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On the Fekete–Szegö type functionals for functions which are convex in the direction of the imaginary axis

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Abstract. In this paper we consider two functionals of the Fekete–Szegö type: $\Phi_f(\mu) = a_2 a_4 - \mu a_3{}^2$ and $\Theta_f(\mu) = a_4 - \mu a_2 a_3$ for analytic functions $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$, $z \in \Delta$, ($\Delta = \{z \in \mathbb{C} : |z| < 1\}$) and for real numbers μ. For f which is univalent and convex in the direction of the imaginary axis, we find sharp bounds of the functionals $\Phi_f(\mu)$ and $\Theta_f(\mu)$. It is possible to transfer the results onto the class $\mathcal{K}_{\mathbb{R}}(i)$ of functions convex in the direction of the imaginary axis with real coefficients as well as onto the class $\mathcal{T}$ of typically real functions. As corollaries, we obtain bounds of the second Hankel determinant in $\mathcal{K}_{\mathbb{R}}(i)$ and $\mathcal{T}$.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}^*$ denote the class of starlike functions, i.e. functions $f \in \mathcal{A}$ such that $\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$ for all $z \in \Delta$. Given $\beta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, a function $f \in \mathcal{A}$ is called close-to-convex with argument β with respect to g if

$$\operatorname{Re} \frac{e^{i\beta} z f'(z)}{g(z)} > 0, \quad z \in \Delta. \tag{2}$$

The class of all functions satisfying (2) is denoted by $\mathscr{C}_\beta(g)$. Let $\mathscr{C}$ denote the family of all close-to-convex functions (see [10]). Hence,

$$\mathscr{C} = \bigcup_{\beta\in(-\pi/2,\pi/2)} \bigcup_{g\in\mathscr{S}^*} \mathscr{C}_\beta(g).$$

All functions in $\mathscr{S}^*$ and $\mathscr{C}$ are univalent. It is not the case for the class $\mathscr{T}$ of typically real functions. A function f of the form (1) is in $\mathscr{T}$ if the condition $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$ holds for all $z \in \Delta$. All coefficients of any $f \in \mathscr{T}$ are real. It results in the symmetry of $f(\Delta)$ with respect to the real axis. It is worth recalling that there exists a unique correspondence between the functions in $\mathscr{T}$ and $\mathscr{P}_{\mathbb{R}}$.

By $\mathscr{P}$ we denote the class of analytic functions p with a positive real part in Δ, having the Taylor series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n . \tag{3}$$

The subclass of $\mathscr{P}$ consisting of those functions which have real coefficients is denoted by $\mathscr{P}_{\mathbb{R}}$. The above correspondence between $\mathscr{P}_{\mathbb{R}}$ and $\mathscr{T}$ is as follows

$$p \in \mathscr{P}_{\mathbb{R}} \Leftrightarrow \frac{z}{1-z^2} p(z) \in \mathscr{T} . \tag{4}$$

One of the main problems in the geometric theory of analytic functions is connected with Michael Fekete and Gabor Szegö. In [6] they considered the expression $|a_3 - \mu a_2{}^2|$, which is now called the Fekete–Szegö functional. If $\mu = 1$, then this expression is referred to as the classical Fekete–Szegö functional. It is worth recalling that Fekete and Szegö obtained in [6] the exact bound of $|a_3 - \mu a_2{}^2|$ for the class $\mathscr{S}$ of all univalent functions in Δ and, in this way, they disproved the famous Littlewood-Paley conjecture about the coefficients of odd functions in $\mathscr{S}$.

The problem of estimating the Fekete–Szegö functional, while f is in a given subclass A of the class $\mathscr{A}$, is still of great interest for many mathematicians. This problem was difficult to solve for functions which are close-to-convex. It was completely solved by Koepf in [11].

Let us consider two functionals which are constructed in a similar way as the Fekete–Szegö functional. Namely, for a fixed real number μ, let us define

$$\Phi_f(\mu) \equiv a_2 a_4 - \mu a_3{}^2 \tag{5}$$

and

$$\Theta_f(\mu) \equiv a_4 - \mu a_2 a_3 . \tag{6}$$

The functionals $\Phi_f(\mu)$ and $\Theta_f(\mu)$ are generalizations of two expressions: $a_2 a_4 - a_3{}^2$ and $a_4 - a_2 a_3$. The first one is known as the second Hankel determinant and it was examined in many papers. The investigation of Hankel determinants for analytic functions was started by Pommerenke (see [23, 24]). Following Pommerenke, many mathematicians published their results concerning this determinant for various subclasses of the class $\mathscr{S}$ of univalent functions (see, for example [7–9, 12, 15, 21], and quite recently [2, 3, 25]) and for multivalent functions (see [20]). The bound of $a_2 a_4 - a_3{}^2$ for $\mathscr{T}$ was obtained in [29]. Hayami and Owa were the first who studied $\Phi_f(\mu)$. They discussed an even more general functional $a_n a_{n+2} - \mu a_{n+1}{}^2$ for the class $\mathscr{Q}(\alpha)$ consisting of those functions $f \in \mathscr{A}$ such that $\operatorname{Re}(f(z)/z) > \alpha$. Details will be given in Section 5.

The functional $\Theta_f(\mu)$ is a particular case of the generalized Zalcman functional. At the end of 1960's Lawrence Zalcman conjectured that if $f \in \mathscr{S}$ is of the form (1), then $|a_n{}^2 - a_{2n-1}| \leq (n-1)^2$ for $n \geq 2$ with equality for rotations of the Koebe function $k(z) = \frac{z}{(1-z)^2}$. This problem was strictly connected with the famous Bieberbach conjecture. If it were true, the Bieberbach conjecture would also be true. After 50 years from formulating, the Zalcman conjecture is still an open problem. We know only that it holds for some subclasses of univalent functions, for example for $\mathscr{S}^*$ (Brown and Tsao, [1]) or for $\mathscr{C}$ (Ma [18] and Li and Ponnusamy [16]). On the other hand, Krushkal proved the Zalcman conjecture for the whole class $\mathscr{S}$, but only for some initial values of n (see [13, 14]). In the meantime, more general versions of the Zalcman functionals $\lambda a_n{}^2 - a_{2n-1}$ and $\lambda a_n a_m - a_{n+m-1}$ have appeared. The latter, called the generalized Zalcman functional, was investigated by Ma for $f \in \mathscr{S}^*$ (see [19]) and by Efraimidis and Vukotić for f in the Hurwitz class (f is given by (1) and it satisfies the inequality $\sum_{n=2}^{\infty} n|a_n| \le 1$) and for f in the Noshiro–Warschawski class $\mathscr{R}$ as well as for f in the closed convex hull of convex functions (see [5]).

The functionals Φ_f and Θ_f for f in $\mathscr{S}^*$ and in the class $\mathscr{K}$ of convex functions were discussed in [30]. The estimates of Φ_f and Θ_f for functions in $\mathscr{C}_0(k)$ were published in [28].

In this paper we shall estimate $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$ in the subclass $\mathscr{C}_0(h)$ of $\mathscr{C}$ where $h(z) = \frac{z}{1-z^2}$. It follows from the definition of $\mathscr{C}_0(g)$ that

$$f \in \mathscr{C}_0(h) \Leftrightarrow \operatorname{Re}\left(1-z^2\right) f'(z) > 0\,. \tag{7}$$

The importance of this class is a consequence of the following two facts.

Theorem 1. *If $f \in \mathscr{C}_0(h)$, then f is convex in the direction of the imaginary axis.*

Proof of Theorem 1. According to the result of Royster and Ziegler [27], f is convex in the direction of the imaginary axis if and only if there exist $\mu \in [0, 2\pi]$ and $\nu \in [0, \pi]$ such that $\operatorname{Re}[-ie^{i\mu}(1-2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu})f'(z)] > 0$ for all $z \in \Delta$.

It is enough to take $\mu = \nu = \pi/2$ to obtain the right hand side of the condition in (7). □

Theorem 2. *Let all coefficients of f given by* (1) *be real. Then,*

$$f \in \mathscr{C}_0(h) \Leftrightarrow f \in \mathscr{K}_{\mathbb{R}}(i)\,.$$

In the above, $\mathscr{K}_{\mathbb{R}}(i)$ denotes the class of functions of the form (1) which are convex in the direction of the imaginary axis and have all real coefficients. Robertson [26] proved that

$$f \in \mathscr{K}_{\mathbb{R}}(i) \Leftrightarrow zf'(z) \in \mathscr{T}\,. \tag{8}$$

Proof of Theorem 2. If f has real coefficients, then the condition in (7) can be rewritten as

$$f \in \mathscr{C}_0(h) \Leftrightarrow f'(z) = \frac{1}{1-z^2}\,p(z)\,, \tag{9}$$

where p is in $\mathscr{P}_{\mathbb{R}}$. From (4), $f \in \mathscr{C}_0(h)$ if and only if $zf'(z) \in \mathscr{T}$. In view of (8), the latter means that $f \in \mathscr{K}_{\mathbb{R}}(i)$, which proves our assertion. □

It is worth recalling that in [26] Robertson also proved that if $f \in \mathscr{K}_{\mathbb{R}}(i)$, not then $\operatorname{Re}(f(z)/z) > 1/2$, or in other words, $f \in \mathscr{Q}(1/2)$.

At the end of this section, we derive the estimate of the coefficients of f in $\mathscr{C}_0(h)$.

Theorem 3. *If $f \in \mathscr{C}_0(h)$ is of the form* (1)*, then $|a_n| \le 1$.*

Proof of Theorem 3. From (9), it follows that

$$na_n = p_1 + p_3 + \ldots + p_{n-1} \quad \text{if } n \text{ is even} \tag{10}$$

and

$$na_n = 1 + p_2 + \ldots + p_{n-1} \quad \text{if } n \text{ is odd}\,. \tag{11}$$

Then, for even n,

$$n|a_n| \le \frac{n}{2}\cdot 2 = n$$

and, for odd n,

$$n|a_n| \le 1 + \frac{n-1}{2}\cdot 2 = n\,.$$

□

Finally, observe that $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$ are invariant under rotation. If f is given by (1) and $f_\varphi(z) = e^{-i\varphi} f(ze^{i\varphi})$, $\varphi \in \mathbb{R}$, then $f_\varphi(z) = z + \sum_{n=2}^{\infty} a_n e^{i(n-1)\varphi} z^n$. Hence

$$\left|\Phi_{f_\varphi}(\mu)\right| = \left|a_2 e^{i\varphi} \cdot a_4 e^{3i\varphi} - \mu \cdot \left(a_3 e^{2i\varphi}\right)^2\right| = \left|\Phi_f(\mu)\right|$$

and

$$\left|\Theta_{f_\varphi}(\mu)\right| = \left|a_4 e^{3i\varphi} - \mu \cdot a_3 e^{2i\varphi} \cdot a_2 e^{i\varphi}\right| = \left|\Theta_f(\mu)\right| .$$

If for every $f \in A$ and every $\varphi \in \mathbb{R}$,

$$f \in A \Leftrightarrow f_\varphi \in A\,, \tag{12}$$

then in the research on $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$, it is not necessary to discuss all functions f of a given class, but only those functions whose coefficients a_2 are non-negative real numbers. If (12) does not hold, then the research is more complicated. In such cases, one can usually obtain results under an additional assumption $a_2 \in \mathbb{R}$. It is the case for the class $\mathscr{C}_0(h)$.

Note that $f \in \mathscr{C}_0(h)$ if and only if $f_\pi \in \mathscr{C}_0(h)$, so $|\Phi_f(\mu)| = |\Phi_{f_\pi}(\mu)|$ and $|\Theta_f(\mu)| = |\Theta_{f_\pi}(\mu)|$. Hence, when deriving the bounds of $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$, we can replace the assumption $a_2 \in \mathbb{R}$ by $a_2 \in \mathbb{R}_+$.

2. Auxiliary lemmas

In order to prove our results, we need a few lemmas concerning functions in the class $\mathscr{P}$.

Lemma 4 ([7]). *If $p \in \mathscr{P}$ and $\mu \in \mathbb{R}$, then the following sharp estimates hold*

(1) $|p_{n+m} - \mu p_m p_n| \le 2$ *for* $n, m = 1, 2, \ldots,$

(2) $|p_2 - \mu p_1{}^2| \le 2.$

Lemma 5 ([17]). *If $p \in \mathscr{P}$, then*

(1) $2p_2 = p_1{}^2 + x(4 - p_1{}^2),$

(2) $4p_3 = p_1{}^3 + 2p_1(4 - p_1{}^2)x - p_1(4 - p_1{}^2)x^2 + 2(4 - p_1{}^2)(1 - |x|^2)y,$

for some x and y such that $|x| \le 1$, $|y| \le 1$.

Since

$$\left|p_2{}^2 - \mu p_1 p_3\right| \le \left|p_2{}^2 - p_4\right| + \left|p_4 - \mu p_1 p_3\right| ,$$

directly from Lemma 4 we conclude the following fact.

Lemma 6. *If $p \in \mathscr{P}$, then the sharp estimate $|p_2{}^2 - \mu p_1 p_3| \le 4$ holds for $\mu \in [0,1]$.*

Let us return to the correspondence between the functions in $\mathscr{C}_0(h)$ and $\mathscr{P}$. Rewriting it as follows

$$\left(1 - z^2\right) f'(z) = p(z), \quad f \in \mathscr{C}_0(h),\ p \in \mathscr{P}, \tag{13}$$

we can express $\Phi_f(\mu)$ and $\Theta_f(\mu)$ for $f \in \mathscr{C}_0(h)$ in terms of the coefficients of $p \in \mathscr{P}$:

$$\Phi_f(\mu) = \frac{1}{8} p_1 \left(p_1 + p_3\right) - \frac{1}{9} \mu \left(1 + p_2\right)^2 \tag{14}$$

and

$$\Theta_f(\mu) = \frac{1}{4} \left(p_1 + p_3\right) - \frac{1}{6} \mu p_1 \left(1 + p_2\right) . \tag{15}$$

3. Estimation of $|\Phi_f(\mu)|$

Theorem 7. *If $f \in \mathscr{C}_0(h)$ is of the form* (1) *and a_2 is a real number, then*

$$\left|\Phi_f(\mu)\right| \le \begin{cases} 1-\mu, & \mu \le 1/2 \\ \mu, & \mu \ge 1/2. \end{cases}$$

Equality holds for the functions $f(z) = \frac{z}{1-z}$ and $f(z) = \frac{z}{1+z}$ if $\mu \le 1/2$ and for $f(z) = \frac{z}{1-z^2}$ if $\mu \ge 1/2$.

Proof of Theorem 7. By Theorem 3, if $\mu \le 0$, then $|a_2 a_4 - \mu {a_3}^2| \le 1-\mu$. Equality holds if $a_2 = a_3 = a_4 = 1$ or $a_2 = -a_3 = a_4 = 1$, i.e. for $f(z) = \frac{z}{1-z}$ or $f(z) = \frac{z}{1+z}$.

If $\mu \ge 9/8$, then

$$a_2 a_4 - \mu {a_3}^2 = -\left[\frac{1}{9}\mu\left({p_2}^2 - \frac{9}{8\mu} p_1 p_3\right) + \tfrac{2}{9}\mu\left(p_2 - \frac{9}{16\mu}{p_1}^2\right) + \frac{1}{9}\mu\right].$$

Hence, by Lemma 6 and Lemma 4,

$$\left|a_2 a_4 - \mu {a_3}^2\right| \le \frac{1}{9}\mu\cdot 4 + \frac{2}{9}\mu\cdot 2 + \frac{1}{9}\mu = \mu,$$

with equality for $p_2 = 2$ and $p_1 = p_3 = 0$, i.e. for $p(z) = \frac{1+z^2}{1-z^2}$. This means that the extremal function $f \in \mathscr{C}_0(h)$ is $f(z) = \frac{z}{1-z^2}$.

Now, we shall derive the bound of $\Phi_f(1/2)$. Applying Lemma 5 and writing p instead of p_1, we have

$$\begin{aligned} &288\Phi_f(1/2) \\ &\quad = 5p^4 + 20p^2 - 16 + 2\left(4-p^2\right)\left(5p^2-8\right)x - \left(4-p^2\right)\left(5p^2+16\right)x^2 + 18p\left(4-p^2\right)\left(1-|x|^2\right)y. \end{aligned}$$

Assume now that a_2 is a real number. As it was said in Introduction, instead of the condition $a_2 \in \mathbb{R}$ we can assume that $a_2 \in \mathbb{R}_+$. Consequently, $p \in [0,2]$. Applying the triangle inequality we obtain

$$288\left|\Phi_f(1/2)\right| \le H(p,r),$$

where $r = |x| \in [0,1]$ and

$$H(p,r) = \left|5p^4 + 20p^2 - 16\right| + 18p\left(4-p^2\right) + 2\left(4-p^2\right)\left|5p^2-8\right| r + \left(4-p^2\right)(2-p)(8-5p)r^2.$$

If $0 \le p \le 8/5$, then $H(p,r) \le H(p,1)$. If $8/5 < p \le 2$, then H is an increasing function of $r \in [0, r_0)$, where

$$r_0 = \frac{5p^2-8}{(2-p)(5p-8)}.$$

It is easy to check that for $p \in (8/5, 2]$ there is $r_0 > 1$, so in this case we also have $H(p,r) \le H(p,1)$.

Hence, for $p \in [0,2]$,

$$H(p,r) \le h(p),$$

where

$$h(p) = \left|5p^4 + 20p^2 - 16\right| + 2\left(4-p^2\right)\left|5p^2-8\right| + \left(4-p^2\right)\left(5p^2+16\right),$$

or equivalently,

$$h(p) = \begin{cases} 72\left(2-p^2\right), & 0 \le p \le p_* \\ 2\left(5p^4 - 16p^2 + 56\right), & p_* \le p \le p_{**} \\ 2\left(-5p^4 + 40p^2 - 8\right), & p_{**} \le p \le 2, \end{cases}$$

where $p_* = \sqrt{6/\sqrt{5}-2} = 0.826\ldots$ and $p_{**} = 4/\sqrt{10} = 1.261\ldots$.

It is easy to check that $h_1(p) = 72(2-p^2)$ decreases for $p \in [0, p_*]$, $h_2(p) = 2(5p^4 - 16p^2 + 56)$ decreases for $p \in [0, p_{**}]$ and $h_3(p) = 2(-5p^4 + 40p^2 - 8)$ increases for $p \in [0,2]$. Moreover, $h(0) = h(2) = 144$. For this reason

$$\max\left\{h(p) :\ p \in [0,2]\right\} = 144,$$

which results in

$$\left|\Phi_f(1/2)\right| \le 1/2\,. \tag{16}$$

A careful analysis of the cases when equality holds in (16) leads to a conclusion that only the functions mentioned at the beginning of this proof, i.e. $f(z)=\frac{z}{1-z}$, $f(z)=\frac{z}{1+z}$ and $f(z)=\frac{z}{1-z^2}$ are extremal functions.

Finally, if $\mu\in(0,1/2)$, then

$$\Phi_f(\mu)=(1-2\mu)a_2a_4+2\mu\left(a_2a_4-\frac{1}{2}{a_3}^2\right).$$

The previous part of this proof yields

$$\left|\Phi_f(\mu)\right|\le(1-2\mu)\cdot 1+2\mu\cdot\tfrac{1}{2}=1-\mu\,.$$

In a similar way, for $\mu\in(1/2,9/8)$,

$$\begin{aligned}&\left|\Phi_f(\mu)\right|\\ &\quad=\left|\frac{1}{5}(9-8\mu)\left(a_2a_4-\frac{1}{2}{a_3}^2\right)+\frac{1}{5}(8\mu-4)\left(a_2a_4-\frac{9}{8}{a_3}^2\right)\right|\\ &\quad\le\frac{1}{5}(9-8\mu)\cdot\frac{1}{2}+\frac{1}{5}(8\mu-4)\cdot\tfrac{9}{8}=\mu\,.\end{aligned}$$

□

Taking $\mu=1$ we have

Corollary 8. *If $f\in\mathscr{C}_0(h)$ is of the form* (1) *and a_2 is a real number, then*

$$\left|a_2a_4-{a_3}^2\right|\le 1\,.$$

Equality holds if $f(z)=\frac{z}{1-z^2}$.

4. Estimation of $|\Theta_f(\mu)|$

At the beginning of this section, observe that

$$\left|\Theta_f(\mu)\right|\le\left|1-\mu\right|\quad\text{for}\quad\mu\le 0\quad\text{or}\quad\mu\ge 3/2\,. \tag{17}$$

Indeed, if $\mu\le 0$, then obviously $|a_4-\mu a_2a_3|\le 1-\mu$. Since

$$a_4-\mu a_2a_3=-\frac{1}{4}\left[\left(\frac{2}{3}\mu-1\right)p_1+\left(\frac{2}{3}\mu p_1p_2-p_3\right)\right],$$

by Lemma 4, for $\mu\ge 3/2$,

$$\left|a_4-\mu a_2a_3\right|\le\frac{1}{4}\left[\left(\frac{2}{3}\mu-1\right)\cdot 2+\frac{8}{3}\mu-2\right]=\mu-1\,.$$

Equalities in both cases hold for $f(z)=\frac{z}{1-z}$ and $f(z)=\frac{z}{1+z}$.

If $\mu\in(0,3/2)$, then, applying Lemma 5 and writing Θ instead of $\Theta_f(\mu)$ and p instead of p_1, we have

$$16\Theta=4(1-\gamma)p+(1-2\gamma)p^3+2(1-\gamma)p\left(4-p^2\right)x-p\left(4-p^2\right)x^2+2\left(4-p^2\right)\left(1-|x|^2\right)y,$$

where

$$\gamma=\frac{2}{3}\mu;\quad\mu\in(0,3/2)\Leftrightarrow\gamma\in(0,1)\,.$$

Let us now assume that the second coefficient of f is a real non-negative number; consequently, $p\in[0,2]$. By the triangle inequality,

$$|\Theta|\le\frac{1}{8}\left(4-p^2\right)\left[\left|\frac{4(1-\gamma)p+(1-2\gamma)p^3}{2\left(4-p^2\right)}+(1-\gamma)px-\frac{1}{2}px^2\right|+1-|x|^2\right]. \tag{18}$$

Now, we need the following result obtained in [4] (see also [22]).

For $\overline{\Delta} = \{z \in \mathbb{C} : |z| \le 1\}$ and for real numbers a, b, c, let

$$Y(a,b,c) = \max\left\{\left|a + bz + cz^2\right| + 1 - |z|^2 : z \in \overline{\Delta}\right\}. \tag{19}$$

Lemma 9. *If $ac \ge 0$, then*

$$Y(a,b,c) = \begin{cases} |a| + |b| + |c|, & |b| \ge 2(1-|c|), \\ 1 + |a| + \frac{b^2}{4(1-|c|)}, & |b| < 2(1-|c|). \end{cases}$$

If $ac < 0$, then

$$Y(a,b,c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1-|c|)}, & -4a\left(1-c^2\right)/c \le b^2 \quad \textit{and} \quad |b| < 2(1-|c|), \\ 1 + |a| + \frac{b^2}{4(1+|c|)}, & b^2 < -4a\left(1-c^2\right)/c \quad \textit{and} \quad b^2 < 4(1+|c|)^2, \\ R(a,b,c), & \textit{otherwise}, \end{cases}$$

where

$$R(a,b,c) = \begin{cases} |a| + |b| - |c|, & |c|(|b| + 4|a|) \le |ab|, \\ -|a| + |b| + |c|, & |ab| \le |c|(|b| - 4|a|), \\ (|a| + |c|)\sqrt{1 - \frac{b^2}{4ac}}, & \textit{otherwise}. \end{cases}$$

Let $\Omega = \{(\gamma, p) : \gamma \in (0,1)\ p \in [0,2]\}$ and let Ω_k, $k = 1, 2, \ldots, 7$ denote the subsets of Ω defined by inequalities which appear in seven succesive cases described in Lemma 9. From (18) and (19),

$$a = \frac{4(1-\gamma)p + (1-2\gamma)p^3}{2(4-p^2)}, \; b = (1-\gamma)p, \; c = -\frac{1}{2}p, \quad (\gamma, p) \in \Omega. \tag{20}$$

With these values of a, b, c, we shall rewrite Lemma 9. We start with checking some of the conditions from this lemma.

Lemma 10. *The curve $a = 0$ in Ω coincides with the image of $[2/3, 1)$ by the descreasing function $p(\gamma) = 2\sqrt{\frac{1-\gamma}{2\gamma-1}}$.*

Lemma 11. *The inequality $b < 2(1+|c|)$ holds for all $(\gamma, p) \in \Omega$.*

Lemma 12. *The inequalities $a > 0$, $b^2 \ge -4a(1-c^2)/c$, $b \ge 2(1+c)$ and $ab \ge -c(b+4a)$ are contradictory in Ω.*

The first two lemmas are easy to verify.

Proof of Lemma 12. The inequalities written above for $(\gamma, p) \in \Omega$ are equivalent to

$$\begin{cases} \gamma < 2/3 \vee \left(\gamma \ge 2/3 \wedge p < 2\sqrt{\frac{1-\gamma}{2\gamma-1}}\right) \\ \gamma \ge \left(\sqrt{5}-1\right)/2 \wedge p \ge \frac{2\sqrt{1-\gamma}}{\gamma} \\ p \ge \frac{2}{2-\gamma} \\ \gamma \ge 2/3 \wedge p \ge \frac{2\sqrt{2(1-\gamma)(2+\gamma)}}{\gamma}. \end{cases}$$

This means that $\gamma \ge 2/3$ and

$$\max\left\{\frac{2\sqrt{1-\gamma}}{\gamma}, \frac{2}{2-\gamma}, \frac{2\sqrt{2(1-\gamma)(2+\gamma)}}{\gamma}\right\} \le p \le 2\sqrt{\frac{1-\gamma}{2\gamma-1}},$$

which is contradictory because

$$\frac{2\sqrt{2(1-\gamma)(2+\gamma)}}{\gamma} > 2\sqrt{\frac{1-\gamma}{2\gamma-1}} \quad \text{for} \quad \gamma \ge 2/3. \qquad \square$$

In other words, Lemma 12 means that $\Omega_5 = \emptyset$.

Since $b \ge 0$ and $c \le 0$, applying Lemma 10, Lemma 11 and Lemma 12, we have

$$Y = Y(a,b,c) = \begin{cases} -a+b-c, & (\gamma,p) \in \Omega_1 \cup \Omega_6, \\ 1-a+\frac{b^2}{4(1+c)}, & (\gamma,p) \in \Omega_2 \cup \Omega_3, \\ 1+a+\frac{b^2}{4(1-c)}, & (\gamma,p) \in \Omega_4, \\ (a-c)\sqrt{1-\frac{b^2}{4ac}}, & (\gamma,p) \in \Omega_7. \end{cases} \tag{21}$$

Combining (21) with (18),

$$|\Theta| \le h(\gamma,p), \quad h(\gamma,p) = \begin{cases} h_1(\gamma,p), & (\gamma,p) \in \Omega_2 \cup \Omega_3, \\ h_2(\gamma,p), & (\gamma,p) \in \Omega_2 \cup \Omega_3, \\ h_3(\gamma,p), & (\gamma,p) \in \Omega_4, \\ h_4(\gamma,p), & (\gamma,p) \in \Omega_7, \end{cases} \tag{22}$$

where

$$\begin{aligned} h_1(\gamma,p) &= \frac{1}{4}p\left[-(1-\gamma)p^2+2-\gamma\right], && (\gamma,p) \in \Omega_1 \cup \Omega_6, \\ h_2(\gamma,p) &= \frac{1}{16}\left[\gamma^2p^3-2\gamma(2-\gamma)p^2-4(1-\gamma)p+8\right], && (\gamma,p) \in \Omega_2 \cup \Omega_3, \\ h_3(\gamma,p) &= \frac{1}{16}\left[-\gamma^2p^3-2\gamma(2-\gamma)p^2+4(1-\gamma)p+8\right], && (\gamma,p) \in \Omega_4, \\ h_4(\gamma,p) &= \frac{1}{16}p\left[-2\gamma p^2+4(2-\gamma)\right]\sqrt{\frac{4(1-\gamma)(2-\gamma)-\gamma^2p^2}{4(1-\gamma)+(1-2\gamma)p^2}}, && (\gamma,p) \in \Omega_7. \end{aligned}$$

Now, we are ready to derive the maximum value of $h(\gamma,p)$ with respect to p, for a fixed value γ if $(\gamma,p) \in \Omega$.

Lemma 13.

$$\max\left\{h_3(\gamma,p) : (\gamma,p) \in \Omega_4\right\} = \begin{cases} 1-\frac{3}{2}\gamma, & \gamma \in \left(0,\left(\sqrt{29}-5\right)/2\right], \\ \frac{1}{54\gamma}\left[2\sqrt{\gamma^2-7\gamma+7}^3+(2\gamma-1)\left(\gamma^2-10\gamma+34\right)\right], & \gamma \in \left[\left(\sqrt{29}-5\right)/2,1\right). \end{cases} \tag{23}$$

Proof of Lemma 13. The set Ω_4 is defined by two inequalities: $a > 0$ and $b^2 < -4a(1-c^2)/c$, which can be written as

$$4(1-\gamma)+(1-2\gamma)p^2 > 0 \quad \text{and} \quad \gamma^2p^2 < 4(1-\gamma).$$

The above is equivalent to

$$\begin{cases} \gamma < 2/3 & \vee \left(\gamma \ge 2/3,\, p^2 < \frac{4(1-\gamma)}{2\gamma-1}\right), \\ \gamma < (\sqrt{5}-1)/2 & \vee \left(\gamma \ge (\sqrt{5}-1)/2,\, p^2 < \frac{4(1-\gamma)}{\gamma^2}\right), \end{cases}$$

hence,

$$\Omega_4 = \left\{(\gamma,p) \in \Omega : \gamma < \left(\sqrt{5}-1\right)/2 \vee \left(\gamma \ge \left(\sqrt{5}-1\right)/2,\, p < \frac{2\sqrt{1-\gamma}}{\gamma}\right)\right\}.$$

The only positive solution of $\frac{\partial h_3}{\partial p} = 0$ is $p_0 = 2(\sqrt{\gamma^2-7\gamma+7}+\gamma-2)/3\gamma$. It is easy to check that $p_0 < 2\sqrt{1-\gamma}/\gamma$. Moreover, if $\gamma \le (\sqrt{29}-5)/2$ then $p_0 \ge 2$. This yields that

$$\max\left\{h_3(\gamma,p) : (\gamma,p) \in \Omega_4\right\} = \begin{cases} h_3(\gamma,2), & \gamma \in \left(0,\left(\sqrt{29}-5\right)/2\right], \\ h_3(\gamma,p_0), & \gamma \in \left[\left(\sqrt{29}-5\right)/2,1\right). \end{cases}$$

Thus the proof of Lemma 13 is completed. □

Remark 14. Observe that the derived maximum value is greater than $1/2$ for all $\gamma \in (0,1)$. It is obvious for $\gamma \in (0,(\sqrt{29}-5)/2]$. For $\gamma \in [(\sqrt{29}-5)/2,1)$, or even for all $\gamma \in (0,1)$, the inequality

$$\frac{1}{54\gamma}\left[2\sqrt{\gamma^2-7\gamma+7}^{\,3}+(2\gamma-1)\left(\gamma^2-10\gamma+34\right)\right]>\frac{1}{2}$$

is equivalent to

$$2\sqrt{\gamma^2-7\gamma+7}^{\,3}>(2-\gamma)\left(2\gamma^2-17\gamma+17\right).$$

After simple algebraic computation we obtain

$$27\left(\gamma^2-8\gamma+8\right)(1-\gamma)^2>0,$$

which is true for all $\gamma \in (0,1)$. This proves our claim.

Lemma 15. *For $(\gamma,p)\in\Omega_1\cup\Omega_6$, we have $h_1(\gamma,p)<\frac{1}{2}$.*

Proof of Lemma 15. The set $\Omega_1\cup\Omega_6$ is described by the following complex condition

$$b\geq 2(1+c)\wedge\left[a\leq 0\vee\left(a>0,\,b^2\geq -4a\left(1-c^2\right)/c,\quad ab\leq -c(b-4a)\right)\right],$$

which is a simple consequence of the conditions given in Lemma 9.

The inequality $b\geq 2(1+c)$ leads to $p\geq 2/(2-\gamma)$. The second operand of the above conjunction is of the form

$$4(1-\gamma)+(1-2\gamma)p^2\leq 0\vee\begin{cases}4(1-\gamma)+(1-2\gamma)p^2>0\\ \gamma^2p^2\geq 4(1-\gamma)\\ 2(1-\gamma)(2-\gamma)\leq\left(-\gamma^2+4\gamma-2\right)p^2.\end{cases}$$

We exclude the case $p=0$, which is impossible because $p\geq 2/(2-\gamma)$. This condition is equivalent to

$$\left(\gamma\geq 2/3,\ p^2\geq\frac{4(1-\gamma)}{2\gamma-1}\right)\vee\begin{cases}\gamma\leq 2/3\vee\left(\gamma>2/3,\,p^2<\frac{4(1-\gamma)}{2\gamma-1}\right)\\ \gamma\geq\left(\sqrt{5}-1\right)/2,\,p^2\geq\frac{4(1-\gamma)}{\gamma^2}\\ \gamma\geq 2/3,\,p^2\geq\frac{2(1-\gamma)(2-\gamma)}{-\gamma^2+4\gamma-2}.\end{cases}$$

Finally,

$$\Omega_1\cup\Omega_6=\left\{(\gamma,p)\in\Omega:\gamma\geq 2/3,\,p\geq\max\left\{\frac{2}{2-\gamma},\,\frac{2\sqrt{1-\gamma}}{\gamma},\,\sqrt{\frac{2(1-\gamma)(2-\gamma)}{-\gamma^2+4\gamma-2}}\right\}\right\}.$$

Moreover,

$$\frac{2}{2-\gamma}\leq\frac{2\sqrt{1-\gamma}}{\gamma}\leq\frac{2(1-\gamma)(2-\gamma)}{-\gamma^2+4\gamma-2}\quad\text{for}\quad\gamma\in\left[2/3,\gamma_*\right]$$

and

$$\frac{2(1-\gamma)(2-\gamma)}{-\gamma^2+4\gamma-2}\leq\frac{2\sqrt{1-\gamma}}{\gamma}\leq\frac{2}{2-\gamma}\quad\text{for}\quad\gamma\in\left[\gamma_*,1\right),$$

where $\gamma_*=0.704\ldots$ is the only solution of $\gamma^3-4(1-\gamma)^2=0$.

The only positive solution of

$$\frac{\partial h_1}{\partial p}=0\text{is }p_0=\sqrt{(2-\gamma)/3(1-\gamma)}.$$

It is easy to check that $p_0\in[0,2]$ if $\gamma\in(0,10/11]$ and $p_0\geq 2/(2-\gamma)$ if $\gamma\geq\gamma_{**}$, where the number $\gamma_{**}=0.884\ldots$ is the only solution of $(2-\gamma)^3-12(1-\gamma)=0$ in $(0,1)$. This means that $h_1(\gamma,p)$ is a decreasing function of p in $[0,2]$ for $\gamma\in[2/3,\gamma_{**})$ and an increasing function of p in $[0,2]$ for $\gamma\in[10/11,1)$. If $\gamma\in(\gamma_{**},10/11)$, then $h_1(\gamma,p)$ increases for $p\in[0,p_0)$ and decreases for $p\in(p_0,2]$.

Combining the above facts we can write

$$h_1(\gamma,p)\le\begin{cases} h_1\left(\gamma,\sqrt{\frac{2(1-\gamma)(2-\gamma)}{-\gamma^2+4\gamma-2}}\right), & \gamma\in\left[2/3,\gamma_*\right],\\ h_1\left(\gamma,\frac{2}{2-\gamma}\right), & \gamma\in\left[\gamma_*,\gamma_{**}\right],\\ h_1\left(\gamma,p_0\right), & \gamma\in\left[\gamma_{**},10/11\right],\\ h_1\left(\gamma,2\right), & \gamma\in[10/11,1).\end{cases}$$

The monotonicity of h_1 as a function of p implies that

$$\begin{aligned} h_1(\gamma,p)\le h_1\left(\gamma,\frac{2}{2-\gamma}\right) &= \frac{1}{2}-\frac{2(1-\gamma)}{(2-\gamma)^3} && \text{for}\quad \gamma\in\left[2/3,\gamma_{**}\right],\\ h_1(\gamma,p)\le h_1(\gamma,p_0) &= \frac{1}{6}(2-\gamma)\sqrt{\frac{2-\gamma}{3(1-\gamma)}} && \text{for}\quad \gamma\in\left[\gamma_{**},10/11\right],\\ h_1(\gamma,p)\le h_1(\gamma,2) &= \frac{1}{2}(3\gamma-2) && \text{for}\quad \gamma\in[10/11,1).\end{aligned}$$

It is clear that $h_1(\gamma,p)<1/2$ in the first and the third case. The function

$$\rho\,:\,\left[\gamma_{**},10/11\right]\ni\gamma\mapsto\frac{1}{6}(2-\gamma)\sqrt{\frac{2-\gamma}{3(1-\gamma)}}$$

is increasing, so $\rho(\gamma)\le\rho(10/11)=4/11<1/2$. Therefore,

$$h_1(\gamma,p)<\frac{1}{2}\quad\text{for all}\quad\gamma\in[2/3,1).$$

□

Lemma 16. *For $(\gamma,p)\in\Omega_2\cup\Omega_3$, we have $h_2(\gamma,p)\le\frac{1}{2}$.*

Proof of Lemma 16. Since

$$h_2(\gamma,p)=\frac{1}{2}+\frac{1}{16}p\left[\gamma^2p^2-2\gamma(2-\gamma)p-4(1-\gamma)\right]$$

and the expression in brackets is always less than 0 in Ω, so this proves our claim even in the whole set Ω. □

Lemma 17. *For $(\gamma,p)\in\Omega_7$, we have $h_4(\gamma,p)<\frac{1}{2}$.*

Proof. Proof of Lemma 17 From Lemma 9 it follows that Ω_7 is defined by the inequalities

$$a>0,\ b^2\ge-4a\left(1-c^2\right)/c,\ b\ge2(1+c),\ -c(b-4a)\le ab\le-c(b+4a),$$

which can be written as

$$\begin{cases} 4(1-\gamma)+(1-2\gamma)p^2>0\\ \gamma^2p^2\ge4(1-\gamma)\\ (2-\gamma)p\ge2\\ 2(1-\gamma)(2-\gamma)\ge\left(-\gamma^2+4\gamma-2\right)p^2\ \vee\ p=0\\ 2(1-\gamma)(2+\gamma)\ge\gamma^2p^2\ \vee\ p=0.\end{cases}$$

This results in

$$\begin{cases} \frac{2\sqrt{1-\gamma}}{\gamma}\le p\le2, & \gamma\in\left(\frac{1}{2}\left(\sqrt{5}-1\right),2/3\right]\\ \frac{2\sqrt{1-\gamma}}{\gamma}\le p\le\frac{2(1-\gamma)(2-\gamma)}{-\gamma^2+4\gamma-2}, & \gamma\in[2/3,1).\end{cases}$$

If $\gamma=2/3$, then $h_4(2/3,p)=\sqrt{3}(4p-p^3)/18$, $p\in[\sqrt{3},2]$ is a decreasing function, so $h_4(2/3,p)$ $\le h_4(2/3,\sqrt{3})=1/6$.

Now, let us define

$$g_1(\gamma,p)=-2\gamma p^3+4(2-\gamma)p \quad \text{and} \quad g_2(\gamma,p)=\sqrt{\frac{4(1-\gamma)(2-\gamma)-\gamma^2p^2}{4(1-\gamma)+(1-2\gamma)p^2}}\,.$$

With this notation $h_4(\gamma,p)=\frac{1}{16}g_1(\gamma,p)g_2(\gamma,p)$.

Observe that the homography

$$\left[\frac{4(1-\gamma)}{\gamma^2},\frac{2(1-\gamma)(2-\gamma)}{-\gamma^2+4\gamma-2}\right]\ni x\mapsto \frac{4(1-\gamma)(2-\gamma)-\gamma^2x}{4(1-\gamma)+(1-2\gamma)x}$$

is increasing if $\gamma\in(\sqrt{5}-1)/2,2/3)$ and decreasing if $\gamma\in(2/3,1)$. Hence,

$$g_2(\gamma,p)\le g_2\left(\gamma,\frac{2\sqrt{1-\gamma}}{\gamma}\right)=\frac{\gamma}{\sqrt{1-\gamma}} \quad \text{if} \quad \gamma\in\left(\frac{1}{2}\left(\sqrt{5}-1\right),\frac{2}{3}\right) \tag{24}$$

and

$$g_2(\gamma,p)\le g_2\left(\gamma,\frac{2(1-\gamma)(2-\gamma)}{-\gamma^2+4\gamma-2}\right)=2-\gamma \quad \text{if} \quad \gamma\in(2/3,1)\,. \tag{25}$$

Let $\gamma\in(\frac{1}{2}(\sqrt{5}-1),2/3)$. Since the only positive solution $p_0=\sqrt{2(2-\gamma)/3\gamma}$ of $\frac{\partial g_1}{\partial p}=0$ is less than $2\sqrt{1-\gamma}/\gamma$, so

$$g_1(\gamma,p)\le g_1\left(\gamma,\frac{2\sqrt{1-\gamma}}{\gamma}\right)=\frac{8\left(-\gamma^2+4\gamma-2\right)}{\gamma^2}\sqrt{1-\gamma}$$

and, by (24),

$$h_4(\gamma,p)\le\tfrac{1}{2}\left[4-\left(\gamma+\frac{2}{\gamma}\right)\right]\,.$$

Since for γ in $(\frac{1}{2}(\sqrt{5}-1),2/3)$ the expression in brackets increases, we have

$$h_4(\gamma,p)<\tfrac{1}{6}\,.$$

Let now $\gamma\in(2/3,1)$. Since $2-\gamma<2\gamma$, we have $g_1(\gamma,p)\le 2\gamma(4p-p^3)$. Taking the greatest value of $4p-p^3$ in the whole interval $(0,1)$, we obtain $g_1(\gamma,p)\le 32\gamma/3\sqrt{3}$. Applying (25),

$$h_4(\gamma,p)\le\frac{2}{3\sqrt{3}}\gamma(2-\gamma)<\frac{2}{3\sqrt{3}}\,.$$

Consequently, for all $(\gamma,p)\in\Omega_7$, there is $h_4(\gamma,p)<1/2$. □

Taking into account (17), Lemmas 13 - 17 and Remark 14 and substituting $\gamma=2\mu/3$, we obtain

Theorem 18. *If $f\in\mathscr{C}_0(h)$ is of the form* (1) *and a_2 is a real number, then*

$$\left|\Theta_f(\mu)\right|\le\begin{cases}1-\mu, & \mu\le\mu_0\\ F(\mu), & \mu\in\left[\mu_0,3/2\right]\\ \mu-1, & \mu\ge 3/2\,,\end{cases}$$

where $\mu_0=3(\sqrt{29}-5)/4=0.288\ldots$ *and*

$$F(\mu)=\frac{1}{486\mu}\left[\sqrt{63-42\mu+4\mu^2}^{\,3}+(4\mu-3)\left(153-30\mu+2\mu^2\right)\right]\,. \tag{26}$$

For $\mu=1$,

Corollary 19. *If $f\in\mathscr{C}_0(h)$ is of the form* (1) *and a_2 is a real number, then*

$$|a_4-a_2a_3|\le\frac{125}{243}\,.$$

5. Applications

As it was stated earlier, the bounds of $\Phi_f(\mu)$ and $\Theta_f(\mu)$ found in the previous sections are sharp. We shall show that all corresponding extremal functions have real coefficients. This observation and Theorem 2 make it possible to transfere these estimates onto the class $\mathscr{K}_{\mathbb{R}}(i)$. Subsequently, the relation between $\mathscr{K}_{\mathbb{R}}(i)$ and $\mathscr{T}$, i.e. Formula (8), results in the analogous estimates for typically real functions.

Observe that almost all extremal functions in the theorems proved in Sections 3 and 4 have real coefficients. It is the case for functions $f(z)=\frac{z}{1-z}$ and $f(z)=\frac{z}{1+z}$ which appear in Theorem 7 and in Theorem 18 as well as for the function $f(z)=\frac{z}{1-z^2}$ which is the extremal function in Theorem 7. Only one extremal function is not known explicitly, i.e. the function $f\in\mathscr{C}_0(h)$ which gives the bound $F(\mu)$ in Theorem 18. We know only that Y given by (19) with a,b,c defined by (20) is equal to $1+a+b^2/4(1-c)$.

Consider now Y constrained to real $x\in\overline{\Delta}$, i.e. $Y=\max\{k(x): x\in[-1,1]\}$, where $k(x)=|a+bx+cx^2|+1-x^2$. Let $a+bx+cx^2>0$. Then the function $k(x)=1+a+bx-(1-c)x^2$ attains its maximum value for $x_0=b/2(1-c)$, and this value is equal to $1+a+b^2/4(1-c)$, providing that $1-c>0$, $-1\le x_0\le 1$ and $a+bx_0+c{x_0}^2>0$. We know that for $(\gamma,p)\in\Omega_4$ there is $a>0$, $b>0$, $c<0$. Hence, the first inequality holds. The second and the third inequalities reduce to $b\le 2(1-c)$ and $b^2(2-c)\ge -4a(1-c)^2$ respectively, which is true for $(\gamma,p)\in\Omega_4$.

The above means that the maximum value of Θ for $f\in\mathscr{C}_0(h)$ and $(\gamma,p)\in\Omega$ is achieved if $p_1\in\mathbb{R}$ and p_2, p_3 are as in Lemma 5 with $x=x_0\in\mathbb{R}$. For this reason, p_2 is also real, as well as p_3, since it is enough to take $y=1$ to obtain the estimation in (18).

Summing up, the extremal function $p\in\mathscr{P}$ which maximizes the expression Θ is of the form $p(z)=1+p_1z+p_2z^2+\ldots$ with $p_1,p_2,p_3\in\mathbb{R}$. But

$$P(z)=\frac{1}{2}\left[p(z)+\overline{p(\overline{z})}\right]=1+p_1z+p_2z^2+p_3z^3+\mathrm{Re}(p_4)z^4+\ldots\in\mathscr{P}_{\mathbb{R}}\,.$$

This means that there exists a function $P\in\mathscr{P}_{\mathbb{R}}$ and a corresponding function f in $\mathscr{C}_0(h)$ such that for its real coefficients a_2, a_3, a_4 the equality $|a_4-\mu a_2a_3|=F(\mu)$ holds. From this observation and by Theorem 2, we obtain the following Corollaries 20 and 21 with sharp inequalities.

Corollary 20. *If $f\in\mathscr{K}_{\mathbb{R}}(i)$ is of the form* (1)*, then*

$$\left|\Phi_f(\mu)\right|\le\begin{cases}1-\mu, & \mu\le 1/2\\ \mu, & \mu\ge 1/2\,.\end{cases}$$

Corollary 21. *Let F and μ_0 be defined as in Theorem 2. If $f\in\mathscr{K}_{\mathbb{R}}(i)$ is of the form* (1)*, then*

$$\left|\Theta_f(\mu)\right|\le\begin{cases}1-\mu, & \mu\le\mu_0\\ F(\mu), & \mu\in[\mu_0,3/2]\\ \mu-1, & \mu\ge 3/2\,.\end{cases}$$

Let us return to the earlier mentioned result of Hayami and Owa. In [7] they obtained the following inequality for the class $\mathscr{Q}(1/2)$, which is a superclass for $\mathscr{K}_{\mathbb{R}}(i)$.

Theorem 22. *If $f\in\mathscr{Q}(1/2)$ is of the form* (1)*, then*

$$|\Phi_f(\mu)|\le\begin{cases}1-\mu, & \mu\le 0\\ 1, & \mu\in[0,1]\\ \mu, & \mu\ge 1\,,\end{cases}$$

with equality for the functions $f(z)=\frac{z}{1-z}$ ($\mu\le 0$) and $f(z)=\frac{z}{1-z^2}$ ($\mu\ge 1$).

Moreover, they improved the second case of Theorem 22 showing that

Theorem 23. *If $f \in \mathscr{Q}(1/2)$ is of the form* (1)*, then $|\Phi_f(\mu)| \le \mu$ for $\mu \in [3/4, 1]$.*

This result is sharp. For $\mu \in [(2-\sqrt{2})/4, 3/4]$ the bound is $(9-16\mu+8\mu^2)/8(1-\mu)$, but it is not sharp. What is interesting here, they conjectured that the exact bound of $\Phi_f(\mu)$ is the same as in the assertion of Corollary 20. Unfortunately, this conjecture is false. It can be proved that, for example, if $\mu = 1/2$ the sharp inequality $|\Phi_f(1/2)| \le 5/8$ holds for $\mathscr{Q}(1/2)$, contrary to the conjectured value $1/2$. This shows that the results for $\mathscr{Q}(1/2)$ and for $\mathscr{K}_{\mathbb{R}}(i)$ are essentially different.

Now, let us turn to the class $\mathscr{T}$ of typically real functions. Let $g(z) = zf'(z)$. From (8) we know that $f \in \mathscr{K}_{\mathbb{R}}(i)$ if and only if $g \in \mathscr{T}$. If f is of the form (1) and

$$g(z) = z + b_2 z^2 + \dots, \tag{27}$$

then

$$n a_n = b_n \,.$$

Applying this formula in Corollary 20 and in Corollary 21 with $\lambda = 8\mu/9$ and $\lambda = 2\mu/3$, respectively, we obtain the bounds of $\Phi_g(\lambda)$ and $\Theta_g(\lambda)$ while $g \in \mathscr{T}$.

Corollary 24. *If $g \in \mathscr{T}$ is of the form* (27)*, then*

$$\left|\Phi_g(\lambda)\right| \le \begin{cases} 8-9\lambda, & \lambda \le 4/9 \\ 9\lambda, & \lambda \ge 4/9 . \end{cases}$$

The bound is sharp. Equality holds for the functions $g(z) = \frac{z}{(1-z)^2}$ and $g(z) = \frac{z}{(1+z)^2}$ if $\lambda \le 4/9$ and for $f(z) = \frac{z(1+z^2)}{(1-z^2)^2}$ if $\lambda \ge 4/9$.

Corollary 25. *Let*

$$G(\lambda) = \frac{2}{27\lambda}\left[2\sqrt{7-7\lambda+\lambda^2}^{\,3} + (2\lambda-1)\left(34-10\lambda+\lambda^2\right)\right] \tag{28}$$

and $\lambda_0 = (\sqrt{29}-5)/2 = 0.192\dots$. If $g \in \mathscr{T}$ is of the form (27)*, then*

$$\left|\Theta_g(\lambda)\right| \le \begin{cases} 4-6\lambda, & \lambda \le \lambda_0 \\ G(\lambda), & \lambda \in [\lambda_0, 1] \\ 6\lambda-4, & \lambda \ge 1 . \end{cases}$$

The bound is sharp. Equality holds for the functions $g(z) = \frac{z}{(1-z)^2}$ and $g(z) = \frac{z}{(1+z)^2}$ if $\lambda \le \lambda_0$ or $\lambda \ge 1$.

Taking $\lambda = 1$, we conclude that

Corollary 26. *If $g \in \mathscr{T}$ is of the form* (27)*, then*

$$\left|b_2 b_4 - {b_3}^2\right| \le 9 .$$

Equality holds if $g(z) = \frac{z(1+z^2)}{(1-z^2)^2}$.

Corollary 27. *If $g \in \mathscr{T}$ is of the form* (27)*, then*

$$|b_4 - b_2 b_3| \le 2 .$$

Equality holds if $g(z) = \frac{z}{(1-z)^2}$ or $g(z) = \frac{z}{(1+z)^2}$.

The bound from Corollary 26 coincides with the result from [29] which was obtained in a quite different way. The result from Corollary 27 coincides with a particular case of the generalized Zalcman conjecture for the class $\mathscr{T}$ which was proved in [19] by Ma.

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