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ITERATED BROWNIAN MOTION AD LIBITUM IS NOT THE PSEUDO-ARC

JÉRÔME CASSE AND NICOLAS CURIEN

Abstract. The construction of a random continuum $\mathcal{C}$ from independent two-sided Brownian motions as considered in [11] almost surely yields a non-degenerate indecomposable continuum. We show that $\mathcal{C}$ is not-hereditarily indecomposable and, in particular, it is (unfortunately) not the pseudo-arc.

1. Introduction

Iterated Brownian motions ad libitum. Let $(\mathfrak{B}_i)_{i\geqslant 1}$ be a sequence of i.i.d. two-sided Brownian motions (BM), i.e. $(\mathfrak{B}_i(t))_{t\geqslant 0}$ and $(\mathfrak{B}_i(-t))_{t\geqslant 0})$ are independent standard linear Brownian motions started from 0. The nth iterated BM is

$$I^{(n)} = \mathfrak{B}_1 \circ \cdots \circ \mathfrak{B}_n. \tag{1.1}$$

The doubly iterated Brownian motion $I^{(2)}$ has been deeply studied in the 90's. It permits to construct solutions to partial differential equations [9] and lots of results about its probabilistic and analytic properties can be found in [1, 4, 5, 8, 10, 17, 18] and references therein. Of course $I^{(n)}$ is wilder and wilder as n increases (see Figure 1.1) but in [7], second author and Konstantopoulos proved that *the occupation measure* of $I^{(n)}$ over $[0, 1]$ converges as $n \to \infty$ towards a random probability measure Ξ which can be though of as iterated Brownian motions ad libitum. This object has then been studied in [6] by the first author and Marckert, and they gave a description of Ξ using invariant measure of an iterated functions system (IFS). However, many distributional properties of Ξ remain open.

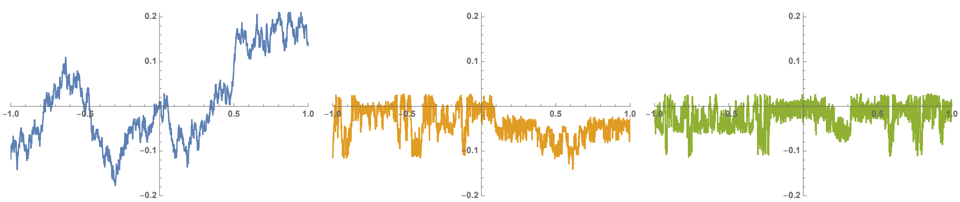

Figure 1.1. Simulations of $I^{(1)}, I^{(2)}$ and $I^{(3)}$, the first three iterations of independent two-sided Brownian motions. The article studies the random continuum build out the sequence of $(I^{(n)} : n \geqslant 1)$.

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Continuum and pseudo-arc. In a recent work, Kiss and Solecki used iterated Brownian motions to define a random *continuum*. Recall that a *continuum* is a nonempty, compact, connected metric space. They were interested by the so-called *pseudo-arc*. The *pseudo-arc* is a homogeneous continuum which is similar to an arc, so similar, that its existence was unclear in the beginning of the last century. A continuum C is

- *reduced to a singleton*, if the cardinality of C is one. It is *non reduced to a singleton* if it contains at least two elements.
- *chainable* (also called *arc-like*, see [16, Theorem 12.11]), if for each $\varepsilon > 0$, there exists a continuous function $f : C \to [0,1]$ such that the pre-images of points under f have diameter less than ε.
- *decomposable*, if there exist A and B two subcontinua of C such that $A, B \neq C$ and $C = A \cup B$. A non decomposable continuum is called *indecomposable*.
- *hereditarily indecomposable* if any of its subcontinuum is indecomposable.

By [3], the *pseudo-arc* is the unique (up to homeomorphisms) chainable and hereditarily indecomposable continuum non reduced to a singleton. In particular, any subcontinuum (non reduced to a singleton) of a pseudo-arc is a pseudo-arc. Its name "pseudo-arc" comes from this property because arcs have the same property, in the sense that any subcontinuum (non reduced to a singleton) of an arc is an arc. For more information on pseudo-arc, we refer the interested reader to the second paragraph of [16, Chapter XII] and to [2, 3, 12, 14]. Sadly, it is very complicated to get a "drawing" of the pseudo-arc due to its complicated crocked structure, see [16, Exercise 1.23] and [13]. Following the works of Bing, one can wonder whether the pseudo-arc is typical among arc-like continua and ask whether there is a natural probabilistic construction of the pseudo-arc.

Let us recall the construction of continua from inverse limits used in [11], see [16, Section II.2] for details. Suppose we are given a sequence

$$\cdots \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1$$

where for any $i \geqslant 1$, the metric space (X_i, d_i) is compact and $f_i : X_{i+1} \to X_i$ is a continuous surjective function. Then the *inverse limit* of $(\{X_i, f_i\})_{i \geqslant 1}$ is the subspace of $\prod_{i \geqslant 1} X_i$ defined by

$$\varprojlim(f_i, X_i : i \geqslant 1) = \left\{ (x_i)_{i \geqslant 1} \in \prod_{i \geqslant 1} X_i : f_i(x_{i+1}) = x_i \right\}. \tag{1.2}$$

In the application below X_i are compact intervals of $\mathbb{R}$ and in this case, by [16, Theorems 2.4 and 12.19], the inverse limit is a chainable continuum. In [11], Kiss and Solecki constructed a system as above using two-sided independent Brownian motions $(\mathfrak{B}_i : i \geqslant 1)$. More precisely, they proved that for any interval J of $\mathbb{R}$ with $0 \in J$ and $J \neq \{0\}$, the following limit exists almost surely

$$\mathcal{I}_i = \lim_{m \to \infty} \mathfrak{B}_i\left(\mathfrak{B}_{i+1}\left(\dots\left(\mathfrak{B}_{i+m}\left(J\right)\right)\dots\right)\right), \tag{1.3}$$

and does not depend on J, so that we can consider the random chainable continuum $\mathcal{C}$ obtained as the inverse limit of the system

$$\cdots \xrightarrow{\mathfrak{B}_3} \mathcal{I}_3 \xrightarrow{\mathfrak{B}_2} \mathcal{I}_2 \xrightarrow{\mathfrak{B}_1} \mathcal{I}_1.$$

Kiss and Solecki proved [11, Theorem 1] that the random chainable continuum $\mathcal{C}$ is almost surely non-degenerate and indecomposable. This note answers negatively the obvious question the preceding result triggers:

Theorem 1.1. — *Almost surely, the random continuum $\mathcal{C}$ is* not *hereditarily indecomposable (hence is not the pseudo-arc).*

The proof below could be adapted to prove that a random continuum constructed similarly from a sequence of i.i.d. *reflected* Brownian motions is neither a pseudo-arc, answering a question in [11, Section 4.3.1]. Although almost surely not homeomorphic to the pseudo-arc, the random continuum $\mathcal{C}$ is interesting in itself and one could ask about its topological property, e.g. we wonder whether the topology of $\mathcal{C}$ is almost surely constant and if it is easy to characterise.

2. Finding good intervals

In the rest of the article the Brownian motions $\mathfrak{B}_i$ are fixed and we recall the definition of $\mathcal{I}_i$ in (1.3) and of the continuum $\mathcal{C}$. We will show that Theorem 1.1 follows from the proposition below stated in terms of images of intervals under the flow of independent Brownian motions whose proof occupy the remaining of the article:

Proposition 2.1. — *For any $\varepsilon > 0$ small enough, with probability at least*

$$p_\varepsilon = \prod_{i=1}^{\infty} 1 - 2\left(\varepsilon^{(5/4)^{i-1}}\right)^{1/8} > 0,$$

there exist two sequences $(U_i)_{i\geqslant 1}$ and $(V_i)_{i\geqslant 1}$ of subintervals of $\mathbb{R}$ such that, for any $i \geqslant 1$, the five following conditions are satisfied

1. $U_i, V_i \subset \mathcal{I}_i$ *where $\mathcal{I}_i$ is defined in* (1.3),
2. $U_i \not\subseteq V_i$ *and* $V_i \not\subseteq U_i$,
3. $U_i \cap V_i \neq \emptyset$,
4. $U_i = \mathfrak{B}_i(U_{i+1})$ *and* $V_i = \mathfrak{B}_i(V_{i+1})$,
5. $|U_i|, |V_i| \leqslant \varepsilon^{(5/4)^{i-1}}$.

Proof of Theorem 1.1 given Proposition 2.1. In the proof, since we are always working with the functions $\mathfrak{B}_i$ we write $\varprojlim(W_i : i \geqslant 1)$ for the inverse limit previously denoted by $\varprojlim(\mathfrak{B}_i, W_i : i \geqslant 1)$ for any sequence of intervals $W_1, W_2, \dots$ such that $W_{i+1} \xrightarrow{\mathfrak{B}_i} W_i$. On the event described in the above proposition we have with probability at least $p_\varepsilon > 0$:

- For any $i \geqslant 1$, $\mathfrak{B}_i(U_{i+1} \cup V_{i+1}) = U_i \cup V_i$ (point 4) and $U_i \cup V_i \subset \mathcal{I}_i$ (point 1) and $U_i \cup V_i$ is an interval (point 3), so by Lemma 2.6 of [16], $\varprojlim(U_i \cup V_i : i \geqslant 1)$ is a subcontinuum of $\mathcal{C}$.
- By Lemma 2.6 of [16], both $\varprojlim(U_i : i \geqslant 1)$ and $\varprojlim(V_i : i \geqslant 1)$ are also subcontinua of $\varprojlim(U_i \cup V_i : i \geqslant 1)$.
- Let $x = (x_i)_{i\geqslant 1} \in \varprojlim(U_i \cup V_i : i \geqslant 1)$, then
  - either, for any i, we have $x_i \in U_i \cap V_i$, and so $x \in \varprojlim(U_i : i \geqslant 1)$ and $x \in \varprojlim(V_i : i \geqslant 1)$,
  - or there exists $j \geqslant 1$ such that $x_j \in U_j$ and $x_j \notin V_j$, but then by point 4 we have $x_i \in U_i$ for all $i \geqslant j$ and so $x \in \varprojlim(U_i : i \geqslant 1)$,

   – or there exists $j \geqslant 1$ such that $x_j \notin U_j$ and $x_j \in V_j$ and similarly we deduce that $x \in \varprojlim(V_i : i \geqslant 1)$.

  Hence, $\varprojlim(U_i \cup V_i : i \geqslant 1) \subset \varprojlim(U_i : i \geqslant 1) \cup \varprojlim(V_i : i \geqslant 1)$ and the reverse inclusion is obvious.

- $\varprojlim(U_i \cup V_i : i \geqslant 1) \neq \varprojlim(U_i : i \geqslant 1)$ nor $\varprojlim(U_i \cup V_i : i \geqslant 1) \neq \varprojlim(V_i : i \geqslant 1)$ by combining point 2 and point 4.

All of these points imply that $\varprojlim(U_i \cup V_i : i \geqslant 1)$ is a decomposable subcontinuum of $\mathcal{C} = \varprojlim(\mathcal{I}_i : i \geqslant 1)$. That implies that $\mathcal{C}$ is not a pseudo-arc with probability at least p_ε for any $\varepsilon > 0$. As $p_\varepsilon \to 1$ when $\varepsilon \to 0$, it is not a pseudo-arc with probability one. □

2.1. **Construction of a decomposable subcontinuum using good shape excursions.** Let us now explain the idea behind the construction of the intervals of Proposition 2.1. This relies on the concept of excursions with a good shape. Imagine that we have a sequence of non trivial intervals $[u_i, v_i] \subset [0,1]$ such that $\mathfrak{B}_i([u_{i+1}, v_{i+1}]) = [u_i, v_i]$ and furthermore that $\mathfrak{B}_i(u_{i+1}) = u_i$ and $\mathfrak{B}_i(v_{i+1}) = v_i$ and $\mathfrak{B}_i(t) \in (u_i, v_i)$ for $t \in (u_{i+1}, v_{i+1})$. In words, over the time interval $[u_{i+1}, v_{i+1}]$, the Brownian motion $\mathfrak{B}_i$ makes an excursion from u_i to v_i. We say that this excursion has a *good shape* if it stays in the pentomino of Figure 2.1.

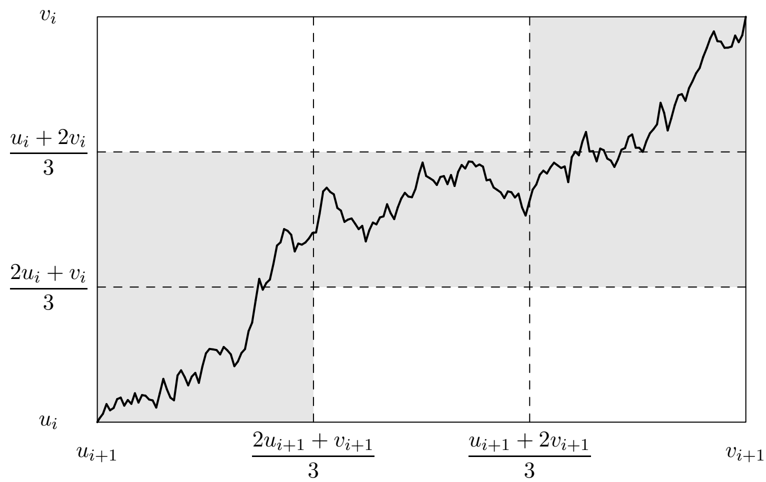

Figure 2.1. An excursion from u_i to v_i over the time interval $[u_{i+1}, v_{i+1}]$ has a good shape if it stays in the light grey region.

If we have such a sequence of intervals and excursions, then one can define a sequence of intervals U_i, V_i by setting for any $i \geqslant 1$,

$$U_i = \lim_{n\to\infty} \underbrace{(\mathfrak{B}_i \circ \mathfrak{B}_{i+1} \circ \cdots \circ \mathfrak{B}_{i+n-1}) \left(\left[u_{i+n}, \frac{u_{i+n} + 2v_{i+n}}{3} \right] \right)}_{U_{i,n}} \text{ and}$$

$$V_i = \lim_{n\to\infty} (\mathfrak{B}_i \circ \mathfrak{B}_{i+1} \circ \cdots \circ \mathfrak{B}_{i+n-1}) \left(\left[\frac{2u_{i+n} + v_{i+n}}{3}, v_{i+n} \right] \right).$$

First, these two limits exist a.s. and are closed intervals a.s. because they are limits of a sequence of decreasing closed intervals. Indeed, because $\mathfrak{B}_{i+n}$ performs a good shape excursion from u_{i+n} to v_{i+n} over $[u_{i+n+1}, v_{i+n+1}]$ we have

$$\mathfrak{B}_{i+n}\left(\left[u_{i+n+1}, \frac{u_{i+n+1}+2v_{i+n+1}}{3}\right]\right) \subset \left[u_{i+n}, \frac{u_{i+n}+2v_{i+n}}{3}\right], \text{ and so}$$
$$U_{i,n+1} \subset U_{i,n},$$

and $U_{i,n}$ are intervals because the BM is continuous a.s. It is then an easy matter to check that the interval constructed above satisfies points 2-4 of Proposition 2.1. Our task is thus to construct the sequence u_i, v_i so that $\mathfrak{B}_i$ performs a good shape excursion from u_i to v_i over $[u_{i+1}, v_{i+1}]$ and to ensure points 1 and 5 of Proposition 2.1. The key idea is to look for these intervals in the vicinity of 0 because any given small interval close to 0 has MANY pre-images close to 0 by a Brownian motion. These many pre-images enable us to select one with a good shape.

2.2. **Pre-images of a small interval by a Brownian motion.** In the following lemma the dependence in i is superfluous but we keep it to make the connection with the preceding discussion easier to understand.

Lemma 2.2. — *Let a_i be any real positive number small enough. Fix $[u_i, v_i] \subset [0, a_i]$. Then with probability at least*

$$1 - 2a_i^{1/8}$$

we can find $[u_{i+1}, v_{i+1}] \subset [0, a_i^{5/4}]$ so that $\mathfrak{B}_i$ performs an excursion with a good shape from u_i to v_i over the time interval $[u_{i+1}, v_{i+1}]$.

Proof. — Fix $0 < u_i < v_i$ and consider the successive excursions $\mathcal{E}_1, \mathcal{E}_2, \ldots$ that the Brownian motion $\mathfrak{B}_i$ performs from u_i to v_i over the respective time intervals $[u_{i+1}^{(1)}, v_{i+1}^{(1)}], [u_{i+1}^{(2)}, v_{i+1}^{(2)}], \cdots$. By the Markov property of Brownian motion and standard argument in excursion theory, these excursions are i.i.d. We claim that

$$r = \mathbb{P}(\mathcal{E} \text{ has a good shape}) > 0.$$

Indeed, since the law of Brownian motion has full support in the space of continuous functions (with the topology of uniform convergence over all compacts of $\mathbb{R}_+$), the first excursion from u_i to v_i might be close to any prescribed continuous function and in particular, the probability to have a good shape is strictly positive. See Figure 2.2.

Hence, the probability that at least one of the k first excursions has a good shape is at least

$$1 - (1-r)^k.$$

To control the number of excursions from u_i to v_i performed up to time a_i by $\mathfrak{B}_i$, we introduce the auxiliary stopping times defined by $w_{i+1}^{(1)} = \inf\{t \geqslant 0 : \mathfrak{B}_i(t) = u_i\}$ and for $k \geqslant 2$

$$w_{i+1}^{(k)} = \inf\{t \geqslant v_{i+1}^{(k-1)} : \mathfrak{B}_i(t) = u_i\}.$$

Hence $w_{i+1}^{(1)} < v_{i+1}^{(1)} < w_{i+1}^{(2)} < v_{i+1}^{(2)} < \cdots$ are the successive hitting times of u_i, v_i, u_i, v_i by $\mathfrak{B}_i$, see Figure 2.3. For $a \geqslant 0$, we let $\mathcal{T}_a = \inf\{t \geqslant 0 : \mathfrak{B}_i(t) = a\}$ the hitting time of a by a standard linear Brownian motion. It is classic (see e.g. [15,

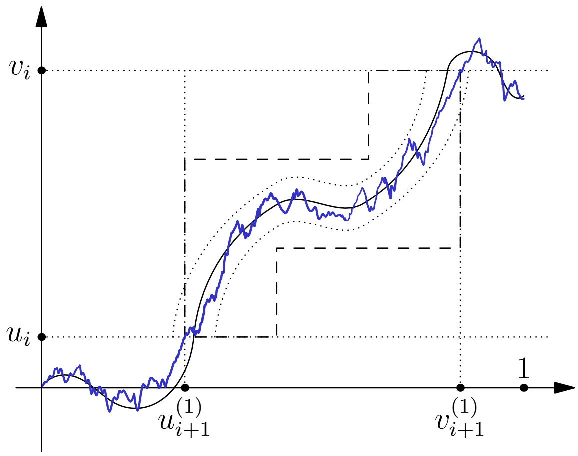

FIGURE 2.2. For any given continuous function f starting from 0 and any $\varepsilon > 0$, the Brownian motion may stay within distance $\varepsilon > 0$ of f up to time 1 with a positive probability. Choosing f carefully, we deduce that the first excursion from u_i to v_i has a good shape with positive probability.

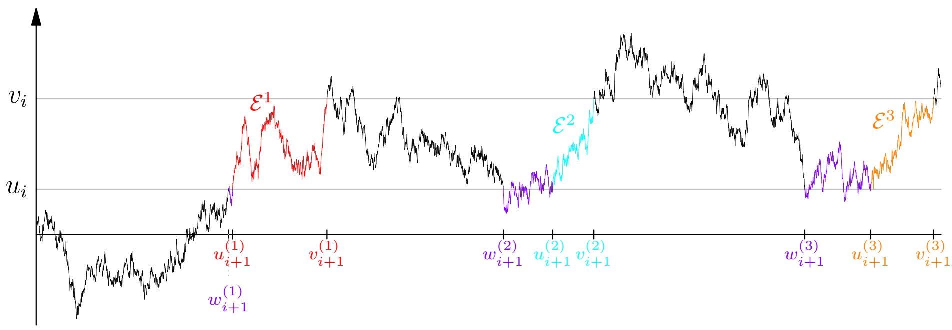

FIGURE 2.3. In red, blue and orange, the excursion from u_i to v_i we consider.

Theorem 2.35]) that for $a > 0$ we have $\mathcal{T}_a = a^2 \cdot \mathcal{T}_1$ in law where $\mathcal{T}_1$ is distributed according to the Lévy law

$$\mathcal{T}_1 \underset{(d)}{=} \frac{\mathrm{d}t}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\right) \mathbf{1}_{t>0}.$$

In our case, applying the strong Markov property at time $w_{i+1}^{(1)} < v_{i+1}^{(1)} < w_{i+1}^{(2)} < v_{i+1}^{(2)} < \cdots$ and using invariance by symmetry we deduce that we have the equalities in distribution

$$w_{i+1}^{(1)} \overset{(d)}{=} \mathcal{T}_{u_i}, \quad v_{i+1}^{(1)} \overset{(d)}{=} \mathcal{T}_{u_i+|v_i-u_i|}, \quad w_{i+1}^{(2)} \overset{(d)}{=} \mathcal{T}_{u_i+2|v_i-u_i|},$$
$$\ldots, \qquad v_{i+1}^{(k)} \overset{(d)}{=} \mathcal{T}_{u_i+(2k-1)|v_i-u_i|},$$

for $k \geqslant 2$. Since $\mathcal{T}_{u_i+(2k-1)|v_i-u_i|} \leqslant \mathcal{T}_{2ka_i}$, the probability that the first k excursions of $\mathfrak{B}_i$ occurs before $a_i^{5/4}$ is at least

$$\mathbb{P}\left(\mathcal{T}_{2ka_i} < a_i^{5/4}\right) \overset{\text{(by scaling)}}{=} \mathbb{P}\left(\mathcal{T}_1 < \left(\frac{1}{2k\, a_i^{3/8}}\right)^2\right) \geqslant 1 - \sqrt{\frac{2}{\pi}} 2ka_i^{3/8}.$$

The last inequality holds because, for any $x \geqslant 0$,

$$\begin{aligned}\mathbb{P}\left(\mathcal{T}_1 < \frac{1}{x^2}\right) &= \int_0^{1/x^2} \frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\right) \mathrm{d}t = \int_x^\infty \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u \\ &= 1 - \int_{-x}^x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u \geqslant 1 - \int_{-x}^x \frac{1}{\sqrt{2\pi}} \mathrm{d}u = 1 - \sqrt{\frac{2}{\pi}} x.\end{aligned}$$

Gathering-up the above remarks and taking $k = \lfloor a_i^{-1/4} \rfloor$, we deduce that the probability to do not find an excursion from u_i to v_i with a good shape in $[0, a_i^{5/4}]$ is bounded above by

$$(1-r)^{\lfloor a_i^{-1/4} \rfloor} + 2\sqrt{\frac{2}{\pi}} \lfloor a_i^{-1/4} \rfloor a_i^{3/8} \leqslant 2a_i^{-1/8} \text{ (for } a_i \text{ small enough).} \qquad \square$$

3. Proof of Proposition 2.1

Let $(\mathfrak{B}_i)_{i \geqslant 1}$ be a sequence of i.i.d. two-sided Brownian motions, and ε be any real positive number small enough. For any $i \geqslant 1$, take $a_i = \varepsilon^{(5/4)^{i-1}}$.

Firstly, we put $[u_1, v_1] = [0, \varepsilon] = [0, a_1]$, by Lemma 2.2, with probability at least $1 - 2a_1^{1/8}$, there exists an interval $[u_2, v_2] \subset [0, a_1^{5/4}] = [0, a_2]$ such that $\mathfrak{B}_1$ performs a good shape excursion from u_1 to v_1 over the time interval $[u_2, v_2]$. Now, we apply Lemma 2.2 to $[u_2, v_2] \subset [0, a_2]$, etc. At the end, with probability at least

$$\prod_{i=1}^\infty 1 - 2\left(\varepsilon^{(5/4)^{i-1}}\right)^{1/8},$$

we obtain a sequence of non trivial intervals $([u_i, v_i])_{i \geqslant 1}$ such that for any i, $\mathfrak{B}_i$ makes a good shape excursion from u_i to v_i over $[u_{i+1}, v_{i+1}]$. By Section 2.1, we can then construct two sequences of intervals U_i, V_i that satisfy points 2-4 of Proposition 2.1. Moreover, by construction, $U_i, V_i \subset [u_i, v_i] \subset [0, a_i]$, hence point 5 is also satisfied.

Finally, to obtain point 1, just remark that, for any $i, n \geqslant 1$, $[u_{i+n}, v_{i+n}] \subset [0, a_{i+n}] \subset [0, 1]$, so

$$\begin{aligned}U_i &= \lim_{n\to\infty} (\mathfrak{B}_i \circ \mathfrak{B}_{i+1} \circ \cdots \circ \mathfrak{B}_{i+n}) \left(\left[u_{i+n+1}, \frac{u_{i+n+1} + 2v_{i+n+1}}{3}\right]\right) \\ &\subset \lim_{n\to\infty} (\mathfrak{B}_i \circ \mathfrak{B}_{i+1} \circ \cdots \circ \mathfrak{B}_{i+n}) ([0,1]) = \mathcal{I}_i \text{ (by (1.3)).}\end{aligned}$$

Similarly, $V_i \subset \mathcal{I}_i$. $\square$

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