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Comptes Rendus

Mathématique

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Volume 359, issue 3 (2021), p. 229-236

https://doi.org/10.5802/crmath.151

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Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
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Homological algebra / *Algèbre homologique*

The mod p Margolis homology of the Dickson–Mùi algebra

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In memory of my parent

Abstract. Let $E^n = (\mathbb{Z}/p)^n$ be regarded as the translation group on itself. It is considered as a subgroup of the symmetric group $\mathbb{S}_{p^n}$ on p^n letters. We completely compute the mod p Margolis homology of the Dickson–Mùi algebra, i.e. the homology of the image of the restriction $\operatorname{Res}(\mathbb{S}_{p^n}, E^n) : H^*(\mathbb{S}_{p^n};\mathbb{F}_p) \to H^*(E^n;\mathbb{F}_p)$ with the differential to be the Milnor operation Q_j, for p an odd prime and for any n, j. The motivation for this problem is that, the Margolis homology of the Dickson–Mùi algebra plays a key role in study of the Morava K-theory $K(j)^*(B\mathbb{S}_m)$ of the symmetric group $\mathbb{S}_m$ on m letters. The main tool of our work is the notion of "critical" elements. The mod p Margolis homology of the Dickson–Mùi algebra concentrates on even degrees. It is analogous to the mod 2 Margolis homology of the Dickson algebra.

Résumé. Soit $E^n = (\mathbb{Z}/p)^n$ le groupe agissant sur lui même par les translations. On le considère comme sous-groupe du groupe symétrique $\mathbb{S}_{p^n}$ en p^n lettres. Dans cette note on calcule entièrement l'homologie de Margolis modulo p de l'algèbre de Dickson–Mùi, i.e. l'homologie de l'image de la restriction $\operatorname{Res}(\mathbb{S}_{p^n}, E^n)$: $H^*(\mathbb{S}_{p^n};\mathbb{F}_p) \to H^*(E^n;\mathbb{F}_p)$ en choisissant pour différentielles les opérations de Milnor Q_j, pour p un nombre premier impair et pour tout n, j. La motivation pour cette étude est le rôle clé joué par cette homologie dans l'étude de la K-théorie de Morava $K(j)^*(B\mathbb{S}_m)$ du groupe symétrique $\mathbb{S}_m$ en m lettres. L'outil principal de notre travail est la notion d'éléments « critiques ». L'homologie de Margolis mod p de l'algèbre de Dickson–Mùi concentre en degrés pairs. Elle est analogue à l'homologie de Margolis mod 2 de l'algèbre de Dickson.

2020 Mathematics Subject Classification. 55S05, 55S10, 55N99.

Manuscript received 22 July 2020, revised 5 October 2020 and 16 November 2020, accepted 15 November 2020.

A key step toward the determination of the symmetric group's cohomology is to apply the Quillen restriction from this cohomology to the cohomologies of all maximal elementary abelian subgroups of the symmetric group.

Let $E^n = (\mathbb{Z}/p)^n$ be regarded as the translation group on itself. So it is considered as a subgroup of the symmetric group $\mathbb{S}_{p^n}$ on p^n letters. This is the "generic" maximal elementary abelian p-subgroup of the symmetric group $\mathbb{S}_{p^n}$, where the terminology "generic" means that the set $\{E^n \mid n \geq 1\}$ has been used to describe all maximal elementary abelian p-subgroups of any symmetric groups. (See Mùi [7, Prop. II.2.3].) Let us study the restriction $\operatorname{Res}(\mathbb{S}_{p^n}, E^n)$: $H^*(\mathbb{S}_{p^n},;\mathbb{F}_p) \to H^*(E^n;\mathbb{F}_p)$ induced in cohomology by the canonical inclusion $E^n \subset \mathbb{S}_{p^n}$. We have

$$H^*(E^n;\mathbb{F}_p) = \begin{cases} \mathbb{F}_p[y_1,\dots,y_n], & p = 2, \\ \mathbb{E}(x_1,\dots,x_n) \otimes \mathbb{F}_p[y_1,\dots,y_n], & p > 2, \end{cases}$$

ISSN (electronic) : 1778-3569

where $\deg(y_i) = 1$ for $p = 2$, and $\deg(x_i) = 1, \deg(y_i) = 2$ for p an odd prime $(1 \le i \le n)$. Here $\mathbb{E}(x_1,\dots,x_n)$ and $\mathbb{F}_p[y_1,\dots,y_n]$ denote respectively the exterior algebra and the polynomial algebra on the given generators.

The Weyl group, which is the quotient of the normalizer by the centralizer, of the maximal elementary abelian subgroup E^n in $\mathbb{S}_{p^n}$ is the general linear group $GL_n = GL(n,\mathbb{F}_p)$. It is well-known that the image of the restriction $\operatorname{Res}(\mathbb{S}_{p^n}, E^n)$ is a subalgebra of the invariant algebra under the Weyl group action $H^*(E^n;\mathbb{F}_p)^{GL_n}$. According to H. Mùi [7, Thm. II.6.1 and Thm. II.6.2], the image of the restriction $\operatorname{Res}(\mathbb{S}_{p^n}, E^n)$ is the Dickson algebra $D_n = \mathbb{F}_p[y_1,\dots,y_n]^{GL_n}$ for $p = 2$, and a subalgebra of the algebra $(\mathbb{E}(x_1,\dots,x_n)\otimes\mathbb{F}_p[y_1,\dots,y_n])^{GL_n}$ for p an odd prime, where GL_n acts canonically on $\mathbb{F}_p[y_1,\dots,y_n]$ and on $\mathbb{E}(x_1,\dots,x_n)\otimes\mathbb{F}_p[y_1,\dots,y_n]$.

For p an odd prime, let us denote $DM_n := \operatorname{Im}\operatorname{Res}(\mathbb{S}_{p^n}, E^n)$ and call it the n-th Dickson–Mùi algebra. It should be noted that $DM_n \ne (\mathbb{E}(x_1,\dots,x_n)\otimes\mathbb{F}_p[y_1,\dots,y_n])^{GL_n}$(see H. Mùi [7, I.4.17 & II.6.1] or Theorem 1 below).

Let Q_j be the Milnor operation (see [6]) of degree $2p^j - 1$ in the mod p Steenrod algebra $\mathscr{A}$ inductively defined for $j \ge 0$ as follows

$$Q_0 = \beta,\ Q_{j+1} = P^{p^j}Q_j - Q_jP^{p^j},$$

where β denotes the Bockstein operation. In the article, for p an odd prime, we completely compute the mod p Margolis homology of the Dickson–Mùi algebra DM_n, i.e. the homology of DM_n with the differential to be the Milnor operation Q_j, for every n and j. The solution for the similar problem on the mod 2 Margolis homology of the Dickson algebra has been announced in [4] and published in detail in [2], where we denied the Pengelley–Sinha conjecture on the problem. This conjecture turns out to be false because of the occurence of the so-called critical elements, which are our main creation in the study. The Dickson–Mùi algebra DM_n *is not free in the category of graded commutative algebras*. Therefore, its Margolis homology is completly different and requires new techniques, more care and details than the case of $p = 2$. In particular, Definition 10 of critical elements is distinguished from the one for $p = 2$.

The real goal that we persue in the near future is to compute the Morava K-theory $K(j)^*(B\mathbb{S}_m)$ of the symmetric group $\mathbb{S}_m$ on m letters. It was well known that, the Milnor operation is the first non-zero differential, $Q_j = d_{2p^j-1}$, in the Atiyah–Hirzebruch spectral sequence for computing $K(j)^*(X)$, the Morava K-theory of a space X. So, the Q_j-homology of $H^*(X)$ is the E_{2p^j}-page in the Atiyah–Hirzebruch spectral sequence for $K(j)^*(X)$. (See e.g. Yagita [9, §2], although the fact was well known before this article.) Particularly, the E_{2p^j}-page in the Atiyah–Hirzebruch spectral sequence for $K(j)^*(B\mathbb{S}_{p^n})$ maps to $H_*(DM_n;Q_j)$. This is why the Margolis homology of the Dickson–Mùi algebra is taken into account.

Let us study the n-th Dickson algebra of invariants $D_n = \mathbb{F}_p[y_1,\dots,y_n]^{GL_n}$. Following Dickson [1], we set

$$[e_1,\dots,e_n] = \det\big(y_\ell^{p^{e_k}}\big)_{1\le k,\ell\le n},$$

for non-negative integers $e_1,\dots,e_n$. The right action of $\omega = (\omega_{ij})_{n\times n} \in GL_n$ sends $g \in \mathbb{F}_p[y_1,\dots,y_n]$ to $(g\omega)(y_1,\dots,y_n) = g(\sum_{i=1}^n \omega_{i1}y_i,\dots,\sum_{i=1}^n \omega_{in}y_i)$, while its left action sends g to $(\omega g)(y_1,\dots,y_n) = g(\sum_{j=1}^n \omega_{1j}y_j,\dots,\sum_{j=1}^n \omega_{nj}y_j)$. Since $\omega g = g\omega^t$, where ω^t is the transposed matrix of ω, a polynomial is a right GL_n-invariant if and only if it is a left GL_n-invariant. By Fermat's little theorem, $[e_1,\dots,e_n]\omega = \det(\omega)[e_1,\dots,e_n]$ for $\omega \in GL_n$ (see [1]). Set $L_{n,s} = [0,1,\dots,\widehat{s},\dots,n]$ $(0 \le s \le n)$, where $\widehat{s}$

means s being omitted, and $L_n = L_{n,n}$. The Dickson invariant, defined by $c_{n,s} = L_{n,s}/L_n$ $(0 \le s < n)$, is a GL_n-invariant. It is of degree $2^n - 2^s$ for $p = 2$ and degree $2(p^n - p^s)$ for p an odd prime. Dickson proved in [1] that D_n is a polynomial algebra on the Dickson invariants

$$D_n = \mathbb{F}_p[c_{n,0}, \dots, c_{n,n-1}].$$

Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix with entries a_{ij}'s in the graded commutative algebra $\mathbb{E}(x_1, \dots, x_n) \otimes \mathbb{F}_p[y_1, \dots, y_n]$. The determinant of A is defined by

$$\det A = \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

Remark. As $x_1, \dots, x_n$ are of odd degree, $x_k x_\ell = -x_\ell x_k$ for any k and ℓ, we have

$$\det \begin{pmatrix} x_1 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_1 & \dots & x_n \end{pmatrix} = n! x_1 \cdots x_n,$$

while

$$\det \begin{pmatrix} x_1 & \dots & x_1 \\ \vdots & \ddots & \vdots \\ x_n & \dots & x_n \end{pmatrix} = 0.$$

See H. Mùi [7, p. 324–325] for detail.

Let $e_{k+1}, \dots, e_n$ be non-negative integers. H. Mùi set in [7, p. 330]:

$$\langle k : e_{k+1}, \dots, e_n \rangle = \det \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \dots & \cdot \\ x_1 & x_2 & \dots & x_n \\ y_1^{p^{e_{k+1}}} & y_2^{p^{e_{k+1}}} & \dots & y_n^{p^{e_{k+1}}} \\ \cdot & \cdot & \dots & \cdot \\ y_1^{p^{e_n}} & y_2^{p^{e_n}} & \dots & y_n^{p^{e_n}} \end{pmatrix},$$

where there are exactly k rows $(x_1\, x_2 \dots x_n)$ in the determinant. Further, we set

$$R_{n,s_1,\dots,s_k} = \frac{1}{k!} \langle k : 0, \dots \widehat{s_1}, \dots, \widehat{s_k}, \dots, n-1 \rangle L_n^{p-2}.$$

See H. Mùi [7, p. 330, p. 338]. The right and the left actions of $\omega = (\omega_{ij})_{n\times n} \in GL_n$ respectively sends $f \in \mathbb{E}(x_1, \dots, x_n) \otimes \mathbb{F}_p[y_1, \dots, y_n]$ to

$$(f\omega)(x_1, \dots, x_n, y_1, \dots, y_n) = f\left(\sum_{i=1}^n \omega_{i1} x_i, \dots, \sum_{i=1}^n \omega_{in} x_i, \sum_{i=1}^n \omega_{i1} y_i, \dots, \sum_{i=1}^n \omega_{in} y_i\right),$$
$$(\omega f)(x_1, \dots, x_n, y_1, \dots, y_n) = f\left(\sum_{j=1}^n \omega_{1j} x_j, \dots, \sum_{j=1}^n \omega_{nj} x_j, \sum_{j=1}^n \omega_{1j} y_j, \dots, \sum_{j=1}^n \omega_{nj} y_j\right).$$

Since $\omega f = f\omega^t$, a generalized polynomial f is a right GL_n-invariant if and only if it is a left GL_n-invariant. By Fermat's little theorem, $R_{n,s_1,\dots,s_k}$ is a GL_n-invariant.

Theorem 1 (H. Mùi [7, I.4.17 & II.6.1]). *For p an odd prime and $n > 1$, the Dickson–Mùi algebra DM_n is the subalgebra of the graded commutative algebra $(\mathbb{E}(x_1, \dots, x_n) \otimes \mathbb{F}_p[y_1, \dots, y_n])^{GL_n}$ generated by*

$$R_{n,s}\ (0 \le s < n),\ R_{n,r,s}\ (0 \le r < s < n),\ c_{n,s}\ (0 \le s < n),$$

which satisfy the relations

$$R_{n,s}^2 = 0\ \ (0 \le s < n), \qquad R_{n,r,s}^2 = 0\ \ (0 \le r < s < n),$$
$$R_{n,r} R_{n,s} = -R_{n,r,s} c_{n,0}\ \ (0 \le r < s < n).$$

The action of the Steenrod algebra on the Dickson–Mùi one is basically computed in [3] and [5]. We are interested in the following element:

$$A_{j,n,s} = [0,\dots,\widehat{s},\dots,n-1,j]/L_n \in D_n, \quad (\text{for } 0 \le s < n \le j).$$

Proposition 2. *Let p be an odd prime.*

(i) *For $0 \le s < n$, and arbitrary j,*

$$Q_j(c_{n,s}) = 0.$$

(ii) *For $0 \le s < n$,*

$$Q_j(R_{n,s}) = \begin{cases} (-1)^s c_{n,0}, & j = s, \\ (-1)^{n-1} c_{n,0} A_{j,n,s}, & j \ge n, \\ 0, & \textit{otherwise.} \end{cases}$$

(iii) *For $0 \le r < s < n$,*

$$Q_j(R_{n,r,s}) = \begin{cases} (-1)^{r-1} R_{n,s}, & j = r, \\ (-1)^s R_{n,r}, & j = s, \\ (-1)^n \{A_{j,n,r}R_{n,s} - R_{n,r}A_{j,n,s}\}, & j \ge n, \\ 0, & \textit{otherwise.} \end{cases}$$

Following [1, 7], we set $V_n = L_n/L_{n-1} = \prod_{\lambda_i \in \mathbb{F}_p}(\lambda_1 y_1 + \cdots + \lambda_{n-1}y_{n-1} + y_n)$. Generalizing the formulas by Dickson [1] on the inductive definition for $c_{n,s}$ and on the expansion of V_n in terms of $c_{n-1,0},\dots,c_{n-1,n-2}$, we have

Proposition 3.

(i) *$A_{j,n,s} \ne 0$ in D_n for $0 \le s < n \le j$. Further,*

$$A_{j,n,s} = A^p_{j-1,n-1,s-1} + A^p_{j-1,n,n-1} c_{n-1,s} V_n^{p-1}.$$

(ii) *For $0 \le n-1 \le j$,*

$$A_{j,n,n-1}V_n = (-1)^{n-1}\left\{\sum_{s=0}^{n-2}(-1)^s A_{j,n-1,s} y_n^{p^s} + (-1)^{n-1} y_n^{p^j}\right\}.$$

Here, by convention, $A_{j,n,-1} = 0$, $A_{n-1,n,n-1} = 1$, $c_{n-1,n-1} = 1$.

Lemma 4.

(i) $c_{n,s} = \begin{cases} 0 \mod (y_n,\dots,y_{n-r}), & 0 \le s \le r < n, \\ \ne 0 \mod (y_n,\dots,y_{n-r}), & 0 \le r < s < n. \end{cases}$

Consequently $(c_{n,0},\dots,c_{n,r}) = (y_n,\dots,y_{n-r}) \cap D_n$.

(ii) $A_{j,n,s} = \begin{cases} 0 \mod (c_{n,0},\dots,c_{n,r}), & 0 \le s \le r < n, \\ \ne 0 \mod (c_{n,0},\dots,c_{n,r}), & 0 \le r < s < n. \end{cases}$

Lemma 5. *$A_{j,n,r}$ and $A_{j,n,s}$ are coprime in D_n for $0 \le r \ne s < n$.*

The next two theorems compute the j-th Margolis homology of DM_n for the unstable cases, i.e. for either $n = 1$ or $1 < n$ and $0 \le j < n$.

Theorem 6. *For $n = 1$, $j \ge 0$, and $c_{1,0} = y^{p-1}$,*

$$H_*(DM_1; Q_j) \cong \mathbb{F}_p[c_{1,0}]/\left(c_{1,0}^{\frac{p^j+p-2}{p-1}}\right).$$

Theorem 7. *For p an odd prime, $1 < n$ and $0 \le j < n$, the j-th Margolis homology of DM_n is isomorphic as $\mathbb{F}_p$-vector spaces to the quotient of the algebra*

$$\mathbb{E}(R_{n,r,s} \mid 0 \le r < s < n;\ r \ne j, s \ne j) \otimes \mathbb{F}_p[c_{n,1}, \ldots, c_{n,n-1}]$$

subject to the relations

$$\begin{aligned} &R_{n,r,s}R_{n,t,u} = 0, \quad \text{if } \{r,s\} \cap \{t,u\} \ne \emptyset, \\ &R_{n,r,s}R_{n,t,u} = -R_{n,r,t}R_{n,s,u} = R_{n,r,u}R_{n,s,t}, \ (0 \le r < s < t < u < n). \end{aligned}$$

Example 8. In the example, the equality $R_{n,r}R_{n,s} = -c_{n,0}R_{n,r,s}$ is essential.

(a) We show why the exponent of $c_{n,0}$ in the denominator of Lemma 9 increases as k does. If $k = 3$, then $\left[\frac{3-1}{2}\right] + 1 = 2$. For $0 \le r < s < t < n \le j$,

$$\begin{aligned} Q_j(R_{n,r}R_{n,s}R_{n,t}) &= Q_j(R_{n,r})R_{n,s}R_{n,t} - R_{n,r}Q_j(R_{n,s})R_{n,t} + R_{n,r}R_{n,s}Q_j(R_{n,t}) \\ &= (-1)^{n-1}c_{n,0}A_{j,n,r}R_{n,s}R_{n,t} - (-1)^{n-1}R_{n,r}c_{n,0}A_{j,n,s}R_{n,t} \\ &\quad + (-1)^{n-1}R_{n,r}R_{n,s}c_{n,0}A_{j,n,t}. \end{aligned}$$

$$\frac{1}{c_{n,0}^2}Q_j(R_{n,r}R_{n,s}R_{n,t}) = -\left\{(-1)^{n+1}R_{n,s,t}A_{j,n,r} + (-1)^{n+2}R_{n,r,t}A_{j,n,s} + (-1)^{n+3}R_{n,r,s}A_{j,n,t}\right\} \in DM_n.$$

(b) If k is even, then the following equality proves Lemma 11 (ii):

$$\frac{1}{c_{n,0}^{\left[\frac{k-1}{2}\right]+1}}Q_j(R_{n,s_1}\cdots R_{n,s_k}) = (-1)^{\frac{k}{2}}Q_j(R_{n,s_1,s_2}\cdots R_{n,s_{k-1},s_k}) \in \operatorname{Im} Q_j.$$

Let D_n^{ex} be the ideal of DM_n generated by $R_{n,0}, \ldots, R_{n,n-1}, R_{n,r,s}$ $(0 \le r < s < n)$. Remarkably, $R_{n,s}$ is of odd degree, while $R_{n,r,s}$ is of even degree. Note that D_n^{ex} is not a Q_j-submodule of DM_n, but $\operatorname{Im} Q_j \cap D_n^{\text{ex}}$ and $\operatorname{Ker} Q_j \cap D_n^{\text{ex}}$ are, since Q_j vanishes on these modules. The evident equality $DM_n = D_n \oplus D_n^{ex}$ implies

$$\begin{aligned} \operatorname{Ker} Q_j &= D_n \oplus (\operatorname{Ker} Q_j \cap D_n^{\text{ex}}), \\ \operatorname{Im} Q_j &= (\operatorname{Im} Q_j \cap D_n) \oplus (\operatorname{Im} Q_j \cap D_n^{\text{ex}}). \end{aligned}$$

In the sequel, when j and n are fixed, the elements $c_{n,s}$, $R_{n,s}$, and $A_{j,n,s}$ will respectively be denoted c_s, R_s, and A_s $(0 \le s < n)$ for abbreviation.

From Theorem 1, using the argument of Example 8, we see that if $\alpha > \left[\frac{k-1}{2}\right] + 1$ then $\frac{1}{c_0^{\alpha}}Q_j(R_{s_1}\cdots R_{s_k})$ does not belong to DM_n for $n \le j$.

Lemma 9. *$Q_j(R_{s_1}\ldots R_{s_k}) = \sum_{i=1}^{k}(-1)^{n+i}R_{s_1}\ldots\widehat{R}_{s_i}\ldots R_{s_k}c_0A_{s_i}$ is divisible by $c_0^{\left[\frac{k-1}{2}\right]+1}$ but not $c_0^{\left[\frac{k-1}{2}\right]+2}$ for $n \le j$. Particularly,*

$$\frac{1}{c_0^{\left[\frac{k-1}{2}\right]+1}}Q_j(R_{s_1}\cdots R_{s_k}) = \frac{1}{c_0^{\left[\frac{k-1}{2}\right]}}\sum_{i=1}^{k}(-1)^{n+i}R_{s_1}\ldots\widehat{R}_{s_i}\ldots R_{s_k}A_{s_i} \in D_n^{\text{ex}},$$

for $0 \le s_1 < \cdots < s_k < n \le j$, $1 < k$.

The critical elements defined below are the main ingredient in determination of $\operatorname{Im} Q_j \cap D_n^{\text{ex}}$ and $\operatorname{Ker} Q_j \cap D_n^{\text{ex}}$ for $2 \le n \le j$.

Definition 10. *For $0 \le s_1 < \cdots < s_k < n \le j$, $1 < k$, the element*

$$h_{s_1,\ldots,s_k} = \frac{1}{c_0^{\left[\frac{k-1}{2}\right]+1}}Q_j(R_{s_1}\cdots R_{s_k})$$

is called critical *if k is odd. Here $[\ell]$ is the biggest integer with $[\ell] \le \ell$.*

Lemma 11. *For* $0 \le s_1 < \cdots < s_k < n \le j$ *and* $1 < k$,

(i) $h_{s_1,\ldots,s_k} \in \operatorname{Ker} Q_j$;

(ii) *If* k *is even, then* $h_{s_1,\ldots,s_k} \in \operatorname{Im} Q_j$, *equivalently* $[h_{s_1,\ldots,s_k}] = 0 \in H_*(DM_n; Q_j)$;

(iii) *If* k *is odd, then* $h_{s_1,\ldots,s_k}$ *is not divisible by* c_0 *in* DM_n; *Particularly,* $[h_{s_1,\ldots,s_k}] \neq 0 \in H_*(DM_n; Q_j)$.

The partial derivation and the integral on a direction are our main tools in determination of $\operatorname{Ker} Q_j \cap D_n^{\mathrm{ex}}$ and $\operatorname{Im} Q_j \cap D_n^{\mathrm{ex}}$ for $2 \le n \le j$.

Definition 12. *Let* $s_1, \ldots, s_k$ *be pairwise distinct, with* $0 \le s_1, \ldots, s_k < n$. *The* s*-th partial derivation* $\partial_s : DM_n \to DM_n$ *is the morphism defined for* $0 \le s < n$ *by*

$$\partial_s\left(\frac{1}{c_0^\alpha} R_{s_1} \cdots R_{s_k} Z\right) = \begin{cases} (-1)^{n+i} \frac{1}{c_0^\alpha} R_{s_1} \cdots \widehat{R}_{s_i} \cdots R_{s_k} c_0 A_{s_i} Z, & s = s_i, \\ 0, & \textit{otherwise}, \end{cases}$$

for $\alpha \le [k/2]$ *and* $Z \in D_n$.

If $\partial_s(R_{s_1} \cdots R_{s_k}) \neq 0$, then s should be one of the indices $s_1, \ldots, s_k$. Obviously, $\operatorname{Im} \partial_s \subset c_0 A_s(DM_n)$. Proposition 2 implies

Lemma 13. *Let* $s_1, \ldots, s_k$ *be pairwise distinct, with* $0 \le s_1, \ldots, s_k < n \le j$, *and* $Z \in D_n$. *Then*

$$Q_j\left(\frac{1}{c_0^\alpha} R_{s_1} \cdots R_{s_k} Z\right) = \sum_{s=0}^{n-1} \partial_s\left(\frac{1}{c_0^\alpha} R_{s_1} \cdots R_{s_k} Z\right),$$

for $\alpha \le [k/2]$ *and* $Z \in D_n$.

Definition 14. *The integral on the* r*-th direction*

$$I_r : c_0 A_r(DM_n) \to DM_n,$$

for $0 \le r < n$, *is the morphism given by:*

$$I_r\left(\frac{1}{c_0^\alpha} R_{s_1} \cdots R_{s_k} c_0 A_r Z\right) = \begin{cases} (-1)^{n-1} \frac{1}{c_0^\alpha} R_r R_{s_1} \cdots R_{s_k} Z, & r \neq s_1, \ldots, s_k, \\ 0, & \textit{otherwise}, \end{cases}$$

where $s_1, \ldots, s_k$ *are pairwise distinct,* $0 \le s_1, \ldots, s_k < n$, $0 \le k$, $\alpha \le [k/2]$, $Z \in D_n$.

Lemma 15. *Let* $s_1, \ldots, s_k$ *be pairwise distinct, with* $0 \le s_1, \ldots, s_k < n$, $0 < s \le n$, $\alpha \le [k/2]$, *and* $Z \in D_n$. *Then*

(i) $$I_s \partial_s\left(\tfrac{1}{c_0^\alpha} R_{s_1} \cdots R_{s_k} Z\right) = \begin{cases} \frac{1}{c_0^\alpha} R_{s_1} \cdots R_{s_k} Z, & s \in \{s_1, \ldots, s_k\}, \\ 0, & \textit{otherwise}. \end{cases}$$

(ii) $$\partial_s I_s\left(\tfrac{1}{c_0^\alpha} R_{s_1} \cdots R_{s_k} c_0 A_s Z\right) = \begin{cases} \frac{1}{c_0^\alpha} R_{s_1} \cdots R_{s_k} c_0 A_s Z, & s \neq s_1, \ldots, s_k, \\ 0, & \textit{otherwise}. \end{cases}$$

Let $D_n = \mathbb{F}_p[c_0, c_1, \ldots, c_{n-1}]$ and $\overline{D}_n = \mathbb{F}_p[c_1, \ldots, c_{n-1}]$. Denote by hc_0D_n and $h\overline{D}_n$ the submodules of DM_n generated by the generators $\{h_{s_1,\ldots,s_k} \mid 0 \le s_1 < \cdots < s_k < n, 1 < k \text{ odd}\}$ over c_0D_n and $\overline{D}_n$ respectively.

Theorem 16. *For* p *an odd prime and* $2 \le n \le j$,

$$\operatorname{Ker} Q_j \cap D_n^{\mathrm{ex}} = \left(\operatorname{Im} Q_j \cap D_n^{\mathrm{ex}}\right) + h\overline{D}_n,$$

where $\left(\operatorname{Im} Q_j \cap D_n^{\mathrm{ex}}\right) \cap h\overline{D}_n = hc_0D_n \cap h\overline{D}_n$.

The critical elements are not linear independent over $D_n = \mathbb{F}_p[c_0, \ldots, c_{n-1}]$.

Lemma 17. *For* $0 \leq s_1 < \cdots < s_k < n$, $2 < k$.

$$\sum_{i=1}^{k} (-1)^i h_{s_1,\ldots,\widehat{s_i},\ldots,s_k} A_{s_i} = 0.$$

In particular, the sum in Theorem 16 is not a direct sum.

Let $\pi : D_n \to \overline{D}_n$ be the projection, whose kernel is $c_0 D_n$. We denote $\pi(Z)$ by $\overline{Z}$ for abbreviation. So $Z - \overline{Z} \in c_0 D_n$ for $Z \in D_n$. By Lemma 17,

$$\sum_{i=1}^{k} (-1)^i h_{s_1,\ldots,\widehat{s_i},\ldots,s_k} \big(A_{s_i} - \overline{A}_{s_i}\big) = -\sum_{i=1}^{k} (-1)^i h_{s_1,\ldots,\widehat{s_i},\ldots,s_k} \overline{A}_{s_i}.$$

This is a non-zero element in the intersection $hc_{n,0} D_n \cap h\overline{D}_n$. Therefore,

$$\big(\operatorname{Im} Q_j \cap D_n^{\text{ex}}\big) \cap h\overline{D}_n \supset hc_0 D_n \cap h\overline{D}_n \neq \{0\}.$$

Example 18. For $k = 3$ and $0 \leq r < s < t < n$:

$$\begin{aligned}&(-1)^1 h_{s,t} A_r + (-1)^2 h_{r,t} A_s + (-1)^3 h_{r,s} A_t \\ &\quad = (-1)^{n+1}\{(-1)^1 (R_t A_s - R_s A_t) A_r + (-1)^2 (R_t A_r - R_r A_t) A_s + (-1)^3 (R_s A_r - R_r A_s) A_t\} = 0.\end{aligned}$$

The following is the main result of our article.

Theorem 19. *Let p be an odd prime. The mod p Margolis homology of the Dickson–Mùi algebra DM_n for $2 \leq n \leq j$ is given by*

$$H_*(DM_n; Q_j) \cong \frac{D_n}{\big(c_0 A_0, \ldots, c_0 A_{n-1}\big)} \bigoplus \frac{h\overline{D}_n}{hc_0 D_n \cap h\overline{D}_n}.$$

The theorem implies that the mod p Margolis homology of the Dickson–Mùi algebra concentrates on even degrees, as the degrees of the critical elements are even. It should be noted that the mod 2 Margolis homology of the Dickson algebra also concentrates on even degrees. (See [2,4].)

Example 20. For $j = n \geq 2$, by definition of $A_{j,n,s}$, we have

$$A_s = A_{n,n,s} = [0,\ldots,\widehat{s},\ldots,n-1,j]/L_n = c_s, \ \ (0 \leq s < n).$$

So the critical element, which also depends on n and j, is explicitly given by

$$h_{s_1,\ldots,s_k} = \frac{1}{c_0^{\left[\frac{k-1}{2}\right]}} \sum_{i=1}^{k} (-1)^{n+i} R_{s_1} \ldots \widehat{R}_{s_i} \ldots R_{s_k} c_{s_i} \in D_n^{\text{ex}},$$

for $0 \leq s_1 < \cdots < s_k < n$, $1 < k$ odd.

Theorem 19 yields

$$\begin{aligned}H_*(DM_n; Q_n) &\cong \frac{D_n}{\big(c_0^2, c_0 c_1, \ldots, c_0 c_{n-1}\big)} \bigoplus \frac{h\overline{D}_n}{hc_0 D_n \cap h\overline{D}_n} \\ &= \overline{\mathbb{E}(c_0)} \bigoplus \mathbb{F}_2[c_1,\ldots,c_{n-1}] \bigoplus \frac{h\overline{D}_n}{hc_0 D_n \cap h\overline{D}_n}.\end{aligned}$$

When k is even and $k > 2$, by Lemma 17, $h_{s_2,\ldots,s_k} c_0 = \sum_{i=2}^{k} (-1)^i h_{0,s_2,\ldots,\widehat{s_i}\ldots,s_k} c_{s_i}$ is a nonzero element in $hc_0 D_n \cap h\overline{D}_n$ for $0 = s_1 < \cdots < s_k < n$, while $\sum_{i=1}^{k} (-1)^i h_{s_1,\ldots,\widehat{s_i},\ldots,s_k} c_{s_i} = 0$ is a linear relationship of the critical elements over $\overline{D}_n$ for $0 < s_1 < \cdots < s_k < n$.

Conjecture 21. *For* $2 \leq n \leq j$,

$$hc_0 D_n \cap h\overline{D}_n = \operatorname{Span}\left\{ \overline{H}_S = \sum_{i=1}^{k} (-1)^i h_{s_1,\ldots,\widehat{s_i},\ldots,s_k} \overline{A}_{s_i} \right\},$$

where $S = (s_1,\ldots,s_k)$ runs over the sequences with $0 \leq s_1 < \cdots < s_k < n$, $2 < k$ *even.*

The contains of this note will be published in detail elsewhere.

Acknowledgments

The author warmly thanks Dev Sinha for an interesting discussion [8] on the subject during the Vietnam-US Mathematical joint Metting, Quynhon June 10-13, 2019. He is deeply indebted to H. Miller, H. Mùi, and L. Schwartz for generous discussions on the goal of the problem and the references of this article. The author would like to thank the referee for many helpful suggestions and comments, which have led to improvement of the note's exposition.

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