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Group Theory / *Théorie des groupes*

A question of Malinowska on sizes of finite nonabelian simple groups in relation to involution sizes

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Abstract. Let $I_n(G)$ denote the number of elements of order n in a finite group G. Malinowska recently asked "what is the smallest positive integer k such that whenever there exist two nonabelian finite simple groups S and G with prime divisors $p_1, \cdots, p_k$ of $|G|$ and $|S|$ satisfying $2 = p_1 < \cdots < p_k$ and $I_{p_i}(G) = I_{p_i}(S)$ for all $i \in \{1, \cdots, k\}$, we have that $|G| = |S|$?". This paper resolves Malinowska's question.

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1. Introduction

In 1979, Herzog [6] conjectured that two finite simple groups containing the same number of involutions have the same order. Zarrin [9], in 2018, disproved Herzog's conjecture with a counterexample. Then he conjectured that "if S is a non-abelian simple group and G a group such that $I_2(G) = I_2(S)$ and $I_p(G) = I_p(S)$ for some odd prime divisor p, then $|G| = |S|$". Zarrin's conjecture was disproved by the recent author in [1]. In an attempt to reformulate the mentioned conjecture of Zarrin (considering the works in [1,6,9]), Malinowska [8] asked: "what is the smallest positive integer k such that whenever there exist two nonabelian finite simple groups S and G with prime divisors $p_1, \cdots, p_k$ of $|G|$ and $|S|$ satisfying $2 = p_1 < \cdots < p_k$ and $I_{p_i}(G) = I_{p_i}(S)$ for all $i \in \{1, \cdots, k\}$, we have that $|G| = |S|$?".

A simple computational check in GAP [5] or Magma [3] (see [1] for example) tells us that $I_2(A_8) = 315 = I_2(PSL(3,4))$, $I_7(A_8) = 5760 = I_7(PSL(3,4))$ and $|A_8| = 20160 = |PSL(3,4)|$. Also, the only prime divisors of 20160 are 2, 3, 5 and 7, and that $I_3(A_8) = 1232 < 2240 = I_3(PSL(3,4))$ and $I_5(A_8) = 1344 < 8064 = I_5(PSL(3,4))$. One might think that the smallest such integer k in Malinowska's question is 2 (since $k > 1$ by hypothesis). This is not true since the results in [1] implies that $k > 2$. In particular, $PSL(4,3)$ and $PSL(3,9)$ are nonabelian simple groups. Assume for contradiction that $k = 2$. Then there exist prime divisors p_1 and p_2 of the orders of the groups $PSL(4,3)$ and $PSL(3,9)$ such that $2 = p_1 < p_2 = 13$ and $I_2(PSL(4,3)) = 7371 = I_2(PSL(3,9))$, $I_{13}(PSL(4,3)) = 1866240 = I_{13}(PSL(3,9))$ but $|PSL(4,3)| = 6065280 < 42456960 = |PSL(3,9)|$. Thus, the smallest value of k in Malinowska's question cannot be 2. The goal of this paper is to resolve Malinowska's question.

For clarity, we reinstate Malinowska's question as follows: "What is the smallest positive integer k such that whenever there exist two nonabelian finite simple groups S and G that satisfy Hypothesis(k), we have that $|G| = |S|$?". We note that Hypothesis(k) says that there exist prime divisors $p_1, \cdots, p_k$ of $|G|$ and $|S|$ satisfying $2 = p_1 < \cdots < p_k$ such that G and S have the same numbers of elements of order p_i for each $i \in [1,k]$.

For the rest of this section, we state the main result of this paper.

Theorem 1. *If G and S are two non-isomorphic finite simple nonabelian groups such that $|G| = |S|$, then Hypothesis(k) does not hold for any $k > 2$.*

2. Proof of Theorem 1

Before we give a proof of Theorem 1, we first recall some important results.

Theorem 2 ([7, Theorem 5.1]). *If G and S are non-isomorphic finite nonabelian simple groups such that $|S| = |G|$, then either $G = PSL_3(4)$ and $S = PSL_4(2)$ or $G = \Omega_{2n+1}(q)$ and $S = PSp_{2n}(q)$ for some odd prime power q, and some $n \geq 3$.*

As at 1955, Artin [2] gave a prototype of Theorem 2 for all the finite nonabelian simple groups known then. The groups ($PSL_3(4)$ and $PSL_4(2)$ or $\Omega_{2n+1}(q)$ and $PSp_{2n}(q)$ for some odd prime power q, and some $n \geq 3$) mentioned in Theorem 2 above were also given in Artin's paper since they were known at that time. As more finite simple groups were discovered, Tits et al. in many papers reaffirmed that no other pair of finite simple groups other than the ones found by Artin satisfies the hypothesis of Theorem 2.

Remark 3.

(i) The groups $\Omega_{2n+1}(q)$ and $PSp_{2n}(q)$ are simple for all odd prime power q and $n > 2$. Moreover,
$$\left|\Omega_{2n+1}(q)\right| = \frac{q^{n^2}\prod_{i=1}^{n}\left(q^{2i}-1\right)}{2} = \left|PSp_{2n}(q)\right|.$$

(ii) The involutions in $\Omega_{2n+1}(q)$ and $PSp_{2n}(q)$ arise from subspace configurations of the F_q-space V that $O(V)$ or $Sp(V)$ naturally acts on. The group $\Omega_{2n+1}(q)$ has n conjugacy classes of involutions and the group $PSp_{2n}(q)$ has $\lfloor \frac{n}{2} \rfloor + 1$ conjugacy classes of involutions. One can then use [4, Table 4.5.1] or otherwise to obtain the values of $I_2(\Omega_{2n+1}(q))$ and $I_2(PSp_{2n}(q))$ according as $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$. We give a summary of such results in Proposition 4 below.

Proposition 4. *Let q be any odd prime power and $n \geq 3$. Then $I_2(\Omega_{2n+1}(q)) \neq I_2(PSp_{2n}(q))$. In particular, the following holds:*

(a) *if $q \equiv 3 \mod 4$, then $I_2(\Omega_{2n+1}(q)) > I_2(PSp_{2n}(q))$;*
(b) *if $q \equiv 1 \mod 4$, then $I_2(\Omega_{2n+1}(q)) < I_2(PSp_{2n}(q))$.*

Proof. Follows from computations using [4, Table 4.5.1]. □

We now give a proof of Theorem 1.

Proof of Theorem 1. Let G and S be two non-isomorphic finite simple nonabelian groups such that $|G| = |S|$. Suppose there exists an integer $k > 2$ satisfying Hypothesis(k). In the light of Theorem 2, $(G,S) \in \{(PSL_3(4), PSL_4(2)), (\Omega_{2n+1}(q), PSp_{2n}(q))\}$ for some odd prime power q, and $n \geq 3$. The exposition above tells us that G and S cannot be $PSL_3(4)$ and $PSL_4(2)$. So the only possibilty is that G and S are $\Omega_{2n+1}(q)$ and $PSp_{2n}(q)$ for some odd prime power q, and $n \geq 3$. By Proposition 4, $I_2\big(\Omega_{2n+1}(q)\big) \neq I_2(PSp_{2n}(q))$ for all such q and n; a contradiction to Hypothesis(k). Therefore $k \ngtr 2$. □

An immediate consequence of Theorems 1 and 2 is the following:

Corollary 5. *$k = 3$ in Malinowska's question.*

3. An observation

We close this paper with Observation 6 below which was motivated by the following question: Are there non-isomorphic simple groups S and G with $I_n(G) = I_n(S)$ for all orders of elements n? The answer to this question is clearly 'yes' if we remove the word "simple" from the question. For instance, the Heisenberg group mod 3 (given by the presentation: $\langle x, y, z \mid x^3 = y^3 = z^3 = [x,z] = [y,z] = 1, [x,y] = z \rangle$) and the elementary abelian 3-group of rank 3 are the only two groups of size 27 whose exponent is 3; so they give rise to a positive answer to the question when the word "simple" is removed from the question. However, if we leave the question 'as it is', then an answer is given in the following result:

Observation 6. *There are no non-isomorphic simple groups S and G with $I_n(G) = I_n(S)$ for all orders of elements n.*

Proof. Suppose there are two non-isomorphic simple groups S and G with $I_n(G) = I_n(S)$ for all orders of elements n. Clearly, $|S| = |G|$. In the light of Theorem 2, either $G = PSL(3,4)$ and $S = PSL(4,2)$ *or* $G = \Omega_{2n+1}(q)$ and $S = PSp_{2n}(q)$ for some odd prime power q, and some $n > 2$. But

$$I_3(PSL(4,2)) = 1232 < 2240 = I_3(PSL(3,4)).$$

So G and S cannot be $PSL(3,4)$ and $PSL(4,2)$; whence we take $G = \Omega_{2n+1}(q)$ and $S = PSp_{2n}(q)$ for some odd prime power q, and some $n > 2$. This choice is not possible by Proposition 4. Thus, no such non-isomorphic simple groups exist. □

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