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TOWARDS FORMAL BAER CRITERIA

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Abstract. Baer's criterion helps to identify the injective objects in a category of modules by reducing the problem of map extension to a certain subclass of morphisms. Due to its notorious reliance on Zorn's lemma, it is inherently non-constructive. However, we put Baer's criterion on constructive grounds by couching it in point-free terms. Classical principles which will be developed alongside readily allow to gain back the conventional version. Several case studies further indicate a fair applicability.

1. Introduction

Categorical techniques have long had a certain reputation as promoting "abstract nonsense" which perhaps reflects an initial resistance to their introduction. If on the one hand this resistance is due to an increased level of abstraction, on the other hand the effective content of basic categorical concepts seems barred. For example, recall that an object C in a category $\mathcal{C}$ is said to be *injective* if, for any monomorphism $m : A \hookrightarrow B$ and morphism $f : A \to C$, we can find $g : B \to C$ (namely an *extension*) such that $f = g \circ m$. This notion involves quantification over the morphisms of the given category $\mathcal{C}$; more often than not, this is not a set, let alone an effectively given one.

Concrete enough instances sometimes give way to elementary, yet equivalent criteria for injectivity. E.g., in the category of abelian groups, the injective objects are the divisible ones [36], and in the category of distributive lattices, Sikorski's theorem can be used to identify the injective objects precisely as the complete Boolean algebras [37, 5, 6]. Such results have to accommodate *real* and *ideal*, so their inherent non-constructivity is hardly surprising. Due to their notorious reliance on Zorn's lemma, presumably the most common incarnation of the axiom of choice in abstract algebra, their computational import is rather difficult to pin down; from a philosophical perspective, a strong ontological commitment is forced upon the algebraist.

Transfinite methods and the ideal objects they bring into being are prevalent in contemporary algebra, but viewpoints shaped by the partial realisation of Hilbert's programme gain momentum [12, 16]. In this paper we attempt to put Baer's criterion, which allows to detect injective objects in categories of modules, on constructive grounds by couching it in point-free terms. The strategy we adopt has been informed by the localic version of the Hahn-Banach theorem, due to which the latter be interpreted as a conservation result of formal theories [9, 10, 25, 11, 33].

Our take on Baer's criterion carries over a related treatment of Sikorski's theorem [29] by way of Scott's entailment relations [35], which considerably owes to [9]. However, we require a generalized concept of entailment relation, as will briefly

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be outlined below; alongside we require new choice principles which reconcile our formal criterion with its customary counterpart.

Discussion proceeds as follows. In Section 2 we briefly review the common form and proof of Baer's criterion. Section 3 summarizes the key tools for our approach. Section 4 contains the main result of this paper (Theorem 4.3), some instances of which will be studied in Section 5.

On method and foundations. The main content of this paper is meant to be elementary and intuitive, but can be formalised in a suitable fragment of Aczel's Constructive Set Theory **CZF** [2, 3] which is based on intuitionistic logic and does not contain the axiom of power set, let alone the axiom of choice (AC). Accordingly, sometimes certain assumptions have to be made explicit which otherwise would be trivial in classical set theory. Recall that a set S is said to be *discrete* if $(\forall a, b \in S)(a = b \vee a \neq b)$. By a *finite* set we understand a set that can be written as $\{ a_1, \ldots, a_n \}$ for some $n \geqslant 0$. The set of all finite subsets of a set S will be denoted by $\mathrm{Fin}(S)$, and the class of *all* subsets of S by $\mathrm{Pow}(S)$. From formal topology [32] we borrow the overlap symbol: $U \between V$ is to say that the sets U and V have an element in common. In order to make precise the connection between our results and their customary counterparts, we have to invoke AC in the guise of Zorn's lemma. We may then as well use classical logic, so we switch in these cases to **ZFC**, signalling this appropriately.

2. Baer's criterion

Let us, for sake of reference and comparison, review the well-known and time-honored argument for Baer's criterion, to be found in many a textbook on the subject matter; we closely follow [36]. Throughout, let R be a (not necessarily commutative) ring with 1, and let M, A, B be (left) R-modules, where A is a submodule of B. We say that M is *injective with respect to ideals of R* if, for every R-homomorphism $\mu : I \to M$ defined on an ideal I of R, there is $\nu : R \to M$ such that $\nu(r) = \mu(r)$ for every $r \in I$.

Lemma 2.1. — *Suppose that M is injective with respect to ideals of R. Let $f : A \to M$ be an R-homomorphism. For every $b \in B$ there is $f' : A + Rb \to M$ extending f.*

Proof. — Consider the ideal $(A : b) = \{ r \in R \mid rb \in A \}$ and define $\mu : (A : b) \to M$ by mapping r to $f(rb)$. This μ is an R-homomorphism. According to the assumption, there is an extension $\nu : R \to M$ of μ. Now put

$$f' : A + Rb \to M, \quad a + rb \mapsto f(a) + \nu(r).$$

This f' is a well-defined R-homomorphism, extending f. □

The following is Baer's criterion.

Theorem 2.2 (**ZFC**). — *The following are equivalent.*

(1) *M is injective.*
(2) *M is injective with respect to ideals of R.*

Proof. — Of course the second item is necessary for the first. To show that it is sufficient, consider a *maximal* extension of the homomorphism in question, the existence of which is guaranteed by Zorn's lemma. That a maximal extension is indeed *total* is a direct consequence of Lemma 2.1. □

Recall that by Baer's criterion every divisible abelian group is injective [36], which in turn was shown **ZFA**-equivalent to AC by Blass [8, Theorem 2.1].[1] The main result of our paper, Theorem 4.3, will lead to a potential replacement of injectivity with respect to ideals by an elementary, yet classically equivalent criterion (Theorem 4.3) which does not involve R-homomorphisms.

3. Geometric entailment relations

3.1. **Sequents.** The strategy pursued in this paper has deeply been influenced by Coquand and Cederquist's treatment of ideal objects by means of Scott's multi-conclusion entailment relations [9], as well as by the methods and philosophy of formal topology [32, 13]. Our approach requires a slight adaptation of [9, 14] which in this section will briefly be outlined. While a first case study has already been carried out in the context of Krull's maximal ideal theorem [34], a thorough and systematic treatment, in particular with regard to inductive generation, will be given elsewhere [38].

Let S be a set. By a *geometric entailment relation* on S we understand a class relation

$$\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Pow}(S)$$

between finite and arbitrary subsets of S. This relation is required to be *reflexive*, *monotone*, and *transitive* in the following sense, respectively:[2]

$$\frac{U \between V}{U \vdash V}\ (\mathrm{R}) \qquad \frac{U \vdash V}{U, U' \vdash V, V'}\ (\mathrm{M}) \qquad \frac{U \vdash V \quad (\forall b \in V)(\,U, b \vdash W\,)}{U \vdash W}\ (\mathrm{T})$$

If $U \in \mathrm{Fin}(S)$ is such that $U \vdash$, i.e., $U \vdash \emptyset$, then it is said to be *inconsistent*. An arbitrary subset V of S is *consistent* if none of its finite subsets is inconsisent. The consistent subsets of S form a directed-complete partially ordered class (a *dcpo* [1]), which is due to the fact that we maintain the restriction on finite antecedents, and which will be important for heuristics later on.

As regards our use of inference lines, following a certain tradition in formal topology, we adopt the convention that the defining conditions "be understood as requirements of validity: if the premises hold, also the conclusion must hold" [13].

The recommended reading of $U \vdash V$ is the same as for conventional, finitary entailment relations [35], i.e., as that of a Gentzen sequent, or rather as

$$\bigwedge_{a \in U} P(a) \to \bigvee_{b \in V} P(b)$$

where P is a distinguished predicate on S. However, the succedent V must now be read as an infinitary disjunction, so what we intend to capture are *geometric sequents* [23], and thus our choice of terminology. By "entailment relation" will henceforth always be meant "geometric entailment relation".

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[1] **ZFA** denotes Zermelo-Fraenkel set theory with atoms.

[2] The usual shorthand notation is at work, i.e., we put a comma to abbreviate set union, and suppress brackets where it should read a singleton set.

An *ideal element* of an entailment relation $\vdash$ on S is a subset α of S which is closed under $\vdash$, i.e., such that if $U \vdash V$ and $U \subseteq \alpha$, then $\alpha \between V$. The class of all ideal elements of $\vdash$, its *spectrum*, will be denoted

$$\mathrm{Spec}(\vdash).$$

It is a consequence of the prime ideal theorem for distributive lattices that finitary entailment relations à la Scott [35] are completely determined by their ideal elements, which is to say that

$$U \vdash V \qquad \text{whenever} \qquad (\forall \alpha \in \mathrm{Spec}(\vdash))(U \subseteq \alpha \implies \alpha \between V), \tag{3.1}$$

which of course is an equivalence. In general, this (3.1) is *not* the case for entailment relations as considered here, due to the well-known fact from point-free topology that locales need not be spatial [22]. In particular, we cannot infer from $\emptyset \nvdash \emptyset$ that $\mathrm{Spec}(\vdash)$ is inhabited.[3] There is, however, a complete and constructive topological semantics available, as will briefly be outlined in the following section.

3.2. **Inductive generation and generic model.** The entailment relations at work in the present paper are generated by sets of initial entailments. We thus spend a few words on inductive generation, to which we have been led by combining the approach pioneered in formal topology [13] with related strategies developed in sequent calculus [26, 27].

By an *axiom set* on S we understand a set $\mathcal{E}$ of pairs (A, B), where $A \in \mathrm{Fin}(S)$ and $B \in \mathrm{Pow}(S)$. An entailment relation $\vdash_{\mathcal{E}}$ is *inductively generated* by $\mathcal{E}$ if it is the smallest among reflexive class relations $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Pow}(S)$ that satisfy *transitivity on axioms*, viz.

$$\frac{(\forall b \in B)\, U, b \vdash V}{U, A \vdash V}$$

where $(A, B) \in \mathcal{E}$, and which in fact renders $\vdash_{\mathcal{E}}$ an entailment relation. A subset α of S is an ideal element of $\vdash_{\mathcal{E}}$ already if it is closed under $\mathcal{E}$, i.e., such that for all $(A, B) \in \mathcal{E}$, if $A \subseteq \alpha$ then $\alpha \between B$.

To every inductively generated entailment relation $\vdash_{\mathcal{E}}$ we can associate a *set-generated frame* F [1] with generating subset G,[4] together with a mapping $f : S \to G$ such that, for all $U \in \mathrm{Fin}(S)$ and $V \in \mathrm{Pow}(S)$,

$$U \vdash_{\mathcal{E}} V \quad \iff \quad \bigwedge_{a \in U} f(a) \leqslant \bigvee_{b \in V} f(b), \tag{3.2}$$

and with the evident universal property analogous to the fundamental result [9] of Cederquist and Coquand for Scott's entailment relations: if F' is a set-generated frame with generating subset G' and $f' : S \to G'$ is an interpretation, i.e., satisfies the corresponding left-to-right implication of (3.2), then there is a uniquely determined frame morphism $h : F \to F'$ such that $h \circ f = f'$. Moreover, both the (set-generated) completely prime filters [1] of F and the frame morphisms $F \to \mathrm{Pow}(1)$

[3] For instance, the locale of surjections $\mathbb{N} \to \mathbb{R}$, which can be generated through a geometric entailment relation, does not have any points, yet is non-trivial [23, C1.2.8].

[4] This is to say that there is a subset G of F such that, for every $x \in F$, $G_x = \{\, a \in G \mid a \leqslant x \,\}$ is a *set* and $x = \bigvee G_x$. The requirement that F be set-generated takes account of size issues in constructive set theory: no non-degenerate class-frame can be proved to have a set of elements in **CZF** [18, 17, 19]. Moreover, to construct this frame, we adopt the Regular Extension Axiom REA on top of Constructive Set Theory **CZF**.

correspond bijectively with the ideal elements of $\vdash_{\mathcal{E}}$. In order to obtain this frame, we put an inductively generated formal topology on top of $\vdash_{\mathcal{E}}$, following closely the strategy employed in the finitary case [14]. The generated frame is a point-free presentation of the space $\mathrm{Spec}(\vdash_{\mathcal{E}})$ of ideal elements equipped with the *finite information topology* [20], which can be investigated through its syntactic underpinning, i.e., by way of the entailment relation at hand.

3.3. **Lying-over.** This section contains important tools insofar as we are interested in regaining the customary counterparts to our results later on. It will be crucial to find direct, non-inductive descriptions at least for empty-conclusion instances, i.e., for the finite inconsistent subsets. While in practice an educated guess often suffices to describe a condition on finite sets which yields inconsistency, more often than not a proof of necessity requires some effort, hinging on the following criteria.

Throughout, let S be a set with geometric entailment relation $\vdash$. Given $\mathcal{H} \subseteq \mathrm{Fin}(S)$ and $W \subseteq S$, we adopt predicate notation and write

$$\mathcal{H}(W) \;\equiv\; \mathrm{Fin}(W) \between \mathcal{H}.$$

We say that $\mathcal{H}$ is *monotone* if $\mathcal{H}(W)$ and whenever $\mathcal{H}(V)$ and $V \subseteq W$.

Definition 3.1. — We say that a monotone $\mathcal{H} \subseteq \mathrm{Fin}(S)$ is *hereditary* if, for every $U, V \in \mathrm{Fin}(S)$ and $W \in \mathrm{Pow}(S)$,

$$\frac{U \vdash V \quad (\forall b \in V)\, \mathcal{H}(W, b)}{\mathcal{H}(W, U)} \tag{3.3}$$

For inductively generated entailment relations it suffices for (3.3) to check axioms in place of $U \vdash V$ in order for $\mathcal{H}$ to be hereditary at large.

Lemma 3.2 (**ZFC**). — *The following are equivalent.*

(1) $\mathrm{Inc} := \{\, U \in \mathrm{Fin}(S) \mid U \vdash \emptyset \,\}$ *is hereditary.*
(2) *Every maximal consistent subset of S is an ideal element.*
(3) *Every consistent subset of S is contained in an ideal element.*

Proof. — $(1 \to 2)$ Suppose that the finite inconsistent subsets form a hereditary family Inc, and let α be maximal with respect to set inclusion among consistent subsets of S. Suppose that $U \vdash V$ and $U \subseteq \alpha$, yet $\alpha \cap V = \emptyset$. Maximality implies $\mathrm{Inc}(\alpha, b)$ for every $b \in V$, and so $\mathrm{Inc}(\alpha, U)$ which is to say that $\mathrm{Inc}(\alpha)$, a contradiction.

$(2 \to 3)$ Suppose that every maximal consistent subset of S is an ideal element. Let W by a consistent subset of S. As the consistent subsets containing W form a dcpo, ZL yields a maximal element which by assumption is an ideal element.

$(3 \to 1)$ Suppose that every consistent subset of S is contained in an ideal element. Let $U \vdash V$ and consider W such that $\mathrm{Inc}(W, b)$ for every $b \in V$. If $\mathrm{Fin}(W, U) \cap \mathrm{Inc} = \emptyset$, then $W \cup U$ is consistent, and so is contained in an ideal element α. It follows that there is $b \in \alpha \cap V$. But then there is a finite subset U of W such that $U, b \vdash$ which however contradicts $\alpha \in \mathrm{Spec}(\vdash)$. □

Definition 3.3. — Let S and S' be sets equipped with entailment relations $\vdash$ and $\vdash'$, respectively. By an *interpretation* we understand a function $\iota : S \to S'$ such that, for all $U \in \mathrm{Fin}(S)$ and $V \in \mathrm{Pow}(S)$,

$$U \vdash V \quad \Longrightarrow \quad \iota(U) \vdash' \iota(V). \tag{3.4}$$

Every such ι induces a mapping of ideal elements

$$\iota^* : \mathrm{Spec}(\vdash') \to \mathrm{Spec}(\vdash), \quad \alpha \mapsto \iota^{-1}(\alpha).$$

We say that ι is *weakly conservative* if, for all $a_1, \ldots, a_k \in S$,

$$\iota(a_1), \ldots, \iota(a_k) \vdash' \quad \Longrightarrow \quad a_1, \ldots, a_k \vdash \tag{3.5}$$

the converse of which is simply the empty conclusion case of interpretation (3.4).

If f denotes a subset inclusion $S \subseteq S'$, then f^* amounts to restriction, which is to say that $f^*(\alpha) = \alpha \cap S$. If moreover $S = S'$, then $\vdash \subseteq \vdash'$ entails $\mathrm{Spec}(\vdash') \subseteq \mathrm{Spec}(\vdash)$. In this manner, several extension theorems can be captured in terms of entailment relations [9, 29]; in fact, we will find Baer's criterion as another instance below. The respective classical counterparts can be regained by means of the following versatile "lying-over" principle.

Proposition 3.4 (**ZFC**). — *Suppose that ι is weakly conservative. If*

$$\{\, U \in \mathrm{Fin}(T) \mid U \vdash' \emptyset \,\}$$

is hereditary, then for every $\alpha \in \mathrm{Spec}(\vdash)$ there is $\beta \in \mathrm{Spec}(\vdash')$ such that $\alpha \subseteq \iota^*(\beta)$.

Proof. — If $\alpha \in \mathrm{Spec}(\vdash)$, then by weak conservation the image $\iota(\alpha)$ is consistent with respect to $\vdash'$. According to Lemma 3.2, there is an ideal element $\beta \in \mathrm{Spec}(\vdash')$ such that $\iota(\alpha) \subseteq \beta$. ☐

Two remarks on Proposition 3.4 are in order before we proceed. Note first that an interesting case arises if $\mathrm{Spec}(\vdash)$ happens to be *flat*, i.e., such that

$$(\forall \alpha, \beta \in \mathrm{Spec}(\vdash))(\alpha \subseteq \beta \implies \alpha = \beta),$$

or, in other words, if every ideal element of $\vdash$ is maximal with respect to set inclusion. The lying-over principle then asserts that the induced mapping ι^* is surjective! Examples include the pull-back of maximal ideals along integral ring extensions [34], as well as the results addressed in the present paper, where flatness is forced by way of totality.

Note further that there is no way around AC proper towards Proposition 3.4. In fact, Proposition 3.4 can be used to derive Krull's maximal ideal theorem [34], and so is in fact **ZF**-equivalent to AC by way of [21].

4. Formal Baer criterion

4.1. **Spectra of morphisms.** With the proper tools in place, we are now ready to embark on the main part of this paper. Let R be a ring with 1. Until further notice, R need not be commutative. Let A and M be (left-)modules over R. We take the set $S = A \times M$ for our domain of discourse. On S we consider the entailment relation $\vdash_{A \to M}$ which is generated by all instances of the following axioms:

$$(a, m), (a, n) \vdash_{A \to M} \qquad \text{(s } [m \neq n])$$

$$(a, m), (b, n) \vdash_{A \to M} (ra + sb, rm + sn) \tag{h}$$

$$\vdash_{A \to M} (0_A, 0_M) \tag{0}$$

$$\vdash_{A \to M} \{\, (a, m) \mid m \in M \,\} \tag{t}$$

with side condition on the axiom of *single values* (s) as indicated in square brackets. Every R-homomorphism, set-theoretically conceived, is an ideal element of this entailment relation; these are precisely the ideal elements if M is discrete (which we do not assume at the outset), in which case

$$\mathrm{Spec}(\vdash_{A\to M}) = \mathrm{Hom}_R(A, M).$$

The intended reading of an entailment of the form

$$(a_1, m_1), \ldots, (a_k, m_k) \vdash_{A\to M} \{ (b_j, n_j) \mid j \in J \}$$

is that for a generic, or yet to be determined R-homomorphism $f : A \to M$,

if $f(a_1) = m_1$ *... and ...* $f(a_k) = m_k$*, then there is* $j \in J$ *such that* $f(b_j) = n_j$*.*

Accordingly, we say that $\vdash_{A\to M}$ is the entailment relation of R-homomorphism $A \to M$. Subscripts indicate which modules we refer to. For instance, with the ring R in place of A, we obtain the entailment relation $\vdash_{R\to M}$ of R-homomorphism $R \to M$. Working with $\vdash_{A\to M}$ amounts to putting focus on partial specifications of module homomorphisms $f : A \to M$. Partiality rarely occurs in classical algebra, where instead it is preferred to ensure totality by restriction to substructures. While classically the property of injectivity involves extendability from sub- to ambient structures, our approach encourages a rephrasing: demand that partial maps have total extensions. Concrete instances, especially within a set-theoretic framework, give access only to a finite amount of data U about a certain morphism f, but $\vdash_{A\to M}$ helps to extract further information. E.g., if $(a, m), (b, n) \in U$, then f should contain $(ra + sb, rm + sn)$ due to (h). However, totality's (t) infinite disjunction presents a severe obstruction.

Along the lines of the frequently encountered proof, we turn to the finitary property of consistency, for which Zorn's lemma brings into being a maximal witness—alas, is the latter an ideal element, i.e., a total function? This crucially hinges on the constructive "one-step" extension principle (Lemma 2.1), often put to use towards a contradiction.[5] Turning the classical argument on its head, we are led to an algebraic description of *in*consistency, as follows.

4.2. **Multiple values.** In order to provide an explicit, non-inductive description for subsets of $A \times M$ which are inconsistent with respect to $\vdash_{A\to M}$, we consider, for finite subsets $U = \{ (a_1, m_1), \ldots, (a_k, m_k) \}$ of $A \times M$, the following predicate:

$$\mathrm{mv}_{A\to M}(U) \;\equiv\; (\exists r_1, \ldots, r_k \in R)\Big(\sum_{i=1}^{k} r_i a_i = 0_A \,\wedge\, \sum_{i=1}^{k} r_i m_i \neq 0_M \Big). \tag{4.1}$$

This predicate is meant to express that U "has multiple values", i.e., the span of U forces 0_A to take a non-zero value in M, whence U cannot be considered to approximate an R-homomorphism $A \to M$.

Lemma 4.1. — *If* $\mathrm{mv}_{A\to M}(U)$*, then* U *is inconsistent, i.e.,* $U \vdash_{A\to M} \emptyset$*.*

Proof. — Disassemble (4.1) along axioms (s), (h), and (0). □

We identify mv with its extension in $\mathrm{Fin}(S)$, i.e.,

$$\mathrm{mv}_{A\to M} = \{ U \in \mathrm{Fin}(S) \mid \mathrm{mv}_{A\to M}(U) \}$$

[5]Examples abound in classical mathematics; for a related discussion cf. [7]. Note further that not even bare sets need to be injective unless excluded middle is assumed [4].

which is certainly monotone (Section 3.3). Furthermore, we extend this predicate in the canonical manner to arbitrary subsets W of S by stipulating

$$\mathrm{mv}_{A\to M}(W) \;\equiv\; \mathrm{Fin}(W) \between \mathrm{mv}_{A\to M}.$$

Again we keep track about the modules we are working with through subscripts. For instance, with R in place of A as above, we write $\mathrm{mv}_{R\to M}$. This predicate stands to reason in the present context: a subset U is inconsistent if its span assigns a non-zero value to 0_A. It is straightforward to show that $\mathrm{mv}_{A\to M}$ is hereditary for the axiom of single-values (s), as well as for the structural axioms (h) and (0). As regards totality (t), let us first do heuristics. We need to show that, for subsets W of $A \times M$ and elements a of A,

$$\frac{(\forall m \in M)\, \mathrm{mv}_{A\to M}(W,(a,m))}{\mathrm{mv}_{A\to M}(W)} \tag{4.2}$$

Intuitively, if W were considered to approximate an R-homomorphism $A \to M$, but cannot coherently be extended by assigning a value in M to a, then W itself must bear witness to violating (s). Consider the classical contrapositive form of (4.2) for a certain subset W of S,

$$\neg\mathrm{mv}_{A\to M}(W) \;\to\; (\exists m \in M)\, \neg\mathrm{mv}_{A\to M}(W,(a,m)). \tag{4.3}$$

Freely employing excluded middle, it suffices to verify the latter in order to show that $\mathrm{mv}_{A\to M}$ is hereditary for (t), to which end we could resort to the one-step extension principle (Lemma 2.1) at work in the classical proof of Baer's criterion: in fact, every consistent subset of $A \times M$ gives rise to a partial R-homomorphism defined on a certain submodule of A; conversely, every partial homomorphism, construed as a subset of $A \times M$ is consistent with respect to $\vdash_{A\to M}$. In **ZFC** we can thus pin down the import of $\mathrm{mv}_{A\to M}$ as follows.

Proposition 4.2 (**ZFC**). — *Let B and M be R-modules. then $\mathrm{mv}_{B\to M}$ is hereditary if and only if any morphism $A \to M$, where A is a submodule of B, extends to a morphism $B \to M$.*

Proof. — On top of the preceding discussion, note that there is an interpretation of entailment relations $\vdash_{A\to M} \to \vdash_{B\to M}$ along which R-homomorphisms $A \to M$ extend according to Proposition 3.4. ☐

In particular, and still in **ZFC**, the module M is injective with respect to ideals—and thus injective at large—if and only if $\mathrm{mv}_{R\to M}$ is hereditary. We are thus led to the following elementary version of Baer's criterion: heredity of $\mathrm{mv}_{R\to M}$ carries over to $\mathrm{mv}_{A\to M}$!

Theorem 4.3. — *Let A and M be R-modules. If $\mathrm{mv}_{R\to M}$ is hereditary, then so is $\mathrm{mv}_{A\to M}$.*

Proof. — We concentrate on showing that $\mathrm{mv}_{A\to M}$ is hereditary with respect to totality (t) if so is $\mathrm{mv}_{R\to M}$; the remaining axioms are straightforward to check. Accordingly, let W be an arbitrary subset of $A \times M$, let $a \in A$, and suppose that $\mathrm{mv}_{A\to M}(W,(a,m))$ for every $m \in M$. Let W_R consist of all pairs $(r,m) \in R \times M$ such that there are $r_1,\dots,r_k \in R$ and $(a_1,m_1),\dots,(a_k,m_k) \in W$ with

$$\sum_{i=1}^{k} r_i a_i = ra \qquad \text{and} \qquad \sum_{i=1}^{k} r_i m_i = m.$$

We claim that $\mathrm{mv}_{R\to M}(W_R,(1,m))$ for every $m\in M$. For if m is an arbitrary element of M, then, because of $\mathrm{mv}_{A\to M}(W,(a,m))$, there are $r_0,r_1,\dots,r_k\in R$ and $(a_1,m_1),\dots,(a_k,m_k)\in W$ such that

$$r_0a+\sum_{i=1}^k r_ia_i=0 \qquad \text{and} \qquad r_0m+\sum_{i=1}^k r_im_i\neq 0.$$

The former implies $(-r_0,\sum_{i=1}^k r_im_i)\in W_R$ by definition of W_R. It is immediate that

$$\mathrm{mv}_{R\to M}((-r_0,\sum_{i=1}^k r_im_i),(1,m)),$$

and thus $\mathrm{mv}_{R\to M}(W_R,(1,m))$ holds indeed. As $\mathrm{mv}_{R\to M}$ is hereditary, it follows that $\mathrm{mv}_{R\to M}(W_R)$. Therefore, there are $(r_1,m_1),\dots,(r_\ell,m_\ell)\in W_R$ and $s_1,\dots,s_\ell\in R$ such that

$$\sum_{j=1}^\ell s_jr_j=0 \qquad \text{and} \qquad \sum_{j=1}^\ell s_jm_j\neq 0.$$

Moreover, for every (r_j,m_j) there are $r_1^j,\dots,r_{k_j}^j\in R$ and $(a_1^j,m_1^j),\dots,(a_{k_j}^j,m_{k_j}^j)\in W$ such that

$$\sum_{i=1}^{k_j} r_i^ja_i^j=r_ja \qquad \text{and} \qquad \sum_{i=1}^{k_j} r_i^jm_i^j=m_j.$$

It remains to put this information together and compute, on the one hand,

$$\sum_{j=1}^\ell\sum_{i=1}^{k_j} s_jr_i^ja_i^j=\sum_{j=1}^\ell s_j(\sum_{i=1}^{k_j} r_i^ja_i^j)=\sum_{j=1}^\ell s_jr_ja=0,$$

and on the other hand

$$\sum_{j=1}^\ell\sum_{i=1}^{k_j} s_jr_i^jm_i^j=\sum_{j=1}^\ell s_j(\sum_{i=1}^{k_j} r_i^jm_i^j)=\sum_{j=1}^\ell s_jm_j\neq 0,$$

which together witness $\mathrm{mv}_{A\to M}(W)$. □

As a consequence of Theorem 4.3, we can further give a direct, non-inductive description of the canonical finitary subrelation of $\vdash_{A\to M}$ under the assumption that $\mathrm{mv}_{R\to M}$ is hereditary.

Corollary 4.4. — *Let A be an R-module and suppose that $\mathrm{mv}_{R\to M}$ is hereditary. The following are equivalent.*

1. $(a_1,m_1),\dots,(a_k,m_k)\vdash_{A\to M}(b_1,n_1),\dots,(b_\ell,n_\ell)$
2. *For any choice of $n_1'\neq n_1,\dots,n_\ell'\neq n_\ell$, there are $r_1,\dots,r_k,s_1,\dots,s_\ell\in R$ such that*

$$\sum_{i=1}^k r_ia_i+\sum_{j=1}^\ell s_jb_j=0_A \quad \textit{and} \quad \sum_{i=1}^k r_im_i+\sum_{j=1}^\ell s_jn_j'\neq 0_M.$$

Proof. — Notice that

$$U \vdash_{A\to M} V, (b,n) \quad \text{if and only if} \quad (\forall n' \neq n)(U, (b,n') \vdash_{A\to M} V). \tag{4.4}$$

which is easy to see through (s), (t), and cut (T), cf. [29]. The description of finitary entailments thus reduces to empty-conclusion instances, for which Lemma 4.1 and Theorem 4.3 combine. □

It is in order to recap. Adopting heredity as syntactical substitute for injectivity allows to phrase and prove a formal version of Baer's criterion. Heredity has been employed twice to this end: first to fix semantics, i.e., to deduce that, classically, maximal consistent subsets are ideal elements (Lemma 3.2); second, to prove that (t) is not relevant when characterizing inconsistent subsets (Theorem 4.3). Incidentally, the latter can be encoded as a *simultaneous collapse* [15]: totality (t) is in fact redundant when restricting attention to empty-conclusion sequents.

Proposition 4.5. — *Let A be an R-module and suppose that $\vdash_{R\to M}$ is hereditary. Consider the entailment relation $\vdash'_{A\to M}$ which is obtained by discarding (t). For every $U \in \mathrm{Fin}(A \times M)$,*

$$\underbrace{U \vdash'_{A\to M} \emptyset}_{\mathrm{mv}_{A\to M}(U)} \quad \iff \quad U \vdash_{A\to M} \emptyset,$$

Proof. — On the one hand, $\vdash'_{A\to M}$ certainly interprets in $\vdash_{A\to M}$, being a subrelation of the latter. Conversely, the proof of Theorem 4.3 shows how to resolve dependency on (t). □

Remark 4.6 (**ZFC**). — Suppose that $\mathrm{mv}_{R\to M}$ is hereditary. In this case the finitary instances of $\vdash_{A\to M}$ as considered in Corollary 4.4 are completely determined by ideal elements (3.1). In view of (4.4), it suffices to check this for $V = \emptyset$, to which end we may concentrate on the contrapositive: but if $U \nvdash$, then, with ZL and Lemma 3.2, there is indeed $\alpha \in \mathrm{Spec}(\vdash_{A\to M})$ containing U.

5. Examples

5.1. **Torsion-free modules.** It is well-known that every injective module is divisible [36]; we shall now revisit one of the partial converses, following [36], but replace injectivity with heredity. Let R be a commutative ring with 1. Recall that R is *integral* if every element a of R is null or *regular* [24], the latter of which is to say that $\{ r \in R \mid ra = 0 \} = 0$. An R-module M is *divisible* if $aM = M$ for every regular element a of R. It is *torsion-free* if, for each regular $a \in R$ and each $m \in M$, if $am = 0$, then $m = 0$.

The following is an elementary counterpart of [36, Proposition 2.7].

Proposition 5.1. — *Let R be integral, and let M be a torsion-free divisible R-module. Then $\mathrm{mv}_{R\to M}$ is hereditary.*

Proof. — Consider $W \subseteq R\times M$, let $a \in R$, and suppose that $\mathrm{mv}_{R\to M}(W, (a,m))$ for every $m \in M$. Start with $m = 0$ which by assumption gives $r_0, r_1, \ldots, r_k \in R$ and $(a_1, m_1), \ldots, (a_k, m_k) \in W$ such that

$$r_0 a + \sum_{i=1}^{k} r_i a_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{k} r_i m_i \neq 0.$$

If $r_0 = 0$, nothing need be done. If r_0 is regular, consider

$$m = -\sum_{i=1}^{k} r_i m_i.$$

Since M is divisible, there is $n \in M$ such that

$$m = r_0 n.$$

Since $\mathrm{mv}_{R\to M}(W,(a,n))$ there are $s_0, s_1, \dots, s_\ell \in R$ and $(b_1,n_1),\dots,(b_\ell,n_\ell) \in W$ such that

$$s_0 a + \sum_{j=1}^{\ell} s_j b_j = 0 \qquad \text{and} \qquad s_0 n + \sum_{j=1}^{k} s_j n_j \neq 0.$$

Putting together the available data, we obtain on the one hand

$$\sum_{j=1}^{\ell} r_0 s_j b_j - \sum_{i=1}^{k} s_0 r_i a_i = r_0 \Big(\sum_{j=1}^{\ell} s_j b_j + s_0 a \Big) = 0.$$

On the other hand, since r_0 is regular and M is torsion-free,

$$\sum_{j=1}^{\ell} r_0 s_j n_j - \sum_{i=1}^{k} s_0 r_i m_i = \sum_{j=1}^{\ell} r_0 s_j n_j + s_0 m = r_0 \Big(\sum_{j=1}^{\ell} s_j n_j + s_0 n\Big) \neq 0. \qquad \square$$

5.2. **Linear forms and quotient spaces.** Through Corollary 4.4 and Proposition 5.1 we next aim at a constructive interpretation of the statement that every vector space embeds in its double dual (Corollary 5.5 below). To this end, let K be a discrete field, i.e., a ring such that every element is null or invertible. Let M be a K-vector space. Consider the entailment relation $\vdash_{M\to K}$ of linear form. In other words, this entailment relation captures the dual space of M,

$$\mathrm{Spec}(\vdash_{M\to K}) = M^*.$$

In view of our previous results, we have a direct, non-inductive description as follows.

Proposition 5.2. — *The following are equivalent.*

(1) $(m_1, a_1), \dots, (m_k, a_k) \vdash_{M\to K} (n_1, b_1), \dots, (n_\ell, b_\ell)$
(2) *For any choice of $b'_1 \neq b_1, \dots, b'_\ell \neq b_\ell$, there are $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_\ell$ such that*

$$\sum_{i=1}^{k} \lambda_i m_i + \sum_{j=1}^{\ell} \mu_j n_j = 0 \quad \textit{and} \quad \sum_{i=1}^{k} \lambda_i a_i + \sum_{j=1}^{\ell} \mu_j b'_j = 1.$$

Proof. — Instantiate Corollary 4.4 and Proposition 5.1. $\square$

Proposition 5.2 can be used to reduce other and more intricate entailment relations by way of suitable weakly conservative interpretations. To give an example, let V be a set of vectors of M. Let $N = \mathrm{span}(V)$ be the linear hull of V in M.

We consider the entailment relation $\vdash_{M\to K,N=0}$ on $M \times K$ which is inductively generated by all instances of the following axioms:

$$(m,a),(m,b) \vdash_{M\to K,N=0} \qquad \text{(s } [a \neq b])$$
$$(m,a),(n,b) \vdash_{M\to K,N=0} (\lambda m + \mu n, \lambda a + \mu b) \qquad \text{(h)}$$
$$\vdash_{M\to K,N=0} \{\, (m,a) \mid a \in K \,\} \qquad \text{(t)}$$
$$\vdash_{M\to K,N=0} (n,0) \qquad (n\ [n \in V])$$

with side conditions indicated in square brackets. The ideal elements of this entailment relation precisely are the linear forms which vanish on N. Considering initial entailments, it is clear that the canonical mapping $[\cdot] : M \to M/N$ induces an interpretation of entailment relations:

$$\iota_{[\cdot]} : (M \times K, \vdash_{M\to K,N=0}) \to (M/N \times K, \vdash_{M/N\to K}), \quad (m,a) \mapsto ([m],a).$$

Proposition 5.3. — *$\iota_{[\cdot]}$ is weakly conservative.*

Proof. — Let $(m_1,a_1),\dots,(m_k,a_k) \in M \times K$ and suppose that

$$([m_1],a_1),\dots,([m_k],a_k) \vdash_{M/N\to K} .$$

In other words, there are $\lambda_1,\dots,\lambda_k \in K$ for which

$$\sum_{i=1}^{k} \lambda_i m_i \in N \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i a_i = 1.$$

Hence we can find $n_1,\dots,n_\ell \in N$ along with $\mu_1,\dots,\mu_\ell \in K$ such that

$$\sum_{i=1}^{k} \lambda_i m_i = \sum_{j=1}^{\ell} \mu_j n_j$$

which implies

$$(m_1,a_1),\dots,(m_k,a_k),(n_1,0),\dots,(n_\ell,0) \vdash_{M\to K}$$

and so

$$(m_1,a_1),\dots,(m_k,a_k) \vdash_{M\to K,N=0}$$

by successive cut with corresponding instances of the additional axiom (n). ☐

Proposition 5.4. — *For every $m \in M$, the following are equivalent.*

(1) $m \in N$.
(2) $\vdash_{M\to K,N=0} (m,0)$.

Proof. — The second item is an axiom in case $m \in N$. Conversely, if $\vdash_{M\to K,N=0} (m,0)$, then $\vdash_{M/N\to K} ([m],0)$ by interpretation. It follows that $([m],1) \vdash_{M/N\to K}$ which with Proposition 5.2 implies $[m] = 0$, i.e., $m \in N$. ☐

From a semantical point of view, Proposition 5.4 has the following interpretation: in order to show that some $m \in M$ is covered by a certain set V of vectors of M, it suffices to show that $\alpha(m) = 0$ for every linear form $\alpha : M \to K$ that vanishes on V. This is related to a well-known principle in functional analysis [31, 3.5 Remark].

Corollary 5.5. — *For every $m \in M$, the following are equivalent.*

(1) $m = 0$.
(2) $\vdash_{M\to K} (m,0)$

Proof. — Instantiate Proposition 5.4 with $N = 0$. □

Keeping in mind Remark 4.6, this corollary is a constructive counterpart of the classical fact that the only element of M which maps to 0 under *every* linear form $\alpha : M \to K$ is null, i.e.,

$$\bigcap_{\alpha \in M^*} \ker \alpha = 0.$$

Recall that this amounts to say that the canonical mapping

$$\iota : M \to M^{**}, \quad m \mapsto (\, \mathrm{ev}_m : \alpha \mapsto \alpha(m) \,)$$

embeds M in its double dual.

6. Conclusion

The present paper has its origins in our endeavour to put certain extension theorems on constructive grounds [29]. For the methods developed in [30, 29] to carry over to the present setting, it was necessary first to develop a generalized concept of entailment relation, dropping the restriction on finite sets of succedents [38]. An outline of this has been given in Section 3.

Guided by Baer's criterion, which allows to identify injective modules by reducing the problem to the extension of maps defined on ideals of the underlying ring, we have been able to trace back injectivity to a comparatively elementary, computational property in Section 4. From a classical point of view, a suitable variant of AC immediately gives back the conventional result.

Applications seem to indicate that a further pursuit might bear fruits. For instance, where homology abounds with injective resolutions, the development of constructive methods may profit from formal Baer criteria. In linear algebra, suitable axioms allow to treat (symmetric, skew-symmetric, alternating) bilinear forms by means of entailment relations. Our approach to injectivity further raises the question as to how algebraically compact modules [28] might find a treatment in terms of entailment relations. However, it remains to be seen whether *concrete* computational use can be made of our formal Baer criterion. Last but not least, a reapproach through *dynamical methods* [15] is likely to shed further light.

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[6] The opinions expressed in this paper are those of the authors.

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