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REVIEW ON SPECTRAL ASYMPTOTICS FOR THE SEMICLASSICAL BOCHNER LAPLACIAN OF A LINE BUNDLE

LÉO MORIN

Abstract. We first give a short introduction to the Bochner Laplacian on a Riemannian manifold, and explain why it acts locally as a magnetic Laplacian. Then we review recent results on the semiclassical properties of semi-excited spectrum with inhomogeneous magnetic field, including Weyl estimates and eigenvalue asymptotics. These results show under specific assumptions that the spectrum is well described by a familly of operators whose symbols are space-dependent Landau levels. Finally we discuss the strength and limitations of these theorems, in terms of possible crossings between Landau levels.

1. Introduction

1.1. **Motivations and context.** The spectral theory of the magnetic Laplacian, and Bochner Laplacian, has given rise to many interesting questions. First motivated by the Ginzburg-Landau theory, bound states of the magnetic Laplacian $(ih\mathrm{d} + A)^*(ih\mathrm{d} + A)$ on a Riemannian manifold in the semiclassical limit $h \to 0$ were studied in many works (see the books [7, 23]), and appeared to have very various behaviours according to the variations of the magnetic field $B = \mathrm{d}A$. If we are given a magnetic field B which is closed but not exact, there is no potential A and we cannot define the magnetic Laplacian in the same way. However, the Bochner Laplacian $\frac{1}{p^2}\Delta^{L^p}$ appears to be the suitable generalization in this case, since it acts *locally* as a magnetic Laplacian. In this context the semiclassical parameter is $p = h^{-1}$. The structure of its spectrum appears to be deeply related to holomorphic structures, Kodaira Laplacians (or renormalized Bochner Laplacians more generaly) and geometric quantization, as explained for instance in [8, 2] or more recently [3, 15, 16, 18, 19].

Even though homogeneous fields already raise interesting questions (see for instance in [20, 5, 3, 17, 15]) we focus here on the non-homogeneous case. The first main technique to study the semiclassical spectrum of magnetic Laplacians consisted in the construction of approximated eigenfunctions (see for instance the works of Helffer-Mohamed [13] and Helffer-Kordyukov [9, 10, 11]). More recently, an other approach was developed, which consists in an approximation of the operator itself, using semiclassical tools such as microlocalisation estimates and Birkhoff normal forms. Raymond-Vũ Ngọc [24] used these techniques to describe the semi-excited spectrum of magnetic Laplacians on the Euclidean $\mathbf{R}^2$ with non-degenerate magnetic field, and to relate semiclassical spectrum with the classical dynamics. This work was generalized to arbitrary manifolds in [21], where metric and higher dimensions create new problems. Few results are known in the case of degenerate magnetic fields, however it is a natural question since this is always the case in odd dimensions. In [12] (on $\mathbf{R}^3$) and [22] (on arbitrary manifolds) it is shown in the case of magnetic wells, that the degeneracy of the field induces a new classical

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motion and significantly modify the behaviour of the spectrum in the semiclassical limit. In section 3 below, we review these eigenvalue asymptotics and show how to apply them to the Bochner Laplacian. In any case, higher dimensions creates specific problems which we describe in section 4 and give limitations to apply these techniques outside magnetic wells, even with non-degenerate fields. This issue is somehow related to the one appearing in the work of Charles [4] where he proves local Weyl laws for Bochner Laplacians. We review these results in section 2, explain their strength and limitations and the link with the above mentionned works in section 4.

1.2. **The Bochner Laplacian on a line bundle.** Let (M,g) be a compact oriented manifold of dimension $d>1$. We consider a complex line bundle $L\to M$ over M, endowed with a Hermitian metric h. In other words, we associate to each $x\in M$ a 1-dimensional complex vector space L_x, and a Hermitian product h_x on L_x. L is a $d+1$-dimensional manifold such that $L=\bigcup_{x\in M}L_x$. A smooth section of L (or L-valued function) is a smooth function $s:M\to L$ such that $s(x)\in L_x$. It is the generalisation of the notion of function $f:M\to\mathbf{C}$, but here the target space can vary with $x\in M$. Similarly, L-valued k-forms are sections of $\wedge^k T^*M\otimes L$. We denote by $\mathcal{C}^\infty(M,L)$ the set of smooth sections of L, and $\Omega^k(M,L)$ the set of smooth L-valued k-forms.

We take ∇^L a Hermitian connection on (L,h). It is the generalisation of the exterior derivative d. The underlying idea is that the "derivative" of a L-valued function should be L-valued too. $\nabla^L:\Omega^k(M,L)\to\Omega^{k+1}(M,L)$ satisfies:

$$\nabla^L(s\alpha)=\nabla^L s\wedge\alpha+s\mathrm{d}\alpha,\quad \forall s\in\mathcal{C}^\infty(M,L),\quad \alpha\in\Omega^k(M,\mathbf{C}), \tag{1.1}$$

$$\mathrm{d}h(s_1,s_2)=h(\nabla^L s_1,s_2)+h(s_1,\nabla^L s_2),\quad \forall s_1,s_2\in\mathcal{C}^\infty(M,L). \tag{1.2}$$

One can prove that $(\nabla^L)^2:\Omega^0(M,L)\to\Omega^2(M,L)$ acts as a multiplication. There exists a real closed 2-form B on M such that:

$$(\nabla^L)^2 s=iBs,\quad \forall s\in\mathcal{C}^\infty(M,L). \tag{1.3}$$

Example: The trivial line bundle. The line bundle $L=M\times\mathbf{C}$, such that $L_x=\{x\}\times\mathbf{C}$ is called the trivial line bundle. We identify sections $s\in\mathcal{C}^\infty(M,L)$ with functions $f\in\mathcal{C}^\infty(M)$ by $s(x)=(x,f(x))$. Similarly, L-valued k-forms are identified with $\mathbf{C}$-valued k-forms, and we recover the usual differential objects on M. If L is endowed with the Hermitian product $h_x((x,z_1),(x,z_2))=z_1\overline{z}_2$, we call (L,h) the trivial Hermitian line bundle. We write $h(z_1,z_2)$ for short. Hermitian connections on the trivial line bundle are given by $\nabla_\alpha=\mathrm{d}+i\alpha$ where $\alpha\in\Omega^1(M,\mathbf{R})$ and d is the exterior derivative. The curvature of ∇_α is $\nabla_\alpha^2=i\mathrm{d}\alpha$, as shown by the easy but enlightening calculation:

$$\begin{aligned}\nabla_\alpha^2 f&=(\mathrm{d}+i\alpha)(\mathrm{d}f+if\alpha)=\mathrm{d}^2f+i\alpha\wedge\mathrm{d}f+i\mathrm{d}(f\alpha)+if\alpha\wedge\alpha\\&=i\alpha\wedge\mathrm{d}f+i\mathrm{d}f\wedge\alpha+if\mathrm{d}\alpha=if\mathrm{d}\alpha.\end{aligned} \tag{1.4}$$

Let us describe now the Bochner Laplacian Δ^L associated to a Hermitian connection ∇^L on a Hermitian complex line bundle (L,h). First note that the spaces $\mathcal{C}^\infty(M,L)=\Omega^0(M,L)$ and $\Omega^1(M,L)$ are endowed with L^2-norms. The norm of a section $s\in\mathcal{C}^\infty(M,L)$ is:

$$\|s\|^2=\int_M h_x(s(x),s(x))\mathrm{d}\nu_g(x), \tag{1.5}$$

where $\mathrm{d}\nu_g$ denotes the volume form of the oriented Riemannian manifold (M, g). We denote by $\mathsf{L}^2(M, L)$ the completion of $\mathcal{C}^\infty(M, L)$ for this norm. The definition of the norm of a L-valued 1-form α is a little more involved. First, using a partition of unity, it is enough to define it for $\alpha \in \Omega^1(U, L)$ where U is a small open subset of M. If U is small enough, there exists a section $e \in \mathcal{C}^\infty(U, L)$ such that $h_x(e(x), e(x)) = 1$. Then for any $\alpha \in \Omega^1(U, L)$, there exists a unique $X \in TM$ such that $\alpha_x(\bullet) = g_x(X_x, \bullet)e_x$ (we identify 1-forms with tangent vectors using the metric g). We define:

$$\|\alpha\|^2 = \int_M g_x(X_x, X_x)\mathrm{d}\nu_g(x). \tag{1.6}$$

The completion of $\Omega^1(M, L)$ for this norm is denoted by $\mathsf{L}^2\Omega^1(M, L)$: it is the space of square-integrable L-valued 1-forms. These norms are associated with scalar products, denoted by brackets $\langle ., . \rangle$.

The formal adjoint of $\nabla^L : \Omega^0(M, L) \to \Omega^1(M, L)$ for these scalar products is denoted by $(\nabla^L)^* : \Omega^1(M, L) \to \Omega^0(M, L)$. The Bochner Laplacian Δ^L is the self-adjoint extension of $(\nabla^L)^*\nabla^L$. It is the operator associated with the quadratic form:

$$Q(s_1, s_2) = \langle \nabla^L s_1, \nabla^L s_2 \rangle. \tag{1.7}$$

We denote by $\mathsf{Dom}(\Delta^L)$ its domain. $\mathcal{C}^\infty(M, L)$ is a dense subspace of $\mathsf{Dom}(\Delta^L)$ and:

$$\langle \Delta^L s_1, s_2 \rangle = \langle \nabla^L s_1, \nabla^L s_2 \rangle, \quad \forall s_1, s_2 \in \mathsf{Dom}(\Delta^L). \tag{1.8}$$

Since M is compact, one can prove that Δ^L has compact resolvent, and we denote by

$$\lambda_1(\Delta^L) \leqslant \lambda_2(\Delta^L) \leqslant ... \tag{1.9}$$

the non-decreasing sequence of its eigenvalues. We will use the following notation for the Weyl counting function:

$$N(\Delta^L, \lambda) := \sharp\{j; \lambda_j(\Delta^L) \leqslant \lambda\}.$$

In this paper, we are interested in the semiclassical limit, i.e. the high curvature limit "$B \to +\infty$". We can increase the curvature B using tensor products of L. For any $p \in \mathbf{N}$, we denote by $L^p = L \otimes ... \otimes L$ the p-th tensor power of L. L^p is still a complex line bundle over M, with $L_x^p = L_x \otimes ... \otimes L_x$. It is endowed with the Hermitian product $h_x^p(s_1 \otimes ... \otimes s_p, s_1 \otimes ... \otimes s_p) = \Pi_{i=1}^p h_x(s_i, s_i)$. The connection ∇^L induces a Hermitian connection ∇^{L^p} on L^p by the Leibniz rule:

$$\nabla^{L^p}(s_1 \otimes ... \otimes s_p) = (\nabla^L s_1) \otimes ... \otimes s_p + ... + s_1 \otimes ... \otimes (\nabla^L s_p).$$

The curvature of ∇^{L^p} is

$$(\nabla^{L^p})^2 = ipB. \tag{1.10}$$

Hence, the high curvature limit is $p \to +\infty$. We want to investigate the behaviour of $\lambda_j(\Delta^{L^p})$ and the corresponding eigensections in the limit $p \to +\infty$.

1.3. **The Bochner Laplacian is locally a magnetic Laplacian.** If U is any open subset of M such that there exists a non-vanishing section $e \in \mathcal{C}^\infty(U, L)$, then any $s \in \mathcal{C}^\infty(U, L)$ can be written $s = ue$ for some $u \in \mathcal{C}^\infty(M)$. Hence,

$$\nabla s = (\nabla e)u + e(\mathrm{d}u) = e[(\mathrm{d} + iA)u],$$

with $\nabla e = eiA$. Moreover,

$$\begin{aligned}\nabla^2 s &= \nabla e \wedge [(\mathrm{d} + iA)u] + e\mathrm{d}[(\mathrm{d} + iA)u] \\ &= e(iA \wedge \mathrm{d}u) + e(iA \wedge iA)u + e\mathrm{d}^2 u + ieu\mathrm{d}A + e\mathrm{d}u \wedge iA \\ &= ieu\mathrm{d}A = (i\mathrm{d}A)s,\end{aligned}$$

and thus $B = \mathrm{d}A$. Hence ∇ acts locally as $\mathrm{d}+iA$, and Δ^L as the magnetic Laplacian $(\mathrm{d} + iA)^*(\mathrm{d} + iA)$.

1.4. **Remarks on the quantization of a magnetic field.** If we are given a closed 2-form B (the magnetic field), the quantization question constist in finding a quantum operator associated to B. If B is exact, this question is answered by the semiclassical magnetic Laplacian $(\hbar\mathrm{d} + iA)^*(\hbar\mathrm{d} + iA)$, with $B = \mathrm{d}A$. Here, $\hbar > 0$ is the semiclassical parameter (Planck's constant) and the semiclassical limit is $\hbar \to 0$.

If B is not exact, but if there exists an Hermitian line bundle with Hermitian connection such that $\nabla^2 = iB$, then the Bochner Laplacian $\nabla^*\nabla$ acts locally as the magnetic Laplacian and hence it is a good candidate. Moreover, we have *locally*

$$\Delta^{L^p} = (\mathrm{d} + ipA)^*(\mathrm{d} + ipA) = p^2(\frac{1}{p}\mathrm{d} + iA)^*(\frac{1}{p}\mathrm{d} + iA),$$

so that the semiclassical parameter is now $\hbar = \frac{1}{p}$ (Also notice the p^2 factor which is important for the eigenvalue asymptotics). The limit $\hbar \to 0$ is equivalent to $p \to +\infty$ exept that the semiclassical parameter becomes discrete ($p \in \mathbf{N}$).

A new question arises : When does such an Hermitian line bundle exist ? Weil's Theorem states that it exists if and only if B satisfies the prequantization condition:

$$[B] \in 2\pi\mathbf{Z}, \tag{1.11}$$

where $[B]$ denotes the cohomology class of B. This condition also enlightens the discreteness of the semiclassical parameter. Indeed, if one wants to quantize the magnetic field $\frac{1}{\hbar}B$, then one must have $\left[\frac{1}{\hbar}B\right] \in 2\pi\mathbf{Z}$, and thus $\frac{1}{p} \in \mathbf{Z}$, unless $[B] = 0$ which means that B is exact (and thus we can use the magnetic Laplacian).

1.5. **Local data.** For every fixed $x \in M$, B_x is a skew-symmetric bilinear form on T_xM. One can use the scalar product g_x to define the associated endomorphism $\mathbf{B}_x$ which satisfies

$$g_x(\mathbf{B}_x U, V) = B_x(U, V), \quad \forall U, V \in T_xM. \tag{1.12}$$

This endomorphism is g_x-skewsymmetric, and we denote by $\beta_1(x) \geqslant \cdots \geqslant \beta_s(x) > 0$ the absolute values of its non-zero eigenvalues counted with multiplicities. Actually s depends on x and the rank of $\mathbf{B}_x$ is $2s \leqslant d$.

One can measure the "intensity" of the magnetic field using the function $b : M \to \mathbf{R}_+$ defined by:

$$b(x) = \sum_{j=1}^{s(x)} \beta_j(x). \tag{1.13}$$

This function is continuous on M, but not smooth in general. However, note that it is smooth on a neighborhood of any point x_0 where the $(\beta_j(x_0))_{1\leqslant j\leqslant s}$ are simple (if s is locally constant near x_0).

2. Weyl Laws

A global Weyl law for the Bochner Laplacian was proven by Demailly with no further assumptions on B. The magnetic field has a very different effect on this Weyl law than an electric potential. This law states that the spectrum of $p^{-1}\Delta^{L^p}$ is an agregate of the spectra of Landau Hamiltonians $\square_y$. For $y \in M$, $\square_y$ is a magnetic Laplacian with constant field $\mathbf{B}_y$ on the tangent space T_yM. Its spectrum is

$$\Sigma_y = \mathsf{sp}(\square_y) = \left\{ \sum_{j=1}^{s} (2n_j + 1)\beta_j(y); \quad n \in \mathbf{N}^s \right\} . \tag{2.1}$$

In the following we denote by $b_n(y) = \sum_{j=1}^{s}(2n_j + 1)\beta_j(y)$ its eigenvalues.

Theorem 2.1 (Demailly [6]). — *There is a countable set $D \subset \mathbf{R}$ such that for $\lambda \in \mathbf{R} \setminus D$,*

$$N(p^{-1}\Delta^{L^p}, \lambda) \sim \frac{2^{s-n}\pi^{-n/2}p^{n/2}}{\Gamma(\frac{n}{2} - s + 1)} \int_M \beta_1(x)\cdots\beta_s(x) \sum_{n\in\mathbf{N}^s} (\lambda - b_n(x))_+^{\frac{n}{2}-s} \, \mathrm{d}\nu_g(x) \,,$$

in the limit $p \to +\infty$.

The main idea of the proof is to locally approximate the magnetic field and the metric by constants. Note that the remainder in this estimate is only $o(p^{n/2})$. One could also consider the Schrödinger operator $p^{-1}\Delta^{L^p} + V(x)$, in which case $V(x)$ should be added to $b_n(x)$ in the asymptotic formula.

Recently Charles [4] proved a refinement of this result in the case of non-degenerate magnetic fields. His work shows that the spectrum of $p^{-1}\Delta_p$ has clusters, and that the number of eigenvalues in each cluster is given by a geometric quantity, the Riemann-Roch number $RR(Y)$ of some vector bundle Y over M. The clusters are the connected components of $\Sigma = \bigcup_{n\in\mathbf{N}^s} b_n(M)$.

Theorem 2.2 (Charles [4]). — *Assume that B is non-degenerate i.e. $d = 2s$ and let $a, b \in \mathbf{R} \setminus \Sigma$ with $a < b$. Then when p is sufficiently large,*

$$\sharp\mathsf{sp}(p^{-1}\Delta^{L^p}) \cap [a,b] = \begin{cases} RR(L^p \otimes F) & \text{if } [a,b] \cap \Sigma \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

where F is the vector bundle with fibers

$$F_y = \mathsf{Ran}\ \mathbf{1}_{[a,b]}(\square_y)\,, \quad y \in M\,.$$

Remark 2.3. — This result is consistent with Theorem 2.1 because the Riemann-Roch number depends polynomialy on p with leading term

$$RR(L^p \otimes F) = \mathsf{rank} F \left(\frac{p}{2\pi}\right)^s \int_M \frac{B^s}{s!} + \mathcal{O}(p^{s-1})\,.$$

Remark 2.4. — Related results also appear in [16] (Theorem 1.4).

Remark 2.5. — This result has the following limitations. Firstly, it could be that $\mathbf{R} \setminus \Sigma$ is a half-line, in which case Theorem 2.2 gives nothing. Secondly, one would like to understand the contribution of each $b_n(M)$ in the spectrum, but here they are gathered according to the connected components of Σ.

Charles also proved the following pointwise Weyl law. Denote by $(\psi_{j,p})_{j\geqslant 1}$ a normalized eigenbasis of Δ^{L^p} such that $\Delta^{L^p}\psi_{j,p} = \lambda_j\psi_{j,p}$. For any $y \in M$ and $a < b$ define

$$N(y,a,b,p) = \sum_{j\ :\ p^{-1}\lambda_j\in[a,b]} |\psi_{j,p}(y)|^2 .$$

Theorem 2.6 (Charles [4]). — *Assume that B is non degenerate. For any $\Lambda \in \mathbf{R}\setminus\Sigma$, $y \in M$ and a, $b \in\,]-\infty,\Lambda]\setminus\Sigma_y$ such that $a < b$, the following holds. If $[a,b]\cap\Sigma_y$ is empty then $N(y,a,b,p) = \mathcal{O}(p^{-\infty})$. Otherwise, we have an asymptotic expansion:*

$$N(y,a,b,p) = \left(\frac{k}{2\pi}\right)^s \sum_{\lambda\in\Sigma_y\cap[a,b]}\sum_{\ell=0}^{\infty} m_{\ell,\lambda}p^{-\ell} + \mathcal{O}(p^{-\infty}),$$

where the coefficients $m_{\ell,\lambda}$ do not depend on a, b, p. $m_{0,\lambda}$ is the multiplicity of the eigenvalue λ of $\square_y$.

Charles uses a specific Toeplitz quantization to see $p^{-1}\Delta^{L^p}$ as an operator with symbol $\square_y$. Since the spectrum of $\square_y$ is Σ_y, we recover the idea of [21]. However, one needs symbols like $\mathbf{1}_{[a,b]}(\square_y)$ to depend smoothly on y, and this is where the assumption $\Lambda \in \mathbf{R}\setminus\Sigma$ appear.

3. Eigenvalue asymptotics in magnetic wells

3.1. **Reduction to local models.** One can prove that the eigensections of the Bochner Laplacian are localized near the minimum points of b and deduce that the lower part of its spectrum is given by magnetic Laplacians on neighborhoods of the minimal points of b. We can use this to get asymptotic expansions of the eigenvalues of Δ^{L^p}, under the following assumptions.

- (A1) $b \in \mathcal{C}^\infty(M)$, and the minimal value of b is only reached at non degenerate points $x_1,\cdots,x_N \in M$. We denote by $b_0 = b(x_j) = \min_{x\in M} b$.
- (A2) The rank of $\mathbf{B}$ is constant on small neighborhoods $U_1,\cdots,U_N$ of $x_1,\cdots,x_N$. We denote by $2s_j$ the rank of $\mathbf{B}_{x_j}$.
- (A3) We assume $b_0 > 0$, which is equivalent to say that $s_j > 0$ for any j.

As noticed in several papers (for instance [13, 23] and Section A below), one can prove using Agmon-like estimates that the eigensections of Δ^{L^p} associated to low-lying eigenvalues are exponentially localized near $\{x \in M, \quad b(x) = b_0\}$, in the limit $p \to +\infty$. Now let us present the local model operators on U_j.

Recall that the 2-form B is closed: $\mathrm{d}B = 0$. Hence, if the open sets U_j are small enough, B is exact on U_j: there exists $A_j \in \Omega^1(U_j)$ such that $B = \mathrm{d}A_j$ on U_j. We denote by $\mathcal{L}_p^{(j)}$ the Dirichlet realization of $(d+ipA_j)^*(d+ipA_j)$ on $\mathsf{L}^2(U_j)$. It is the self-adjoint operator associated to the following sesquilinear form on $\mathcal{C}_0^\infty(U_j)$:

$$Q_j(u,v) = \int_M (\mathrm{d}u + ipA_ju)\overline{(\mathrm{d}v + ipA_jv)}\mathrm{d}\nu_g. \tag{3.1}$$

We prove the following Theorem in Appendix A.

Theorem 3.1. — *Let $\alpha \in (0,\frac{1}{2})$. Under assumptions (A1) and (A3), if $\eta,\varepsilon > 0$ are small enough, then:*

$$\lambda_k(\Delta^{L^p}) = \lambda_k\left(\mathcal{L}_p^{(1)}\oplus\ldots\oplus\mathcal{L}_p^{(N)}\right) + \mathcal{O}(\exp(-\varepsilon p^\alpha)), \tag{3.2}$$

uniformly with respect to $k \in [1, K_p]$, where

$$K_p = \min\left(N(\Delta^{L^p}, (b_0+\eta)p), N(\mathcal{L}_p^{(1)} \oplus ... \oplus \mathcal{L}_p^{(N)}, (b_0+\eta)p)\right),$$

and $N(\mathcal{A}, \lambda)$ denotes the number of eigenvalues of an operator $\mathcal{A}$ below λ, counted with multiplicities.

As a corollary, we can deduce spectral asymptotics for Δ^{L^p} from already-known results for $\mathcal{L}_p^{(j)}$. Let us recall some of these results here.

3.2. **The full-rank case.** Under the assumptions $(A1)-(A2)-(A3)$, we fix a $j \in \{1, \cdots, N\}$, and we denote by $B_j = \mathrm{d}A_j$. Hence, B_j is just the restriction of B to the small open set U_j, where it admits a primitive A_j. $\mathcal{L}_h^{(j)}$ is the magnetic Laplacian with Dirichlet boundary conditions on U_j, with magnetic field B_j. We first focus on the full-rank case, when the rank of B_j is maximal: $2s_j = d$. We define $r_j \in \mathbf{N}$ by the condition

$$\forall n \in \mathbf{Z}^{s_j}, \quad 0 < \sum_{\ell=1}^{s_j} |n_\ell| < r_j \Rightarrow \sum_{\ell=1}^{s_j} n_\ell \beta_\ell(x_j) \neq 0. \tag{3.3}$$

Note that, if the $\beta_\ell(x_j)$ are pairwise distinct, we can choose $r_j \geqslant 3$. Moreover, if the open set U_j is small enough we have, for all $x \in U_j$ and $n \in \mathbf{Z}^{s_j}$,

$$0 < \sum_{\ell=1}^{s_j} |n_\ell| < r_j \implies \sum_{\ell=1}^{s_j} n_\ell \beta_\ell(x) \neq 0. \tag{3.4}$$

The following Theorem is proved in [21].

Theorem 3.2. — *We assume $(A1)-(A2)-(A3)$ with $2s_j = d$ and $r \geqslant 3$ in (3.3). Let $\eta, \varepsilon > 0$ small enough. Then there exists a symplectomorphism $\psi : U_j \to T^*\mathbf{R}^{d/2}$ such that:*

$$\frac{1}{p^2}\lambda_k(\mathcal{L}_p^{(j)}) = \lambda_k\left(\bigoplus_{n \in \mathbf{N}^d} \mathcal{N}_p^{[j,n]}\right) + \mathcal{O}(p^{-r_j/2+\varepsilon}), \tag{3.5}$$

uniformly with respect to $k \in [1, \tilde{K}_p]$, where $\mathcal{N}_p^{[j,n]}$ is a pseudo-differential operator with principal symbol:

$$\sigma(\mathcal{N}_p^{[j,n]}) = \frac{1}{p}\sum_{\ell=1}^{s_j}(2n_\ell+1)\beta_\ell \circ \psi^{-1}(x,\xi),$$

and

$$\tilde{K}_p = \min\left(N(\mathcal{L}_p^{(j)}, (b_0+\eta)p), N(\oplus_n \mathcal{N}_p^{[j,n]}, (b_0+\eta)p^{-1})\right).$$

Hence, we have a description of the semi-excited states of $\mathcal{L}_p^{(j)}$. We this result one can recover a Weyl law, and deduce asymptotic expansions of the first eigenvalues.

Corollary 3.3. — *Assume $(A1)-(A2)-(A3)$, and for any $j \in \{1, \cdots N\}$ that $s_j = d/2$, that $(\beta_\ell(x_j))_{1 \leqslant \ell \leqslant N}$ are pairwise distinct, and $r := \min_j r_j \geqslant 5$. Then, for any $k \in \mathbf{N}$ and $\varepsilon > 0$,*

$$\lambda_k(\Delta^{L^p}) = b_0 p + \sum_{i=0}^{r-5} \alpha_{i,k} p^{-i/2} + \mathcal{O}(p^{2-r/2+\varepsilon}),$$

for some coefficients $\alpha_{i,k} \in \mathbf{R}$.

This result follows from the asymptotic expansions in [21]. Kordyukov proved similar expansions in [14] using a different method.

Remark 3.4. — We also have geometric interpretations of the coefficients. First, the full expansion comes from the effective operator $\mathcal{N}_p^{[j,0]}$, which is the reduction of $\mathcal{L}_p^{(j)}$ to the lowest energy of the Harmonic oscillator describing the classical cyclotron motion. Moreover, $\alpha_{0,k}$ is given by an eigenvalue of an other Harmonic oscillator whose symbol is the Hessian of b at x_j (for some $1 \leqslant j \leqslant N$): it describes a slow drift of the classical particle arround x_j. If the eigenvalues of this oscillator are simple, then a Birkhoff normal form can be used to show that $\alpha_{i,k} = 0$ if i is odd.

3.3. **The constant-rank case.** In the non-full-rank case, the kernel of B (which corresponds to the directions of the field lines), has a great influence on the spectrum of Δ^{L^p}. Fix $1 \leqslant j \leqslant N$. If the rank of B_j is constant, equal to $2s_j$, then its kernel as dimension $k_j = d - 2s_j$. The partial Hessian of b at x_j, in the directions of the Kernel of B_j, is non-degenerate. we denote by

$$\nu_{j,1}^2, \cdots, \nu_{j,k_j}^2 \tag{3.6}$$

its eigenvalues. For simplicity, we will make the following non-resonance assumptions (however, we can deal with resonances using a resonance order r as in the full-rank case).

(A4) For every j, $(\beta_\ell(x_j))_{1\leqslant \ell \leqslant s_j}$ are non-resonnant:

$$\forall n \in \mathbf{Z}^{s_j}, \quad n \neq 0 \implies \sum_{\ell=1}^{s_j} n_\ell \beta_\ell(x_j) \neq 0.$$

(A5) For every j such that $k_j > 0$, $(\nu_{j,\ell})_{1\leqslant \ell \leqslant k_j}$ are non resonnant:

$$\forall n \in \mathbf{Z}^{k_j}, \quad n \neq 0 \implies \sum_{\ell=1}^{k_j} n_\ell \nu_{j,\ell} \neq 0.$$

Applying the results of [22] to get spectral asymptotics for $\mathcal{L}_h^{(j)}$, we deduce from Theorem 3.1 the following corollary. As far as we know, [22] and its 3-dimensional Euclidean version [12] are the only works proving eigenvalue expansions of magnetic Laplacians with degenerate inhomogeneous fields. Note that in odd dimensions a magnetic field is always degenerate.

Corollary 3.5. — *Assume (A1) – (A2) – (A3) – (A4) – (A5), and let $n \in \mathbf{N}$. Then $\lambda_n(\Delta^{L^p})$ admits a full asymptotic expansion in powers of $p^{-1/2}$:*

$$\lambda_n(\Delta^{L^p}) = b_0 p + \kappa p^{1/2} + \sum_{i\geqslant 0} \alpha_{i,n} p^{-i/2} + \mathcal{O}(p^{-\infty}).$$

Moreover:

1 *If there is at least one j such that $k_j = 0$, then $\kappa = 0$.*
2 *If $\forall j \in \{1, \cdots, N\}, k_j > 0$, then $\kappa = \min_{j=1,\cdots,N} \sum_{\ell=1}^{k_j} \nu_{j,\ell}$.*

4. About Landau levels crossings

The results described above show the influence of Landau levels

$$b_n(x) = \sum_{j=1}^{s}(2n_j + 1)\beta_j(x)\,,$$

on the distribution of Bochner Laplacians' eigenvalues. Let us first assume that B is non-degenerate, i.e. $d = 2s$. One would like to say that the spectrum is an agregate of the spectra of operators $\mathcal{N}^{[n]}$ with symbol $pb_n(x)$. Indeed, this is the meaning of Theorem 3.2 under the assumptions of magnetic wells and non-resonances, and Theorems 2.2 under a global gap assumption. However it would be interesting to discuss to what extend one could generalize this informal statement.

The main problem is due to crossings between Landau levels. Imagine first that there exist $n \neq n'$ and x such that $b_n(x) = b_{n'}(x)$. This is equivalent to say that there is some resonance relation:

$$\sum_{j=1}^{s}(n_j - n_j')\beta_j(x) = 0\,,$$

thus limiting Theorem 3.2, at least if x is close to a magnetic well : We are restricted by *local Landau levels crossings*.

If instead we make the much weaker assumption that there exists x, $y \in M$ and $n \neq n'$ such that $b_n(x) = b_{n'}(y)$, it means that $b_n(M)$ and $b_{n'}(M)$ overlap each other. Thus you cannot isolate $b_n(M)$ from the other Landau levels to apply Theorem 2.2 and count the eigenvalues created by this level : This theorem is restricted by *globlal Landau levels crossings*, but does not require any magnetic well assumption. More importantly, one always need a global gap Λ which separates the Landau levels into two groups in order to apply this theorem, and this could not be possible in many situations. However it seems achievable to adapt the proof in [4] to allow the gap Λ to depend on $x \in M$ thus reducing the problem to local Landau level crossings.

On Figure 4, Landau levels are drawn with various behaviours, to see which Theorem can be applied in which case. Note that this is a very schematic drawing since the Landau levels should have the dimension of M, and we could imagine any kind of crossing between such surfaces.

This is a motivation for further research, to understand how two crossing Landau levels or more can interact and influence the spectrum of a Bochner Laplacian, and compare to the case when they do not cross.

Another problem is raised by $(1,1)$-resonances, i.e when $\beta_i(x_0) = \beta_j(x_0)$ for some $x_0 \in M$ and $i \neq j$. Indeed, further then the creation of many crossings, this could imply that the function β_i is no longer smooth arround x_0, but only Hölder continuous and the whole method in [4, 22] breaks down. More importantly, it is not clear how strong it would be to assume that such resonances never happen on M.

Appendix A. Proof of Theorem 3.1

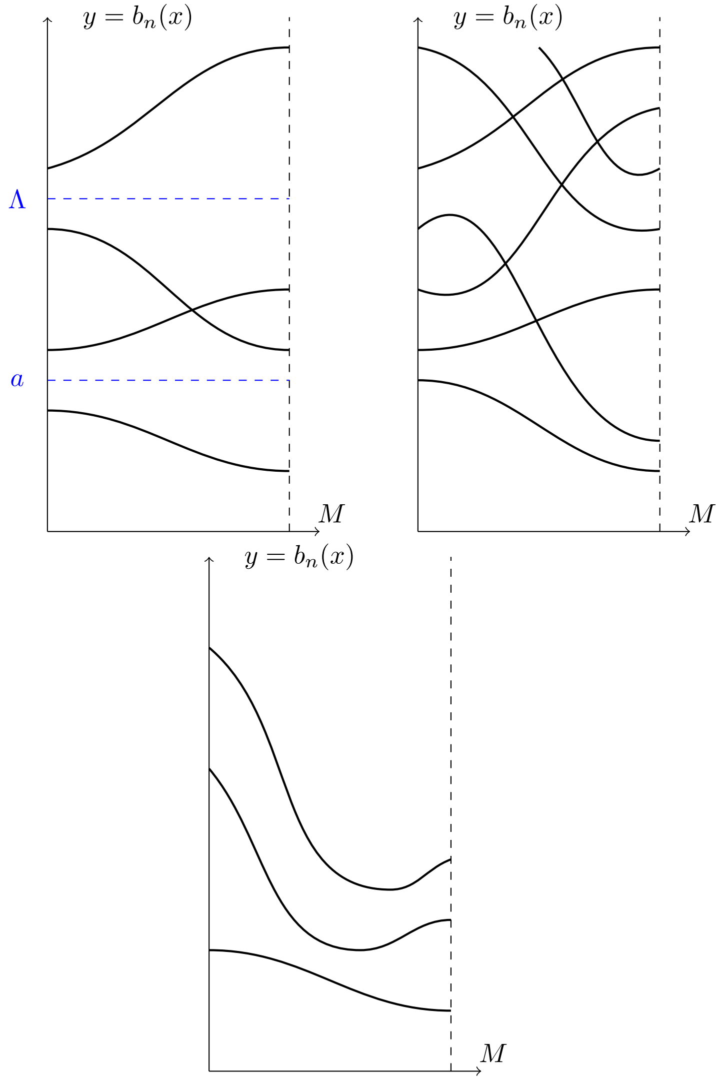

FIGURE 4.1. First drawing : The second and third Landau levels are isolated from the others, but cross each other. We can apply Theorem 2.2 to count the eigenvalues generated by both levels, but not independently. If a magnetic well is close to the crossing point, we cannot use Theorem 3.2 for this well. Second drawing : All the Landau levels are crossing, there is no threshold Λ, we cannot apply Theorem 2.2. Third drawing : The second and third Landau levels have global overlaping, but no local crossing. We can apply Theorem 3.2 but not Theorem 2.2 to isolate the influence of the second level.

A.1. **Agmon-like estimates.** In this section we recall some results on the exponential decay of eigensections of Δ^{L^p}, away from the set $\{x_1, \cdots x_N\}$. We need the following result.

Proposition A.1. — *There exist $p_0 > 0$ and $C_0 > 0$ such that, for $p \geqslant p_0$ and $s \in \mathcal{C}^\infty(M, L)$,*

$$\|\nabla^{L^p} s\|^2 \geqslant p \int_M (b(x) - \frac{C_0}{p})|s(x)|^2 \mathrm{d}x.$$

Remark A.2. — This result was proven by Guillemin-Uribe [8] in the case of full rank B, by Borthwick-Uribe [2] in the constant rank case, and by Ma-Marinescu [18] in a more general setting. A weaker version was also given in [13], with a simpler proof relying on a local approximation of the magnetic field and the metric by constants. This last version would be enough here.

From Proposition A.1 follow Agmon-like decay estimates. The proof given here is taken from [23] and follows the ideas of [1].

Proposition A.3. — *Let $\alpha \in (0, 1/2)$, $\eta > 0$, and $K_\eta = \{b(x) \leqslant b_0 + 2\eta\}$. There exist $C > 0$ and $p_0 > 0$ such that, for all $p \geqslant p_0$ and all eigenpair (λ, ψ) of Δ^{L^p} with $\lambda \leqslant (b_0 + \eta)p$,*

$$\int_M |e^{\mathrm{d}(x, K_\eta) p^\alpha} \psi|^2 \mathrm{d}x \leqslant C\|\psi\|^2.$$

Proof. — Let $\Phi : M \to \mathbf{R}$ be a Lipschitz function. The Agmon formula is:

$$\langle \Delta^{L^p} e^\Phi \psi, e^\Phi \psi\rangle = \lambda \|e^\Phi \psi\|^2 + \|\mathrm{d}\Phi e^\Phi \psi\|^2. \tag{A.1}$$

Using Lemma A.1, we deduce that:

$$\int (pb(x) - C_0 - \lambda - |\mathrm{d}\Phi|^2)|e^\Phi \psi|^2 \mathrm{d}x \leqslant 0.$$

We split this integral into two parts.

$$\int_{K_\eta^c} (pb(x) - C_0 - \lambda - |\mathrm{d}\Phi|^2)|e^\Phi \psi|^2 \mathrm{d}x$$
$$\leqslant \int_{K_\eta} (-pb(x) + C_0 + \lambda + |\mathrm{d}\Phi|^2)|e^\Phi \psi|^2 \mathrm{d}x$$

We choose Φ:

$$\Phi_m(x) = \chi_m(d(x, K_\eta))p^\alpha, \quad \text{for } m > 0,$$

where $\chi_m(t) = t$ for $t < m$, $\chi_m(t) = 0$ for $t > 2m$, and χ_m' uniformly bounded with respect to m. Since $\Phi_m = 0$ on K_η and $pb(x) - C_0 > 0$ for p large enough, we have:

$$\int_{K_\eta^c} (pb(x) - C_0 - \lambda - |\mathrm{d}\Phi_m|^2)|e^{\Phi_m} \psi|^2 \mathrm{d}x \leqslant (b_0 + \eta)p \int_{K_\eta} |\psi|^2 \mathrm{d}x \leqslant Cp\|\psi\|^2.$$

Moreover, since $\lambda \leqslant (b_0 + \eta)p$ and $|\mathrm{d}\Phi_m|^2 \leqslant Cp^{2\alpha}$,

$$\int_{K_\eta^c} (pb(x) - C_0 - (b_0 + \eta)p - Cp^{2\alpha})|e^{\Phi_m} \psi|^2 \mathrm{d}x \leqslant Cp\|\psi\|^2$$

$$p \int_{K_\eta^c} (b(x) - (b_0 + \eta) - C_0 p^{-1} - Cp^{2\alpha - 1})|e^{\Phi_m} \psi|^2 \mathrm{d}x \leqslant Cp\|\psi\|^2,$$

for p large enough. But $b(x) > b_0 + 2\eta$ on K_η^c, so there is a $\delta > 0$ and $p_0 > 0$ such that, for $p \geqslant p_0$:

$$\delta \int_{K^c} |e^{\Phi_m}\psi|^2 \mathrm{d}q \leqslant C\|\psi\|^2.$$

Since $\Phi_m = 0$ on K, we get a new $C > 0$ such that:

$$\|e^{\Phi_m}\psi\|^2 \leqslant C\|\psi\|^2,$$

and we can use Fatou's lemma in the limit $m \to +\infty$ to get the desired inequality. □

Corollary A.4. — *Let $\varepsilon > 0$ and $\chi : M \to [0,1]$ be a smooth cutoff function, being 1 on a small neighborhood of*

$$K_\eta + \varepsilon = \{x; \quad \mathrm{d}(x, K_\eta) < \varepsilon\}.$$

Then, for any eigenpair (λ, ψ) of Δ^{L^p}, with $\lambda \leqslant (b_0 + \eta)p$ we have:

$$\psi = \chi\psi + \mathcal{O}(e^{-\varepsilon p^\alpha})\|\psi\|,$$

and

$$\nabla^{L^p}(\chi\psi) = \nabla^{L^p}\psi + \mathcal{O}(p^{1/2}e^{-\varepsilon p^\alpha})\|\psi\|,$$

uniformly with respect to (λ, ψ).

Proof. — By Theorem A.3, we have:

$$\|(1-\chi)\psi\|^2 \leqslant \int_{(K_\eta+\varepsilon)^c} |\psi|^2 \mathrm{d}q \leqslant \int_M e^{-2\varepsilon p^\alpha} |e^{\mathrm{d}(x,K_\eta)p^\alpha}\psi|^2 \mathrm{d}x \leqslant Ce^{-2\varepsilon p^\alpha}\|\psi\|^2, \tag{A.2}$$

which gives the first estimates. Moreover, we have with $\Phi(x) = \mathrm{d}(x, K_\eta)$,

$$\|e^{\Phi p^\alpha}\nabla^{L^p}\psi\| \leqslant \|\nabla^{L^p}(e^{\Phi p^\alpha}\psi)\| + p^\alpha\|\mathrm{d}\Phi e^{\Phi p^\alpha}\psi\|,$$

and using Agmon's formula A.1 and Theorem A.3:

$$\|\nabla^{L^p}(e^{\Phi p^\alpha}\psi)\|^2 = \lambda\|e^{\Phi p^\alpha}\psi\|^2 + p^{2\alpha}\|\mathrm{d}\Phi e^{\Phi p^\alpha}\psi\|^2 \leqslant C^2 p\|\psi\|^2.$$

Thus,

$$\|e^{\Phi p^\alpha}\nabla^{L^p}\psi\| \leqslant Cp^{1/2}\|\psi\|^2. \tag{A.3}$$

We can use these Agmon estimates on $\nabla^{L^p}\psi$ to get our second result.

$$\|\nabla^{L^p}((1-\chi)\psi)\| \leqslant \|(\nabla^{L^p}\chi)\psi\| + \|(1-\chi)\nabla^{L^p}\psi\| \tag{A.4}$$

The first term is dominated by

$$\|(\nabla^{L^p}\chi)\psi\| \leqslant C\|(1-\overline{\chi})\psi\| \tag{A.5}$$

where $\overline{\chi}$ is a cutoff function such that $\overline{\chi} = 1$ on $K_\eta + \varepsilon$ and $\overline{\chi} = 0$ on $\mathsf{supp}(1-\chi)$. We can apply (A.2) to $\overline{\chi}$ to get:

$$\|(\nabla^{L^p}\chi)\psi \leqslant Ce^{-\varepsilon p^\alpha}\|\psi\|. \tag{A.6}$$

The second term of (A.4) is dominated as in (A.2), using (A.3):

$$\|(1-\chi)\nabla^{L^p}\psi\| \leqslant Cp^{1/2}e^{-\varepsilon p^\alpha}\|\psi\|. \tag{A.7}$$

Finally, (A.4) with (A.6) and (A.7) yields

$$\|\nabla^{L^p}((1-\chi)\psi)\| \leqslant Cp^{1/2}e^{-\varepsilon p^\alpha}\|\psi\|.$$

□

A.2. **Comparison of the spectrum of Δ^{L^p} and $\mathcal{L}_p^{(j)}$.** Here we prove Theorem 3.1. Recall that the minimum b_0 of b is reached at $x_1, \cdots, x_N$ in a non-degenerate way. For $\eta > 0$ small enough, the compact set $K_\eta = \{b(x) \leqslant b_0 + \eta\}$ has N disjoint connected components $K_\eta^{(j)}$ such that $x_j \in K_\eta^{(j)}$. We fix the value of η, and we take U_j a neighborhood of $K_\eta^{(j)}$. For $\varepsilon > 0$ sufficiently small, $K_\eta^{(j)} + 2\varepsilon \subset U_j$.

We denote by B_j the restriction of B to U_j. $\mathcal{L}_p^{(j)}$ is the Dirichlet realisation of $(\mathrm{d} + ipA_j)^*(\mathrm{d} + ipA_j)$, with $A_j \in \Omega^1(U_j, L)$ such that $B_j = \mathrm{d}A_j$. It is the self adjoint operator associated to the quadratic form:

$$Q_j(u,v) = \int_{U_j} (\mathrm{d} + ipA_j)u\overline{(\mathrm{d} + ipA_j)v}\mathrm{d}x, \quad \forall u, v \in \mathsf{H}_0^1(U_j). \tag{A.8}$$

Let us denote by

$$K_p = \min\left[N(\Delta^{L^p}, (b_0+\eta)p); N\left(\bigoplus_{j=1}^N \mathcal{L}_p^{(j)}, (0, b_0+\eta)p\right)\right]. \tag{A.9}$$

We split the proof of Theorem 3.1 into two Lemmas.

Lemma A.5. — *Let $\alpha \in (0, 1/2)$. We have:*

$$\lambda_k(\bigoplus_{j=1}^N \mathcal{L}_p^{(j)}) \leqslant \lambda_k(\Delta^{L^p}) + \mathcal{O}(\exp(-\varepsilon p^\alpha)),$$

uniformly with respect to $k \in [1, K_p]$.

Proof. — We prove this using the min-max principle. For $k \in [1, J_p]$, let ψ_k be the normalized eigenfunction associated to $\lambda_k(\Delta^{L^p})$. We will define the quasimode $u_{j,k} \in \mathcal{C}_0^\infty(U_j)$ using a local trivialisation of L^p on U_j. Let $e_j \in \mathcal{C}^\infty(U_j, L)$ be the non-vanishing local section of L such that, for any $u \in \mathcal{C}^\infty(U_j)$,

$$\nabla^{L^p}(ue_j) = [(\mathrm{d} + ipA_j)u]\, e_j. \tag{A.10}$$

Let $\chi_j \in \mathcal{C}_0^\infty(U_j)$ be a smooth cutoff function, such that $\chi_j = 1$ on $K_\eta^{(j)} + \varepsilon$. We define $u_{j,k} \in \mathcal{C}_0^\infty(U_j)$ by $\chi_j \psi_k = u_{j,k} e_j$, and

$$u_k = u_{1,k} \oplus ... \oplus u_{N,k}.$$

Then

$$\langle \bigoplus_j \mathcal{L}_p^{(j)} u_k, u_k\rangle = \sum_{j=1}^N \langle \mathcal{L}_p^{(j)} u_{j,k}, u_{j,k}\rangle = \sum_{j=1}^N \|(\mathrm{d} + ipA_j)u_{j,k}\|^2.$$

Moreover, by (A.10),

$$\|(\mathrm{d} + ipA_j)u_{j,k}\|^2 = \int_{U_j} |(\mathrm{d} + ipA_j)u_{j,k}|^2 \mathrm{d}x = \int_{U_j} |\nabla^{L^p}(\chi_j\psi_k)|^2 \mathrm{d}x.$$

Now, $\chi = \sum_{j=1}^N \chi_j$ satisfies the assumptions of Corollary A.4 (with 2ε instead of ε). Thus,

$$\langle \bigoplus_j \mathcal{L}_p^{(j)} u_k, u_k\rangle = \int_M |\nabla^{L^p}(\chi\psi_k)|^2 \mathrm{d}x = \|\nabla^{L^p}\psi_k\|^2 + \mathcal{O}(p^{1/2}e^{-2\varepsilon p^\alpha})\|\psi_k\|,$$

uniformly with respect to k. ψ_k being the eigensection associated to $\lambda_k(\Delta^{L^p})$, it remains:

$$\langle \bigoplus_j \mathcal{L}_p^{(j)} u_k, u_k \rangle = \Big(\lambda_k(\Delta^{L^p}) + \mathcal{O}(p1/2 e^{-2\varepsilon p^\alpha}) \Big) \|\psi_k\|.$$

This is true for every $k \in [1, K_p]$. Hence, for $1 \leqslant i \leqslant k \leqslant K_p$ we have

$$\langle \bigoplus_j \mathcal{L}_p^{(j)} u_i, u_i \rangle \leqslant \Big(\lambda_k(\Delta^{L^p}) + \mathcal{O}(p^{1/2} e^{-2\varepsilon p^\alpha}) \Big) \|\psi_k\|,$$

and the Lemma follows from the min-max principle, because the vector space ranged by $(u_i)_{1\leqslant i \leqslant k}$ is k-dimensional (and $p^{1/2}e^{-2\varepsilon p^\alpha} = \mathcal{O}(e^{-\varepsilon p^\alpha})$). □

The reverse inequality is proven similarly.

Lemma A.6. — *Let $\alpha \in (0, 1/2)$. We have:*

$$\lambda_k(\Delta^{L^p}) \leqslant \lambda_k(\bigoplus_{j=1}^N \mathcal{L}_p^{(j)}) + \mathcal{O}(\exp(-\varepsilon p^\alpha)),$$

uniformly with respect to $k \in [1, K_p]$.

Proof. — The k-th eigenvalue of $\bigoplus_{j=1}^N \mathcal{L}_p^{(j)}$ is given by an eigenpair (μ_k, u_k) of $\mathcal{L}_p^{(j_k)}$ for some $j_k \in \{1, \cdots, N\}$. Let $\chi_k \in \mathcal{C}_0^\infty(U_{j_k})$ be a cutoff function equal to 1 on $K_\eta^{(j_k)} + 2\varepsilon$. Then, Agmon estimates (Theorem A.3) for $\mathcal{L}_p^{(j)}$ imply that

$$(\mathrm{d} + ipA)u_k = (\mathrm{d} + ipA)(\chi_k u_k) + \mathcal{O}(e^{-\varepsilon p^\alpha})\|u_k\|$$

uniformly with respect to k. We define $s_k = \chi_k u_k e_{j_k}$, where e_{j_k} satisfies (A.10), and we extend s_k by 0 outside U_{j_k}. Then,

$$\begin{aligned} \langle \Delta^{L^p} s_k, s_k \rangle &= \int_{U_{j_k}} |(\mathrm{d} + ipA)\chi_k u_k|^2 \mathrm{d}x \\ &= \int_{U_{j_k}} |(\mathrm{d} + ipA) u_k|^2 \mathrm{d}x + \mathcal{O}(e^{-\varepsilon p^\alpha}) \\ &= \mu_k \|u_k\|^2 + \mathcal{O}(e^{-\varepsilon p^\alpha}). \end{aligned}$$

Hence the min-max principle implies

$$\lambda_k(\Delta^{L^p}) \leqslant \mu_k + \mathcal{O}(e^{-\varepsilon p^\alpha}),$$

which is the desired inequality. □

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