7}^{(a,b)}$	$\begin{pmatrix} e^{at} & * & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{-(a+b)t} \end{pmatrix}$	conjugate to $W_{3,1}$	$\cong \mathrm{Aff}(\mathbb{R})$ algebraic $a=b$ definable $a\neq b$
$W_{2,8}$	$\begin{pmatrix} e^t & * & te^t \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^t \end{pmatrix}$	$W_{3,2}$	$\cong \mathrm{Aff}(\mathbb{R})$ definable
$W_{2,9}$	$\begin{pmatrix} e^{-2t} & 0 & * \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}$	$W_{3,3}$	$\cong \mathrm{Aff}(\mathbb{R})$ definable
$W_{2,10}$	$\begin{pmatrix} e^t & e^t s & e^t \frac{s^2}{2} \\ 0 & 1 & s \\ 0 & 0 & e^{-t} \end{pmatrix}$	$W_{2,10}$	$\cong \mathrm{Aff}(\mathbb{R})$ algebraic

Note that $W_{2,3} = W_{2,7}^{a=b}$.

The full chart of local limits in dimension 2 is

$$\begin{tikzcd}
\mathbf{W_{2,1}} \arrow[r] & \mathbf{W_{2,3}} \arrow[r] & \mathbf{W_{2,6}} \arrow[r] \arrow[dr] & \mathbf{W_{2,4}} \\
\mathbf{W_{2,2}} \arrow[ur] & W_{2,7}^{(a,b)} \arrow[ur] & & \mathbf{W_{2,5}} \\
W_{2,10} \arrow[uurr, bend left] & W_{2,8} \arrow[uur] & & \\
 & W_{2,9} \arrow[uuur] & &
\end{tikzcd}$$

We have put the abelian groups in *bold* to distinguish them. Excluding limits of $W_{2,1}$ the computations for abelian groups appear in [12, 17]. We first give the computations for the remainder of the arrows which do appear in the chart. To finish the limits of the abelian groups, we see $W_{2,1} \to W_{2,3}$ by $diag\langle n, 1, \frac{1}{n}\rangle$.

Next we compute all of the limits of the nonabelian groups. The next two limits are done by first conjugating by a permutation matrix to move the free element to the upper right corner, and then applying the sequence shown.

$$W_{2,7}^{(a,b)} \to W_{2,6}: \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & \frac{-a+b}{-a-2b}n \\ 0 & 0 & 1 \end{pmatrix} \quad W_{2,8} \to W_{2,6}: \begin{pmatrix} n & 0 & \frac{-2n}{9} \\ 0 & 1 & \frac{-1}{3} \\ 0 & 0 & \frac{1}{n} \end{pmatrix}. \quad W_{2,9} \to W_{2,6}: \begin{pmatrix} \frac{9}{1-n^3} & \frac{3n^3}{1-n^3} & 1 \\ 0 & n & 0 \\ 0 & 0 & \frac{1-n^3}{9n} \end{pmatrix} \quad W_{2,10} \to W_{2,6}: \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It remains to prove nonexistence of the missing arrows. By Proposition 11 limits of $W_{2,1}$ are abelian groups.

To finish the argument, we need to explain why there are no missing arrows. It remains to check the non-abelian groups. All the missing arrows from non-abelian groups would originate from $W_{2,10}$, and Proposition 10 does not allow an arrow to any of $W_{2,3}, W_{2,7}, W_{2,8}, W_{2,9}$, because the corresponding characteristic polynomials are not contained in $\mathrm{Char}(W_{2,10})$.

Dimension 3

Name	Group	Normalizer	Properties
$W_{3,1}$	$\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$	$W_{3,1}$	$\cong \mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ algebraic
$W_{3,2}$	$\begin{pmatrix} e^t & * & * \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$	$W_{4,2}$	$\cong \mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ algebraic
$W_{3,3}$	$\begin{pmatrix} e^{-2t} & 0 & * \\ 0 & e^t & * \\ 0 & 0 & e^t \end{pmatrix}$	$W_{4,3}$	$\cong \mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ algebraic
$W_{3,4}$	$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$	$W_{5,3}$	Heisenberg algebraic
$W_{3,5}^{(a,b)}$	$\begin{pmatrix} e^{at} & * & * \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{-(a+b)t} \end{pmatrix}$	$W_{4,2}$	$\cong \mathbb{R}^2 \rtimes \mathbb{R}$ $\mathbb{R}$ acts as $W_{1,1}$ definable
$W_{3,6}^{(a,b)}$	$\begin{pmatrix} e^{at} & 0 & * \\ 0 & e^{bt} & * \\ 0 & 0 & e^{-(a+b)t} \end{pmatrix}$	$W_{4,3}$	$\cong \mathbb{R}^2 \rtimes \mathbb{R}$ $\mathbb{R}$ acts as $W_{1,1}$ definable

$W_{3,7}$	$\begin{pmatrix} e^t & e^t s & * \\ 0 & 1 & s \\ 0 & 0 & e^{-t} \end{pmatrix}$	$W_{3,7}$	$\cong \mathbb{R}^2 \rtimes \mathbb{R}$ $\mathbb{R}$ acts as $W_{1,1}$ algebraic
$W^{(z)}_{3,8}$	$\begin{pmatrix} e^{zt} & \mathbb{C} \\ 0 & e^{-2\Re(z)t} \end{pmatrix}, z \in \mathbb{C}$	$W_{4,4}$	$\cong \mathbb{R}^2 \rtimes \mathbb{R}$ $\mathbb{R}$ acts as $W_{1,2}$ limits 3 dimensional
$W^{(z)}_{3,9}$	$\begin{pmatrix} e^{-2\Re(z)t} & \mathbb{C} \\ 0 & e^{zt} \end{pmatrix}, z \in \mathbb{C}$	$W_{4,5}$	$\cong \mathbb{R}^2 \rtimes \mathbb{R}$ $\mathbb{R}$ acts as $W_{1,2}$ limits 3 dimensional
$W_{3,10}$	$\begin{pmatrix} e^t & te^t & * \\ 0 & e^t & * \\ 0 & 0 & e^{-2t} \end{pmatrix}$	$W^{(1,1)}_{4,6}$	$\cong \mathbb{R}^2 \rtimes \mathbb{R}$ $\mathbb{R}$ acts as $W_{1,3}$ definable
$W_{3,11}$	$\begin{pmatrix} e^{-2t} & * & * \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}$	$W^{(-2,1)}_{4,6}$	$\cong \mathbb{R}^2 \rtimes \mathbb{R}$ $\mathbb{R}$ acts as $W_{1,3}$ definable
$W_{3,12}$	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$W_{4,1}$	$\cong \mathrm{SL}_2(\mathbb{R})$ algebraic
$W_{3,13}$	$\mathrm{SO}(2,1)$	$W_{3,13}$	$\cong \mathrm{SL}_2(\mathbb{R})$ algebraic
$W_{3,14}$	$\mathrm{SO}(3)$	$W_{3,14}$	$\cong \mathrm{SO}(3)$ algebraic

We compactify notation to write a group and its transpose in the same line, for example: $W_{3,8/9}$. We show the full chart of local limits is as follows:

$$\begin{tikzcd}
 & W_{3,1} \arrow[d] & W_{3,7} \arrow[d] & & W_{3,13} \arrow[d] & W_{3,14} \arrow[dl] & \\
W^{(a,b)}_{3,5/6} \arrow[drrr] & W_{3,2/3} \arrow[drr] & W^{b=0}_{3,5/6} \arrow[dr] & W^{(z)}_{3,8/9} \arrow[d] & W^{(i)}_{3,8/9} \arrow[dl] & W_{3,10/11} \arrow[dll] & W_{3,12} \arrow[dlll] \\
 & & & W_{3,4} & & &
\end{tikzcd}$$

Recall [5] compute limits of $W^i_{3,8} = \mathrm{SO}(2,1) = W_{3,13}$ and [9] calculate limits of $\mathrm{SO}(3) = W_{3,14}$. We first give the computations of the remainder of the limits where we write computations for the transpose in the same line.

$$W_{3,12} \to W_{3,4}: \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad W_{3,10/11} \to W_{3,4}: \begin{pmatrix} n & 0 & 0 \\ 0 & \frac{1}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad W^z_{3,8/9} \to W_{3,4}: \begin{pmatrix} n & 0 & 0 \\ 0 & \frac{1}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad W^{(a,b)}_{3,5/6} \to W_{3,4}: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

the last limit includes $W_{3,2}$ and $W_{3,3}$ as singular cases, $W_{3,2/3} = W^{(a=b)}_{3,5/6} \to W_{3,4}$. Finally,

$$W_{3,7} \to W^{b=0}_{3,5/6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & \frac{1}{n} \end{pmatrix} \qquad W_{3,1} \to W^{(a=b)}_{3,5/6} \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now it remains to prove nonexistence of the missing arrows. By Theorem 5 limits of $W_{3,7}$ are algebraic. Proposition 10 rules out the rest of the options for limits of $W_{3,7}$ except for $W_{3,5/6}$.

Proposition 12. *The only possible values of* (a,b) *for the groups* $W^{a,b}_{3,5/6}$ *which can appear as limits of* $W_{3,7}$ *are conjugate to* $W_{3,5/6}$ *with* $b=0$.

Proof. Using the standard Iwasawa decomposition argument, any limit of $W_{3,7}$ is conjugate to a limit under an element of the Borel, B. Conjugating by an element of B leaves the diagonal invariant. Thus the limit must either be unipotent, or is conjugate to an upper triangular group with diagonal $\langle e^t,1,e^{-t}\rangle$. In the case $W_{3,5}$ the possibilities for the limit group are $a+b=0$ or $b=0$ which are conjugate by a permutation. For $W_{3,6}$ we see $a=0$ and $b=0$ are conjugate. □

Next $W_{3,1}$ is algebraic, and so its limits must be algebraic, and by Proposition 10 its limits must have real weights. Thus $W_{3,12}$ and $W_{3,8/9}$ cannot be limits because they have complex wieghts. So the only possible limits are the algebraic groups in $W^{(a,b)}_{3,5/6}$ where $a,b\in\mathbb{Q}$. By Proposition 10 the only possibilities are the singular $W^{(a=b)}_{3,5/6}$.

Proposition 13. *The group* $W_{3,1}$ *limits only to singular groups among* $W^{(a,b)}_{3,5/6}$.

Proof. The subgroup $W_{3,1}$ is contained in the Borel B. Since $\mathrm{SL}_3(\mathbb{R})=\mathrm{SO}(3)B$ and $\mathrm{SO}(3)$ is compact, it suffices to consider conjugating only by sequences of elements $g_n\in B$. Notice $B=W_{3,1}N'$ where $W_{3,1}$ contains A, the subgroup of diagonal matrices, and $N'=\{I+tE_{1,2}+sE_{2,3}\,|\,t,s\in\mathbb{R}\}$ (notice N' is not a subgroup). So, it suffices to consider $g_n=I+t_nE_{1,2}+s_nE_{2,3}\in N'$, and to assume that such a sequence is unbounded, i.e $t_n\to\infty$ or $s_n\to\infty$. Now, if

$$h_n=\begin{pmatrix}a_n & 0 & d_n\\ 0 & b_n & 0\\ 0 & 0 & c_n\end{pmatrix}\quad\text{with}\quad a_nb_nc_n=1$$

is a sequence of elements in $W_{3,1}$ so that $h_n^{g_n}$ converges then the entries

$$\left(h_n^{g_n}\right)_{1,2}=s_n(a_n-b_n),\ \left(h_n^{g_n}\right)_{2,3}=t_n(b_n-c_n)$$

converge. Thus either $a_n-b_n\to0$ or $b_n-c_n\to0$ depending on $s_n\to\infty$ or $t_n\to\infty$. This shows that on the diagonal of the limit two of the entries are the same. The only possible limits which are algebraic, conjugate into B and have two equal entries on the diagonal are $W^{(a=b)}_{3,5/6}$. Indeed,

$$\begin{array}{cc} W_{3,1}\to W^{a=b}_{3,5} & W_{3,1}\to W^{a=b}_{3,6}\\ \begin{pmatrix}1&n&0\\0&1&0\\0&0&1\end{pmatrix} & \begin{pmatrix}1&0&0\\0&1&n\\0&0&1\end{pmatrix}.\end{array}$$

□

It is not possible for $W^z_{3,8/9}$ with $z\neq i$ to be a limit of another group, since $W^z_{3,8/9}$ for $z\neq i$ is not algebraic and $W_{3,13}=\mathrm{SO}(2,1)$ and $W_{3,14}=\mathrm{SO}(3)$ are algebraic. Also $W^z_{3,8/9}$ cannot be a limit of $W_{3,1}$ or $W_{3,7}$ by Proposition 10. Finally, it remains to check that $W_{3,4}$ is the only possible limit of $W^z_{3,8/9}$. Since Proposition 10 implies limits of $W^z_{3,8/9}$ are unipotent, the only possibility is the limit we computed above to $W_{3,4}$.

Dimension 4

Name	Group	Normalizer	Properties
$W_{4,1}$	$\begin{pmatrix} GL_2 & 0 \\ 0 & \det^{-1} \end{pmatrix}$	$W_{4,1}$	$\cong \mathrm{GL}_2(\mathbb{R})$ algebraic
$W_{4,2}$	$\begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$	$W_{4,2}$	$\cong \mathrm{Aff}(\mathbb{R}^2)$ algebraic
$W_{4,3}$	$\begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$	$W_{4,3}$	$\cong \mathrm{Aff}(\mathbb{R}^2)$ algebraic
$W_{4,4}$	$\begin{pmatrix} \mathbb{C}^* & \mathbb{C} \\ 0 & \det^{-1} \end{pmatrix}$	$W_{4,4}$	$\cong \mathbb{C}^* \ltimes \mathbb{C}$ algebraic
$W_{4,5}$	$\begin{pmatrix} \det^{-1} & \mathbb{C} \\ 0 & \mathbb{C}^* \end{pmatrix}$	$W_{4,5}$	$\cong \mathbb{C}^* \ltimes \mathbb{C}$ algebraic
$W_{4,6}^{(a,b)}$	$\begin{pmatrix} e^{at} & * & * \\ 0 & e^{bt} & * \\ 0 & 0 & e^{-(a+b)t} \end{pmatrix}$	$W_{5,3}$	$\cong \mathrm{Heis}(\mathbb{R}) \rtimes W_{1,1}$ $a=b$ algebraic $a \neq b$ definable

The chart of local limits is as follows.

$$\begin{tikzcd}
W_{4,1} \arrow[drr] & W_{4,2} \arrow[dr] & W_{4,3} \arrow[d] & W_{4,4} \arrow[dl] & W_{4,5} \arrow[dll] \\
W_{4,6}^{a\neq b} & & W_{4,6}^{a=b} & &
\end{tikzcd}$$

$W_{4,1} \to W_{4,6}^{a=b}$: $\begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $W_{4,2} \to W_{4,6}^{a=b}$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ $W_{4,3} \to W_{4,6}^{a=b}$: $\begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $W_{4,4} \to W_{4,6}^{a=b}$: $\begin{pmatrix} n & 0 & 0 \\ 0 & \frac{1}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $W_{4,5} \to W_{4,6}^{a=b}$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & \frac{1}{n} \end{pmatrix}$

By Theorem 5 $W_{4,6}^{(a,b)}$ is the only possible limit of the first 5 groups, and limits of the first 5 groups are algebraic. Notice $W_{4,6}^{(a,b)}$ is algebraic for $a,b \in \mathbb{Q}$, but we claim only the singular groups $W_{4,6}^{(a=b)}$ are possible as limits of the first 5 groups. Apply the Iwasawa decomposition $G = KNA$, and notice we only need to conjugate by elements of $B = NA$, since K is compact then conjugating by elements of K will not change the limit.

To show that

$$W_{4,3} \not\to W_{4,6}^{(a\neq b)},$$

we note that $B = W_{4,3}U$, where $U = \{Id + t \cdot E_{1,2}\}$. So it suffices to consider conjugating by sequences in U. But this is what we computed in the limits of $W_{4,2/3}$ above. Similarly $W_{4,2} \not\to W_{4,6}^{(a\neq b)}$.

To show that $W_{4,1} \not\to W_{4,6}^{a\neq b}$. Using the standard Iwasawa decomposition argument, it is easy to verify that

$$G = W_{4,1} N' \,\mathrm{SO}(3) \text{ where } N' = \left(\begin{smallmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right).$$

Since $\mathrm{SO}(3)$ is compact, it suffices to check for limits of conjugates of $W_{4,1}$ by elements of N'. However, this is exactly the limit computed above.

Finally $W_{4,4}, W_{4,5} \not\to W_{4,6}^{a\neq b}$ by Proposition 10, since if $W_{4,4}, W_{4,5}$ have real weights then two of them must be equal.

Dimension 5

Recall a subgroup is parabolic if and only if it contains the Borel subgroup. In our case, a subgroup of $G = \mathrm{SL}_3(\mathbb{R})$ is parabolic if it contains $B = W_{5,3}$. The space G/B is compact and hence $G/N_G(H)$ is compact for every subgroup $H \le G$ with a parabolic normaliser $P = N_G(H)$. For such a subgroup, the subspace of its conjugates is homeomorphic to $G/N_G(H)$ and hence is compact in $\mathrm{Sub}(G)$, and in particular closed. Thus subgroups with parabolic normalizers cannot locally converge to non-conjugate subgroups. Since all subgroups of dimension 5 are have parabolic normalisers, the chart of local limits consists of isolated points.

Name	Group	Normalizer	Properties
$W_{5,1}$	$\begin{pmatrix} SL_2 & & * \\ & & * \\ 0 & 0 & 1 \end{pmatrix}$	$W_{6,1}$	$\cong \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ algebraic
$W_{5,2}$	$\begin{pmatrix} 1 & * & * \\ 0 & & \\ 0 & SL_2 & \end{pmatrix}$	$W_{6,2}$	$\cong \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ algebraic
$W_{5,3}$	$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$	$W_{5,3}$	$\cong$ Borel algebraic

Dimension 6

The subspace of conjugates of a parabolic subgroup is closed in the Chabauty compactification. So the chart of limits is two isolated points.

Name	Group	Normalizer	Properties
$W_{6,1}$	$\begin{pmatrix} GL_2 & * \\ 0 & \det^{-1} \end{pmatrix}$	$W_{6,1}$	algebraic
$W_{6,2}$	$\begin{pmatrix} \det^{-1} & * \\ 0 & GL_2 \end{pmatrix}$	$W_{6,2}$	algebraic

Dimensions 7 and 8

Only $\mathrm{SL}_3(\mathbb{R})$ is of dimension 8. There are no subgroups of dimension 7.

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