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ON HARDER-NARASIMHAN FILTRATIONS AND THEIR COMPATIBILITY WITH TENSOR PRODUCTS

CHRISTOPHE CORNUT

Abstract. We attach buildings to modular lattices of finite length and show that they yield a natural framework for a metric version of the Harder-Narasimhan formalism. We establish a sufficient condition for the compatibility of Harder-Narasimhan filtrations with tensor products and verify our criterion in various cases coming from p-adic Hodge theory.

Contents

1. Introduction

The Harder-Narasimhan formalism, as set up for instance by André in [1], requires a category C with kernels and cokernels, along with rank and degree functions

$$\mathrm{rank} : \mathrm{sk}\,\mathsf{C} \to \mathbb{N} \quad \text{and} \quad \deg : \mathrm{sk}\,\mathsf{C} \to \mathbb{R}$$

on the skeleton of C, subject to various axioms. It then functorially equips every object X of C with a Harder-Narasimhan filtration $\mathcal{F}_{HN}(X)$ by strict subobjects. This categorical formalism is very nice and useful, but it does not say much about

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what $\mathcal{F}_{HN}(X)$ really is. The build-in characterization of this filtration only involves the restriction of the rank and degree functions to the poset $\mathrm{Sub}(X)$ of strict subobjects of X, and a first aim of this paper is to pin down the relevant formalism.

André's axioms on $(\mathsf{C}, \mathrm{rank})$ imply that the poset $\mathrm{Sub}(X)$ is a modular lattice of finite length [14]. Thus, starting in section 2 with an arbitrary modular lattice $\mathcal{X}$ of finite length, we introduce a space $\mathbf{F}(\mathcal{X})$ of $\mathbb{R}$-filtrations on $\mathcal{X}$. This looks first like a combinatorial object with building-like features: apartments, facets and chambers. The choice of a rank function on $\mathcal{X}$ equips $\mathbf{F}(\mathcal{X})$ with a distance d, and we show that $(\mathbf{F}(\mathcal{X}), d)$ is a complete, CAT(0)-metric space, whose underlying topology and geodesic segments do not depend upon the chosen rank function. The choice of a degree function on $\mathcal{X}$ amounts to the choice of a concave function on $\mathbf{F}(\mathcal{X})$, and we show that a closely related continuous function has a unique minimum $\mathcal{F} \in \mathbf{F}(\mathcal{X})$: this is the Harder-Narasimhan filtration for the triple $(\mathcal{X}, \mathrm{rank}, \deg)$. The fact that modular lattices provide a natural framework for the Harder-Narasimhan theory was discovered independently by Hugues Randriambololona, see [20, §1].

In section 3, we derive our own Harder-Narasimhan formalism for categories from this Harder-Narasimhan formalism for modular lattices. It differs slightly from André's: we are perhaps a bit more flexible in our axioms on C, but a bit more demanding in our axioms for the rank and degree functions.

When the category C is also equipped with a k-linear tensor product, is the Harder-Narasimhan filtration compatible with this auxiliary structure? Many cases have already been considered and solved by ad-hoc methods, often building on Totaro's pioneering work [22], which itself relied on tools borrowed from Mumford's Geometric Invariant Theory [18]. Trying to understand and generalize the latest installment of this trend [17], we came up with some sort of axiomatized version of its overall strategy in which the GIT tools are replaced by tools from convex metric geometry. This is exposed in section 4, which gives a numerical criterion for the compatibility of HN-filtrations with various tensor product constructions. Our approach simultaneously yields some results towards exactness of HN-filtrations, which classically required separate proofs, often using Haboush's theorem [15].

In the last section, we verify our criterion in three cases (which could be combined as explained in section 4.3.2): filtered vector spaces (5.1), normed vector spaces (5.2) and normed φ-modules (5.3). The first case has been known for some times, see for instance [9]. The second case seems to be new, and it applies for instance to the isogeny category of sthukas with one paw, as considered in Scholze's Berkeley course or in [2]. The third case is a mild generalization of [17, 3.1.1].

Acknowledgements. — I would like to thank Brandon Levin and Carl Wang-Erickson for their explanations on [17]. In my previous attempts to deal with the second and third of the above cases, a key missing step was part (3) of the proof of proposition 5.6. The related statement appears to be lemma 3.6.6 of [17]. Finally, I would like to end this introduction with a question: in all three cases, the semi-stable objects of slope 0 form a full subcategory C^0 of C which is a neutral k-linear tannakian category.

What are the corresponding Tannaka groups?

2. The Harder-Narasimhan formalism for modular lattices

2.1. **Basic notions.** We refer to [14] for all things pertaining to basic lattice theory.

2.1.1. A *lattice* is a partially ordered set (a poset) $(X, \leqslant)$ such that every pair of elements $(x, y) \in X$ has a *meet* $x \vee y := \sup\{x, y\}$ and a *join* $x \wedge y := \inf\{x, y\}$. It is *bounded* if it has both a minimal element 0_X and a maximal element 1_X. It is *distributive* (resp. *modular*) if and only if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in X$ (resp. for all $x, y, z \in X$ with $z \leqslant x$). A *subposet* of X is a subset equipped with the induced partial order, a *sublattice* is a subposet stable under the meet and join operators of X, and a *chain* in X is a totally ordered subposet. A chain of *length* ℓ is a finite chain of order $\ell + 1$ and the *length* of X is the supremum of the length of its finite chains (with values in $\mathbb{N} \cup \{\infty\}$). An element x of a bounded lattice X is *join-irreducible* if $x \neq 0_X$ and $x = y \vee z$ implies $x = y$ or $x = z$; it is an *atom* if $x \neq 0_X$ and $y \leqslant x$ implies $y = 0_X$ or $y = x$. We denote by $\mathrm{Atom}(X) \subset \mathrm{Ji}(X)$ the set of atoms and join-irreducible elements of X. A *complement* of x is an element y of X such that $x \wedge y = 0_X$ and $x \vee y = 1_X$. A *complemented* lattice is a bounded lattice in which every element as a complement. A *boolean* lattice is a complemented distributive lattice. A non-decreasing map between bounded lattices is a *lattice map* (resp. a *$\{0, 1\}$-map*) if it is compatible with the meet and join operators (resp. with the minimal and maximal elements). For $x \leqslant y$ in X, we denote by $[x, y]$ or $\frac{y}{x}$ the subposet $\{z \in X : x \leqslant z \leqslant y\}$ of X.

2.1.2. Let X be a fixed bounded modular lattice of finite length r. An *apartment* in X is a maximal distributive sublattice S of X. Any such S is finite [21, Theorem 4.28], of length r [16, Corollary 2], with also $|\mathrm{Ji}(S)| = r$ by [14, Corollary 108]. The formula $c_i = c_{i-1} \wedge s_i$ yields a bijection between the set of all maximal chains $C = \{c_0 < \cdots < c_r\}$ in S and the set of all bijections $i \mapsto s_i$ from $\{1, \cdots, r\}$ to $\mathrm{Ji}(S)$ whose inverse $s_i \mapsto i$ is non-decreasing. The theorem of Birkhoff and Dedekind [14, Theorem 363] asserts that any two chains in X are contained in some apartment.

2.1.3. A *degree function* on X is a function $\deg : X \to \mathbb{R}$ such that

$$\deg(0_X) = 0 \quad \text{and} \quad \deg(x \vee y) + \deg(x \wedge y) \geqslant \deg(x) + \deg(y)$$

for every x, y in X. We say that it is *exact* if also $-\deg$ is a degree function, i.e.

$$\deg(x \vee y) + \deg(x \wedge y) = \deg(x) + \deg(y)$$

for every x, y in X. A *rank function* on X is an increasing exact degree function. Thus a rank function on X is a function $\mathrm{rank} : X \to \mathbb{R}_+$ such that $\mathrm{rank}(0_X) = 0$,

$$\mathrm{rank}(x \vee y) + \mathrm{rank}(x \wedge y) = \mathrm{rank}(x) + \mathrm{rank}(y)$$

for every x, y in X and $\mathrm{rank}(x) < \mathrm{rank}(y)$ if $x < y$. The *standard rank function* is given by $\mathrm{rank}(x) = \mathrm{height}(x)$, the length of any maximal chain in $[0_X, x]$.

2.1.4. For a chain $C = \{c_0 < \cdots < c_s\}$ in X, set

$$\mathrm{Gr}_C^{\bullet} \overset{\mathrm{def}}{=} \prod_{i=1}^{s} \mathrm{Gr}_C^i \quad \text{with} \quad \mathrm{Gr}_C^i \overset{\mathrm{def}}{=} [c_{i-1}, c_i].$$

For the direct product partial order on $\mathrm{Gr}_C^{\bullet}$ defined by

$$(x_1, \cdots, x_s) \leqslant (y_1, \cdots, y_s) \overset{\mathrm{def}}{\iff} \forall i \in \{1, \cdots, s\} : \quad x_i \leqslant y_i,$$

this is again plainly a modular lattice of finite length $\leqslant r$, which is even a finite boolean lattice of length r if C is maximal. We denote by $\varphi_C : X \to \mathrm{Gr}_C^\bullet$ the non-decreasing $\{0,1\}$-map which sends $x \in X$ to $\varphi_C(x) = ((x \wedge c_i) \vee c_{i-1})_{i=1}^s$. The restriction of φ_C to any apartment containing C is a lattice $\{0,1\}$-map.

2.1.5. For $\deg : X \to \mathbb{R}$, $\mathrm{rank} : X \to \mathbb{R}_+$ and C as above, we still denote by

$$\deg : \mathrm{Gr}_C^\bullet \to \mathbb{R} \quad \text{and} \quad \mathrm{rank} : \mathrm{Gr}_C^\bullet \to \mathbb{R}_+$$

the induced degree and rank functions on $\mathrm{Gr}_C^\bullet$ defined by

$$\deg\left((z_i)_{i=1}^s\right) \stackrel{\text{def}}{=} \sum_{i=1}^s \deg(z_i) - \deg(c_{i-1})$$

$$\mathrm{rank}\left((z_i)_{i=1}^s\right) \stackrel{\text{def}}{=} \sum_{i=1}^s \mathrm{rank}(z_i) - \mathrm{rank}(c_{i-1})$$

for $z_i \in \mathrm{Gr}_C^i = [c_{i-1}, c_i]$. If C is a $\{0,1\}$-chain, i.e. $c_0 = 0_X$ and $c_s = 1_X$, then

$$\deg(x) \leqslant \deg\left(\varphi_C(x)\right) \quad \text{and} \quad \mathrm{rank}(x) = \mathrm{rank}\left(\varphi_C(x)\right)$$

for every x in X. Indeed since $x \wedge c_{i-1} = (x \wedge c_i) \wedge c_{i-1}$ for all $i \in \{1, \cdots, s\}$,

$$\underbrace{\textstyle\sum_{i=1}^s \deg(x \wedge c_i) - \deg(x \wedge c_{i-1})}_{= \deg(x)} \quad \leqslant \quad \underbrace{\textstyle\sum_{i=1}^s \deg((x \wedge c_i) \vee c_{i-1}) - \deg(c_{i-1})}_{= \deg\left(\varphi_C(x)\right)}$$

with equality if and only if for every $i \in \{1, \cdots, s\}$,

$$\deg(x \wedge c_i) + \deg(c_{i-1}) = \deg\left((x \wedge c_i) \vee c_{i-1}\right) + \deg(x \wedge c_{i-1}).$$

This occurs for instance if deg is exact on the sublattice of X spanned by $C \cup \{x\}$.

2.1.6. In particular, a rank function on X is uniquely determined by its values on any maximal chain $C = \{c_0 < \cdots < c_r\}$ of X. Indeed for every $x \in X$,

$$\mathrm{rank}(x) = \sum_{\substack{i \in \{1, \cdots, r\} \\ (x \wedge c_i) \vee c_{i-1} = c_i}} \mathrm{rank}(c_i) - \mathrm{rank}(c_{i-1}).$$

If C is a maximal chain in X, the degree map on $\mathrm{Gr}_C^\bullet$ is exact and

$$\deg(x) \leqslant \sum_{\substack{i \in \{1, \cdots, r\} \\ (x \wedge c_i) \vee c_{i-1} = c_i}} \deg(c_i) - \deg(c_{i-1})$$

for every $x \in X$. In particular, $\deg : X \to \mathbb{R}$ is bounded above.

2.1.7. We started with a modular lattice of finite length, but the definition of a rank function makes sense for an arbitrary bounded lattice X. We claim that:

Lemma 2.1. — *A bounded lattice X is modular of finite length if and only if it has an integer-valued rank function* $\mathrm{rank} : X \to \mathbb{N}$.

Proof. — One direction is obvious: if X is modular of finite length, then the standard rank function $\mathrm{height} : X \to \mathbb{N}$ works. Suppose conversely that $\mathrm{rank} : X \to \mathbb{N}$ is a rank function. Then $\mathrm{rank}(1_X)$ bounds the length of any chain in X, thus X already has finite length. For modularity, we have to show that for every $a, b, c \in X$ with $a \leqslant c$, $(a \vee b) \wedge c = a \vee (b \wedge c)$. Replacing c by $c' = (a \vee b) \wedge c$, we may assume that $a \leqslant c \leqslant a \vee b$, thus $a \vee b = c \vee b$. Replacing a by $a' = a \vee (b \wedge c)$, we may assume that also $a \wedge b = c \wedge b$. In other words, we have to show that if $a \leqslant c$, $a \wedge b = c \wedge b$ and $a \vee b = c \vee b$, then $a = c$. But these assumptions imply that

$$\begin{array}{rclcl} \mathrm{rank}(a) + \mathrm{rank}(b) & = & \mathrm{rank}(a \vee b) + \mathrm{rank}(a \wedge b) & & \\ & = & \mathrm{rank}(c \vee b) + \mathrm{rank}(c \wedge b) & = & \mathrm{rank}(c) + \mathrm{rank}(b) \end{array}$$

thus $\mathrm{rank}(a) = \mathrm{rank}(c)$ and indeed $a = c$ since otherwise $\mathrm{rank}(a) < \mathrm{rank}(c)$. □

2.1.8. An apartment S of X is *special* if S is a (finite) boolean lattice.

Lemma 2.2. — *Suppose that X is complemented. Then any chain C in X is contained in a special apartment S of X.*

Proof. — Indeed, we may assume that $C = \{c_0 < \cdots < c_r\}$ is maximal. Since X is complemented, an induction on the length r of X shows that there is another maximal chain $C' = \{c'_0 < \cdots < c'_r\}$ in X such that c'_{r-i} is a complement of c_i for all $i \in \{0, \cdots, r\}$ – we then say that C' is *opposed* to C. We claim that any apartment S of X containing C and C' is special. Indeed, if $\mathrm{Ji}(S) = \{x_1, \cdots, x_r\}$ with $c_i = c_{i-1} \vee x_i$ for all $i \in \{1, \cdots, r\}$, then $c'_i = c'_{i-1} \vee x_{r+1-i}$ for all $i \in \{1, \cdots, r\}$, thus $x_i \mapsto i$ and $x_i \mapsto r + 1 - i$ are non-decreasing bijections $\mathrm{Ji}(S) \to \{1, \cdots, r\}$, so $\mathrm{Ji}(S)$ is unordered and S is indeed boolean by [14, II.1.2]. □

2.2. **$\mathbb{R}$-filtrations.** Let again X be a modular lattice of finite length r.

2.2.1. An $\mathbb{R}$-*filtration* on X is a function $f : \mathbb{R} \to X$ which is non-increasing, exhaustive, separated and left continuous: $f(\gamma_1) \geqslant f(\gamma_2)$ for $\gamma_1 \leqslant \gamma_2$, $f(\gamma) = 1_X$ for $\gamma \ll 0$, $f(\gamma) = 0_X$ for $\gamma \gg 0$ and $f(\gamma) = \inf\{f(\eta) : \eta < \gamma\}$ for $\gamma \in \mathbb{R}$. We set

$$f_+(\gamma) \overset{\text{def}}{=} \sup\{f(\eta) : \eta > \gamma\} \leqslant f(\gamma) \quad \text{and} \quad \mathrm{Gr}_f^\gamma \overset{\text{def}}{=} [f_+(\gamma), f(\gamma)] \,.$$

Note that $f_+(\gamma)$ is indeed well-defined since $f(\mathbb{R})$ is a (finite) chain in X. Equivalently, an $\mathbb{R}$-filtration on X is a pair $(C, \underline{\gamma})$ where $C = \{c_0 < \cdots < c_s\}$ is a $\{0,1\}$-chain in X (i.e. with $c_0 = 0_X$, $c_s = 1_X$) and $\underline{\gamma} = (\gamma_1 > \cdots > \gamma_s)$ is a decreasing sequence in $\mathbb{R}$. The correspondence $f \leftrightarrow (C, \underline{\gamma})$ is characterized by

$$C = F(f) \overset{\text{def}}{=} f(\mathbb{R}) \quad \text{and} \quad \underline{\gamma} = \mathrm{Jump}(f) \overset{\text{def}}{=} \left\{\gamma \in \mathbb{R} : \mathrm{Gr}_f^\gamma \neq 0\right\},$$

where $\mathrm{Gr}_f^\gamma \neq 0$ means $f_+(\gamma) \neq f(\gamma)$. Thus for every $\gamma \in \mathbb{R}$,

$$f(\gamma) = \begin{cases} c_0 = 0_X & \text{for } \gamma > \gamma_1, \\ c_i & \text{for } \gamma_{i+1} < \gamma \leqslant \gamma_i,\ i \in \{1, \cdots, s-1\}, \\ c_s = 1_X & \text{for } \gamma \leqslant \gamma_s. \end{cases}$$

2.2.2. We denote by $\mathbf{F}(X)$ the set of all $\mathbb{R}$-filtrations on X. We say that $f, f' \in \mathbf{F}(X)$ are in the same *facet* if $F(f) = F(f')$. We write $F^{-1}(C) \overset{\text{def}}{=} \{f : f(\mathbb{R}) = C\}$ for the facet defined by a chain C; thus Jump yields a bijection from $F^{-1}(C)$ to

$$\mathbb{R}^s_> \overset{\text{def}}{=} \{(\gamma_1, \cdots, \gamma_s) \in \mathbb{R}^s : \gamma_1 > \cdots > \gamma_s\}, \quad s = \mathrm{length}(C).$$

The *closed facet* of C is $\mathbf{F}(C) = \{f : f(\mathbb{R}) \subset C\}$, isomorphic to

$$\mathbb{R}^s_\geqslant \overset{\text{def}}{=} \{(\gamma_1, \cdots, \gamma_s) \in \mathbb{R}^s : \gamma_1 \geqslant \cdots \geqslant \gamma_s\}\,.$$

We call *chambers* (open or closed) the facets of the maximal C's.

2.2.3. For any $\mu \in \mathbb{R}$, we denote by $X(\mu)$ the unique element of $F^{-1}(\{0_X, 1_X\})$ such that $\mathrm{Jump}(X(\mu)) = \mu$, i.e. $X(\mu)(\gamma) = 1_X$ for $\gamma \leqslant \mu$ and $X(\mu)(\gamma) = 0_X$ for $\gamma > \mu$. We define a scalar multiplication and a symmetric addition map

$$\mathbb{R}_+ \times \mathbf{F}(X) \to \mathbf{F}(X) \quad \text{and} \quad \mathbf{F}(X) \times \mathbf{F}(X) \to \mathbf{F}(X)$$

by the following formulas: for $\lambda > 0$, $f, g \in \mathbf{F}(X)$ and $\gamma \in \mathbb{R}$,

$$(\lambda \cdot f)(\gamma) \stackrel{\text{def}}{=} f(\lambda^{-1}\gamma) \quad \text{and} \quad (f+g)(\gamma) \stackrel{\text{def}}{=} \bigvee \{f(\gamma_1) \wedge g(\gamma_2) : \gamma_1 + \gamma_2 = \gamma\},$$

while for $\lambda = 0$, we set $0 \cdot f = X(0)$. Note that the formula defining $f + g$ indeed makes sense since $f(\mathbb{R})$ and $g(\mathbb{R})$ are finite. One checks easily that

$$\begin{aligned} X(\mu_1) + X(\mu_2) &= X(\mu_1 + \mu_2) \\ \lambda \cdot X(\mu) &= X(\lambda\mu) \\ \lambda \cdot (f+g) &= \lambda \cdot f + \lambda \cdot g \\ \text{and} \quad (f + X(\mu))(\gamma) &= f(\gamma - \mu) \end{aligned}$$

for every $\mu_1, \mu_2, \mu \in \mathbb{R}$, $\lambda \in \mathbb{R}_+$, $f, g \in \mathbf{F}(X)$ and $\gamma \in \mathbb{R}$.

2.2.4. *Examples.* If $(X, \leqslant) = \{c_0 < \cdots < c_r\}$ is a finite chain, the formula

$$f_i^\sharp \stackrel{\text{def}}{=} \sup \{\gamma \in \mathbb{R} : c_i \leqslant f(\gamma)\}$$

yields a bijection $f \mapsto f^\sharp$ between $(\mathbf{F}(X), \cdot, +)$ and the closed cone

$$\mathbb{R}^r_\geqslant \stackrel{\text{def}}{=} \{(\gamma_1, \cdots, \gamma_r) \in \mathbb{R}^r : \gamma_1 \geqslant \cdots \geqslant \gamma_r\}.$$

Note that the left continuity of f implies that for all $i \in \{1, \cdots, r\}$, also

$$f_i^\sharp = \max \{\gamma \in \mathbb{R} : c_i \leqslant f(\gamma)\}.$$

More generally if $(X, \leqslant)$ is a finite distributive lattice (and thus also a bounded modular lattice of finite length, so that $\mathbf{F}(X)$ is well-defined), the formula

$$\begin{aligned} f^\sharp(x) &\stackrel{\text{def}}{=} \sup \{\gamma : x \leqslant f(\gamma)\} \\ &= \max \{\gamma : x \leqslant f(\gamma)\} \end{aligned}$$

yields a bijection $f \mapsto f^\sharp$ between $(\mathbf{F}(X), \cdot, +)$ and the cone of all non-increasing functions $f^\sharp : \mathrm{Ji}(X) \to \mathbb{R}$, where $\mathrm{Ji}(X) \subset X$ is the subposet of all join-irreducible elements of X (compare with [14, II.1.3]). The inverse bijection is given by

$$f(\gamma) = \bigvee \{x \in \mathrm{Ji}(X) : f^\sharp(x) \geqslant \gamma\}.$$

In particular if $(X, \leqslant)$ is a finite boolean lattice, $\mathrm{Ji}(X) = \mathrm{Atom}(X)$ is the unordered finite set of atoms in X and the above formula yields a bijection between $(\mathbf{F}(X), \cdot, +)$ and the finite dimensional $\mathbb{R}$-vector space of all functions $\mathrm{Atom}(X) \to \mathbb{R}$.

2.2.5. *Functoriality.* Let $\varphi : X \to Y$ be a non-decreasing $\{0,1\}$-map between bounded modular lattices of finite length. Then φ induces a map

$$\mathbf{F}(\varphi) : \mathbf{F}(X) \to \mathbf{F}(Y), \qquad f \mapsto \varphi \circ f.$$

Plainly for every $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}_+$ and $f \in \mathbf{F}(X)$,

$$\mathbf{F}(\varphi)(X(\mu)) = Y(\mu) \quad \text{and} \quad \mathbf{F}(\varphi)(\lambda \cdot f) = \lambda \cdot \mathbf{F}(\varphi)(f).$$

If moreover φ is a lattice map, i.e. if it is compatible with the meet and join operations on X and Y, then $\mathbf{F}(\varphi)$ is also compatible with the addition maps:

$$\mathbf{F}(\varphi)(f+g) = \mathbf{F}(\varphi)(f) + \mathbf{F}(\varphi)(g).$$

2.2.6. An *apartment* of $\mathbf{F}(X)$ is a subset of the form $\mathbf{F}(S)$, where S is an apartment of X, i.e. a maximal distributive sublattice of X. Thus $(\mathbf{F}(S), \cdot, +)$ is isomorphic to the cone of non-increasing maps $\mathrm{Ji}(S) \to \mathbb{R}$ by 2.2.4. The map $S \mapsto \mathbf{F}(S)$ is a bijection between apartments in X and $\mathbf{F}(X)$. The apartment $\mathbf{F}(S)$ is a finite disjoint union of facets of $\mathbf{F}(X)$, indexed by the $\{0,1\}$-chains in S. By [14, Theorem 363], for any $f, g \in \mathbf{F}(X)$, there is an apartment $\mathbf{F}(S)$ which contains f and g.

We also write $0 \in \mathbf{F}(X)$ for the trivial $\mathbb{R}$-filtration $X(0)$ on X. It is a neutral element for the addition map on $\mathbf{F}(X)$. More precisely, for every $f, g \in \mathbf{F}(X)$, $f + g = f$ if and only if $g = 0$: this follows from a straightforward computation in any apartment $\mathbf{F}(S)$ containing f and g. We say that two $\mathbb{R}$-filtrations f and f' are *opposed* if $f + f' = 0$. If f belongs to a special apartment $\mathbf{F}(S)$ (i.e. one with S boolean), then there is a unique $f' \in \mathbf{F}(S)$ which is opposed to f. Thus if X is complemented, any $f \in \mathbf{F}(X)$ has at least one opposed $\mathbb{R}$-filtration by lemma 2.2.

2.2.7. For any chain C in X, the $\{0,1\}$-map $\varphi_C : X \to \mathrm{Gr}_C^\bullet$ induces a map

$$r_C : \mathbf{F}(X) \to \mathbf{F}(\mathrm{Gr}_C^\bullet), \qquad r_C \overset{\text{def}}{=} \mathbf{F}(\varphi_C).$$

If S is an apartment of X which contains C, the restriction of φ_C to S is a lattice $\{0,1\}$-map and the restriction of r_C to $\mathbf{F}(S)$ is compatible with the addition maps.

If C is maximal, then $\mathrm{Gr}_C^\bullet = \prod_{i=1}^r \mathrm{Gr}_C^i$ is a finite boolean lattice and

$$\mathrm{Atom}(\mathrm{Gr}_C^\bullet) = \{c_1^*, \cdots, c_r^*\}$$

with c_i^* corresponding to the atom c_i of $\mathrm{Gr}_C^i = \{c_{i-1}, c_i\}$. For $C \subset S \subset X$ as above, the $\{0,1\}$-lattice map $\varphi_C|_S : S \to \mathrm{Gr}_C^\bullet$ then induces a bijection

$$\mathrm{Ji}(\varphi_C|_S) : \mathrm{Atom}(\mathrm{Gr}_C^\bullet) = \mathrm{Ji}(\mathrm{Gr}_C^\bullet) \xrightarrow{\simeq} \mathrm{Ji}(S)$$

mapping c_i^* to s_i, characterized by $c_i = c_{i-1} \vee s_i$ for all $i \in \{1, \cdots, r\}$. Then

$$r_C : \mathbf{F}(S) \to \mathbf{F}(\mathrm{Gr}_C^\bullet)$$

maps a non-increasing function $f^\sharp : \mathrm{Ji}(S) \to \mathbb{R}$ to the corresponding function $f^\sharp \circ \mathrm{Ji}(\varphi_C|_S) : \mathrm{Atom}(\mathrm{Gr}_C^\bullet) \to \mathbb{R}$. In particular, it is injective.

2.2.8. The rank function height : $X \to \{0, \cdots, r\}$ is a non-decreasing $\{0,1\}$-map, it thus induces a function $\mathbf{t} := \mathbf{F}(\text{height})$ which we call the *type map*:

$$\mathbf{t} : \mathbf{F}(X) \to \mathbf{F}(\{0, \cdots, r\}) = \mathbb{R}^r_{\geqslant}.$$

The restriction of $\mathbf{t}$ to an apartment $\mathbf{F}(S)$ maps $f^\sharp : \mathrm{Ji}(S) \to \mathbb{R}$ to

$$\mathbf{t}(f^\sharp) = (\gamma_1 \geqslant \cdots \geqslant \gamma_r) \quad \text{with} \quad |\{i : \gamma_i = \gamma\}| = \left|\{x : f^\sharp(x) = \gamma\}\right|.$$

The restriction of $\mathbf{t}$ to a closed chamber $\mathbf{F}(C)$ is a cone isomorphism (i.e. a bijection compatible with the scalar operations and addition maps).

2.2.9. The set $\mathbf{F}(X)$ is itself a lattice, with meet and join given by

$$(f \wedge g)(\gamma) \stackrel{\text{def}}{=} f(\gamma) \wedge g(\gamma) \quad \text{and} \quad (f \vee g)(\gamma) \stackrel{\text{def}}{=} f(\gamma) \vee g(\gamma)$$

for every $f, g \in \mathbf{F}(X)$ and $\gamma \in \mathbb{R}$. Moreover, there is a natural lattice embedding

$$X \hookrightarrow \mathbf{F}(X), \qquad x \mapsto x(-) \quad \text{with} \quad x(\gamma) \stackrel{\text{def}}{=} \begin{cases} 1_X & \text{if } \gamma \leqslant 0, \\ x & \text{if } 0 < \gamma \leqslant 1, \\ 0_X & \text{if } 1 < \gamma. \end{cases}$$

It maps 0_X to $X(0)$ and 1_X to $X(1)$. Viewing X as a sublattice of $\mathbf{F}(X)$, the addition map on $\mathbf{F}(X)$ sends $(x, y) \in X^2$ to the $\mathbb{R}$-filtration $x + y \in \mathbf{F}(X)$ given by

$$(x+y)(\gamma) = \begin{cases} 1_X & \text{if } \gamma \leqslant 0, \\ x \vee y & \text{if } 0 < \gamma \leqslant 1, \\ x \wedge y & \text{if } 1 < \gamma \leqslant 2, \\ 0_X & \text{if } 2 < \gamma. \end{cases}$$

For every $f \in \mathbf{F}(X)$ with $\mathrm{Jump}(f) \subset \{\gamma_1, \cdots, \gamma_N\}$ where $\gamma_1 < \cdots < \gamma_N$, we have

$$f = \gamma_1 \cdot 1_X + \sum_{i=2}^{N} (\gamma_i - \gamma_{i-1}) \cdot f(\gamma_i).$$

Since the addition map on $\mathbf{F}(X)$ is not associative, the above sum is a priori not well-defined. However, all of its summands belong to the closed facet $\mathbf{F}(C)$ of f (with $C = f(\mathbb{R})$), and the formula is easily checked inside this commutative monoid.

2.2.10. A *degree function* on $\mathbf{F}(X)$ is a function $\langle \star, - \rangle : \mathbf{F}(X) \to \mathbb{R}$ such that for $\lambda \in \mathbb{R}_+$ and $f, g \in \mathbf{F}(X)$, (1) $\langle \star, \lambda f \rangle = \lambda \langle \star, f \rangle$, (2) $\langle \star, f + g \rangle \geqslant \langle \star, f \rangle + \langle \star, g \rangle$ and (3) $\langle \star, f + g \rangle = \langle \star, f \rangle + \langle \star, g \rangle$ if $f(\mathbb{R}) \cup g(\mathbb{R})$ is a chain. We claim that:

Lemma 2.3. — *Restriction from $\mathbf{F}(X)$ to its sublattice $X \hookrightarrow \mathbf{F}(X)$ yields a bijection between degree functions on $\mathbf{F}(X)$ and degree functions on X.*

Proof. — If $\langle \star, - \rangle : \mathbf{F}(X) \to \mathbb{R}$ is a degree function on $\mathbf{F}(X)$, then for any $x, y \in X$,

$$\langle \star, x \vee y \rangle + \langle \star, x \wedge y \rangle \stackrel{(a)}{=} \langle \star, x \vee y + x \wedge y \rangle \stackrel{(b)}{=} \langle \star, x + y \rangle \stackrel{(c)}{\geqslant} \langle \star, x \rangle + \langle \star, y \rangle$$

using (3) for (a), the equality $x + y = x \vee y + x \wedge y$ in $\mathbf{F}(X)$ for (b), and (2) for (c). Since also $\langle \star, 0_X \rangle = 0$ by (1), it follows that $x \mapsto \langle \star, x \rangle$ is a degree function on X: our map is thus well-defined. It is injective since any function $\deg : X \to \mathbb{R}$ with

$\deg(0_X) = 0$ has a unique extension to a function $\langle \star, - \rangle : \mathbf{F}(X) \to \mathbb{R}$ satisfying (1) and (3), which is given by the following formula: for any $f \in \mathbf{F}(X)$,

$$\langle \star, f \rangle = \sum_{\gamma \in \mathbb{R}} \gamma \cdot \deg\left(\mathrm{Gr}_f^\gamma\right) \quad \text{with} \quad \mathrm{Gr}_f^\gamma = [f_+(\gamma), f(\gamma)]$$

where $\deg([x, y]) = \deg(y) - \deg(x)$ for $x \leqslant y$ in X. Equivalently,

$$\langle \star, f \rangle = \gamma_1 \cdot \deg(1_X) + \sum_{i=2}^{N} (\gamma_i - \gamma_{i-1}) \cdot \deg(f(\gamma_i))$$

whenever $\mathrm{Jump}(f) \subset \{\gamma_1, \cdots, \gamma_N\}$ with $\gamma_1 < \cdots < \gamma_N$.

It remains to establish that if we start with a degree function on X, this unique extension also satisfies our concavity axiom (2). Note that the last formula for $\langle \star, f \rangle$ then shows that for any $\{0, 1\}$-chain C in X,

$$\langle \star, f \rangle \leqslant \langle \star, r_C(f) \rangle$$

with equality if the initial degree function is exact on the sublattice of X spanned by $C \cup f(\mathbb{R})$. Here $r_C(f) = \varphi_C \circ f$ in $\mathbf{F}(\mathrm{Gr}_C^\bullet)$ and $\langle \star, - \rangle : \mathbf{F}(\mathrm{Gr}_C^\bullet) \to \mathbb{R}$ is the extension, as defined above, of the degree function $\deg : \mathrm{Gr}_C^\bullet \to \mathbb{R}$ induced by our initial degree function on X. Now for $f, g \in \mathbf{F}(X)$, pick an apartment S of X containing $f(\mathbb{R}) \cup g(\mathbb{R})$ and a maximal chain $C \subset S$ containing $(f + g)(\mathbb{R})$. Then

$$\langle \star, f + g \rangle = \langle \star, r_C(f + g) \rangle \quad \text{with} \quad r_C(f + g) = r_C(f) + r_C(g)$$

since deg is exact on the chain $C \supset (f + g)(\mathbb{R})$ and $f, g \in \mathbf{F}(S)$ with $C \subset S$. Since also $\langle \star, f \rangle \leqslant \langle \star, r_C(f) \rangle$ and $\langle \star, g \rangle \leqslant \langle \star, r_C(g) \rangle$, it is sufficient to establish that

$$\langle \star, r_C(f) + r_C(g) \rangle \geqslant \langle \star, r_C(f) \rangle + \langle \star, r_C(g) \rangle.$$

We may thus assume that X is a finite Boolean lattice equipped with an exact degree function, in which case the function $\langle \star, - \rangle : \mathbf{F}(X) \to \mathbb{R}$ is actually linear:

$$\langle \star, f \rangle = \sum_{a \in \mathrm{Atom}(X)} f^\sharp(a) \deg(a).$$

This finishes the proof of the lemma. $\square$

2.3. **Metrics.** Let now $\mathrm{rank} : X \to \mathbb{R}_+$ be a rank function on X.

2.3.1. We equip $\mathbf{F}(X)$ with a symmetric pairing

$$\langle -, - \rangle : \mathbf{F}(X) \times \mathbf{F}(X) \to \mathbb{R}, \quad \langle f_1, f_2 \rangle \stackrel{\text{def}}{=} \sum_{\gamma_1, \gamma_2 \in \mathbb{R}} \gamma_1 \gamma_2 \cdot \mathrm{rank}\left(\mathrm{Gr}_{f_1, f_2}^{\gamma_1, \gamma_2}\right)$$

with notations as above, where for any $f_1, f_2 \in \mathbf{F}(X)$ and $\gamma_1, \gamma_2 \in \mathbb{R}$,

$$\mathrm{Gr}_{f_1, f_2}^{\gamma_1, \gamma_2} \stackrel{\text{def}}{=} \frac{f_1(\gamma_1) \wedge f_2(\gamma_2)}{(f_{1,+}(\gamma_1) \wedge f_2(\gamma_2)) \vee (f_1(\gamma_1) \wedge f_{2,+}(\gamma_2))}.$$

Note that with these definitions and for any $\lambda \in \mathbb{R}_+$,

$$\langle \lambda f_1, f_2 \rangle = \lambda \langle f_1, f_2 \rangle = \langle f_1, \lambda f_2 \rangle.$$

If $\mathrm{Jump}(f_\nu) \subset \{\gamma_1^\nu, \cdots, \gamma_{s_\nu}^\nu\}$ with $\gamma_1^\nu < \cdots < \gamma_{s_\nu}^\nu$ and $x_j^\nu = f_\nu(\gamma_j^\nu)$ for $\nu \in \{1, 2\}$,

$$\langle f_1, f_2 \rangle = \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} \gamma_i^1 \gamma_j^2 \cdot \mathrm{rank}\left(\frac{x_i^1 \wedge x_j^2}{(x_{i+1}^1 \wedge x_j^2) \vee (x_i^1 \wedge x_{j+1}^2)}\right)$$

with the convention that $x^\nu_{s_\nu+1} = 0_X$. Thus with $r_{i,j} = \mathrm{rank}\left(x^1_i \wedge x^2_j\right)$, also

$$\begin{aligned}\langle f_1, f_2\rangle &= \sum_{i=1}^{s_1}\sum_{j=1}^{s_2} \gamma^1_i\gamma^2_j\left(r_{i,j} - r_{i+1,j} - r_{i,j+1} + r_{i+1,j+1}\right)\\ &= \sum_{i=2}^{s_1}\sum_{j=2}^{s_2}\left(\gamma^1_i - \gamma^1_{i-1}\right)\left(\gamma^2_j - \gamma^2_{j-1}\right) r_{i,j} + \gamma^1_1\gamma^2_1 r_{1,1}\\ &\quad + \sum_{i=2}^{s_1}(\gamma^1_i - \gamma^1_{i-1})\gamma^2_1 r_{i,1} + \sum_{j=2}^{s_2}\gamma^1_1(\gamma^2_j - \gamma^2_{j-1}) r_{1,j}\end{aligned}$$

2.3.2. Let $\varphi : X \to Y$ be a non-decreasing $\{0,1\}$-map between bounded modular lattices of finite length such that the rank function on X is induced by a rank function on Y. Then for the pairing on $\mathbf{F}(Y)$,

$$\begin{aligned}\langle \varphi\circ f_1, \varphi\circ f_2\rangle &= \sum_{i=2}^{s_1}\sum_{j=2}^{s_2}\left(\gamma^1_i - \gamma^1_{i-1}\right)\left(\gamma^2_j - \gamma^2_{j-1}\right) r'_{i,j}\\ &\quad + \gamma^1_1\gamma^2_1 r'_{1,1} + \sum_{i=2}^{s_1}(\gamma^1_i - \gamma^1_{i-1})\gamma^2_1 r'_{i,1} + \sum_{j=2}^{s_2}\gamma^1_1(\gamma^2_j - \gamma^2_{j-1}) r'_{1,j}\end{aligned}$$

where $r'_{i,j} = \mathrm{rank}\left(\varphi(x^1_i) \wedge \varphi(x^2_j)\right)$. Since $\varphi(x^1_i \wedge x^2_j) \leqslant \varphi(x^1_i) \wedge \varphi(x^2_j)$ with equality when i or j equals 1, $r'_{i,j} \geqslant r_{i,j}$ with equality when i or j equals 1, thus

$$\langle f_1, f_2\rangle \leqslant \langle \varphi\circ f_1, \varphi\circ f_2\rangle\,.$$

If $\varphi(z_1 \wedge z_2) = \varphi(z_1) \wedge \varphi(z_2)$ for all $z_\nu \in f_\nu(\mathbb{R})$, for instance if the restriction of φ to the sublattice of X generated by $f_1(\mathbb{R}) \cup f_2(\mathbb{R})$ is a lattice map, then

$$\langle f_1, f_2\rangle = \langle \varphi\circ f_1, \varphi\circ f_2\rangle\,.$$

In particular, this holds whenever $f_1(\mathbb{R}) \cup f_2(\mathbb{R})$ is a chain.

2.3.3. For a $\{0,1\}$-chain $C = \{c_0 < \cdots < c_s\}$ in X, we equip $\mathrm{Gr}^\bullet_C = \prod_{i=1}^s \mathrm{Gr}^i_C$ with the induced rank function as explained in 2.1.5. Applying the previous discussion to the rank-compatible $\{0,1\}$-map $\varphi_C : X \to \mathrm{Gr}^\bullet_C$ (which restricts to a lattice map on any apartement S of X containing C), we obtain the following lemma.

Lemma 2.4. — *Let C be a $\{0,1\}$-chain. Then for every $f_1, f_2 \in \mathbf{F}(X)$,*

$$\langle f_1, f_2\rangle \leqslant \langle r_C(f_1), r_C(f_2)\rangle$$

with equality if C, f_1 and f_2 are contained in a common apartement of $\mathbf{F}(X)$.

2.3.4. This yields another formula for the pairing on $\mathbf{F}(X)$: for every apartment $\mathbf{F}(S)$, there is a function $\delta_S : \mathrm{Ji}(S) \to \mathbb{R}_{>0}$ such that for every $f_1, f_2 \in \mathbf{F}(S)$,

$$\langle f_1, f_2\rangle = \sum_{x\in \mathrm{Ji}(S)} f^\sharp_1(x) f^\sharp_2(x)\cdot \delta_S(x)$$

where $f^\sharp : \mathrm{Ji}(S) \to \mathbb{R}$ is the non-increasing map attached to $f \in \mathbf{F}(S)$. Indeed, pick a maximal chain $C \subset S$. Then $\langle f_1, f_2\rangle = \langle r_C(f_1), r_C(f_2)\rangle$. But the pairing on

$\mathbf{F}(\mathrm{Gr}_C^\bullet)$ is easily computed, and it is a positive definite symmetric bilinear form: for g_1 and g_2 in $\mathbf{F}(\mathrm{Gr}_C^\bullet)$ corresponding to functions $g_1^\sharp$ and $g_2^\sharp : \mathrm{Atom}(\mathrm{Gr}_C^\bullet) \to \mathbb{R}$,

$$\langle g_1, g_2 \rangle = \sum_{a \in \mathrm{Atom}(\mathrm{Gr}_C^\bullet)} g_1^\sharp(a) g_2^\sharp(a) \,\mathrm{rank}(a).$$

For $g_\nu = r_C(f_\nu) = \varphi_C \circ f_\nu$, we have seen that $g_\nu^\sharp = f_\nu^\sharp \circ \mathrm{Ji}(\varphi_C|S)$, where $\mathrm{Ji}(\varphi_C|_S)$ is the bijection $\mathrm{Atom}(\mathrm{Gr}_C^\bullet) \simeq \mathrm{Ji}(S)$. This proves our claim, with $\delta_S(x) = \mathrm{rank}(a)$ if $\mathrm{Ji}(\varphi_C|_S)(a) = x$. If $C = \{c_0 < \cdots < c_r\}$, then $\mathrm{Ji}(S) = \{x_1, \cdots, x_r\}$ with $c_i = c_{i-1} \wedge x_i$ and $\delta_S(x_i) = \mathrm{rank}(c_i) - \mathrm{rank}(c_{i-1})$ for all $i \in \{1, \cdots, r\}$.

2.3.5. The next lemma says that our pairing is concave.

Lemma 2.5. — *For every f, g and h in $\mathbf{F}(X)$, we have*

$$\langle f, g+h \rangle \geqslant \langle f, g \rangle + \langle f, h \rangle$$

with equality if f, g and h belong to a common apartement of $\mathbf{F}(X)$.

Proof. — Indeed, choose S, C and S' as follows: S is an apartment of X containing $g(\mathbb{R})$ and $h(\mathbb{R})$, C is a maximal chain in S containing $(g+h)(\mathbb{R}) \subset S$, and S' is an apartment of X containing $f(\mathbb{R})$ and C. If f, g and h belong to a common apartement, we may and do also require that $S = S'$. In all cases,

$$\langle f, g+h \rangle \overset{(1)}{=} \langle r_C(f), r_C(g+h) \rangle \quad \text{and} \quad r_C(g+h) \overset{(2)}{=} r_C(g) + r_C(h)$$

since respectively (1) $C \subset S'$ and f, $g+h$ belong to $\mathbf{F}(S')$ and (2) $C \subset S$ and g, h belong to $\mathbf{F}(S)$. Since C is maximal, $\mathrm{Gr}_C^\bullet$ is boolean, $\mathbf{F}(\mathrm{Gr}_C^\bullet)$ is an $\mathbb{R}$-vector space and the pairing on $\mathbf{F}(\mathrm{Gr}_C^\bullet)$ is a positive definite symmetric bilinear form, thus

$$\langle r_C(f), r_C(g) + r_C(h) \rangle \overset{(3)}{=} \langle r_C(f), r_C(g) \rangle + \langle r_C(f), r_C(h) \rangle .$$

Our claim now follows from (1), (2) and (3) since also by lemma 2.4,

$$\langle r_C(f), r_C(g) \rangle \geqslant \langle f, g \rangle \quad \text{and} \quad \langle r_C(f), r_C(h) \rangle \geqslant \langle f, g \rangle$$

with equality if, along with g, h and C, also f belongs to $\mathbf{F}(S)$. $\square$

2.3.6. It follows that for every $f \in \mathbf{F}(X)$, the function $g \mapsto \langle f, g \rangle$ is a degree function on $\mathbf{F}(X)$. The corresponding degree function on X maps $x \in X$ to

$$\deg_f(x) \overset{\text{def}}{=} \sum_{\gamma \in \mathbb{R}} \gamma \,\mathrm{rank}\left(\mathrm{Gr}_{f \wedge x}^\gamma\right) \quad \text{with} \quad \mathrm{Gr}_{f \wedge x}^\gamma \overset{\text{def}}{=} [f_+(\gamma) \wedge x, f(\gamma) \wedge x] .$$

For $f = X(1)$, we retrieve the rank: $\deg_{X(1)}(x) = \mathrm{rank}(x)$. For $f \in \mathbf{F}(X)$,

$$\deg(f) \overset{\text{def}}{=} \langle X(1), f \rangle = \sum_{\gamma \in \mathbb{R}} \gamma \,\mathrm{rank}\left(\mathrm{Gr}_f^\gamma\right)$$

is the natural degree function on $\mathbf{F}(X)$ and the formula

$$\deg(f+g) \geqslant \deg(f) + \deg(g)$$

follows either from 2.3.5 or from 2.2.10.

2.3.7. For $f, g \in \mathbf{F}(X)$, $\langle f, f\rangle \geqslant 0$ and $2\langle f, g\rangle \leqslant \langle f, f\rangle + \langle g, g\rangle$: this follows from the formula in 2.3.4. We may thus define

$$\|f\| \stackrel{\text{def}}{=} \sqrt{\langle f, f\rangle} \quad \text{and} \quad d(f,g) \stackrel{\text{def}}{=} \sqrt{\|f\|^2 + \|g\|^2 - 2\langle f, g\rangle}.$$

For every $\{0,1\}$-chain C in X, $\|r_C(f)\| = \|f\|$ and

$$d\left(r_C(f), r_C(g)\right) \leqslant d(f,g)$$

with equality if there is an apartment $\mathbf{F}(S)$ with $C \subset S$ and $f, g \in \mathbf{F}(S)$. Also,

$$\begin{aligned} \|f\| = d(0_X, f), \quad \|tf\| = t\|f\|, \quad d(tf, tg) = td(f,g) \\ \text{and} \quad \|f+g\|^2 = \|f\|^2 + \|g\|^2 + 2\langle f, g\rangle \end{aligned}$$

for every $f, g \in \mathbf{F}(X)$ and $t \in \mathbb{R}_+$. The first three formulas are obvious, and the last one follows from the additivity of the symmetric pairing on any apartment. If f and f' are opposed in $\mathbf{F}(X)$, then $\|f\| = \|f'\| = \frac{1}{2}d(f, f')$ and $\langle f, f'\rangle = -\|f\|^2$.

2.3.8. We refer to [5] for all things pertaining to geodesic and CAT(0)-spaces.

PROPOSITION 2.6. — *The function $d : \mathbf{F}(X) \times \mathbf{F}(X) \to \mathbb{R}_{\geqslant 0}$ is a CAT(0)-distance.*

Proof. — If X is a finite boolean lattice, then d is the euclidean distance attached to the positive definite symmetric bilinear form (in short: scalar product) $\langle -, -\rangle$ on the $\mathbb{R}$-vector space $\mathbf{F}(X)$, which proves the proposition. For the general case:

$$\forall f, g \in \mathbf{F}(X): \qquad d(f,g) = 0 \implies f = g.$$

Indeed, choose an apartment with $f, g \in \mathbf{F}(S)$, a maximal chain $C \subset S$. Then $d(r_C(f), r_C(g)) = 0$, thus $r_C(f) = r_C(g)$ since d is a (euclidean) distance on $\mathbf{F}(\mathrm{Gr}_C^\bullet)$ and $f = g$ since the restriction $r_C|_{\mathbf{F}(S)} : \mathbf{F}(S) \to \mathbf{F}(\mathrm{Gr}_C^\bullet)$ is injective.

$$\forall f, g, h \in \mathbf{F}(X): \qquad d(f,h) \leqslant d(f,g) + d(g,h).$$

Indeed, choose an apartment with $f, h \in \mathbf{F}(S)$, a maximal chain $C \subset S$. Then

$$\begin{aligned} d(f,h) &= d(r_C(f), r_C(h)) \\ &\leqslant d(r_C(f), r_C(g)) + d(r_C(g), r_C(h)) \\ &\leqslant d(f,g) + d(g,h). \end{aligned}$$

Thus d is a distance, and a similar argument shows that $(\mathbf{F}(X), d)$ is a geodesic metric space. More precisely, for every $g, h \in \mathbf{F}(X)$ and $t \in [0,1]$, if

$$g_t = (1-t)g + th$$

is the sum of $(1-t) \cdot g$ and $t \cdot h$ in $\mathbf{F}(X)$, then $d(g, g_t) = t \cdot d(g,h)$, thus $t \mapsto g_t$ is a geodesic segment from g to h in $\mathbf{F}(X)$. Note also that

$$\|g_t\|^2 = (1-t)^2\|g\|^2 + t^2\|h\|^2 + 2t(1-t)\langle g, h\rangle.$$

For the CAT(0)-inequality, we finally have to show that for every $f \in \mathbf{F}(X)$,

$$d(f, g_t)^2 + t(1-t)d(g,h)^2 \leqslant (1-t)d(f,g)^2 + td(f,h)^2.$$

Given the previous formula for $\|g_t\|^2$, this amounts to

$$\langle f, g_t\rangle \geqslant (1-t)\langle f, g\rangle + t\langle f, h\rangle$$

which is the already established concavity of $\langle f, -\rangle$. □

2.3.9. Let $d_{\mathrm{Std}} : \mathbf{F}(X) \times \mathbf{F}(X) \to \mathbb{R}$ be the distance attached to the standard rank function $x \mapsto \mathrm{height}(x)$ on X. By 2.1.5, there are constants $A > a > 0$ such that $a \leqslant \mathrm{rank}(y) - \mathrm{rank}(x) \leqslant A$ for every $x < y$ in X. It then follows from 2.3.4 that there are constants $B > b > 0$ such that $b\, d_{\mathrm{Std}}(f,g) \leqslant d(f,g) \leqslant B\, d_{\mathrm{Std}}(f,g)$ for every $f, g \in \mathbf{F}(X)$. The topology induced by d on $\mathbf{F}(X)$ thus does not depend upon the chosen rank function. We call it the *canonical topology*. Being complete for the induced distance, apartments and closed chambers are closed in $\mathbf{F}(X)$.

Proposition 2.7. — *The metric space $(\mathbf{F}(X), d)$ is complete.*

Proof. — We may assume that $d = d_{\mathrm{Std}}$. The type function $\mathbf{t} : \mathbf{F}(X) \to \mathbb{R}^r_{\geqslant}$ defined in 2.2.8 is then non-expanding for the standard euclidean distance d on $\mathbb{R}^r_{\geqslant}$: this follows from 2.3.2 applied to height : $X \to \{0, \cdots, r\}$. In fact, for any maximal chain C in any apartment S of X, the composition of the isometric embeddings

$$\mathbf{F}(C) \hookrightarrow \mathbf{F}(S) \xhookrightarrow{r_c} \mathbf{F}(\mathrm{Gr}^\bullet_C) \simeq \mathbb{R}^r$$

with the non-expanding type map $\mathbf{t} : \mathbf{F}(\mathrm{Gr}^\bullet_C) \to \mathbb{R}^r_{\geqslant}$ is an isometry $\mathbf{F}(C) \simeq \mathbb{R}^r_{\geqslant}$. It follows that for every pair of types (t_1, t_2) in $\mathbb{R}^r_{\geqslant}$,

$$\left\{ d(f_1, f_2) \,\middle|\, \begin{array}{l} f_\nu \in \mathbf{F}(S) \\ \mathbf{t}(f_\nu) = t_\nu \end{array} \right\} \subset \left\{ d(f_1, f_2) \,\middle|\, \begin{array}{l} f_\nu \in \mathbf{F}(\mathrm{Gr}^\bullet_C) \\ \mathbf{t}(f_\nu) = t_\nu \end{array} \right\}$$

and both sets are finite with the same minimum $d(t_1, t_2)$, thus also

$$\left\{ d(f_1, f_2) \,\middle|\, \begin{array}{l} f_\nu \in \mathbf{F}(X) \\ \mathbf{t}(f_\nu) = t_\nu \end{array} \right\} \subset \left\{ d(f_1, f_2) \,\middle|\, \begin{array}{l} f_\nu \in \mathbf{F}(\mathrm{Gr}^\bullet_C) \\ \mathbf{t}(f_\nu) = t_\nu \end{array} \right\}$$

is finite with minimum $d(t_1, t_2)$. In particular, there is a constant $\epsilon(t_1, t_2) > 0$ such that for every $f_1, f_2 \in \mathbf{F}(X)$ with $\mathbf{t}(f_1) = t_1$ and $\mathbf{t}(f_2) = t_2$,

$$d(f_1, f_2) = d(t_1, t_2) \quad \text{or} \quad d(f_1, f_2) \geqslant d(t_1, t_2) + \epsilon(t_1, t_2).$$

Let now $(f_n)_{n \geqslant 0}$ be a Cauchy sequence in $\mathbf{F}(X)$. Then $t_n = \mathbf{t}(f_n)$ is a Cauchy sequence in $\mathbb{R}^r_{\geqslant}$, so it converges to a type $t \in \mathbb{R}^r_{\geqslant}$. Fix $N \in \mathbb{N}$ such that

$$d(f_n, f_m) < \tfrac{1}{3}\epsilon(t,t) \quad \text{and} \quad d(t_n, t) \leqslant \tfrac{1}{3}\epsilon(t,t)$$

for all $n, m \geqslant N$. For each $n \geqslant N$, pick a maximal chain C_n containing $f_n(\mathbb{R})$ and let g_n be the unique element of the closed chamber $\mathbf{F}(C_n)$ such that $\mathbf{t}(g_n) = t$. Then $d(f_n, g_n) = d(t_n, t)$ since f_n and g_n belong to $\mathbf{F}(C_n)$. Note that if g'_n is any other element of $\mathbf{F}(X)$ such that $\mathbf{t}(g'_n) = t$ and $d(f_n, g'_n) = d(t_n, t)$, then

$$d(g_n, g'_n) \leqslant d(g_n, f_n) + d(f_n, g'_n) = 2d(t_n, t) \leqslant \tfrac{2}{3}\epsilon(t,t) < \epsilon(t,t),$$

therefore $g_n = g'_n$. Similarly for every $n, m \geqslant N$,

$$d(g_n, g_m) \leqslant d(g_n, f_n) + d(f_n, f_m) + d(f_m, g_m) < \epsilon(t,t)$$

thus $g_n = g_m$. Call $g \in \mathbf{F}(X)$ this common value. Then

$$d(f_n, g) = d(f_n, g_n) = d(t_n, t)$$

thus $f_n \to g$ in $\mathbf{F}(X)$ since $t_n \to t$ in $\mathbb{R}^r_{\geqslant}$. □

2.3.10. Let $\deg : X \to \mathbb{R}$ be a degree function on X and let $\langle \star, - \rangle : \mathbf{F}(X) \to \mathbb{R}$ be its unique extension to a degree function on $\mathbf{F}(X)$, as explained in 2.2.10.

Proposition 2.8. — *Suppose that* $\lim f_n = f$ *in* $\mathbf{F}(X)$. *Then*

$$\limsup \langle \star, f_n \rangle \leqslant \langle \star, f \rangle .$$

If moreover $\deg(X)$ *is bounded, then* $\langle \star, - \rangle : \mathbf{F}(X) \to \mathbb{R}$ *is continuous.*

Remark 2.9. — The first assertion says that $\langle \star, - \rangle$ is upper semi-continuous.

Proof. — Let $C = f(\mathbb{R}) = \{c_0 < \cdots < c_s\}$. In the previous proof, we have seen that for every sufficiently large n, any maximal chain C_n containing $f_n(\mathbb{R})$ also contains C. Since our degree function is exact on the chain C_n,

$$\langle \star, f_n \rangle = \langle \star, r_C(f_n) \rangle \quad \text{and} \quad \langle \star, f \rangle = \langle \star, r_C(f) \rangle .$$

Since $d\left(r_C(f_n), r_C(f)\right) \leqslant d(f_n, f)$, also $\lim r_C(f_n) = r_C(f)$ in $\mathbf{F}(X)$. Now on

$$\mathbf{F}(\mathrm{Gr}_C^\bullet) = \prod_{i=1}^{s} \mathbf{F}(\mathrm{Gr}_C^i) \quad \text{with} \quad \mathrm{Gr}_C^i = [c_{i-1}, c_i]$$

the distance and degree are respectively given by

$$d\left((a_i), (b_i)\right)^2 = \sum_{i=1}^{s} d_i(a_i, b_i)^2 \quad \text{and} \quad \langle \star, (a_i) \rangle = \sum_{i=1}^{s} \langle \star_i, a_i \rangle$$

where d_i and $\langle \star_i, - \rangle$ are induced by the corresponding rank and degree functions

$$\mathrm{rank}_i(z) = \mathrm{rank}(z) - \mathrm{rank}(c_{i-1}) \quad \text{and} \quad \deg_i(z) = \deg(z) - \deg(c_{i-1})$$

for $z \in \mathrm{Gr}_C^i$. All this reduces us to the case where $f = X(\mu)$ for some $\mu \in \mathbb{R}$. Now

$$\langle \star, f_n \rangle = \gamma_{n,1} \deg(1_X) + \sum_{i=2}^{s_n} (\gamma_{n,i} - \gamma_{n,i-1}) \deg\left(f_n(\gamma_{n,i})\right)$$

with $\mathrm{Jump}(f_n) = \{\gamma_{n,1} < \cdots < \gamma_{n,s_n}\}$. Since $\lim \mathbf{t}(f_n) = \mathbf{t}(f) = (\mu, \cdots, \mu)$ in $\mathbb{R}^r_{\geqslant}$,

$$\lim \gamma_{n,1} = \mu \quad \text{and} \quad \limsup \{\gamma_{n,i} - \gamma_{n,i-1} : 2 \leqslant i \leqslant s_n\} = 0.$$

Since finally $\deg(X)$ is bounded above, we obtain

$$\limsup \langle \star, f_n \rangle \leqslant \mu \deg(1_X) = \langle \star, f \rangle$$

and $\lim \langle \star, f_n \rangle = \langle \star, f \rangle$ if $\deg(X)$ is also bounded below. □

2.3.11. For $f \in \mathbf{F}(X)$, the degree function $\langle f, - \rangle : \mathbf{F}(X) \to \mathbb{R}$ is continuous since

$$\langle f, g \rangle = \frac{1}{2} \left(\|f\|^2 + \|g\|^2 - d(f,g)^2 \right) .$$

This also follows from proposition 2.8 since for every $x \in X$,

$$\left|\deg_f(x)\right| = |\langle f, x \rangle| \leqslant \|f\| \, \|x\|$$

with $\|x\|^2 = \mathrm{rank}(x) \leqslant \mathrm{rank}(1_X)$, but a bit more is actually true:

Proposition 2.10. — *The degree function*

$$\langle f, - \rangle : \mathbf{F}(X) \to \mathbb{R}$$

is $\|f\|$*-Lipschitzian.*

Proof. — We have to show that $|\langle f,h\rangle - \langle f,g\rangle| \leqslant \|f\| \cdot d(g,h)$ for every $g,h \in \mathbf{F}(X)$. Pick an apartment S of X with $g,h \in \mathbf{F}(S)$ and set $g_t = (1-t)g + th \in \mathbf{F}(S)$ for $t \in [0,1]$. Since $\mathbf{F}(S)$ is the union of finitely many closed (convex) chambers, there is an integer $N > 0$, a finite sequence $0 = t_0 < \cdots < t_N = 1$ and maximal chains $C_1, \cdots, C_N$ in S such that for every $1 \leqslant i \leqslant N$ and $t \in [t_{i-1}, t_i]$, g_t belongs the closed chamber $\mathbf{F}(C_i)$. Set $g_i = g_{t_i}$ for $i \in \{0, \cdots, N\}$. Since

$$|\langle f,h\rangle - \langle f,g\rangle| = \left| \sum_{i=1}^{N} \langle f, g_i\rangle - \langle f, g_{i-1}\rangle \right| \leqslant \sum_{i=1}^{N} |\langle f, g_i\rangle - \langle f, g_{i-1}\rangle|$$

and $d(g,h) = \sum_{i=1}^{N} d(g_{i-1}, g_i)$, we may assume that $g,h \in \mathbf{F}(C)$ for some maximal chain C in X. Now choose an apartment S of X containing C and $f(\mathbb{R})$ and let f', g', h' be the images of f,g,h under $r_C : \mathbf{F}(X) \to \mathbf{F}(\mathrm{Gr}_C^{\bullet})$. Then

$$\begin{array}{rcl} \langle f,h\rangle & = & \langle f',h'\rangle \\ \langle f,g\rangle & = & \langle f',g'\rangle \end{array} \quad \text{and} \quad \begin{array}{rcl} \|f\| & = & \|f'\| \\ d(g,h) & = & d(g',h') \end{array}$$

since $f,g,h \in \mathbf{F}(S)$ with $C \subset S$. This reduces us further to the case of a finite boolean lattice X, where $\mathbf{F}(X)$ is a euclidean space and our claim is trivial. □

2.4. **HN-filtrations.** Suppose now that our modular lattice X is also equipped with a degree function $\deg : X \to \mathbb{R}$ and let $\langle \star, - \rangle : \mathbf{F}(X) \to \mathbb{R}$ be its unique extension to a degree function on $\mathbf{F}(X)$, as explained in 2.2.10.

2.4.1. We say that X is *semi-stable of slope* $\mu \in \mathbb{R}$ if and only if for every $x \in X$, $\deg(x) \leqslant \mu \operatorname{rank}(x)$ with equality for $x = 1_X$. More generally for every $x \leqslant y$ in X, we say that the interval $[x,y]$ is *semi-stable of slope* μ if and only if it is semi-stable of slope μ for the induced rank and degree functions, i.e. for every $z \in [x,y]$,

$$\deg(z) \leqslant \mu \left(\operatorname{rank}(z) - \operatorname{rank}(y)\right) + \deg(y)$$

with equality for $z = y$. Note that for $x = y$, $[x,y] = \{x\}$ is semi-stable of slope μ for every $\mu \in \mathbb{R}$. For any $x < y$, the *slope* of $[x,y]$ is defined by

$$\mu([x,y]) = \frac{\deg(y) - \deg(x)}{\operatorname{rank}(y) - \operatorname{rank}(x)} \in \mathbb{R}.$$

2.4.2. For any $x,y,z \in X$ with $x < y < z$, we have

$$\mu([x,z]) = \frac{\operatorname{rank}(z) - \operatorname{rank}(y)}{\operatorname{rank}(z) - \operatorname{rank}(x)} \mu([y,z]) + \frac{\operatorname{rank}(y) - \operatorname{rank}(x)}{\operatorname{rank}(z) - \operatorname{rank}(x)} \mu([x,y])$$

thus one of the following cases occurs:

$$\begin{array}{rl} & \mu([x,y]) < \mu([x,z]) < \mu([y,z]), \\ \text{or} & \mu([x,y]) > \mu([x,z]) > \mu([y,z]), \\ \text{or} & \mu([x,y]) = \mu([x,z]) = \mu([y,z]). \end{array}$$

Lemma 2.11. — *Suppose that $x \leqslant x' \leqslant y'$ and $x \leqslant y \leqslant y'$ with $[x,y]$ semi-stable of slope μ and $[x',y']$ semi-stable of slope μ'. If $\mu > \mu'$, then also $y \leqslant x'$.*

Proof. — Suppose not, i.e. $x' < y \vee x'$ and $y \wedge x' < y$. Then

$$\mu \overset{(1)}{\leqslant} \mu([y \wedge x', y]) \overset{(2)}{\leqslant} \mu([x', y \vee x']) \overset{(3)}{\leqslant} \mu'$$

since (1) $y \wedge x'$ belongs to $[x,y]$ which is semi-stable of slope μ, (3) $y \vee x'$ belongs to $[x',y']$ which is semi-stable of slope μ', and (2) follows from the definition of μ. □

2.4.3. The main result of this section is the following proposition.

PROPOSITION 2.12. — *For any $\mathcal{F} \in \mathbf{F}(X)$, the following conditions are equivalent.*

(1) *For every $f \in \mathbf{F}(X)$, $\|\mathcal{F}\|^2 - 2\langle\star, \mathcal{F}\rangle \leqslant \|f\|^2 - 2\langle\star, f\rangle$.*
(2) *For every $f \in \mathbf{F}(X)$, $\langle\star, f\rangle \leqslant \langle\mathcal{F}, f\rangle$ with equality for $f = \mathcal{F}$.*
(3) *For every $\gamma \in \mathbb{R}$, $\mathrm{Gr}^{\gamma}_{\mathcal{F}}$ is semi-stable of slope γ.*

Moreover, there is a unique such $\mathcal{F}$, and $\|\mathcal{F}\|^2 = \langle\star, \mathcal{F}\rangle$.

Proof. — It is sufficient to establish (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and the existence (resp. uniqueness) of an $\mathcal{F} \in \mathbf{F}(X)$ satisfying (1) (resp. (3)). We start with the following claim.

There is a constant $A > 0$ such that $\langle\star, f\rangle \leqslant A\|f\|$. Indeed, pick any maximal chain C in X. Then $\langle\star, f\rangle \leqslant \langle\star, r_C(f)\rangle$ and $\|f\| = \|r_C(f)\|$ for every $f \in \mathbf{F}(X)$. But on the finite dimensional $\mathbb{R}$-vector space $\mathbf{F}(\mathrm{Gr}^\bullet_C)$, $\langle\star, -\rangle : \mathbf{F}(\mathrm{Gr}^\bullet_C) \to \mathbb{R}$ is a linear form while $\|-\| : \mathbf{F}(\mathrm{Gr}^\bullet_C) \to \mathbb{R}_+$ is a euclidean norm. Our claim easily follows.

Existence in (1). Since $\langle\star, f\rangle \leqslant A\|f\|$, the function $f \mapsto \|f\|^2 - 2\langle\star, f\rangle$ is bounded below. Let (f_n) be any sequence in $\mathbf{F}(X)$ such that $\|f_n\|^2 - 2\langle\star, f_n\rangle$ converges to $I = \inf\left\{\|f\|^2 - 2\langle\star, f\rangle : f \in \mathbf{F}(X)\right\}$. By the CAT(0)-inequality,

$$2\left\|\tfrac{1}{2}f_n + \tfrac{1}{2}f_m\right\|^2 + \tfrac{1}{2}d(f_n, f_m)^2 \leqslant \|f_n\|^2 + \|f_m\|^2.$$

By concavity of $f \mapsto \langle\star, f\rangle$,

$$\left\langle\star, \tfrac{1}{2}f_n + \tfrac{1}{2}f_m\right\rangle \geqslant \tfrac{1}{2}\langle\star, f_n\rangle + \tfrac{1}{2}\langle\star, f_m\rangle.$$

We thus obtain

$$\begin{aligned} 2I + \tfrac{1}{2}d(f_n, f_m)^2 &\leqslant 2\left(\left\|\tfrac{1}{2}f_n + \tfrac{1}{2}f_m\right\|^2 - 2\left\langle\star, \tfrac{1}{2}f_n + \tfrac{1}{2}f_m\right\rangle\right) + \tfrac{1}{2}d(f_n, f_m)^2 \\ &\leqslant \left(\|f_n\|^2 - 2\langle\star, f_n\rangle\right) + \left(\|f_m\|^2 - 2\langle\star, f_m\rangle\right). \end{aligned}$$

It follows that (f_n) is a Cauchy sequence in $\mathbf{F}(X)$, and therefore converges to some $\mathcal{F} \in \mathbf{F}(X)$. Then $\|f_n\| \to \|\mathcal{F}\|$ and $\langle\star, f_n\rangle \to \frac{1}{2}\left(\|\mathcal{F}\|^2 - I\right)$. By proposition 2.8, $\|\mathcal{F}\|^2 - 2\langle\star, \mathcal{F}\rangle \leqslant I$ thus actually $\|\mathcal{F}\|^2 - 2\langle\star, \mathcal{F}\rangle = I$ by definition of I.

(1) *implies* (2). Suppose (1). Then for any $f \in \mathbf{F}(X)$ and $t \geqslant 0$,

$$\|\mathcal{F}\|^2 - 2\langle\star, \mathcal{F}\rangle \leqslant \|\mathcal{F} + tf\|^2 - 2\langle\star, \mathcal{F} + tf\rangle.$$

Since $\|\mathcal{F} + tf\|^2 = \|\mathcal{F}\|^2 + t^2\|f\|^2 + 2t\langle\mathcal{F}, f\rangle$ and $\langle\star, \mathcal{F} + tf\rangle \geqslant \langle\star, \mathcal{F}\rangle + t\langle\star, f\rangle$,

$$0 \leqslant t^2\|f\|^2 + 2t\left(\langle\mathcal{F}, f\rangle - \langle\star, f\rangle\right).$$

Since this holds for every $t \geqslant 0$, indeed $\langle\star, f\rangle \leqslant \langle\mathcal{F}, f\rangle$. On the other hand,

$$\|\mathcal{F}\|^2 - 2\langle\star, \mathcal{F}\rangle \leqslant \|t\mathcal{F}\|^2 - 2\langle\star, t\mathcal{F}\rangle = t^2\|\mathcal{F}\|^2 - 2t\langle\star, \mathcal{F}\rangle$$

for all $t \geqslant 0$, therefore also $\|\mathcal{F}\|^2 = \langle\star, \mathcal{F}\rangle$.

(2) *implies* (3). Suppose (2). Let s be the number of jumps of $\mathcal{F}$ and set

$$\mathcal{F}(\mathbb{R}) = \{c_0 < \cdots < c_s\} \quad \text{and} \quad \mathrm{Jump}(\mathcal{F}) = \{\gamma_1 > \cdots > \gamma_s\}.$$

For $i \in \{1, \cdots, s\}$ and θ sufficiently close to γ_i, let $f_{i,\theta}$ be the unique $\mathbb{R}$-filtration on X such that $f_{i,\theta}(\mathbb{R}) = \mathcal{F}(\mathbb{R})$ and $\mathrm{Jump}(f_{i,\theta}) \setminus \{\theta\} = \mathrm{Jump}(\mathcal{F}) \setminus \{\gamma_i\}$. Then

$$\begin{aligned} \langle \star, f_{i,\theta} \rangle - \theta \deg\left(\mathrm{Gr}_{\mathcal{F}}^{\gamma_i}\right) &= \langle \star, \mathcal{F} \rangle - \gamma_i \deg\left(\mathrm{Gr}_{\mathcal{F}}^{\gamma_i}\right) \\ \text{and} \quad \langle \mathcal{F}, f_{i,\theta} \rangle - \theta\gamma_i \operatorname{rank}\left(\mathrm{Gr}_{\mathcal{F}}^{\gamma_i}\right) &= \langle \mathcal{F}, \mathcal{F} \rangle - \gamma_i^2 \operatorname{rank}\left(\mathrm{Gr}_{\mathcal{F}}^{\gamma_i}\right). \end{aligned}$$

Since $\langle \star, f_{i,\theta} \rangle \leqslant \langle \mathcal{F}, f_{i,\theta} \rangle$ and $\langle \star, \mathcal{F} \rangle = \langle \mathcal{F}, \mathcal{F} \rangle$, it follows that

$$(\theta - \gamma_i)\left(\gamma_i \operatorname{rank}\left(\mathrm{Gr}_{\mathcal{F}}^{\gamma_i}\right) - \deg\left(\mathrm{Gr}_{\mathcal{F}}^{\gamma_i}\right)\right) \geqslant 0.$$

Since this holds for every θ close to γ_i, it must be that $\gamma_i = \mu\left(\mathrm{Gr}_{\mathcal{F}}^{\gamma_i}\right)$. Now for any $c_{i-1} < z < c_i$ and a sufficiently small $\epsilon > 0$, let $f_{i,z,\epsilon}$ be the unique $\mathbb{R}$-filtration on X such that $f_{i,z,\epsilon}(\mathbb{R}) = \mathcal{F}(\mathbb{R}) \cup \{z\}$ and $\mathrm{Jump}(f_{i,z,\epsilon}) = \mathrm{Jump}(\mathcal{F}) \cup \{\gamma_i + \epsilon\}$. Then

$$\begin{aligned} \langle \star, f_{i,z,\epsilon} \rangle &= \langle \star, \mathcal{F} \rangle + \epsilon \deg\left(\tfrac{z}{c_{i-1}}\right) \\ \text{and} \quad \langle \mathcal{F}, f_{i,z,\epsilon} \rangle &= \langle \mathcal{F}, \mathcal{F} \rangle + \epsilon\gamma_i \operatorname{rank}\left(\tfrac{z}{c_{i-1}}\right). \end{aligned}$$

Since again $\langle \star, f_{i,z,\epsilon} \rangle \leqslant \langle \mathcal{F}, f_{i,z,\epsilon} \rangle$ and $\langle \star, \mathcal{F} \rangle = \langle \mathcal{F}, \mathcal{F} \rangle$, we obtain

$$\deg\left(\tfrac{z}{c_{i-1}}\right) \leqslant \gamma_i \operatorname{rank}\left(\tfrac{z_i}{c_{i-1}}\right).$$

Thus $\mathrm{Gr}_{\mathcal{F}}^{\gamma_i}$ is indeed semi-stable of slope γ_i for all $i \in \{1, \cdots, s\}$.

Unicity in (3). Suppose that $\mathcal{F}$ and $\mathcal{F}'$ both satisfy (3) and set

$$\{\gamma_1 > \cdots > \gamma_s\} = \mathrm{Jump}(\mathcal{F}) \cup \mathrm{Jump}(\mathcal{F}'), \quad \gamma_0 = \gamma_1 + 1.$$

We show by ascending induction on $i \in \{0, \cdots, s\}$ and descending induction on $j \in \{i, \cdots, s\}$ that $\mathcal{F}(\gamma_i) \leqslant \mathcal{F}'(\gamma_j)$. For $i = 0$ or $j = s$ there is nothing to prove since $\mathcal{F}(\gamma_0) = 0_X$ and $\mathcal{F}'(\gamma_s) = 1_X$. Suppose now that $1 \leqslant i \leqslant j < s$ and we already now $\mathcal{F}(\gamma_{i-1}) \leqslant \mathcal{F}'(\gamma_{i-1})$ and $\mathcal{F}(\gamma_i) \leqslant \mathcal{F}'(\gamma_{j+1})$. Then $\mathcal{F}(\gamma_i) \leqslant \mathcal{F}'(\gamma_j)$ by lemma 2.11. Thus $\mathcal{F}(\gamma_i) \leqslant \mathcal{F}'(\gamma_i)$ for all $i \in \{1, \cdots, s\}$. By symmetry $\mathcal{F} = \mathcal{F}'$. □

Definition 2.13. — We call $\mathcal{F} \in \mathbf{F}(X)$ the *Harder-Narasimhan filtration* of $(X, \deg)$.

2.4.4. *Example.* For $f \in \mathbf{F}(X)$ and the degree function $\deg_f(x) = \langle f, x \rangle$ on X, the Harder-Narasimhan filtration $\mathcal{F} \in \mathbf{F}(X)$ of $(X, \deg_f)$ minimizes

$$g \mapsto \|f\|^2 + \|g\|^2 - 2\langle f, g \rangle = d(f,g)^2$$

thus plainly $\mathcal{F} = f$. More generally suppose that Y is a $\{0,1\}$-sublattice of X with the induced rank function. Then $\mathbf{F}(Y) \hookrightarrow \mathbf{F}(X)$ is an isometric embedding, with a non-expanding retraction, namely the convex projection $p : \mathbf{F}(X) \twoheadrightarrow \mathbf{F}(Y)$ of [5, II.2.4]. Then for any $f \in \mathbf{F}(X)$, $y \mapsto \langle f, y \rangle$ is a degree function on Y and the corresponding Harder-Narasimhan filtration $\mathcal{F} \in \mathbf{F}(Y)$ equals $p(f)$. In particular,

$$\langle f, g \rangle \leqslant \langle p(f), g \rangle$$

for every $f \in \mathbf{F}(X)$ and $g \in \mathbf{F}(Y)$ with equality for $g = p(f)$.

2.4.5. If X is complemented and $\deg : X \to \mathbb{R}$ is exact, the Harder-Narasimhan filtration may also be characterized by the following weakening of condition (2):

(2′) *For every* $f \in \mathbf{F}(X)$, $\langle \star, f\rangle \leqslant \langle \mathcal{F}, f\rangle$.

We have to show that for any $\mathcal{F} \in \mathbf{F}(X)$ satisfying (2′), $\langle \star, \mathcal{F}\rangle \geqslant \langle \mathcal{F}, \mathcal{F}\rangle$. Since X is complemented, there is an $\mathbb{R}$-filtration $\mathcal{F}'$ on X which is opposed to $\mathcal{F}$. Since deg is exact, $f \mapsto \langle \star, f\rangle$ is additive, thus $\langle \star, \mathcal{F}\rangle + \langle \star, \mathcal{F}'\rangle = \langle \star, \mathcal{F} + \mathcal{F}'\rangle = 0$ and indeed

$$\langle \star, \mathcal{F}\rangle = -\langle \star, \mathcal{F}'\rangle \geqslant -\langle \mathcal{F}, \mathcal{F}'\rangle = \langle \mathcal{F}, \mathcal{F}\rangle .$$

This also shows that then $\langle \star, \mathcal{F}'\rangle = \langle \mathcal{F}, \mathcal{F}'\rangle$ for any $\mathcal{F}' \in \mathbf{F}(X)$ opposed to $\mathcal{F}$.

3. The Harder-Narasimhan formalism for categories (after André)

3.1. **Basic notions.** Let C be a category with a null object 0, with kernels and cokernels. Let $\operatorname{sk}\mathsf{C}$ be the *skeleton* of C: the isomorphism classes of objects in C.

3.1.1. Let X be an object of C. Recall that a *subobject* of X is an isomorphism class of monomorphisms with codomain X. We write $x \hookrightarrow X$ for the subobject itself or any monomorphism in its class. We say that $f : x \hookrightarrow X$ is *strict* if f is a kernel. Equivalently, f is strict if and only if $f = \operatorname{im}(f)$. Dually, we have the notions of *quotients* and *strict quotients*, and $f \mapsto \operatorname{coker} f$ yields a bijection between strict subobjects and strict quotients of X, written $x \mapsto X/x$. A *short exact sequence* is a pair of composable morphisms f and g such that $f = \ker g$ and $g = \operatorname{coker} f$: it is thus of the form $0 \to x \to X \to X/x \to 0$ for some strict subobject x of X.

3.1.2. The class of all strict subobjects of X will be denoted by $\operatorname{Sub}(X)$. It is partially ordered: $(f : x \hookrightarrow X) \leqslant (f' : x' \hookrightarrow X)$ if and only if there is a morphism $h : x \to x'$ such that $f = f' \circ h$. Note that the morphism h is then unique, and is itself a strict monomorphism, realizing x as a strict subobject of x'. Conversely, a strict subobject x of x' yields a subobject of X which is not necessarily strict.

3.1.3. The pull-back of a strict monomorphism $x \hookrightarrow X$ by any morphism $Y \to X$ exists, and it is a strict monomorphism $y \hookrightarrow Y$: it is the kernel of $Y \to X \to X/x$. Dually, the push-out of a strict epimorphism $X \twoheadrightarrow X/x$ by any morphism $X \to Y$ exists, and it is a strict epimorphism $Y \to Y/y$: it is the cokernel of $x \hookrightarrow X \to Y$.

3.1.4. Suppose that C is essentially small and the fiber product of any pair of strict monomorphisms $x \hookrightarrow X$ and $y \hookrightarrow X$ (which exists by 3.1.3) induces a *strict* monomorphism $x \times_X y \to X$. Then $\operatorname{Sub}(X)$ is a set and $(\operatorname{Sub}(X), \leqslant)$ is a bounded lattice, with maximal element X and minimal element 0. The meet of $x, y \in \operatorname{Sub}(X)$ is the image of $x \times_X y \to X$, also given by the less symmetric formulas

$$x \wedge y = \ker(x \to X/y) = \ker(y \to X/x).$$

The join of x, y is the kernel of the morphism from X to the amalgamated sum of $X \to X/x$ and $X \to X/y$, also given by the less symmetric formulas

$$x \vee y = \ker\left(X \to \operatorname{coker}\left(x \to X/y\right)\right) = \ker\left(X \to \operatorname{coker}\left(y \to X/x\right)\right).$$

3.1.5. A *degree function* on C is a function $\deg : \operatorname{sk}\mathsf{C} \to \mathbb{R}$ which is additive on short exact sequences and such that if $f : X \to Y$ is any morphism in C, then $\deg(\operatorname{coim} f) \leqslant \deg(\operatorname{im} f)$. It is *exact* if $-\deg : \operatorname{sk}\mathsf{C} \to \mathbb{R}$ is also a degree function on C. A *rank function* on C is an exact degree function $\operatorname{rank} : \operatorname{sk}\mathsf{C} \to \mathbb{R}_+$ such that for every $X \in \operatorname{sk}\mathsf{C}$, $\operatorname{rank}(X) = 0$ if and only if $X = 0$.

3.1.6. Under the assumptions of 3.1.4, if $\deg : \mathrm{sk}\,\mathsf{C} \to \mathbb{R}$ is a degree function on C, then for every object X of C, $x \mapsto \deg(x)$ is a degree function on $\mathrm{Sub}(X)$. Indeed, for every $x, y \in \mathrm{Sub}(X)$, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & x \wedge y \arrow[r] & x \arrow[r] \arrow[d, "f"] & x/x \wedge y \arrow[r] \arrow[d, "\overline{f}"] & 0 \\
0 & Q \arrow[l] & X/y \arrow[l, "\pi"] & I \arrow[l] & 0 \arrow[l] \\
0 & X/x \vee y \arrow[l] \arrow[u, "\overline{g}"] & X \arrow[l] \arrow[u, "g"] & x \vee y \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where $I = \mathrm{im}(f)$ and $Q = \mathrm{coker}(f) = \mathrm{im}(\pi \circ g)$ with $x/x \wedge y = \mathrm{coim}(f)$ and $X/x \vee y = \mathrm{coim}(\pi \circ g)$. It follows that

$$\begin{array}{rrcl}
& \deg(x) - \deg(x \wedge y) = \deg(x/x \wedge y) & \leqslant & \deg I = \deg(X/y) - \deg(Q) \\
\text{and} & \deg(X) - \deg(x \vee y) = \deg(X/x \vee y) & \leqslant & \deg Q
\end{array}$$

thus since also $\deg(X) = \deg(X/y) + \deg(y)$,

$$\deg(x) + \deg(y) \leqslant \deg(x \wedge y) + \deg(x \vee y).$$

If $\deg : \mathrm{sk}\,\mathsf{C} \to \mathbb{R}$ is exact, so is $\deg : \mathrm{Sub}(X) \to \mathbb{R}$. If $\mathrm{rank} : \mathrm{sk}\,\mathsf{C} \to \mathbb{R}_+$ is a rank function, then so is $\mathrm{rank} : \mathrm{Sub}(X) \to \mathbb{R}_+$.

3.1.7. Suppose that C satisfies the assumptions of 3.1.4 and admits an integer-valued rank function $\mathrm{rank} : \mathrm{sk}\,\mathsf{C} \to \mathbb{N}$. We then have the following properties:

- *C is modular of finite length in the following sense: for every object X of C, the lattice $(\mathrm{Sub}(X), \leqslant)$ of strict subobjects of X is modular of finite length.*

This follows from 2.1.7. We write $\mathrm{length}(X)$ for the length of $\mathrm{Sub}(X)$.

- *For every $X \in \mathsf{C}$ and any x in $\mathrm{Sub}(X)$, the following maps are mutually inverse rank-preserving isomorphisms of lattices:*

$$\begin{array}{rcl}
[0, x] & \longleftrightarrow & \mathrm{Sub}(x) \\
y & \longmapsto & y \\
\mathrm{im}(z \to X) & \longleftarrow\!\shortmid & z
\end{array}
\qquad \textit{and} \qquad
\begin{array}{rcl}
[x, X] & \longleftrightarrow & \mathrm{Sub}(X/x) \\
y & \longmapsto & \mathrm{im}(y \to X/x) \\
\ker\left(X \to (X/x)/z\right) & \longleftarrow\!\shortmid & z
\end{array}$$

- *For any $f : Z \to Y$ in C with trivial kernel and cokernel, the following maps are rank-preserving mutually inverse isomorphisms of lattices:*

$$\begin{array}{rcl}
\mathrm{Sub}(Y) & \longleftrightarrow & \mathrm{Sub}(Z) \\
y & \longmapsto & \ker(Z \to Y/y) \\
\mathrm{im}(z \to Y) & \longleftarrow\!\shortmid & z
\end{array}$$

Write (α, β) for any of these pairs of maps. One checks that for y and z as above,

$$\beta \circ \alpha(y) \leqslant y \quad \text{and} \quad z \leqslant \alpha \circ \beta(z).$$

It is therefore sufficient to establish that all of our maps are rank-preserving (the rank on $[x, X]$ maps y to $\mathrm{rank}(y) - \mathrm{rank}(x) = \mathrm{rank}(y/x)$). Writing (α_i, β_i) for the i-th pair, this is obvious for α_1; for β_1, $\mathrm{im}(z \to X)$ and $z = \mathrm{coim}(z \to X)$ have the same rank; for α_2, $\mathrm{im}(y \to X/x)$ and $y/x = \mathrm{coim}(y \to X/x)$ have the same rank;

for β_2, $X \to (X/x)/z$ is an epimorphism, its coimage $X/\beta_2(z)$ and image $(X/x)/z$ thus have the same rank, and so do $\beta_2(z)/x$ and z; for α_3, the cokernel of $Z \to Y/y$ is trivial, thus $Y/y = \mathrm{im}(Z \to Y/y)$ and $Z/\alpha_3(y) = \mathrm{coim}(Z \to Y/y)$ have the same rank, and so do y and $\alpha_3(y)$ since also $\mathrm{rank}(Z) = \mathrm{rank}(Y)$; for β_3, the kernel of $z \to Y$ is trivial, thus $z = \mathrm{coim}(z)$ and $\beta_3(z) = \mathrm{im}(z \to Y)$ have the same rank.

- *The composition of two strict monomorphism (resp. epimorphisms) is a strict monomorphism (resp. epimorphisms).*
- *For every $X \in \mathsf{C}$ and $a \leqslant b$ in $\mathrm{Sub}(X)$, the following maps are mutually inverse rank-preserving isomorphisms of lattices*

$$\begin{aligned} [a,b] &\longleftrightarrow \mathrm{Sub}(b/a) \\ x &\longmapsto \mathrm{im}(x \to b/a) \\ \ker(b \to (b/a)/y) &\longleftarrow\!\dashv y \end{aligned}$$

This follows easily from the previous statements.

- *For any morphism $f : X \to Y$, the induced morphism $\overline{f} : \mathrm{coim}(f) \to \mathrm{im}(f)$ has trivial kernel and cokernel.*

The kernel of $\overline{f}$ always pulls-back through $X \to \mathrm{coim}(f)$ to the kernel of f, so it now also has to be the image of that kernel, which is trivial by definition of $\mathrm{coim}(f)$. Similarly, the image of $\overline{f}$ always pushes-out through $\mathrm{im}(f) \to Y$ to the image of f, so it now has to be this image, i.e. $\mathrm{coker}(\overline{f}) = 0$.

- *The length function* $\mathrm{length} : \mathrm{sk}\,\mathsf{C} \to \mathbb{N}$ *is an integer-valued rank function.*

Indeed, for a short exact sequence $0 \to x \to X \to X/x \to 0$ in C,

$$\begin{aligned} \mathrm{length}(X) &= \mathrm{length}(\mathrm{Sub}(X)) \\ &= \mathrm{length}([0,x]) + \mathrm{length}([x,X]) \\ &= \mathrm{length}(\mathrm{Sub}(x)) + \mathrm{length}(\mathrm{Sub}(X/x)) \\ &= \mathrm{length}(x) + \mathrm{length}(X/x) \end{aligned}$$

and for any morphism $f : X \to Y$, since $\ker(\overline{f}) = 0 = \mathrm{coker}(\overline{f})$,

$$\mathrm{length}(\mathrm{coim}(f)) = \mathrm{length}(\mathrm{Sub}(\mathrm{coim}(f)) = \mathrm{length}(\mathrm{Sub}(\mathrm{im}(f)) = \mathrm{length}(\mathrm{im}(f)).$$

3.1.8. Suppose that C is a proto-abelian category in the sense of André [1, §2]: (1) every morphism with zero kernel (resp. cokernel) is a monomorphism (resp. an epimorphism) and (2) the pull-back of a strict epimorphism by a strict monomorphism is a strict epimorphism and the push-out of a strict monomorphism by a strict epimorphism is a strict monomorphism. In this case, a degree function on C is a function $\deg : \mathrm{sk}\,\mathsf{C} \to \mathbb{R}$ which is additive on short exact sequences and non-decreasing on mono-epi's (=morphisms which are simultaneously monomorphisms and epimorphisms). Our definitions for rank and degree functions on such a category C are thus more restrictive than those of André (beyond the differences between the allowed codomains of these functions): he only requires the slope $\mu = \deg/\mathrm{rank}$ to be non-decreasing on mono-epi's, while we simultaneously require the denominator to be constant and the numerator to be non-decreasing on mono-epi's. In all the examples we know, the rank functions satisfy our assumptions.

3.2. **HN-filtrations.** Let C be an essentially small category with null objects, kernels and cokernels, such that the fiber product of strict subobjects $x, y \hookrightarrow X$ is a strict subobject $x \wedge y \hookrightarrow X$, and let rank : $\mathrm{sk}\,\mathsf{C} \to \mathbb{N}$ be a fixed, integer-valued rank function on C.

3.2.1. For every object X of C, write $\mathbf{F}(X)$ for the set of $\mathbb{R}$-filtrations on the modular lattice $\mathrm{Sub}(X)$. Thus $\mathbf{F}(X) = \mathbf{F}(\mathrm{Sub}(X))$ is the set of "$\mathbb{R}$-filtrations on X by strict subobjects". It is equipped with its scalar multiplication, symmetric addition, its collection of apartments and facet decomposition. The rank function on C moreover induces a rank function on $\mathrm{Sub}(X)$, which equips $\mathbf{F}(X)$ with a scalar product $\langle -, - \rangle$, a norm $\|-\|$, a complete CAT(0)-distance $d(-,-)$, the underlying topology, and the standard degree function deg : $\mathbf{F}(X) \to \mathbb{R}$ which maps $\mathcal{F}$ to

$$\deg(\mathcal{F}) = \langle X(1), \mathcal{F} \rangle = \sum_{\gamma \in \mathbb{R}} \gamma \,\mathrm{rank}\left(\mathrm{Gr}_{\mathcal{F}}^{\gamma}\right).$$

Here $X(\mu)$ is the $\mathbb{R}$-filtration on X with a single jump at μ and we may either view $\mathrm{Gr}_{\mathcal{F}}^{\gamma}$ as an interval in $\mathrm{Sub}(X)$, or as the corresponding strict subquotient of X. For a strict subquotient y/x of X and $\mathcal{F} \in \mathbf{F}(X)$, we denote by $\mathcal{F}_{y/x}$ the induced $\mathbb{R}$-filtration on y/x, given by $\mathcal{F}_{y/x}(\gamma) = (\mathcal{F}(\gamma) \wedge y) \vee x/x = (\mathcal{F}(\gamma) \vee x) \wedge y/x$.

3.2.2. We denote by $\mathsf{F}(\mathsf{C})$ the category whose objects are pairs $(X, \mathcal{F})$ with $X \in \mathsf{C}$ and $\mathcal{F} \in \mathbf{F}(X)$. A *morphism* $(X, \mathcal{F}) \to (Y, \mathcal{G})$ in $\mathsf{F}(\mathsf{C})$ is a morphism $f : X \to Y$ in C such that for any $\gamma \in \mathbb{R}$, $f(\mathcal{F}(\gamma)) \subseteq \mathcal{G}(\gamma)$. Here $f : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ maps x to

$$\mathrm{im}(x \hookrightarrow X \xrightarrow{f} Y)$$

and we have switched to the notation $\subseteq$ for the partial order $\leqslant$ on $\mathrm{Sub}(-)$. The category $\mathsf{F}(\mathsf{C})$ is essentially small, and it also has a zero object, kernels and cokernels. For the above morphism, they are respectively given by $(\ker(f), \mathcal{F}_{\ker(f)})$ and $(\mathrm{coker}(f), \mathcal{G}_{\mathrm{coker}(f)})$. The fiber product of strict monomorphisms is a strict monomorphism. The forgetful functor $\omega : \mathsf{F}(\mathsf{C}) \to \mathsf{C}$ which takes $(X, \mathcal{F})$ to X is exact and induces a lattice isomorphism $\mathrm{Sub}(X, \mathcal{F}) \simeq \mathrm{Sub}(X)$, whose inverse maps x to $(x, \mathcal{F}_x)$. The category $\mathsf{F}(\mathsf{C})$ is equipped with rank and degree functions,

$$\mathrm{rank}(X, \mathcal{F}) \stackrel{\text{def}}{=} \mathrm{rank}(X) \quad \text{and} \quad \deg(X, \mathcal{F}) \stackrel{\text{def}}{=} \deg(\mathcal{F}).$$

Indeed, the first formula plainly defines an integer valued rank function on $\mathsf{F}(\mathsf{C})$, which thus satisfies all the properties of 3.1.7. For any exact sequence

$$0 \to (x, \mathcal{F}_x) \to (X, \mathcal{F}) \to (X/x, \mathcal{F}_{X/x}) \to 0$$

in $\mathsf{F}(\mathsf{C})$, there is an apartment S of $\mathrm{Sub}(X)$ containing $\mathcal{F}(\mathbb{R})$ and $C = \{0, x, 1_X\}$; the corresponding apartment of $\mathbf{F}(X)$ contains $X(1)$ and $\mathcal{F}$, thus by 2.3.3

$$\begin{aligned}
\deg(X, \mathcal{F}) &= \langle X(1), \mathcal{F} \rangle \\
&= \langle r_C\left(X(1)\right), r_C(\mathcal{F}) \rangle \\
&= \langle x(1), \mathcal{F}_x \rangle + \left\langle X/x(1), \mathcal{F}_{X/x} \right\rangle \\
&= \deg(x, \mathcal{F}_x) + \deg(X/x, \mathcal{F}_{X/x}).
\end{aligned}$$

For a morphism $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ with trivial kernel and cokernel, the induced map $f : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ is a rank preserving lattice isomorphism, thus

$$\begin{aligned}\deg(X, \mathcal{F}) &= \gamma_1 \cdot \mathrm{rank}(X) + \textstyle\sum_{i=2}^{s}(\gamma_i - \gamma_{i-1}) \cdot \mathrm{rank}\left(\mathcal{F}(\gamma_i)\right) \\ &= \gamma_1 \cdot \mathrm{rank}(Y) + \textstyle\sum_{i=2}^{s}(\gamma_i - \gamma_{i-1}) \cdot \mathrm{rank}\left(f(\mathcal{F}(\gamma_i))\right) \\ &\leqslant \gamma_1 \cdot \mathrm{rank}(Y) + \textstyle\sum_{i=2}^{s}(\gamma_i - \gamma_{i-1}) \cdot \mathrm{rank}\left(\mathcal{G}(\gamma_i)\right) \\ &= \deg(Y, \mathcal{G}).\end{aligned}$$

where $\{\gamma_1 < \cdots < \gamma_s\} = \mathrm{Jump}(\mathcal{F}) \cup \mathrm{Jump}(\mathcal{G})$. This shows that $\deg : \mathrm{sk}\,\mathsf{F}(\mathsf{C}) \to \mathbb{R}$ is indeed a degree function on $\mathsf{F}(\mathsf{C})$. Note also that with notations as above, we have $\deg(X, \mathcal{F}) = \deg(Y, \mathcal{G})$ if and only if $\mathcal{G}(\gamma) = \mathrm{im}(\mathcal{F}(\gamma) \to Y)$ for every $\gamma \in \mathbb{R}$.

3.2.3. A degree function $\deg : \mathrm{sk}\,\mathsf{C} \to \mathbb{R}$ on C gives rise to a degree function on $\mathrm{Sub}(X)$ for every $X \in \mathsf{C}$, which yields an *Harder-Narasimhan* $\mathbb{R}$-filtration $\mathcal{F}_{HN}(X) \in \mathbf{F}(X)$ on X: the unique $\mathbb{R}$-filtration $\mathcal{F}$ on X (by strict subobjects) such that $\mathrm{Gr}_{\mathcal{F}}^{\gamma}$ is semi-stable of slope γ for every $\gamma \in \mathbb{R}$. Here semi-stability may either refer to the lattice notion of semi-stable intervals in $\mathrm{Sub}(X)$, as defined earlier, or to the corresponding categorical notion: an object Y of C is semi-stable of slope $\mu \in \mathbb{R}$ if and only if $\deg(Y) = \mu\,\mathrm{rank}(Y)$ and $\deg(y) \leqslant \mu\,\mathrm{rank}(y)$ for every strict subobject y of Y. This is equivalent to: $\deg(Y) = \mu\,\mathrm{rank}(Y)$ and $\deg(Y/y) \geqslant \mu\,\mathrm{rank}(y)$ for every strict subobject y of Y. Note that $Y = 0$ is semi-stable of slope μ for every $\mu \in \mathbb{R}$. In general, the slope of a nonzero object X of C is given by

$$\mu(X) = \frac{\deg(X)}{\mathrm{rank}(X)} \in \mathbb{R}.$$

For any $x \in \mathrm{Sub}(X)$ with $x \neq 0$ and $X/x \neq 0$,

$$\mu(X) = \frac{\mathrm{rank}(x)}{\mathrm{rank}(X)}\mu(x) + \frac{\mathrm{rank}(X/x)}{\mathrm{rank}(X)}\mu(X/x)$$

thus either one of the following cases occur:

$$\begin{array}{rl} & \mu(x) < \mu(X) < \mu(X/x), \\ \text{or} & \mu(x) > \mu(X) > \mu(X/x), \\ \text{or} & \mu(x) = \mu(X) = \mu(X/x). \end{array}$$

3.2.4. We claim that the Harder-Narasimhan filtration $X \mapsto \mathcal{F}_{HN}(X)$ is functorial. This easily follows from the next classical lemma, a categorical variant of lemma 2.11.

Lemma 3.1. — *Suppose that A and B are semi-stable of slope $a > b$. Then*

$$\mathrm{Hom}_{\mathsf{C}}(A, B) = 0.$$

Proof. — Suppose $f : A \to B$ is nonzero, i.e. $\mathrm{coim}(f) \neq 0$ and $\mathrm{im}(f) \neq 0$. Then

$$a \overset{(1)}{\leqslant} \mu(\mathrm{coim}(f)) \overset{(2)}{\leqslant} \mu(\mathrm{im}(f)) \overset{(3)}{\leqslant} b$$

since (1) A is semi-stable of slope a, (3) B is semi-stable of slope b, and (2) follows from the definition of μ. This is a contradiction, thus $f = 0$. □

3.2.5. We thus obtain a *Harder-Narasimhan functor*

$$\mathcal{F}_{HN} : \mathsf{C} \to \mathsf{F}(\mathsf{C})$$

which is a section of the forgetful functor $\omega : \mathsf{F}(\mathsf{C}) \to \mathsf{C}$. The original degree function on C may be retrieved from the associated functor $\mathcal{F}_{HN}$ by composing it with the standard degree function on $\mathsf{F}(\mathsf{C})$ which takes $(X, \mathcal{F})$ to $\deg(\mathcal{F})$. The above construction thus yields an injective map from the set of all degree functions on C to the set of all sections $\mathsf{C} \to \mathsf{F}(\mathsf{C})$ of $\omega : \mathsf{F}(\mathsf{C}) \to \mathsf{C}$. A functor in the image of this map is what André calls a *slope filtration on* C [1, §4].

Remark 3.2. — For the rank and degree functions on $\mathsf{C}' = \mathsf{F}(\mathsf{C})$ defined in section 3.2.2, the Harder-Narasimhan filtration is tautological: $\mathcal{F}_{HN}(X, \mathcal{F}) = \mathcal{F}$ in

$$\mathbf{F}(X) = \mathbf{F}(\mathrm{Sub}(X)) = \mathbf{F}(\mathrm{Sub}(X, \mathcal{F})) = \mathbf{F}(X, \mathcal{F}).$$

3.2.6. As mentioned in the introduction, our Harder-Narasimhan formalism for categories is closely related to André's formalism in [1], which indeed was our main source of inspiration. The formalism used by Fargues in [11] is a specialization of André's, with a set-up closer to what we will have in the next section. Other formalisms have been proposed, dealing with categories equipped with auxilliary structures: triangulations in [4], exact sequences and geometric structures in [7].

4. The Harder-Narasimhan formalism on quasi-Tannakian categories

4.1. **Tannakian categories.** Let k be a field and let A be a k-linear tannakian category [10] with unit 1_{A} and ground field $k_A = \mathrm{End}_{\mathsf{A}}(1_A)$, an extension of k. Let also G be a reductive group over k. We denote by $\mathsf{Rep}(G)$ the k-linear tannakian category of algebraic representations of G on finite dimensional k-vector spaces. Finally, let $\omega_{G,\mathsf{A}} : \mathsf{Rep}(G) \to \mathsf{A}$ be a fixed exact and faithful k-linear $\otimes$-functor.

4.1.1. The category A is equipped with a natural integer-valued rank function

$$\mathrm{rank}_{\mathsf{A}} : \mathrm{sk}\,\mathsf{A} \to \mathbb{N}.$$

Indeed, recall that a fiber functor on A is an exact faithful k_{A}-linear $\otimes$-functor

$$\omega_{\mathsf{A},\ell} : \mathsf{A} \to \mathsf{Vect}_\ell$$

for some extension ℓ of k_{A}. The existence of such fiber functors is part of the definition of tannakian categories, and any two such functors $\omega_{\mathsf{A},\ell_1}$ and $\omega_{\mathsf{A},\ell_2}$ become isomorphic over some common extension ℓ_3 of ℓ_1 and ℓ_2 [10, §1.10]: we may thus set

$$\forall X \in \mathrm{sk}\,\mathsf{A} : \qquad \mathrm{rank}_{\mathsf{A}}(X) \stackrel{\mathrm{def}}{=} \dim_\ell \left(\omega_{\mathsf{A},\ell}(X)\right).$$

This equips $\mathrm{Sub}(X)$ with a natural rank function and $\mathbf{F}(X) = \mathbf{F}(\mathrm{Sub}(X))$ with a natural norm, CAT(0)-distance and scalar product – for every object X of A.

4.1.2. The category $\mathsf{F}(\mathsf{A})$ is a quasi-abelian k_{A}-linear rigid $\otimes$-category, with

$$(X_1, \mathcal{F}_1) \otimes (X_2, \mathcal{F}_2) \stackrel{\mathrm{def}}{=} (X_1 \otimes X_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \quad \text{and} \quad (X, \mathcal{F})^* \stackrel{\mathrm{def}}{=} (X^*, \mathcal{F}^*)$$

where $\mathcal{F}_1 \otimes \mathcal{F}_2 \in \mathbf{F}(X_1 \otimes X_2)$ and $\mathcal{F}^* \in \mathbf{F}(X^*)$ are respectively given by

$$(\mathcal{F}_1 \otimes \mathcal{F}_2)(\gamma) \stackrel{\mathrm{def}}{=} \textstyle\sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}_1(\gamma_1) \otimes \mathcal{F}_2(\gamma_2) \quad \text{and} \quad \mathcal{F}^*(\gamma) \stackrel{\mathrm{def}}{=} \left(X/\mathcal{F}_+(\gamma)\right)^*.$$

Note that the formula defining $\mathcal{F}_1 \otimes \mathcal{F}_2$ indeed makes sense, since $\mathcal{F}_1(\mathbb{R})$ and $\mathcal{F}_2(\mathbb{R})$ are finite subsets of $\mathrm{Sub}(X_1)$ and $\mathrm{Sub}(X_2)$, and the $\otimes$-product is exact. For the standard degree function $\deg_{\mathsf{A}} : \mathrm{sk}\,\mathsf{F}(\mathsf{A}) \to \mathbb{R}$ of section 3.2.2,

$$\deg_{\mathsf{A}}(\mathcal{F}_1 \otimes \mathcal{F}_2) = \mathrm{rank}_{\mathsf{A}}(X_1) \cdot \deg_{\mathsf{A}}(\mathcal{F}_2) + \mathrm{rank}_{\mathsf{A}}(X_2) \cdot \deg_{\mathsf{A}}(\mathcal{F}_1)$$

and $\deg_{\mathsf{A}}(\mathcal{F}^*) = -\deg_{\mathsf{A}}(\mathcal{F})$. This can be checked after applying some fiber functor $\omega_{\mathsf{A},\ell} : \mathsf{A} \to \mathsf{Vect}_\ell$ as above: the formulas are easily established in Vect_ℓ.

4.1.3. We denote by $\mathbf{F}(\omega_{G,\mathsf{A}})$ the set of all factorizations

$$\omega_{G,\mathsf{A}} : \mathsf{Rep}(G) \xrightarrow{\mathcal{F}} \mathsf{F}(\mathsf{A}) \xrightarrow{\omega} \mathsf{A}$$

of our given exact $\otimes$-functor $\omega_{G,\mathsf{A}}$ through a k-linear exact $\otimes$-functor

$$\mathcal{F} : \mathsf{Rep}(G) \to \mathsf{F}(\mathsf{A}).$$

Thus for every $\tau \in \mathsf{Rep}(G)$, we have an evaluation map

$$\mathbf{F}(\omega_{G,\mathsf{A}}) \to \mathbf{F}(\omega_{G,\mathsf{A}}(\tau)), \qquad \mathcal{F} \mapsto \mathcal{F}(\tau).$$

For instance, the trivial filtration $0 \in \mathbf{F}(\omega_{G,\mathsf{A}})$ maps $\tau \in \mathsf{Rep}(G)$ to the $\mathbb{R}$-filtration on $\omega_{G,\mathsf{A}}(\tau)$ with a single jump at $\gamma = 0$, i.e. $0(\tau) = \omega_{G,\mathsf{A}}(\tau)(0)$.

THEOREM 4.1. — *The set $\mathbf{F}(\omega_{G,\mathsf{A}})$ is equipped with a scalar multiplication and a symmetric addition map given by the following formulas: for every $\tau \in \mathsf{Rep}(G)$,*

$$(\lambda \cdot \mathcal{F})(\tau) \overset{\text{def}}{=} \lambda \cdot \mathcal{F}(\tau) \quad \text{and} \quad (\mathcal{F} + \mathcal{G})(\tau) \overset{\text{def}}{=} \mathcal{F}(\tau) + \mathcal{G}(\tau).$$

The choice of a faithful representation τ of G equips $\mathbf{F}(\omega_{G,\mathsf{A}})$ with a norm, a distance, and a scalar product given by the following formulas: for $\mathcal{F}, \mathcal{G}$ in $\mathbf{F}(\omega_{G,\mathsf{A}})$,

$$\|\mathcal{F}\|_\tau \overset{\text{def}}{=} \|\mathcal{F}(\tau)\|, \quad d_\tau(\mathcal{F}, \mathcal{G}) \overset{\text{def}}{=} d(\mathcal{F}(\tau), \mathcal{G}(\tau)) \quad \text{and} \quad \langle \mathcal{F}, \mathcal{G} \rangle_\tau \overset{\text{def}}{=} \langle \mathcal{F}(\tau), \mathcal{G}(\tau) \rangle.$$

The resulting metric space $(\mathbf{F}(\omega_{G,\mathsf{A}}), d_\tau)$ is CAT(0) and complete. The underlying metrizable topology on $\mathbf{F}(\omega_{G,\mathsf{A}})$ does not depend upon the chosen τ.

Proof. — If $\mathsf{A} = \mathsf{Vect}_{k_{\mathsf{A}}}$ and $\omega_{G,\mathsf{A}}$ is the standard fiber functor $\omega_{G,k_{\mathsf{A}}}$ which maps a representation τ of G on the k-vector space $V(\tau)$ to the k_{A}-vector space $V(\tau) \otimes k_{\mathsf{A}}$, then $\mathbf{F}(\omega_{G,k_{\mathsf{A}}})$ is the vectorial Tits building of $G_{k_{\mathsf{A}}}$ studied in [8, Chapter 4] where everything can be found. For the general case, pick an extension ℓ of k_{A} and a fiber functor $\omega_{\mathsf{A},\ell} : \mathsf{A} \to \mathsf{Vect}_\ell$ such that $\omega_{\mathsf{A},\ell} \circ \omega_{G,\mathsf{A}}$ is $\otimes$-isomorphic to the standard fiber functor $\omega_{G,\ell}$. Then, for every $\tau \in \mathsf{Rep}(G)$, we obtain a commutative diagram

$$\begin{tikzcd} \mathbf{F}(\omega_{G,\mathsf{A}}) \arrow[r, hook] \arrow[d] & \mathbf{F}(\omega_{G,\ell}) \arrow[d] \\ \mathbf{F}(\omega_{G,\mathsf{A}}(\tau)) \arrow[r, hook] & \mathbf{F}(\omega_{G,\ell}(\tau)) \end{tikzcd}$$

The horizontal maps are injective since $\omega_{\mathsf{A},\ell}$ is exact and faithful. The second vertical map is continuous, and so is therefore also the first one (for the induced topologies). Moreover, both vertical maps are injective if τ is a faithful representation of G by [8, Corollary 87]. For the first claims, we have to show that the functors $\mathsf{Rep}(G) \to \mathsf{F}(\mathsf{A})$ defined by the formulas for $\lambda \cdot \mathcal{F}$ and $\mathcal{F} + \mathcal{G}$ are exact and compatible with tensor products: this can be checked after post-composition with the fiber functor $\omega_{\mathsf{A},\ell}$, see [8, Section 3.11.10]. It follows that for any faithful τ,

$\mathbf{F}(\omega_{G,\mathsf{A}})$ is a convex subset of $\mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$ and $\mathbf{F}(\omega_{G,\ell})$, the function d_τ is a CAT(0)-distance on $\mathbf{F}(\omega_{G,\mathsf{A}})$ and the resulting topology does not depend upon the chosen τ [8, Section 4.2.11]. It remains to establish that $(\mathbf{F}(\omega_{G,\mathsf{A}}), d_\tau)$ is complete, and this amounts to showing that $\mathbf{F}(\omega_{G,\mathsf{A}})$ is closed in $\mathbf{F}(\omega_{G,\ell})$. But if $\mathcal{F}_n \in \mathbf{F}(\omega_{G,\mathsf{A}})$ converges to $\mathcal{F} \in \mathbf{F}(\omega_{G,\ell})$, then for every $\tau \in \mathsf{Rep}(G)$, $\mathcal{F}_n(\tau) \in \mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$ converges to $\mathcal{F}(\tau) \in \mathbf{F}(\omega_{G,\ell}(\tau))$, thus actually $\mathcal{F}(\tau) \in \mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$ since $\mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$ is (complete thus) closed in $\mathbf{F}(\omega_{G,\ell}(\tau))$, therefore indeed $\mathcal{F} \in \mathbf{F}(\omega_{G,\mathsf{A}})$. □

4.1.4. For a faithful representation τ of G, we have just seen that evaluation at τ identifies $\mathbf{F}(\omega_{G,\mathsf{A}})$ with a closed convex subset $\mathbf{F}(\omega_{G,\mathsf{A}})(\tau)$ of $\mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$. Let

$$p : \mathbf{F}(\omega_{G,\mathsf{A}}(\tau)) \twoheadrightarrow \mathbf{F}(\omega_{G,\mathsf{A}})(\tau)$$

be the corresponding convex projection with respect to the natural distance d on $\mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$. For every $\mathcal{F} \in \mathbf{F}(\omega_{G,\mathsf{A}})$ and $f, g \in \mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$, we have

$$d\left(p(f), p(g)\right) \leqslant d\left(f, g\right), \quad \|p(f)\| \leqslant \|f\| \quad \text{and} \quad \langle \mathcal{F}(\tau), f\rangle \leqslant \langle \mathcal{F}(\tau), p(f)\rangle .$$

The first formula comes from [5, II.2.4]. The second follows, with $g = p(g) = 0(\tau)$. The third formula can be proved as in section 2.4.4, see also [8, Section 5.7.7].

4.2. **Quasi-Tannakian categories.** Let now C be an essentially small k-linear quasi-abelian $\otimes$-category with a faithful exact k-linear $\otimes$-functor $\omega_{\mathsf{C},\mathsf{A}} : \mathsf{C} \to \mathsf{A}$ such that for every object X of C, $\omega_{\mathsf{C},\mathsf{A}}$ induces a bijection between strict subobjects of X in C and (strict) subobjects of $\omega_{\mathsf{C},\mathsf{A}}(X)$ in A. We add to this data a degree function $\deg_{\mathsf{C}} : \mathrm{sk}\,\mathsf{C} \to \mathbb{R}$, i.e. a function which is additive on short exact sequences and non-decreasing on mono-epis. Together with the rank function

$$\mathrm{rank}_{\mathsf{C}}(X) \stackrel{\text{def}}{=} \mathrm{rank}_{\mathsf{A}}(\omega_{\mathsf{C},\mathsf{A}}(X)),$$

it yields a Harder-Narasimhan filtration on C, which we view as a functor over A,

$$\mathcal{F}_{HN} : \mathsf{C} \to \mathsf{F}(\mathsf{A}), \qquad \omega \circ \mathcal{F}_{HN} = \omega_{\mathsf{C},\mathsf{A}}.$$

Note that this functor $\mathcal{F}_{HN}$ is usually neither exact, nor a $\otimes$-functor.

4.2.1. We denote by $\mathsf{C}(X)$ the fiber of $\omega_{\mathsf{C},\mathsf{A}} : \mathsf{C} \to \mathsf{A}$ over an object X of A, and for $x \in \mathsf{C}(X)$, we denote by $\langle x, - \rangle : \mathbf{F}(X) \to \mathbb{R}$ the concave degree function on

$$\mathbf{F}(X) = \mathbf{F}(\mathrm{Sub}(X)) = \mathbf{F}(\mathrm{Sub}(x)) = \mathbf{F}(x)$$

induced by our given degree function on C, thereby obtaining a pairing

$$\langle -, - \rangle : \mathsf{C}(X) \times \mathbf{F}(X) \to \mathbb{R}.$$

By proposition 2.12, the Harder-Narasimhan filtration $\mathcal{F}_{HN}(x)$ of x is the unique element $\mathcal{F} \in \mathbf{F}(X)$ with the following equivalent properties:

(1) For every $f \in \mathbf{F}(X)$, $\|\mathcal{F}\|^2 - 2\langle x, \mathcal{F}\rangle \leqslant \|f\|^2 - 2\langle x, f\rangle$.
(2) For every $f \in \mathbf{F}(X)$, $\langle x, f\rangle \leqslant \langle \mathcal{F}, f\rangle$ with equality for $f = \mathcal{F}$.
(3) For every $\gamma \in \mathbb{R}$, $\mathrm{Gr}^\gamma_{\mathcal{F}}(x)$ is semi-stable of slope γ.

In (3), $\mathrm{Gr}^\gamma_{\mathcal{F}}(x) = \mathcal{F}^\gamma(x)/\mathcal{F}^\gamma_+(x)$ where $\mathcal{F}^\gamma(x)$ and $\mathcal{F}^\gamma_+(x)$ are the strict subobjects of x corresponding to the (strict) subobjects $\mathcal{F}(\gamma)$ and $\mathcal{F}_+(\gamma)$ of $X = \omega_{\mathsf{C},\mathsf{A}}(x)$.

4.2.2. We denote by $\mathsf{C}^{\otimes}(\omega_{G,\mathsf{A}})$ the set of all factorizations

$$\omega_{G,\mathsf{A}} : \mathsf{Rep}(G) \xrightarrow{x} \mathsf{C} \xrightarrow{\omega_{\mathsf{C},\mathsf{A}}} \mathsf{A}$$

of our given exact $\otimes$-functor $\omega_{G,\mathsf{A}}$ through a k-linear exact $\otimes$-functor

$$x : \mathsf{Rep}(G) \to \mathsf{C}.$$

Thus for every $\tau \in \mathsf{Rep}(G)$, we have an evaluation map

$$\mathsf{C}^{\otimes}(\omega_{G,\mathsf{A}}) \to \mathsf{C}(\omega_{G,\mathsf{A}}(\tau)), \qquad x \mapsto x(\tau)$$

and the corresponding pairing

$$\langle -, - \rangle_{\tau} : \mathsf{C}^{\otimes}(\omega_{G,\mathsf{A}}) \times \mathbf{F}(\omega_{G,\mathsf{A}}) \to \mathbb{R}, \qquad \langle x, \mathcal{F} \rangle_{\tau} = \langle x(\tau), \mathcal{F}(\tau) \rangle .$$

Note that the latter is concave in the second variable.

Proposition 4.2. — *For $x \in \mathsf{C}^{\otimes}(\omega_{G,\mathsf{A}})$ and any faithful representation τ of G, there is a unique $\mathcal{F}$ in $\mathbf{F}(\omega_{G,\mathsf{A}})$ which satisfies the following equivalent conditions:*

(1) *For every $f \in \mathbf{F}(\omega_{G,\mathsf{A}})$, $\|\mathcal{F}\|_{\tau}^2 - 2\langle x, \mathcal{F}\rangle_{\tau} \leqslant \|f\|_{\tau}^2 - 2\langle x, f\rangle_{\tau}$.*
(2) *For every $f \in \mathbf{F}(\omega_{G,\mathsf{A}})$, $\langle x, f\rangle_{\tau} \leqslant \langle \mathcal{F}, f\rangle_{\tau}$ with equality for $f = \mathcal{F}$.*

Suppose moreover that for every $f \in \mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$ with projection $p(f) \in \mathbf{F}(\omega_{G,\mathsf{A}})(\tau)$,

$$\langle x(\tau), f \rangle \leqslant \langle x(\tau), p(f) \rangle .$$

Then $\mathcal{F}(\tau) = \mathcal{F}_{HN}(x(\tau))$.

Proof. — For the first claim, it is sufficient to establish the implication (1) $\Rightarrow$ (2) for any $\mathcal{F} \in \mathbf{F}(\omega_{G,\mathsf{A}})$, the existence of an $\mathcal{F}$ satisfying (1), and the uniqueness of any $\mathcal{F}$ satisfying (2). The first two of these are proved as in proposition 2.12, replacing everywhere the complete CAT(0)-space $\mathbf{F}(X)$ by $\mathbf{F}(\omega_{G,\mathsf{A}})$ and the concave function $\langle \star, - \rangle$ by $\langle x, - \rangle_{\tau}$. As for uniqueness, if $\mathcal{F}$ and $\mathcal{G}$ both satisfy (2), then

$$\|\mathcal{F}\|_{\tau}^2 = \langle x, \mathcal{F}\rangle_{\tau} \leqslant \langle \mathcal{G}, \mathcal{F}\rangle_{\tau} \quad \text{and} \quad \|\mathcal{G}\|_{\tau}^2 = \langle x, \mathcal{G}\rangle_{\tau} \leqslant \langle \mathcal{F}, \mathcal{G}\rangle_{\tau}$$

therefore $d_{\tau}(\mathcal{F}, \mathcal{G})^2 = \|\mathcal{F}\|_{\tau}^2 + \|\mathcal{G}\|_{\tau}^2 - 2\langle \mathcal{F}, \mathcal{G}\rangle_{\tau} \leqslant 0$ and $\mathcal{F} = \mathcal{G}$. For the last claim,

$$\|\mathcal{F}(\tau)\|^2 - 2\langle x(\tau), \mathcal{F}(\tau)\rangle \leqslant \|p(f)\|^2 - 2\langle x(\tau), p(f)\rangle \leqslant \|f\|^2 - 2\langle x(\tau), f\rangle$$

for every $f \in \mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$ by the first characterization of $\mathcal{F}$, the assumption on (x, τ) and the inequality $\|p(f)\| \leqslant \|f\|$. Thus indeed $\mathcal{F}(\tau) = \mathcal{F}_{HN}(x(\tau))$ by 4.2.1. □

Proposition 4.3. — *Fix $x \in \mathsf{C}^{\otimes}(\omega_{G,\mathsf{A}})$. Suppose that for every faithful representation τ of G and every $f \in \mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$ with projection $p(f) \in \mathbf{F}(\omega_{G,\mathsf{A}})(\tau)$, we have*

$$\langle x(\tau), f \rangle \leqslant \langle x(\tau), p(f) \rangle .$$

Then $\mathcal{F}_{HN}(x) := \mathcal{F}_{HN} \circ x$ is an exact $\otimes$-functor $\mathcal{F}_{HN}(x) : \mathsf{Rep}(G) \to \mathsf{F}(\mathsf{A})$ and for every faithful representation τ of G, $\mathcal{F}_{HN}(x)$ is the unique element $\mathcal{F}$ of $\mathbf{F}(\omega_{G,\mathsf{A}})$ which satisfies the following equivalent conditions:

(1) *For every $f \in \mathbf{F}(\omega_{G,\mathsf{A}})$, $\|\mathcal{F}\|_{\tau}^2 - 2\langle x, \mathcal{F}\rangle_{\tau} \leqslant \|f\|_{\tau}^2 - 2\langle x, f\rangle_{\tau}$.*
(2) *For every $f \in \mathbf{F}(\omega_{G,\mathsf{A}})$, $\langle x, f\rangle_{\tau} \leqslant \langle \mathcal{F}, f\rangle_{\tau}$ with equality for $f = \mathcal{F}$.*
(3) *For every $\gamma \in \mathbb{R}$, $\mathrm{Gr}^{\gamma}_{\mathcal{F}(\tau)}(x(\tau))$ is semi-stable of slope γ.*

Proof. — By the previous proposition, for any faithful τ, the three conditions are equivalent and determine a unique $\mathcal{F}_\tau \in \mathbf{F}(\omega_{G,\mathsf{A}})$ with $\mathcal{F}_\tau(\tau) = \mathcal{F}_{HN}(x)(\tau)$. For any $\sigma \in \mathsf{Rep}(G)$, $\tau' = \tau \oplus \sigma$ is also faithful. By additivity of $\mathcal{F}_{\tau'}$ and $\mathcal{F}_{HN}(x)$,

$$\mathcal{F}_{\tau'}(\tau) \oplus \mathcal{F}_{\tau'}(\sigma) = \mathcal{F}_{\tau'}(\tau') = \mathcal{F}_{HN}(x)(\tau') = \mathcal{F}_\tau(\tau) \oplus \mathcal{F}_{HN}(x)(\sigma)$$

inside $\mathbf{F}(\omega_{G,\mathsf{A}}(\tau)) \times \mathbf{F}(\omega_{G,\mathsf{A}}(\sigma)) \subset \mathbf{F}(\omega_{G,\mathsf{A}}(\tau'))$, therefore

$$\mathcal{F}_\tau(\tau) = \mathcal{F}_{\tau'}(\tau) \quad \text{and} \quad \mathcal{F}_{HN}(x)(\sigma) = \mathcal{F}_{\tau'}(\sigma).$$

Since evaluation at τ is injective, $\mathcal{F}_\tau = \mathcal{F}_{\tau'}$ and $\mathcal{F}_{HN}(x)(\sigma) = \mathcal{F}_\tau(\sigma)$ for every $\sigma \in \mathsf{Rep}(G)$. In particular, $\mathcal{F} = \mathcal{F}_\tau$ does not depend upon τ and $\mathcal{F}_{HN}(x) = \mathcal{F}$ is indeed an exact $\otimes$-functor. This proves the proposition. □

4.3. **Compatibility with $\otimes$-products.** Let us now slightly change our set-up. We keep k and A fixed, view C, $\omega_{\mathsf{C},\mathsf{A}} : \mathsf{C} \to \mathsf{A}$ and $\deg_{\mathsf{C}} : \operatorname{sk}\mathsf{C} \to \mathbb{R}$ as auxiliary data, and we do not fix G or $\omega_{G,\mathsf{A}}$.

4.3.1. A faithful exact k-linear $\otimes$-functor $x : \mathsf{Rep}(G) \to \mathsf{C}$ is *good* if it satisfies the assumption of the previous proposition, when we view it as an element of $\mathsf{C}^{\otimes}(\omega_{G,\mathsf{A}})$ with $\omega_{G,\mathsf{A}} = \omega_{\mathsf{C},\mathsf{A}} \circ x$. Then $\mathcal{F}_{HN}(x) := \mathcal{F}_{HN} \circ x$ is an exact k-linear $\otimes$-functor

$$\mathcal{F}_{HN}(x) : \mathsf{Rep}(G) \to \mathsf{F}(\mathsf{A}).$$

We say that a pair of objects (x_1, x_2) in C is *good* if the following holds. For $i \in \{1,2\}$, set $d_i = \operatorname{rank}_{\mathsf{C}}(x_i)$ and let τ_i and 1_i be respectively the tautological and trivial representations of $GL(d_i)$ on $V(\tau_i) = k^{d_i}$ and $V(1_i) = k$. We require the existence of a *good* exact k-linear $\otimes$-functor

$$x : \mathsf{Rep}\left(GL(d_1) \times GL(d_2)\right) \to \mathsf{C}$$

mapping $\tau_1' = \tau_1 \boxtimes 1_2$ to x_1 and $\tau_2' = 1_1 \boxtimes \tau_2$ to x_2. Then

$$\mathcal{F}_{HN}(x_1 \otimes x_2) = \mathcal{F}_{HN}(x_1) \otimes \mathcal{F}_{HN}(x_2).$$

We say that $(\mathsf{C}, \deg_{\mathsf{C}})$ is *good* if every pair of objects in C is good.

Corollary 4.4. — *If $(\mathsf{C}, \deg_{\mathsf{C}})$ is good, then $\mathcal{F}_{HN} : \mathsf{C} \to \mathsf{F}(\mathsf{A})$ is a $\otimes$-functor.*

4.3.2. Suppose that $(\omega_i : \mathsf{C}_i \to \mathsf{A}, \deg_i)_{i \in I}$ is a finite collection of data as above. Let $\omega : \mathsf{C} \to \mathsf{A}$ be the fibered product of the ω_i's, with fiber $\mathsf{C}(X) = \prod \mathsf{C}_i(X)$ over any object X of A and with homomorphisms given by

$$\operatorname{Hom}_{\mathsf{C}}((x_i),(y_i)) \stackrel{\text{def}}{=} \cap_i \operatorname{Hom}_{\mathsf{C}_i}(x_i, y_i) \quad \text{in} \quad \operatorname{Hom}_{\mathsf{A}}(X, Y)$$

for $(x_i) \in \mathsf{C}(X)$, $(y_i) \in \mathsf{C}(Y)$. Then C is yet another essentially small quasi-abelian k-linear $\otimes$-category equipped with a faithful exact k-linear $\otimes$-functor $\omega : \mathsf{C} \to \mathsf{A}$ which identifies $\operatorname{Sub}((x_i))$ and $\operatorname{Sub}(X)$ for every $(x_i) \in \mathsf{C}(X)$. Fix $\lambda = (\lambda_i) \in \mathbb{R}^I$ with $\lambda_i > 0$ and for every object $x = (x_i)$ of C, set $\deg_\lambda(x) := \sum \lambda_i \deg_i(x_i)$. Then

$$\deg_\lambda : \operatorname{sk}\mathsf{C} \to \mathbb{R}$$

is a degree function on C and for every $X \in \mathsf{A}$, $x = (x_i) \in \mathsf{C}(X)$ and $\mathcal{F} \in \mathbf{F}(X)$,

$$\langle x, \mathcal{F} \rangle = \sum \lambda_i \langle x_i, \mathcal{F} \rangle .$$

Thus an exact k-linear $\otimes$-functor $x : \mathsf{Rep}(G) \to \mathsf{C}$ is good if it has good components $x_i : \mathsf{Rep}(G) \to \mathsf{C}_i$, a pair $((x_i),(y_i))$ in C is good if it has good components (x_i, y_i) in

C_i, and $(\mathsf{C}, \deg_\lambda)$ is good if the $(\mathsf{C}_i, \deg_i)$'s are, in which case the Harder-Narasimhan filtration $\mathcal{F}_{HN} : \mathsf{C} \to \mathsf{F}(\mathsf{A})$ is compatible with tensor products.

4.3.3. Our use of an auxiliary reductive group G to establish the compatibility of Harder-Narasimhan filtrations with tensor products may obscure the main idea, which goes back to at least Totaro's [22]: once the Harder-Narasimhan filtration has been characterized as the (unique) solution of an optimization problem on a space of $\mathbb{R}$-filtrations, the desired compatibility $\mathcal{F}_{HN}(x_1 \otimes x_2) = \mathcal{F}_{HN}(x_1) \otimes \mathcal{F}_{HN}(x_2)$ follows from an inequality of the form $\langle x_1 \otimes x_2, f\rangle \leqslant \langle x_1 \otimes x_2, p(f)\rangle$, for every $\mathbb{R}$-filtration $f \in \mathbf{F}(x_1 \otimes x_2)$, where p is the convex projection of $\mathbf{F}(x_1 \otimes x_2)$ onto the image of the tensor product map $\otimes : \mathbf{F}(x_1) \times \mathbf{F}(x_2) \to \mathbf{F}(x_1 \otimes x_2)$. Note that $p(f)$ is itself the (unique) solution of a different and easier optimization problem. For a strict subobject z of $x_1 \otimes x_2$ mapping to some f in $\mathbf{F}(x_1 \otimes x_2)$ under the embedding of section 2.2.9, a pair of $\mathbb{R}$-filtrations $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{F}(x_1) \times \mathbf{F}(x_2)$ with the property that $\mathcal{F}_1 \otimes \mathcal{F}_2 = p(f)$ in $\mathbf{F}(x_1 \otimes x_2)$ is what would be called a Kempf filtration in [22] or [17]. In our set-up, the tensor product map is the evaluation map $\mathbf{F}(\omega_{G,\mathsf{A}}) \to \mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$ induced by the tensor product representation τ of $G := GL(d_1) \times GL(d_2)$ (with $d_i = \mathrm{rank}(x_i)$). It turns out that in all the examples we know, the proofs of the desired inequalities work equally well for arbitrary G and τ, and the final results thus obtained are stronger: in addition to their compatibility with $\otimes$-products, our Harder-Narasimhan filtrations also have some exactness properties, a feature that usually required further arguments, most notably Haboush's theorem [15]. Of course, our set-up is also tailor-made for the applications that we have in mind.

5. Examples of good C's

5.1. **Filtered vector spaces.**

5.1.1. We consider the following set-up: k is a field, ℓ is an extension of k and

$$\mathsf{A} = \mathsf{Vect}_k \quad \text{and} \quad \mathsf{C} = \mathsf{Fil}_k^\ell \quad \text{with} \quad \left\{ \begin{array}{rcl} \omega(V, \mathcal{F}) & = & V, \\ \mathrm{rank}(V, \mathcal{F}) & = & \dim_k V, \\ \deg(V, \mathcal{F}) & = & \deg(\mathcal{F}). \end{array} \right.$$

Here Fil_k^ℓ is the category of all pairs $(V, \mathcal{F})$ where V is a finite dimensional k-vector space and $\mathcal{F}$ is an $\mathbb{R}$-filtration on $V_\ell := V \otimes_k \ell$, i.e. a collection $\mathcal{F} = (\mathcal{F}^\gamma)_{\gamma \in \mathbb{R}}$ of ℓ-subspaces of V_ℓ such that $\mathcal{F}^\gamma \subset \mathcal{F}^{\gamma'}$ if $\gamma' \leqslant \gamma$, $\mathcal{F}^\gamma = V_\ell$ for $\gamma \ll 0$, $\mathcal{F}^\gamma = 0$ for $\gamma \gg 0$ and $\mathcal{F}^\gamma = \cap_{\gamma' < \gamma} \mathcal{F}^{\gamma'}$ for every $\gamma \in \mathbb{R}$. A *morphism* $f : (V_1, \mathcal{F}_1) \to (V_2, \mathcal{F}_2)$ is a k-linear morphism $f : V_1 \to V_2$ such that $f_\ell(\mathcal{F}_1^\gamma) \subset \mathcal{F}_2^\gamma$ for every $\gamma \in \mathbb{R}$, where $f_\ell : V_{1,\ell} \to V_{2,\ell}$ is the ℓ-linear extension of f. The kernel and cokernel of f are given by $(\ker f, \mathcal{F}_{1,\ker f})$ and $(\mathrm{coker} f, \mathcal{F}_{2,\mathrm{coker} f})$ where $\mathcal{F}_{1,\ker f}^\gamma$ and $\mathcal{F}_{2,\mathrm{coker} f}^\gamma$ are respectively the inverse and direct images of $\mathcal{F}_1^\gamma$ and $\mathcal{F}_2^\gamma$ under $(\ker f)_\ell \hookrightarrow V_{1,\ell}$ and $V_{2,\ell} \twoheadrightarrow (\mathrm{coker} f)_\ell$. The morphism f is strict if and only if $\mathcal{F}_2^\gamma \cap f_\ell(V_{1,\ell}) = f_\ell(\mathcal{F}_1^\gamma)$ for every $\gamma \in \mathbb{R}$. It is a mono-epi if and only if the underlying map $f : V_1 \to V_2$ is an isomorphism. The category Fil_k^ℓ is quasi-abelian, the rank and degree functions are additive on short exact sequences, and they are respectively constant and non-decreasing on mono-epis. More precisely if $f : (V_1, \mathcal{F}_1) \to (V_2, \mathcal{F}_2)$ is a mono-epi, then $\deg \mathcal{F}_1 \leqslant \deg \mathcal{F}_2$ with equality if and only if f is an isomorphism. We thus obtain a HN-formalism on Fil_k^ℓ. There is also a tensor product, given by

$$\begin{aligned}(V_1,\mathcal{F}_1)\otimes(V_2,\mathcal{F}_2) &\stackrel{\text{def}}{=} (V_1\otimes_k V_2,\mathcal{F}_1\otimes\mathcal{F}_2),\\ \text{with}\quad (\mathcal{F}_1\otimes\mathcal{F}_2)^\gamma &\stackrel{\text{def}}{=} \sum_{\gamma_1+\gamma_2=\gamma}\mathcal{F}_1^{\gamma_1}\otimes_\ell\mathcal{F}_2^{\gamma_2}.\end{aligned}$$

We will show that if ℓ is a separable extension of k, the HN-filtration is compatible with $\otimes$-products. This has been known for some time, see for instance [9, I.2], where a counter-example is also given when ℓ is a finite inseparable extension of k. For $k=\ell$, we simplify our notations to $\mathsf{Fil}_k := \mathsf{Fil}_k^k = \mathsf{F}(\mathsf{Vect}_k)$.

5.1.2. Let $\mathbb{F}(G)$ be the smooth k-scheme denoted by $\mathbb{F}^{\mathbb{R}}(G)$ in [8]. Thus

$$\mathbf{F}(G,\ell)\stackrel{\text{def}}{=}\mathbb{F}(G)(\ell)=\mathbf{F}(\omega_{G,\ell})=(\mathsf{Fil}_k^\ell)^{\otimes}(\omega_{G,k})$$

is the vectorial Tits building of G_ℓ, where $\omega_{G,\ell}:\mathsf{Rep}(G)\to\mathsf{Vect}_\ell$ is the standard fiber functor. The choice of a finite dimensional faithful representation τ of G equips these buildings with compatible complete CAT(0)-metrics d_τ whose induced topologies do not depend upon the chosen τ. These constructions are covariantly functorial in G, compatible with products and closed immersions, and covariantly functorial in ℓ. We thus obtain a (strictly) commutative diagram of functors

$$\begin{tikzcd} \mathsf{Red}(k)\times\mathsf{Ext}(k) \arrow[r, "\mathbf{F}(-,-)"] & \mathsf{Top} \\ \mathsf{Red}(G)\times\mathsf{Ext}(k) \arrow[u, hook] \arrow[r, "(\mathbf{F}(-,-){,}d_\tau)"] & \mathsf{CCat}(0) \arrow[u, hook] \end{tikzcd}$$

where $\mathsf{Red}(k)$ is the category of reductive groups over k, $\mathsf{Red}(G)$ is the poset of all (closed) reductive subgroups H of G viewed as a subcategory of $\mathsf{Red}(k)$, $\mathsf{Ext}(k)$ is the category of field extensions ℓ of k, Top is the category of topological spaces and continuous maps, and $\mathsf{CCat}(0)$ is the category of complete CAT(0)-metric spaces and distance preserving maps. For τ, H and ℓ as above, the commutative diagram

$$\begin{tikzcd} (\mathbf{F}(H,k),d_\tau) \arrow[r, hook] \arrow[d, hook] & (\mathbf{F}(G,k),d_\tau) \arrow[d, hook] \\ (\mathbf{F}(H,\ell),d_\tau) \arrow[r, hook] & (\mathbf{F}(G,\ell),d_\tau) \end{tikzcd}$$

is cartesian in $\mathsf{CCat}(0)$ since $\mathbb{F}(H)(k)=\mathbb{F}(H)(\ell)\cap\mathbb{F}(G)(k)$ inside $\mathbb{F}(G)(\ell)$. Using [5, II.2.4], we obtain a usually non-commutative diagram of non-expanding retractions

$$\begin{tikzcd} (\mathbf{F}(H,k),d_\tau) & (\mathbf{F}(G,k),d_\tau) \arrow[l, two heads, "p_k"'] \\ (\mathbf{F}(H,\ell),d_\tau) \arrow[u, two heads, "\pi_H"] & (\mathbf{F}(G,\ell),d_\tau) \arrow[l, two heads, "p_\ell"'] \arrow[u, two heads, "\pi_G"'] \end{tikzcd}$$

where each map sends a point in its source to the unique closest point in its target.

THEOREM 5.1. — *If ℓ is a separable extension of k, the diagrams*

$$\begin{tikzcd} \mathbf{F}(H,k) \arrow[d, hook] & \mathbf{F}(G,k) \arrow[l, two heads, "p_k"'] \arrow[d, hook] \\ \mathbf{F}(H,\ell) & \mathbf{F}(G,\ell) \arrow[l, two heads, "p_\ell"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{F}(H,k) \arrow[r, hook] & \mathbf{F}(G,k) \\ \mathbf{F}(H,\ell) \arrow[u, two heads, "\pi_H"] \arrow[r, hook] & \mathbf{F}(G,\ell) \arrow[u, two heads, "\pi_G"] \end{tikzcd}$$

are commutative, moreover π_G does not depend upon τ and defines a retraction

$$\pi : \mathbf{F}(-,\ell) \twoheadrightarrow \mathbf{F}(-,k)$$

of the embedding $\mathbf{F}(-,k) \hookrightarrow \mathbf{F}(-,\ell)$ of functors from $\mathsf{Red}(k)$ to Top. Finally,

$$\forall (x,y) \in \mathbf{F}(H,\ell) \times \mathbf{F}(G,k) : \qquad \langle x,y\rangle_\tau \leqslant \langle x, p_k(y)\rangle_\tau$$

Proof. — This is essentially formal.

Commutativity of the first diagram. We have to show that for every $x \in \mathbf{F}(G,k)$, $y = p_\ell(x)$ belongs to $\mathbf{F}(H,k) \subset \mathbf{F}(H,\ell)$ – for then indeed $y = p_k(x)$. Since $\mathbf{F}(H,\ell) = \mathbb{F}(H)(\ell)$ and $\mathbb{F}(H)$ is locally of finite type over k, there is a finitely generated subextension ℓ' of ℓ/k such that y belongs to $\mathbb{F}(H)(\ell') = \mathbf{F}(H,\ell')$. Plainly $y = p_{\ell'}(x)$, and we may thus assume that $\ell = \ell'$ is a finitely generated separable extension of ℓ. Then [3, V, §16, $n°7$, Corollaire of Théorème 5] reduces us to the following cases: (1) $\ell = k(t)$ is a purely transcendental extension of k or (2) ℓ is a separable algebraic extension of k. Note that in any case, y is fixed by the automorphism group Γ of ℓ/k. Indeed, Γ acts by isometries on $\mathbf{F}(G,\ell)$ and $\mathbf{F}(H,\ell)$, thus p_ℓ is Γ-equivariant and Γ fixes $y = p_\ell(x)$ since it fixes $x \in \mathbf{F}(G,k)$. This settles the following sub-cases, where k is the subfield of ℓ fixed by Γ: ($1'$) $\ell = k(t)$ with k infinite (where $\Gamma = PGL_2(k)$), and ($2'$) ℓ is Galois over k (where $\Gamma = \mathrm{Gal}(\ell/k)$). If ℓ is merely algebraic and separable over k, let ℓ' be its Galois closure in a suitable algebraic extension. Then ℓ'/ℓ and ℓ'/k are Galois, thus $p_\ell(x) = p_{\ell'}(x) = p_k(x)$ by ($2'$), which settles case (2). Finally if $\ell = k(t)$ with $k = \mathbb{F}_q$ finite, the Frobenius $\sigma(t) = t^q$, also not bijective on ℓ, still induces a distance preserving map on $\mathbf{F}(G,\ell)$ and $\mathbf{F}(H,\ell)$. Thus $d_\tau(x,y) = d_\tau(x,\sigma y)$ since $\sigma x = x$, but then $\sigma y = y$ by definition of $y = p_\ell(x)$, and $y \in \mathbf{F}(G,k)$ as desired.

Final inequality. For $x, y \in \mathbf{F}(H,\ell) \times \mathbf{F}(G,\ell)$, $\langle x,y\rangle_\tau \leqslant \langle x, p_\ell(y)\rangle_\tau$ by [8, 5.7.7] and for $y \in \mathbf{F}(G,k)$, also $p_\ell(y) = p_k(y)$ by commutativity of the first diagram.

Commutativity of the second diagram. For $x \in \mathbf{F}(H,\ell)$ and $y = \pi_G(x) \in \mathbf{F}(G,k)$,

$$d_\tau\left(x,y\right) \geqslant d_\tau\left(p_\ell(x), p_\ell(y)\right) = d_\tau\left(x, p_k(y)\right)$$

since p_ℓ is non-expanding, equal to the identity on $\mathbf{F}(H,\ell)$ and to p_k on $\mathbf{F}(G,k)$ by commutativity of the first diagram. Since $p_k(y) \in \mathbf{F}(H,k) \subset \mathbf{F}(G,k)$, it follows that $p_k(y) = y$ by definition of y. In particular $y \in \mathbf{F}(H,k)$, thus also $y = \pi_H(x)$.

Independence of τ and functoriality. Let G_1 and G_2 be reductive groups over k with faithful representations τ_1 and τ_2. Set $\tau_3 = \tau_1 \boxplus \tau_2$, a faithful representation of $G_3 = G_1 \times G_2$. Then $\mathbb{F}(G_3) = \mathbb{F}(G_1) \times_k \mathbb{F}(G_2)$ and for every extension m of k,

$$(\mathbf{F}(G_3,m), d_{\tau_3}) = (\mathbf{F}(G_1,m), d_{\tau_1}) \times (\mathbf{F}(G_2,m), d_{\tau_2})$$

in $\mathsf{CCat}(0)$. This actually means that for $x_3 = (x_1,x_2)$ and $y_3 = (y_1,y_2)$ in

$$\mathbf{F}(G_3,m) = \mathbf{F}(G_1,m) \times \mathbf{F}(G_2,m)$$

we have the usual Pythagorean formula

$$d_{\tau_3}(x_3, y_3) = \sqrt{d_{\tau_1}(x_1, y_1)^2 + d_{\tau_2}(x_2, y_2)^2}.$$

It immediately follows that

$$\left(\mathbf{F}(G_3, \ell) \overset{\pi_3}{\twoheadrightarrow} \mathbf{F}(G_3, k)\right) = \left(\mathbf{F}(G_1, \ell) \times \mathbf{F}(G_2, \ell) \overset{(\pi_1,\pi_2)}{\twoheadrightarrow} \mathbf{F}(G_1, k) \times \mathbf{F}(G_2, k)\right)$$

where $\pi_i = \pi_{G_i}$ is the retraction attached to τ_i. Applying this to $G_1 = G_2 = G$ and using the commutativity of our second diagram for the diagonal embedding $\Delta : G \hookrightarrow G \times G$, we obtain $\Delta \circ \pi_3 = (\pi_1, \pi_2) \circ \Delta$, where π_3 is now the retraction π_G attached to the faithful representation $\tau_1 \oplus \tau_2 = \Delta^*(\tau_3)$ of G. Thus $\pi_1 = \pi_3 = \pi_2$, i.e. π_G does not depend upon the choice of τ. Using the commutativity of our second diagram for the graph embedding $\Delta_f : G_1 \hookrightarrow G_1 \times G_2$ of a morphism $f : G_1 \to G_2$, we similarly obtain the functoriality of $G \mapsto \pi_G$. □

5.1.3. For $G = GL(V)$, evaluation at the tautological representation τ of G on V identifies $\mathbf{F}(G, -)$ with $\mathbf{F}(V \otimes_k -)$. For any reductive group G with a faithful representation τ on $V = V(\tau)$, the projection $p : \mathbf{F}(V) \twoheadrightarrow \mathbf{F}(G, k)$ of proposition 4.3 becomes the projection $p_k : \mathbf{F}(GL(V), k) \twoheadrightarrow \mathbf{F}(G, k)$ of the previous theorem for the embedding $\tau : G \hookrightarrow GL(V)$. Thus if ℓ is a separable extension of k, then every $x \in \mathbf{F}(G, \ell)$ is good. Similarly for every pair $x_1 = (V_1, \mathcal{F}_1)$ and $x_2 = (V_2, \mathcal{F}_2)$ of objects in Fil_k^ℓ, $\mathbf{F}(GL(V_1) \times GL(V_2), \ell) \simeq \mathbf{F}(V_1 \otimes_k \ell) \times \mathbf{F}(V_2 \otimes_k \ell)$ contains $(\mathcal{F}_1, \mathcal{F}_2)$, which implies that then also $\left(\mathsf{Fil}_k^\ell, \deg\right)$ is good. We thus obtain:

Proposition 5.2. — *Suppose that ℓ is a separable extension of k. Then*

$$\mathcal{F}_{HN} : \mathsf{Fil}_k^\ell \to \mathsf{Fil}_k \text{ is a } \otimes\text{-functor.}$$

For every $x \in \mathbf{F}(G, \ell)$, $\mathcal{F}_{HN}(x) := \mathcal{F}_{HN} \circ x$ belongs to $\mathbf{F}(G, k)$, i.e.

$$\mathcal{F}_{HN}(x) : \mathsf{Rep}(G) \to \mathsf{Fil}_k \text{ is an exact } \otimes\text{-functor.}$$

Moreover, $\mathcal{F}_{HN}(x) = \pi_G(x)$ in $\mathbf{F}(G, k)$.

Proof. — The last assertion follows either from proposition 4.3 (both $\mathcal{F}_{HN}(x)$ and $\pi_G(x)$ minimize $f \mapsto d_\tau(x, f)^2 = \|x\|_\tau^2 + \|f\|_\tau^2 - 2\langle x, f\rangle_\tau$ on $\mathbf{F}(G, k)$) or from the functoriality of π_G (for every $\sigma \in \mathsf{Rep}(G)$, $\pi_G(x)(\sigma) = \mathcal{F}_{HN}(x)(\sigma)$ by 2.4.4). □

Once we know that the projection $\pi_G : \mathbf{F}(G, \ell) \twoheadrightarrow \mathbf{F}(G, k)$ computes the Harder-Narasimhan filtrations, the compatibility of the latter with tensor product constructions also directly follows from the functoriality of $G \mapsto \pi_G$:

Proposition 5.3. — *The Harder-Narasimhan functor $\mathcal{F}_{HN} : \mathsf{Fil}_k^\ell \to \mathsf{Fil}_k$ is compatible with tensor products, symmetric and exterior powers, and duals.*

Proof. — Apply the functoriality of $G \mapsto \pi_G$ to $GL(V_1) \times GL(V_2) \to GL(V_1 \otimes V_2)$, $GL(V) \to GL(\mathrm{Sym}^r V)$, $GL(V) \to GL(\Lambda^r V)$ and $GL(V) \to GL(V^*)$. □

5.2. **Normed vector spaces.**

5.2.1. Let K be a field with a non-archimedean absolute value $|-| : K \to \mathbb{R}_+$ whose valuation ring $\mathcal{O} = \{x \in K : |x| \leqslant 1\}$ is Henselian with residue field ℓ. A K-*norm* on a finite dimensional K-vector space $\mathcal{V}$ is a function $\alpha : \mathcal{V} \to \mathbb{R}_+$ such that $\alpha(v) = 0 \Leftrightarrow v = 0$, $\alpha(v_1 + v_2) \leqslant \max\{\alpha(v_1), \alpha(v_2)\}$ and $\alpha(\lambda v) = |\lambda|\,\alpha(v)$ for every $v, v_1, v_2 \in \mathcal{V}$ and $\lambda \in K$. It is *splittable* if and only if there exists a K-basis $\underline{e} = (e_1, \cdots, e_r)$ of $\mathcal{V}$ such that $\alpha(v) = \max\{|\lambda_i|\,\alpha(e_i)\}$ for all $v = \sum \lambda_i e_i$ in $\mathcal{V}$; we then say that α and $\underline{e}$ are *adapted*, or that $\underline{e}$ is an orthogonal basis of $(\mathcal{V}, \alpha)$. We denote by $\mathbf{B}(\mathcal{V})$ the set of all splittable K-norms on $\mathcal{V}$: it is the extended Bruhat-Tits building of $GL(\mathcal{V})$. If K is locally compact, then every K-norm is splittable [13, Proposition 1.1]. Given two splittable K-norms α and β on $\mathcal{V}$, there is a K-basis $\underline{e}$ of $\mathcal{V}$ which is adapted to both ([6, Appendice] or [19]), we may furthermore assume that $\lambda_i = \log\alpha(e_i) - \log\beta(e_i)$ is non-increasing, and then [8, 6.1 & 5.2.8]

$$\mathbf{d}(\alpha, \beta) \stackrel{\text{def}}{=} (\lambda_1, \cdots, \lambda_r) \in \mathbb{R}^r_{\geqslant} \quad \text{and} \quad \nu(\alpha, \beta) \stackrel{\text{def}}{=} \lambda_1 + \cdots + \lambda_r \in \mathbb{R}$$

do not depend upon the chosen adapted basis $\underline{e}$ of $\mathcal{V}$. The functions

$$\mathbf{d} : \mathbf{B}(\mathcal{V}) \times \mathbf{B}(\mathcal{V}) \to \mathbb{R}^r_{\geqslant} \quad \text{and} \quad \nu : \mathbf{B}(\mathcal{V}) \times \mathbf{B}(\mathcal{V}) \to \mathbb{R}$$

satisfy the following properties [8, 6.1 & 5.2.8]: for every $\alpha, \beta, \gamma \in \mathbf{B}(\mathcal{V})$,

$$\mathbf{d}(\alpha, \gamma) \leqslant \mathbf{d}(\alpha, \beta) + \mathbf{d}(\beta, \gamma) \quad \text{and} \quad \nu(\alpha, \gamma) = \nu(\alpha, \beta) + \nu(\beta, \gamma)$$

where the inequality is with respect to the usual dominance order on the convex cone $\mathbb{R}^r_{\geqslant}$. A splittable K-norm α on $\mathcal{V}$ induces a splittable K-norm $\alpha_{\mathcal{X}}$ on every subquotient $\mathcal{X} = \mathcal{Y}/\mathcal{Z}$ of $\mathcal{V}$, given by the following formula: for every $x \in \mathcal{X}$,

$$\alpha_{\mathcal{X}}(x) \stackrel{\text{def}}{=} \inf\{\alpha(y) : \mathcal{Y} \ni y \mapsto x \in \mathcal{X}\} = \min\{\alpha(y) : \mathcal{Y} \ni y \mapsto x \in \mathcal{X}\}.$$

For a K-subspace $\mathcal{W}$ of $\mathcal{V}$ and any $\alpha, \beta \in \mathbf{B}(\mathcal{V})$, we then have [8, 6.3.3 & 5.2.10]

$$\begin{aligned} \mathbf{d}(\alpha, \beta) &\geqslant \mathbf{d}(\alpha_{\mathcal{W}}, \beta_{\mathcal{W}}) * \mathbf{d}(\alpha_{\mathcal{V}/\mathcal{W}}, \beta_{\mathcal{V}/\mathcal{W}}) \\ \text{and} \quad \nu(\alpha, \beta) &= \nu(\alpha_{\mathcal{W}}, \beta_{\mathcal{W}}) + \nu(\alpha_{\mathcal{V}/\mathcal{W}}, \gamma_{\mathcal{V}/\mathcal{W}}) \end{aligned}$$

where the $*$-operation just re-orders the components.

5.2.2. We denote by Norm_K the quasi-abelian $\otimes$-category of pairs $(\mathcal{V}, \alpha)$ where $\mathcal{V}$ is a finite dimensional K-vector space and α is a splittable K-norm on $\mathcal{V}$ [8, 6.4]. A *morphism* $f : (\mathcal{V}_1, \alpha_1) \to (\mathcal{V}_2, \alpha_2)$ is a K-linear morphism $f : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\alpha_2(f(x)) \leqslant \alpha_1(x)$ for every $x \in \mathcal{V}_1$. Its kernel and cokernels are given by $(\ker(f), \alpha_{1,\ker(f)})$ and $(\mathrm{coker}(f), \alpha_{2,\mathrm{coker}(f)})$. The morphism is strict if and only if

$$\alpha_2(y) = \inf\{\alpha_1(x) : f(x) = y\} = \min\{\alpha_1(x) : f(x) = y\}$$

for every $y \in f(\mathcal{V}_1)$. It is a mono-epi if and only if $f : \mathcal{V}_1 \to \mathcal{V}_2$ is an isomorphism, in which case $\nu(f_*(\alpha_1), \alpha_2) \geqslant 0$ with equality if and only if f is an isomorphism in Norm_K, where $f_*(\alpha_1)$ is the splittable K-norm on $\mathcal{V}_2$ with $f_*(\alpha_1)(f(x)) = \alpha_1(x)$. The tensor product of Norm_K is given by the formula

$$(\mathcal{V}_1, \alpha_1) \otimes (\mathcal{V}_2, \alpha_2) \stackrel{\text{def}}{=} (\mathcal{V}_1 \otimes_K \mathcal{V}_2, \alpha_1 \otimes \alpha_2)$$

where for every $v \in \mathcal{V}_1 \otimes_K \mathcal{V}_2$,

$$(\alpha_1 \otimes \alpha_2)(v) \stackrel{\text{def}}{=} \min\left\{\max\{\alpha_1(v_{1,i})\alpha_2(v_{2,i}) : i\} \,\middle|\, \begin{array}{l} v = \sum_i v_{1,i} \otimes v_{2,i} \\ v_{1,i} \in \mathcal{V}_1,\ v_{2,i} \in \mathcal{V}_2 \end{array}\right\}.$$

This formula indeed defines a splittable K-norm on $\mathcal{V}_1 \otimes \mathcal{V}_2$ by [6, 1.11].

5.2.3. A *lattice*[1] in $\mathcal{V}$ is a finitely generated $\mathcal{O}$-submodule L of $\mathcal{V}$ which spans $\mathcal{V}$ over K. Any such lattice is actually finite and free over $\mathcal{O}$. The *gauge norm* of L is the splittable K-norm $\alpha_L : \mathcal{V} \to \mathbb{R}_+$ defined by

$$\alpha_L(v) \stackrel{\text{def}}{=} \inf \{|\lambda| : v \in \lambda L\}.$$

This construction defines a faithful exact $\mathcal{O}$-linear $\otimes$-functor

$$\alpha_- : \mathsf{Bun}_{\mathcal{O}} \to \mathsf{Norm}_K$$

where $\mathsf{Bun}_{\mathcal{O}}$ is the quasi-abelian $\mathcal{O}$-linear $\otimes$-category of finite free $\mathcal{O}$-modules. A normed K-vector space $(\mathcal{V}, \alpha)$ belongs to the essential image of this functor if and only if $\alpha(\mathcal{V}) \subset |K|$. This essential image is stable under strict subobjects and quotients, and the functor is an equivalence of categories if $|K| = \mathbb{R}_+$.

5.2.4. Suppose that k is a subfield of $\mathcal{O}$. Thus $|k^\times| = 1$ and ℓ is an extension of k. We denote by Norm_k^K the quasi-abelian k-linear $\otimes$-category of pairs (V, α) where V is a finite dimensional k-vector space and α is a splittable K-norm on $V_K := V \otimes_k K$. A *morphism* $f : (V_1, \alpha_1) \to (V_2, \alpha_2)$ is a k-linear morphism $f : V_1 \to V_2$ inducing a morphism $f_K : (V_{1,K}, \alpha_1) \to (V_{2,K}, \alpha_2)$ in Norm_K. Its kernel and cokernel are given by the obvious formulas, the morphism is strict if and only if f_K is so, it is a mono-epi if and only if $f : V_1 \to V_2$ is an isomorphism, in which case $\nu(f_{K,*}(\alpha_1), \alpha_2) \geqslant 0$ with equality if and only if f is an isomorphism in Norm_k^K. The tensor product in Norm_k^K is given by $(V_1, \alpha_1) \otimes (V_2, \alpha_2) := (V_1 \otimes V_2, \alpha_1 \otimes \alpha_2)$ and the forgetful functor $\omega : \mathsf{Norm}_k^K \to \mathsf{Vect}_k$ is a faithful exact k-linear $\otimes$-functor which identifies the poset $\mathrm{Sub}(V, \alpha)$ of strict subobjects of (V, α) in Norm_k^K with the poset $\mathrm{Sub}(V)$ of k-subspaces of $V = \omega(V, \alpha)$. In addition, there are two exact $\otimes$-functors

$$\mathsf{Norm}_k^K \to \mathsf{Norm}_K, \quad (V, \alpha) \mapsto (V_K, \alpha) \text{ or } (V_K, \alpha_{V \otimes \mathcal{O}})$$

where $V \otimes \mathcal{O} = V \otimes_k \mathcal{O}$ is the standard $\mathcal{O}$-lattice in $V_K = V \otimes_k K$. We set

$$\mathrm{rank}(V, \alpha) \stackrel{\text{def}}{=} \dim_k V \quad \text{and} \quad \deg(V, \alpha) \stackrel{\text{def}}{=} \nu(\alpha_{V \otimes \mathcal{O}}, \alpha).$$

These functions are both plainly additive on short exact sequences and respectively constant and non-decreasing on mono-epis. More precisely, if $f : (V_1, \alpha_1) \to (V_2, \alpha_2)$ is a mono-epi, then $f : V_1 \to V_2$ is an isomorphism, $f_{K,*}(\alpha_{V_1 \otimes \mathcal{O}}) = \alpha_{V_2 \otimes \mathcal{O}}$ and

$$\begin{aligned} \deg(V_1, \alpha_1) &= \nu(\alpha_{V_1 \otimes \mathcal{O}}, \alpha_1) \\ &= \nu(\alpha_{V_2 \otimes \mathcal{O}}, f_{K,*}(\alpha_1)) \\ &= \nu(\alpha_{V_2 \otimes \mathcal{O}}, \alpha_2) - \nu(f_{K,*}(\alpha_1), \alpha_2) \leqslant \deg(V_2, \alpha_2) \end{aligned}$$

with equality if and only if f is an isomorphism in Norm_k^K.

[1] *Not to be confused with the eponymous notion from section 2.1.1*

5.2.5. We may thus consider the following set-up

$$\mathsf{A}=\mathsf{Vect}_k\quad\text{and}\quad\mathsf{C}=\mathsf{Norm}_k^K\quad\text{with}\quad\begin{cases}\omega(V,\alpha) &=& V,\\ \mathrm{rank}(V,\alpha) &=& \dim_k V,\\ \deg(V,\alpha) &=& \nu(\alpha_{V\otimes\mathcal{O}},\alpha),\end{cases}$$

giving rise to a HN-formalism on Norm_k^K, with HN-filtration

$$\mathcal{F}_{HN}:\mathsf{Norm}_k^K\to\mathsf{Fil}_k.$$

We will show that if ℓ is a separable extension of k, then for any reductive group G over k, sufficiently many α's in $(\mathsf{Norm}_k^K)^{\otimes}(\omega_{G,k})$ are good for the pair $(\mathsf{Norm}_k^K,\deg)$ itself to be good. In particular, $\mathcal{F}_{HN}$ is then a $\otimes$-functor.

5.2.6. *A variant.* Let Bun_k^K be the category of pairs (V,L) where V is a finite dimensional k-vector space and L is an $\mathcal{O}$-lattice in V_K. With the obvious morphisms and tensor products, this is yet another quasi-abelian k-linear $\otimes$-category, and the k-linear exact $\otimes$-functor $(V,L)\mapsto(V,\alpha_L)$ identifies Bun_k^K with a full subcategory of Norm_k^K, made of those (V,α) such that $\alpha(V_K)\subset|K|$, which is stable under strict subobjects and quotients. The above rank and degree functions on Norm_k^K therefore induce a HN-formalism on Bun_k^K whose corresponding HN-filtration

$$\mathcal{F}_{HN}:\mathsf{Bun}_k^K\to\mathsf{Fil}_k$$

is a $\otimes$-functor if ℓ is a separable extension of k. Note that

$$\deg(V,L)=\sum_{i=1}^{r}\log|\lambda_i|\quad\text{if}\quad V\otimes_k\mathcal{O}=\oplus_{i=1}^r\mathcal{O}e_i\text{ and }L=\oplus_{i=1}^r\mathcal{O}\lambda_ie_i.$$

If K is discretely valued, it is convenient to either normalize its valuation so that $\log|K^\times|=\mathbb{Z}$, or to renormalize the degree function on Norm_k^K, so that its restriction to Bun_k^K takes values in $\mathbb{Z}$. The HN-filtration on Bun_k^K is then a $\mathbb{Q}$-filtration.

5.2.7. For a reductive group G over $\mathcal{O}$, let $\mathbf{B}^e(G_K)$ be the extended Bruhat-Tits building of G_K. There is a canonical injective and functorial map [8, Theorem 132]

$$\boldsymbol{\alpha}:\mathbf{B}^e(G_K)\hookrightarrow\mathsf{Norm}_K^{\otimes}(\omega_{G,K})$$

from the building $\mathbf{B}^e(G_K)$ to the set $\mathsf{Norm}_K^{\otimes}(\omega_{G,K})$ of all factorizations

$$\omega_{G,K}:\mathsf{Rep}(G)\xrightarrow{\alpha}\mathsf{Norm}_K\xrightarrow{\omega}\mathsf{Vect}_K$$

of the standard fiber functor $\omega_{G,K}:\mathsf{Rep}(G)\to\mathsf{Vect}_K$ through an exact $\otimes$-functor

$$\alpha:\mathsf{Rep}(G)\longrightarrow\mathsf{Norm}_K.$$

Here $\mathsf{Rep}(G)$ is the quasi-abelian $\otimes$-category of algebraic representations of G on finite free $\mathcal{O}$-modules. We shall refer to α as a *K-norm on $\omega_{G,K}$*.

5.2.8. For a reductive group G over k, we set $\mathbf{B}^e(G,K) = \mathbf{B}^e(G_K)$. Pre-composition with the base change functor $\mathsf{Rep}(G) \to \mathsf{Rep}(G_{\mathcal{O}})$ then yields a map

$$\mathsf{Norm}_K^{\otimes}(\omega_{G_{\mathcal{O}},K}) \to \mathsf{Norm}_K^{\otimes}(\omega_{G,K})$$

which is injective: a K-norm on $\omega_{G_{\mathcal{O}},K}$ is uniquely determined by its values on arbitrary large finite free subrepresentations of the representation of $G_{\mathcal{O}}$ on its ring of regular functions $\mathcal{A}(G_{\mathcal{O}}) = \mathcal{A}(G) \otimes_k \mathcal{O}$ [8, 6.4.17], and those coming from finite dimensional subrepresentations of $\mathcal{A}(G)$ form a cofinal system. Note that

$$\mathsf{Norm}_K^{\otimes}(\omega_{G,K}) = (\mathsf{Norm}_k^K)^{\otimes}(\omega_{G,k}).$$

We thus obtain a canonical, functorial injective map

$$\boldsymbol{\alpha} : \mathbf{B}^e(G,K) \hookrightarrow (\mathsf{Norm}_k^K)^{\otimes}(\omega_{G,k}).$$

We will show that if ℓ is a separable extension of k, then any α in

$$\mathbf{B}(\omega_G,K) = \boldsymbol{\alpha}(\mathbf{B}^e(G,K)) \subset (\mathsf{Norm}_k^K)^{\otimes}(\omega_{G,k})$$

is good in the sense of section 4.3.

5.2.9. For a reductive group G over k, the extended Bruhat-Tits building $\mathbf{B}^e(G,K)$ of G_K is equipped with an an action of $G(K)$, a $G(K)$-equivariant addition map

$$+ : \mathbf{B}^e(G,K) \times \mathbf{F}(G,K) \twoheadrightarrow \mathbf{B}^e(G,K),$$

a distinguished point $\circ$ fixed by $G(\mathcal{O})$, and the corresponding localization map

$$\mathrm{loc} : \mathbf{B}^e(G,K) \twoheadrightarrow \mathbf{F}(G,\ell).$$

For $f \in \mathbf{F}(G,k) \subset \mathbf{F}(G,K)$, $\mathrm{loc}(\circ + f) = f$ in $\mathbf{F}(G,k) \subset \mathbf{F}(G,\ell)$, i.e.

$$\mathbf{F}(G,k) \xrightarrow{\ \circ+\ } \mathbf{B}^e(G,K) \xrightarrow{\ \mathrm{loc}\ } \mathbf{F}(G,\ell)$$

is the base change map $\mathbf{F}(G,k) \hookrightarrow \mathbf{F}(G,\ell)$. For $G = GL(V)$, the composition

$$\mathbf{B}^e(G,K) \xrightarrow{\boldsymbol{\alpha}} \mathbf{B}(\omega_G,K) \xrightarrow{\mathrm{ev}} \mathsf{Norm}_k^K(V) = \mathbf{B}(V_K)$$

of the isomorphism $\boldsymbol{\alpha}$ with evaluation at the tautological representation of G on V is a bijection from $\mathbf{B}^e(G,K)$ to the set $\mathbf{B}(V_K)$ of all splittable K-norms on V_K. The distinguished point is the gauge norm of $V \otimes \mathcal{O}$, the addition map is given by

$$(\alpha + \mathcal{F})(v) \stackrel{\text{def}}{=} \min\left\{\max\left\{e^{-\gamma}\alpha(v_\gamma) : \gamma \in \mathbb{R}\right\} : v = \sum v_\gamma,\, v_\gamma \in \mathcal{F}^\gamma\right\},$$

and the localization map $\mathrm{loc} : \mathbf{B}(V_K) \to \mathbf{F}(V_\ell)$ sends α to the $\mathbb{R}$-filtration

$$\mathrm{loc}(\alpha)^\gamma \stackrel{\text{def}}{=} \frac{\{v \in V \otimes \mathcal{O} : \alpha(v) \leqslant e^{-\gamma}\} + V \otimes \mathfrak{m}}{V \otimes \mathfrak{m}} \subseteq V_\ell = \frac{V \otimes \mathcal{O}}{V \otimes \mathfrak{m}}$$

where $\mathfrak{m} = \{\lambda \in K : |\lambda| < 1\}$ is the maximal ideal of $\mathcal{O}$. For a general reductive group G over k, the corresponding addition map, distinguished point and localization map on $\mathbf{B}(\omega_G,K)$ are given by the following formulas: for $\tau \in \mathsf{Rep}(G)$,

$$(\alpha + \mathcal{F})(\tau) \stackrel{\text{def}}{=} \alpha(\tau) + \mathcal{F}(\tau), \quad \boldsymbol{\alpha}(\circ)(\tau) \stackrel{\text{def}}{=} \alpha_{\omega_G(\tau)\otimes\mathcal{O}} \quad \text{and} \quad \mathrm{loc}(\alpha)(\tau) \stackrel{\text{def}}{=} \mathrm{loc}(\alpha(\tau)).$$

Lemma 5.4. — *If $\mathcal{O}$ is strictly Henselian, then $\mathbf{B}(\omega_G,K)$ contains the image of*

$$\mathsf{Bun}_{\mathcal{O}}^{\otimes}(\omega_{G,K}) \hookrightarrow \mathsf{Norm}_K^{\otimes}(\omega_{G,K}).$$

If moreover $|K| = \mathbb{R}_+$, then $\boldsymbol{\alpha} : \mathbf{B}^e(G,K) \to \mathsf{Norm}_K^{\otimes}(\omega_{G,K})$ is a bijection.

Proof. — Plainly $\omega_{G,\mathcal{O}} \in \mathsf{Bun}^{\otimes}_{\mathcal{O}}(\omega_{G,K})$ maps to $\boldsymbol{\alpha}(\circ) \in \mathbf{B}(\omega_G, K)$, and since all of our maps are equivariant under $G(K) = \mathrm{Aut}^{\otimes}(\omega_{G,K})$, it is sufficient to establish that $G(K)$ acts transitively on $\mathsf{Bun}^{\otimes}_{\mathcal{O}}(\omega_{G,K})$. Any $L \in \mathsf{Bun}^{\otimes}_{\mathcal{O}}(\omega_{G,K})$ is a faithful exact $\otimes$-functor $L : \mathsf{Rep}(G) \to \mathsf{Bun}_{\mathcal{O}}$. The groupoid of all such functors is equivalent to the groupoid of all G-bundles over $\mathrm{Spec}(\mathcal{O})$, and the latter are classified by the étale cohomology group $H^1_{et}(\mathrm{Spec}(\mathcal{O}), G)$, which is isomorphic to $H^1_{et}(\mathrm{Spec}(\ell), G)$ by [12, XXIV 8.1], which is trivial since ℓ is separably closed. It follows that all L's are isomorphic, i.e. indeed conjugated under $G(K) = \mathrm{Aut}^{\otimes}(\omega_{G,K})$. If also $|K| = \mathbb{R}_+$, then $\mathsf{Bun}_{\mathcal{O}} \to \mathsf{Norm}_K$ is an equivalence of categories, $\mathsf{Bun}^{\otimes}_{\mathcal{O}}(\omega_{G,K}) \to \mathsf{Norm}^{\otimes}_K(\omega_{G,K})$ is a bijection, and thus $\mathbf{B}(\omega_G, K) = \mathsf{Norm}^{\otimes}_K(\omega_{G,K})$. □

5.2.10. The choice of a faithful representation τ of G yields a distance d_τ on $\mathbf{B}^e(G, K)$ [8, 5.2.9], defined by $d_\tau(x, y) := \|\mathcal{F}\|_\tau$ if $y = x + \mathcal{F}$ in $\mathbf{B}^e(G, K)$, where $\|-\|_\tau : \mathbf{F}(G, K) \to \mathbb{R}_+$ is the length function attached to τ. The resulting metric space is CAT(0) [8, Lemma 112], complete when $(K, |-|)$ is discrete [8, Lemma 114], the addition map is non-expanding in both variables [8, 5.2.8], the localization map is non-expanding [8, 6.4.13 & 5.5.9], and the induced topology on $\mathbf{B}^e(G, K)$ does not depend upon the chosen τ. These constructions are covariantly functorial in G, compatible with products and embeddings, and covariantly functorial in $(K, |-|)$. In particular, we thus obtain a (strictly) commutative diagram of functors

$$\begin{tikzcd} \mathsf{Red}(k) \times \mathsf{HV}(k) \arrow[r, "\mathbf{B}^e(-,-)"] & \mathsf{Top} \\ \mathsf{Red}(G) \times \mathsf{HV}(k) \arrow[u, hook] \arrow[r, "(\mathbf{B}^e(-,-), d_\tau)"] & \mathsf{Cat}(0) \arrow[u, hook] \end{tikzcd}$$

where $\mathsf{HV}(k)$ is the category of Henselian valued extensions $(K, |-|)$ of k and $\mathsf{Cat}(0)$ is the category of CAT(0) metric spaces with distance preserving maps.

5.2.11. For a closed subgroup H of G, the commutative diagram of CAT(0)-spaces

$$\begin{tikzcd} (\mathbf{F}(H, k), d_\tau) \arrow[r, hook] \arrow[d, hook] & (\mathbf{F}(G, k), d_\tau) \arrow[d, hook] \\ (\mathbf{B}^e(H, K), d_\tau) \arrow[r, hook] & (\mathbf{B}^e(G, K), d_\tau) \end{tikzcd}$$

is cartesian: for $\mathcal{F} \in \mathbf{F}(G, k)$ such that $\circ + \mathcal{F} \in \mathbf{B}^e(H, K)$, $\mathrm{loc}(\circ + \mathcal{F}) = \mathcal{F}$ belongs to $\mathbf{F}(H, \ell)$, thus $\mathcal{F}$ belongs to $\mathbf{F}(H, k) = \mathbf{F}(G, k) \cap \mathbf{F}(H, \ell)$. The corresponding (a priori non-commutative) diagram of non-expanding retractions

$$\begin{tikzcd} (\mathbf{F}(H, k), d_\tau) & (\mathbf{F}(G, k), d_\tau) \arrow[l, two heads, "p_k"'] \\ (\mathbf{B}^e(H, K), d_\tau) \arrow[u, two heads, "\varpi_H"] & (\mathbf{B}^e(G, K), d_\tau) \arrow[l, two heads, dotted, "p_K"'] \arrow[u, two heads, "\varpi_G"'] \end{tikzcd}$$

has a caveat: since $(\mathbf{B}^e(H, K), d_\tau)$ may not be complete (and $\mathbf{B}^e(H, K)$ perhaps not even closed in $\mathbf{B}^e(G, K)$), we can not directly appeal to [5, II.2.4], but its proof shows that a non-expanding retraction p_K is at least well-defined on the subset

$$\mathbf{B}^e(G,K)' \overset{\text{def}}{=} \left\{ x \in \mathbf{B}^e(G,K) \;\middle|\; \begin{array}{c} \exists y \in \mathbf{B}^e(H,K) \text{ such that} \\ d_\tau(x,y) = \inf\left\{ d_\tau(x,y') : y' \in \mathbf{B}^e(H,K) \right\} \end{array} \right\}.$$

Of course $\mathbf{B}^e(H,K) \subset \mathbf{B}^e(G,K)'$ and $\mathbf{B}^e(G,K)' = \mathbf{B}^e(G,K)$ if $\mathbf{B}^e(H,K)$ is complete, for instance if H is a torus or if $(K, |-|)$ is discrete [8, 5.3.2].

Theorem 5.5. — *If ℓ is a separable extension of k, then*

$$\mathbf{B}^e(G,K)' \textit{ contains } \circ + \mathbf{F}(G,k).$$

Moreover, the diagrams

$$\begin{tikzcd} \mathbf{F}(H,k) \arrow[d, hook] & \mathbf{F}(G,k) \arrow[l, two heads, "p_k"'] \arrow[d, hook] \\ \mathbf{B}^e(H,K) & \mathbf{B}^e(G,K)' \arrow[l, two heads, "p_K"'] \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathbf{F}(H,k) \arrow[r, hook] & \mathbf{F}(G,k) \\ \mathbf{B}^e(H,K) \arrow[u, two heads, "\varpi_H"] \arrow[r, hook] & \mathbf{B}^e(G,K) \arrow[u, two heads, "\varpi_G"'] \end{tikzcd}$$

are commutative, ϖ_G does not depend upon τ and defines a retraction

$$\varpi : \mathbf{B}^e(-,K) \twoheadrightarrow \mathbf{F}(-,k)$$

of the embedding $\mathbf{F}(-,k) \hookrightarrow \mathbf{B}^e(-,K)$ of functors from $\mathsf{Red}(k)$ to Top.

Proof. — This is again essentially formal.

First claim and commutativity of the first diagram. For $\mathcal{F} \in \mathbf{F}(G,k)$ and any element $y \in \mathbf{B}^e(H,K)$,

$$d_\tau(\circ + \mathcal{F}, y) \geqslant d_\tau\left(\mathcal{F}, \mathrm{loc}(y)\right) \geqslant d_\tau\left(\mathcal{F}, p_\ell(\mathcal{F})\right) = d_\tau\left(\mathcal{F}, p_k(\mathcal{F})\right)$$

since loc is non-expanding and $p_\ell = p_k$ on $\mathbf{F}(G,k)$ by theorem 5.1, therefore

$$d_\tau\left(\mathcal{F}, p_k(\mathcal{F})\right) = d_\tau\left(\circ + \mathcal{F}, \circ + p_k(\mathcal{F})\right) = \inf\left\{ d_\tau(\circ + \mathcal{F}, y) : y \in \mathbf{B}^e(H,K) \right\}.$$

This says that $\circ + \mathcal{F} \in \mathbf{B}^e(G,K)'$ with $p_K(\circ + \mathcal{F}) = \circ + p_k(\mathcal{F})$.

Commutativity of the second diagram. For $x \in \mathbf{B}^e(H,K)$ and $\mathcal{F} := \varpi_G(x)$ in $\mathbf{F}(G,k)$, x and $\circ + \mathcal{F}$ belong to $\mathbf{B}^e(G,K)'$, moreover

$$d_\tau(x, \circ + \mathcal{F}) \geqslant d_\tau\left(p_K(x), p_K(\circ + \mathcal{F})\right) = d_\tau\left(x, \circ + p_k(\mathcal{F})\right)$$

by commutativity of the first diagram, thus $\mathcal{F} = p_k(\mathcal{F})$ by definition of $\mathcal{F} = \omega_G(x)$, in particular $\mathcal{F}$ belongs to $\mathbf{F}(H,k)$, from which easily follows that also $\mathcal{F} = \varpi_H(x)$.

Independence of τ and functoriality. Let G_1 and G_2 be reductive groups over k with faithful representations τ_1 and τ_2. Set $\tau_3 := \tau_1 \boxplus \tau_2$, a faithful representation of $G_3 := G_1 \times G_2$. Then

$$\left(\mathbf{B}^e(G_3,K), d_{\tau_3}\right) = \left(\mathbf{B}^e(G_1,K), d_{\tau_1}\right) \times \left(\mathbf{B}^e(G_2,K), d_{\tau_2}\right)$$

in $\mathsf{Cat}(0)$. This actually means that for $x_3 = (x_1, x_2)$ and $y_3 = (y_1, y_2)$ in

$$\mathbf{B}^e(G_3,K) = \mathbf{B}^e(G_1,K) \times \mathbf{B}^e(G_2,K)$$

we have the usual Pythagorean formula

$$d_{\tau_3}(x_3, y_3) = \sqrt{d_{\tau_1}(x_1,y_1)^2 + d_{\tau_2}(x_2,y_2)^2}.$$

It immediately follows that

$$\left(\mathbf{B}^e(G_3,K) \overset{\varpi_3}{\twoheadrightarrow} \mathbf{F}(G_3,k)\right) = \left(\mathbf{B}^e(G_1,K) \times \mathbf{B}^e(G_2,K) \overset{(\varpi_1,\varpi_2)}{\twoheadrightarrow} \mathbf{F}(G_1,k) \times \mathbf{F}(G_2,k)\right)$$

where $\varpi_i := \varpi_{G_i}$ is the retraction attached to τ_i. Applying this to $G_1 = G_2 = G$ and using the commutativity of our second diagram for the diagonal embedding $\Delta : G \hookrightarrow G\times G$, we obtain $\Delta\circ\varpi_3 = (\varpi_1, \varpi_2)\circ\Delta$, where ϖ_3 is now the retraction ϖ_G attached to the faithful representation $\tau_1 \oplus \tau_2 = \Delta^*(\tau_3)$ of G. Thus $\varpi_1 = \varpi_3 = \varpi_2$, i.e. ϖ_G does not depend upon the choice of τ. Using the commutativity of our second diagram for the graph embedding $\Delta_f : G_1 \hookrightarrow G_1 \times G_2$ of a morphism $f : G_1 \to G_2$, we similarly obtain the functoriality of $G \mapsto \varpi_G$. □

5.2.12. With notations as above, the *Busemann scalar product* is the function

$$\langle -, - \rangle_\tau : \mathbf{B}^e(G, K)^2 \times \mathbf{F}(G, K) \to \mathbb{R}$$

which maps $(x, y, \mathcal{F})$ to

$$\langle \overrightarrow{xy}, \mathcal{F} \rangle_\tau \stackrel{\text{def}}{=} \|\mathcal{F}\|_\tau \cdot \lim_{t\to\infty} \left(d_\tau(x, z + t\mathcal{F}) - d_\tau(y, z + t\mathcal{F}) \right).$$

Here z is any fixed point in $\mathbf{B}^e(G, K)$: the limit exists and does not depend upon the chosen z [8, 5.5.8]. For every $x, y, z \in \mathbf{B}^e(G, K)$, $\mathcal{F} \in \mathbf{F}(G, K)$ and $t \geqslant 0$,

$$\langle \overrightarrow{xz}, \mathcal{F} \rangle_\tau = \langle \overrightarrow{xy}, \mathcal{F} \rangle_\tau + \langle \overrightarrow{yz}, \mathcal{F} \rangle_\tau \quad \text{and} \quad \langle \overrightarrow{xy}, t\mathcal{F} \rangle_\tau = t \langle \overrightarrow{xy}, \mathcal{F} \rangle_\tau .$$

As a function of x, $\langle \overrightarrow{xy}, \mathcal{F} \rangle_\tau$ is convex and $\|\mathcal{F}\|_\tau$-Lipschitzian; as a function of y, it is concave and $\|\mathcal{F}\|_\tau$-Lipschitzian; as a function of $\mathcal{F}$, it is usually neither convex nor concave, but it is $d_\tau(x, y)$-Lipschitzian [8, 5.5.11]; as a function of τ, it is additive: if τ' is another faithful representation of G, then

$$\langle \overrightarrow{xy}, \mathcal{F} \rangle_{\tau\oplus\tau'} = \langle \overrightarrow{xy}, \mathcal{F} \rangle_\tau + \langle \overrightarrow{xy}, \mathcal{F} \rangle_{\tau'} .$$

For any $x \in \mathbf{B}^e(G, K)$ and $\mathcal{F} \in \mathbf{F}(G, k)$, we have the following inequality [8, 5.5.9]:

$$\left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau \leqslant \langle \mathrm{loc}(x), \mathcal{F} \rangle_\tau .$$

This is an equality when x belongs to $\mathbf{F}(G, k) \simeq \circ + \mathbf{F}(G, k)$.

Proposition 5.6. — *Suppose that ℓ is a separable extension of k. Let H be a reductive subgroup of G. Then for every $x \in \mathbf{B}^e(H, K)$ and $\mathcal{F} \in \mathbf{F}(G, k)$,*

$$\left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau \leqslant \left\langle \overrightarrow{\circ x}, p_k(\mathcal{F}) \right\rangle_\tau$$

where $p_k : \mathbf{F}(G, k) \twoheadrightarrow \mathbf{F}(H, k)$ is the convex projection attached to d_τ.

Proof. — Set $\mathcal{G} = p_k(\mathcal{F}) \in \mathbf{F}(H, k)$ and pick a splitting of $\mathcal{G}$ [8, Cor. 63], corresponding to an $\mathbb{R}$-filtration $\mathcal{G}' \in \mathbf{F}(H, k)$ opposed to $\mathcal{G}$: for any representation σ of H,

$$\omega_{H,k}(\sigma) = \oplus_{\gamma\in\mathbb{R}} \mathcal{G}(\sigma)^\gamma \cap \mathcal{G}'(\sigma)^{-\gamma}.$$

Let $Q_\mathcal{G} \subset P_\mathcal{G}$ and $Q_{\mathcal{G}'} \subset P_{\mathcal{G}'}$ be the stabilizers of $\mathcal{G}$ and $\mathcal{G}'$ in H and G, so that $(Q_\mathcal{G}, Q_{\mathcal{G}'})$ and $(P_\mathcal{G}, P_{\mathcal{G}'})$ are pairs of opposed parabolic subgroups of H and G, with Levi subgroups $H' := Q_\mathcal{G} \cap Q_{\mathcal{G}'}$ and $G' := P_\mathcal{G} \cap P_{\mathcal{G}'}$. Let $R^u(-)$ denote the unipotent radical. Then for $\star \in \{k, \ell, K\}$, $\mathbf{B}^e(H', K)$, $\mathbf{B}^e(G', K)$, $\mathbf{F}(H', \star)$ and $\mathbf{F}(G', \star)$ are fundamental domains for the actions of $R^u Q_\mathcal{G}(K)$, $R^u P_\mathcal{G}(K)$, $R^u Q_\mathcal{G}(\star)$ and $R^u P_\mathcal{G}(\star)$ on respectively $\mathbf{B}^e(H, K)$, $\mathbf{B}^e(G, K)$, $\mathbf{F}(H, \star)$ and $\mathbf{F}(G, \star)$ [8, 5.2.10].

We denote by the same letter r the corresponding retractions. They are all non-expanding, and the following diagrams are commutative:

$$\begin{tikzcd} \mathbf{B}^e(H,K) \arrow[r, hook] \arrow[d, "r"'] & \mathbf{B}^e(G,K) \arrow[d, "r"] \\ \mathbf{B}^e(H',K) \arrow[r, hook] & \mathbf{B}^e(G',K) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{F}(H,\star) \arrow[r, hook] \arrow[d, "r"'] & \mathbf{F}(G,\star) \arrow[d, "r"] \\ \mathbf{F}(H',\star) \arrow[r, hook] & \mathbf{F}(G',\star) \end{tikzcd}$$

Let $x' := r(x)$ and $\mathcal{F}' := r(\mathcal{F})$, so that $x' \in \mathbf{B}^e(H',K)$, $\mathcal{F}' \in \mathbf{F}(G',k)$. Note that already $\mathcal{G}, \mathcal{G}' \in \mathbf{F}(H',k)$. We will establish the following inequalities:

$$\left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau \overset{(1)}{\leqslant} \left\langle \overrightarrow{\circ x}', \mathcal{F}' \right\rangle_\tau \overset{(2)}{\leqslant} \langle \mathrm{loc}(x'), \mathcal{F}' \rangle_\tau \overset{(3)}{\leqslant} \langle \mathrm{loc}(x'), \mathcal{G} \rangle_\tau \overset{(4)}{=} \left\langle \overrightarrow{\circ x}', \mathcal{G} \right\rangle_\tau \overset{(5)}{=} \left\langle \overrightarrow{\circ x}, \mathcal{G} \right\rangle_\tau .$$

The second inequality was already mentioned just before the proposition.

Proof of (1). Since $\mathcal{F}' = r(\mathcal{F})$, there is a $u \in R^u P_{\mathcal{G}}(k)$ such that $\mathcal{F}' = u\mathcal{F}$. Since $u \in G(k)$ and all of our distances, norms etc... are $G(k)$-invariant, it follows that $\|\mathcal{F}\|_\tau = \|\mathcal{F}'\|_\tau$. Since $u \in G(\mathcal{O})$ fixes $\circ$, $u(\circ + t\mathcal{F}) = \circ + t\mathcal{F}'$ belongs to $\mathbf{B}^e(G',K)$. Since $u \in R^u P_{\mathcal{G}}(K)$, $r(\circ + t\mathcal{F}) = \circ + t\mathcal{F}'$ for all $t \geqslant 0$. Thus

$$\begin{aligned} \left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau &= \|\mathcal{F}\|_\tau \lim_{t\to\infty} \left(t \|\mathcal{F}\|_\tau - d_\tau(x, \circ + t\mathcal{F})\right) \\ &\leqslant \|\mathcal{F}'\|_\tau \lim_{t\to\infty} \left(t \|\mathcal{F}'\|_\tau - d_\tau(x', \circ + t\mathcal{F}')\right) \\ &= \left\langle \overrightarrow{\circ x}', \mathcal{F}' \right\rangle_\tau \end{aligned}$$

since $r : \mathbf{B}^e(G,K) \twoheadrightarrow \mathbf{B}^e(G',K)$ is non-expanding.

Proof of (3). Note that $\mathrm{loc}(x') \in \mathbf{F}(H',\ell)$ and $\mathcal{F}' \in \mathbf{F}(G',k)$. By the last assertion of theorem 5.1, it is sufficient to establish that $p'_k(\mathcal{F}') = \mathcal{G}$ for the convex projection $p'_k : \mathbf{F}(G',k) \twoheadrightarrow \mathbf{F}(H',k)$ – which is usually not equal to the restriction of $p_k : \mathbf{F}(G,k) \twoheadrightarrow \mathbf{F}(H,k)$ to $\mathbf{F}(G',k)$. For $t \gg 0$, $\mathcal{F} + t\mathcal{G} = \mathcal{F}' + t\mathcal{G}$ by [8, 5.6.2]. In particular $\mathcal{F} + t\mathcal{G}$ belongs to $\mathbf{F}(G',k)$ since $\mathcal{F}'$ and $\mathcal{G}$ do. On the other hand,

$$p_k(\mathcal{F} + t\mathcal{G}) = (1+t) p_k\left(\tfrac{1}{1+t}\mathcal{F} + \tfrac{t}{1+t}\mathcal{G}\right) = (1+t)\mathcal{G}$$

using [5, II.2.4] for the second equality. Since this belongs to $\mathbf{F}(H',k)$, actually

$$p'_k(\mathcal{F}' + t\mathcal{G}) = p'_k(\mathcal{F} + t\mathcal{G}) = p_k(\mathcal{F} + t\mathcal{G}) = (1+t)\mathcal{G}.$$

Now observe that $\mathcal{H} \mapsto \mathcal{H} + t\mathcal{G}$ and $\mathcal{H} \mapsto \mathcal{H} + t\mathcal{G}'$ are mutually inverse isometries of $\mathbf{F}(G',k)$ and $\mathbf{F}(H',k)$, thus p'_k commutes with both of them and

$$p'_k(\mathcal{F}') = p'_k(\mathcal{F}' + t\mathcal{G}) + t\mathcal{G}' = (1+t)\mathcal{G} + t\mathcal{G}' = \mathcal{G}.$$

Proof of (4). This follows from [8, 5.5.3].

Proof of (5). Since $x' = r(x)$, there is a $u \in R^u Q_{\mathcal{G}}(K)$ such that $ux = x'$. For $t \gg 0$, u fixes $\circ + t\mathcal{G}$ by [8, 5.4.6]. Then $d_\tau(x', \circ + t\mathcal{G}) = d_\tau(x, \circ + t\mathcal{G})$ and

$$\begin{aligned} \left\langle \overrightarrow{\circ x}', \mathcal{G} \right\rangle_\tau &= \|\mathcal{G}\|_\tau \lim_{t\to\infty} \left(t \|\mathcal{G}\|_\tau - d_\tau(x', \circ + t\mathcal{G})\right) \\ &= \|\mathcal{G}\|_\tau \lim_{t\to\infty} \left(t \|\mathcal{G}\|_\tau - d_\tau(x, \circ + t\mathcal{G})\right) \\ &= \left\langle \overrightarrow{\circ x}, \mathcal{G} \right\rangle_\tau . \end{aligned}$$

This finishes the proof of the proposition. □

Corollary 5.7. — *For every $x \in \mathbf{B}^e(G,K)$, $\mathcal{F} \mapsto \left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau$ is concave on $\mathbf{F}(G,k)$.*

Proof. — We have to show that for any $x \in \mathbf{B}^e(G,K)$ and $\mathcal{F},\mathcal{G} \in \mathbf{F}(G,k)$,

$$\left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau + \left\langle \overrightarrow{\circ x}, \mathcal{G} \right\rangle_\tau \leqslant \left\langle \overrightarrow{\circ x}, \mathcal{F}+\mathcal{G} \right\rangle_\tau .$$

For the diagonal embedding $\Delta : G \hookrightarrow G \times G$, the proposition gives

$$\left\langle \overrightarrow{\circ x}, \mathcal{H} \right\rangle_{\tau \boxplus \tau} \leqslant \left\langle \overrightarrow{\circ x}, p_k(\mathcal{H}) \right\rangle_{\tau \oplus \tau} = 2 \left\langle \overrightarrow{\circ x}, p_k(\mathcal{H}) \right\rangle_\tau$$

for every $\mathcal{H}$ in $\mathbf{F}(G \times G, k) = \mathbf{F}(G,k) \times \mathbf{F}(G,k)$. For $\mathcal{H} = (\mathcal{F},\mathcal{G})$, we have

$$\left\langle \overrightarrow{\circ x}, \mathcal{H} \right\rangle_{\tau \boxplus \tau} = \left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau + \left\langle \overrightarrow{\circ x}, \mathcal{G} \right\rangle_\tau$$

and $p_k(\mathcal{H})$ is the point closest to $(\mathcal{F},\mathcal{G})$ in the diagonally embedded $\mathbf{F}(G,k)$: the middle point $\frac{1}{2}(\mathcal{F}+\mathcal{G}) = \frac{1}{2}\mathcal{F} + \frac{1}{2}\mathcal{G}$ of the geodesic segment $[\mathcal{F},\mathcal{G}]$ of $\mathbf{F}(G,k)$. Thus

$$\left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau + \left\langle \overrightarrow{\circ x}, \mathcal{G} \right\rangle_\tau \leqslant 2 \left\langle \overrightarrow{\circ x}, \tfrac{1}{2}(\mathcal{F}+\mathcal{G}) \right\rangle_\tau = \left\langle \overrightarrow{\circ x}, \mathcal{F}+\mathcal{G} \right\rangle_\tau ,$$

which proves the corollary. $\square$

5.2.13. For $\mathcal{V} \in \mathsf{Vect}_K$ and for the canonical metric on $\mathbf{F}(\mathcal{V})$, there is an explicit formula for the corresponding Busemann scalar product

$$\langle -, - \rangle : \mathbf{B}(\mathcal{V})^2 \times \mathbf{F}(\mathcal{V}) \to \mathbb{R}.$$

which maps $(\alpha,\beta,\mathcal{F})$ to

$$\left\langle \overrightarrow{\alpha\beta}, \mathcal{F} \right\rangle = \|\mathcal{F}\| \cdot \lim_{t\to\infty} \left(d(\alpha, \gamma + t\mathcal{F}) - d(\beta, \gamma + t\mathcal{F}) \right).$$

By [8, 6.4.15], the latter may indeed be computed as

$$\left\langle \overrightarrow{\alpha\beta}, \mathcal{F} \right\rangle = \sum_\gamma \gamma\, \nu \left(\mathrm{Gr}^\gamma_{\mathcal{F}}(\alpha), \mathrm{Gr}^\gamma_{\mathcal{F}}(\beta) \right)$$

where $\mathrm{Gr}^\gamma_{\mathcal{F}}(\alpha)$ and $\mathrm{Gr}^\gamma_{\mathcal{F}}(\beta)$ are the splittable K-norms on $\mathrm{Gr}^\gamma_{\mathcal{F}}(\mathcal{V})$ induced by α and β. If $\mathcal{V} = V_K$ and $\mathcal{F} = f_K$ for some $V \in \mathsf{Vect}_k$ and $f \in \mathbf{F}(V)$, then $\mathrm{Gr}^\gamma_{\mathcal{F}}(\mathcal{V})$ equals $\mathrm{Gr}^\gamma_f(V) \otimes_k K$; if moreover α is the gauge norm of $V \otimes_k \mathcal{O}$, then $\mathrm{Gr}^\gamma_{\mathcal{F}}(\alpha)$ is the gauge norm of $\mathrm{Gr}^\gamma_f(V) \otimes_k \mathcal{O}$. In particular, the pairing of section 4.2.1,

$$\langle -, - \rangle : \mathsf{Norm}^K_k(V) \times \mathbf{F}(V) \to \mathbb{R}, \quad \langle \alpha, f \rangle = \sum \gamma \deg \mathrm{Gr}^\gamma_f(\alpha)$$

is related to the Busemann scalar product by the formula

$$\langle \alpha, f \rangle = \langle \overrightarrow{\alpha_{V\otimes\mathcal{O}}\alpha}, f_K \rangle .$$

5.2.14. The previous formula yields another proof of corollary 5.7, which now works without any assumption on the extension ℓ of k: for every $x \in \mathbf{B}^e(G,K)$, the function $\mathcal{F} \mapsto \left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau$ is concave on $\mathbf{F}(G,k)$ since for $\alpha := \boldsymbol{\alpha}(x) \in \mathbf{B}(\omega_G, K)$,

$$\left\langle \overrightarrow{\circ x}, \mathcal{F} \right\rangle_\tau = \left\langle \overrightarrow{\circ(\tau)x(\tau)}, \mathcal{F}(\tau) \right\rangle = \langle \alpha(\tau), \mathcal{F}(\tau) \rangle$$

and $f \mapsto \langle \alpha(\tau), f \rangle$ is a degree function on $\mathbf{F}(\omega_{G,k}(\tau))$. If ℓ is a separable extension of k, proposition 5.6 implies that every $\alpha \in \mathbf{B}(\omega_G, K)$ is good. On the other hand for every pair of objects (V_1,α_1) and (V_2,α_2) in Norm^K_k and $G := GL(V_1) \times GL(V_2)$,

$$\mathbf{B}^e(G,K) \simeq \mathbf{B}(\omega_G, K) \simeq \mathbf{B}(V_{1,K}) \times \mathbf{B}(V_{2,K})$$

contains (α_1,α_2), therefore $(\mathsf{Norm}^K_k, \deg)$ is then also good. We obtain:

THEOREM 5.8. — *Suppose that ℓ is a separable extension of k. Then*

$$\mathcal{F}_{HN} : \mathsf{Norm}_k^K \to \mathsf{Fil}_k \text{ is a } \otimes\text{-functor.}$$

For every $\alpha \in \mathbf{B}(\omega_G, K)$, $\mathcal{F}_{HN}(\alpha) := \mathcal{F}_{HN} \circ \alpha$ belongs to $\mathbf{F}(G,k)$, i.e.

$$\mathcal{F}_{HN}(\alpha) : \mathsf{Rep}(G) \to \mathsf{Fil}_k \text{ is an exact } \otimes\text{-functor.}$$

For any faithful representation τ of G and $x \in \mathbf{B}^e(G,K)$, $\pi_G(x) := \mathcal{F}_{HN}(\boldsymbol{\alpha}(x))$ is the unique element $\mathcal{F}$ of $\mathbf{F}(G,k)$ which satisfies the following equivalent conditions:

1. *For every $f \in \mathbf{F}(G,k)$, $\|\mathcal{F}\|_\tau^2 - 2\left\langle \overrightarrow{\circ x}, \mathcal{F}\right\rangle_\tau \leqslant \|f\|_\tau^2 - 2\left\langle \overrightarrow{\circ x}, f\right\rangle_\tau$.*
2. *For every $f \in \mathbf{F}(G,k)$, $\left\langle \overrightarrow{\circ x}, f\right\rangle_\tau \leqslant \langle \mathcal{F}, f\rangle_\tau$ with equality for $f = \mathcal{F}$.*
3. *For every $\gamma \in \mathbb{R}$, $\mathrm{Gr}_{\mathcal{F}}^\gamma(\boldsymbol{\alpha}(x))(\tau)$ is semi-stable of slope γ.*

The function $x \mapsto \pi_G(x)$ is non-expanding for d_τ and defines a retraction

$$\pi : \mathbf{B}^e(-,K) \twoheadrightarrow \mathbf{F}(-,k)$$

of the embedding $\mathbf{F}(-,k) \hookrightarrow \mathbf{B}^e(-,K)$ of functors from $\mathsf{Red}(k)$ to Top.

Proof. — Everything follows from proposition 4.3 except the last sentence, which still requires a proof. For $x, y \in \mathbf{B}^e(G,K)$, set $\mathcal{F} := \pi_G(x)$ and $\mathcal{G} := \pi_G(y)$. Then

$$\begin{aligned}
d_\tau(\mathcal{F},\mathcal{G})^2 &= \|\mathcal{F}\|_\tau^2 + \|\mathcal{G}\|_\tau^2 - \langle \mathcal{F},\mathcal{G}\rangle_\tau - \langle \mathcal{G},\mathcal{F}\rangle_\tau \\
&\leqslant \left\langle \overrightarrow{\circ x}, \mathcal{F}\right\rangle_\tau + \left\langle \overrightarrow{\circ y}, \mathcal{G}\right\rangle_\tau - \left\langle \overrightarrow{\circ x}, \mathcal{G}\right\rangle_\tau - \left\langle \overrightarrow{\circ y}, \mathcal{F}\right\rangle_\tau \\
&= \langle \overrightarrow{xy}, \mathcal{G}\rangle_\tau - \langle \overrightarrow{xy}, \mathcal{F}\rangle_\tau \\
&\leqslant d_\tau(x,y) \cdot d_\tau(\mathcal{F},\mathcal{G})
\end{aligned}$$

thus $d_\tau(\mathcal{F},\mathcal{G}) \leqslant d_\tau(x,y)$, i.e. $\pi_G : \mathbf{B}^e(G,K) \to \mathbf{F}(G,k)$ is indeed non-expanding for d_τ. It is plainly functorial in G. For $\mathcal{F}, f \in \mathbf{F}(G,k)$ and $x := \circ + \mathcal{F}$, we have

$$\left\langle \overrightarrow{\circ x}, f\right\rangle_\tau = \langle \mathcal{F}, f\rangle_\tau$$

thus $\pi_G(x) = \mathcal{F}$, i.e. π is indeed a retraction of $\mathbf{F}(-,k) \hookrightarrow \mathbf{B}^e(-,K)$. □

Once we know that the projection $\pi_G : \mathbf{B}^e(G,K) \twoheadrightarrow \mathbf{F}(G,k)$ computes the Harder-Narasimhan filtrations, the compatibility of the latter with tensor product constructions again directly follows from the functoriality of $G \mapsto \pi_G$:

PROPOSITION 5.9. — *The Harder-Narasimhan functor $\mathcal{F}_{HN} : \mathsf{Norm}_k^K \to \mathsf{Fil}_k$ is compatible with tensor products, symmetric and exterior powers, and duals.*

Proof. — Apply the functoriality of $G \mapsto \pi_G$ to $GL(V_1) \times GL(V_2) \to GL(V_1 \otimes V_2)$, $GL(V) \to GL(\mathrm{Sym}^r V)$, $GL(V) \to GL(\Lambda^r V)$ and $GL(V) \to GL(V^*)$. □

Remark 5.10. — We now have three non-expanding retractions of $\mathbf{F}(-,k) \hookrightarrow \mathbf{B}^e(-,K)$: (1) the composition $\pi \circ \mathrm{loc}$ where $\pi : \mathbf{F}(-,\ell) \twoheadrightarrow \mathbf{F}(-,k)$ is the convex projection from theorem 5.1, which computes the Harder-Narasimhan filtration on Fil_k^ℓ; (2) the convex projection $\omega : \mathbf{B}^e(-,K) \twoheadrightarrow \mathbf{F}(-,k)$ from theorem 5.5; (3) the retraction $\pi : \mathbf{B}^e(-,K) \twoheadrightarrow \mathbf{F}(-,k)$ that we have just defined, which computes the Harder-Narasimhan filtration on Norm_k^K. We leave it to the reader to verify that already for $G = PGL(2)$, these three retractions are pairwise distinct.

5.3. Normed φ-modules.

5.3.1. Let $k = \mathbb{F}_q$ be a finite field, K an extension of k, $|-| : K \to \mathbb{R}_+$ a non-archimedean absolute value such that the local k-algebra $\mathcal{O} = \{x \in K : |x| \leqslant 1\}$ is Henselian with residue field ℓ, K^s a fixed separable closure of K with Galois group $\mathrm{Gal}_K = \mathrm{Gal}(K^s/K)$. The category $\mathsf{Rep}_k(\mathrm{Gal}_K)$ of continuous (i.e. with open kernels) representations (V,ρ) of Gal_K on finite dimensional k-vector spaces is a k-linear neutral tannakian category which is equivalent to the category $\mathsf{Vect}_K^{\varphi}$ of étale φ-modules $(\mathcal{V},\varphi_{\mathcal{V}})$ over K. Here $\varphi(x) = x^q$ is the Frobenius of K, $\mathcal{V}$ is a finite dimensional K-vector space and $\varphi_{\mathcal{V}} : \varphi^*\mathcal{V} \to \mathcal{V}$ is a K-linear isomorphism where $\varphi^*\mathcal{V} = \mathcal{V} \otimes_{K,\varphi} K$. The equivalence of categories is given by

$$\begin{array}{rcl} (V,\rho) & \to & \left((V \otimes_k K^s)^{\mathrm{Gal}_K}, \mathrm{Id}_V \otimes \varphi\right) \\ \left((\mathcal{V} \otimes_K K^s)^{\varphi_{\mathcal{V}}\otimes\varphi = \mathrm{Id}}, \gamma \mapsto \mathrm{Id} \otimes \gamma\right) & \leftarrow & (\mathcal{V},\varphi_V) \end{array}$$

5.3.2. We denote by $\mathsf{Norm}_K^{\varphi}$ the quasi-abelian k-linear $\otimes$-category of all triples $(\mathcal{V},\varphi_{\mathcal{V}},\alpha)$ where $(\mathcal{V},\varphi_{\mathcal{V}})$ is an étale φ-module and α is a splittable K-norm on $\mathcal{V}$, with the obvious morphisms and $\otimes$-products. It comes with two exact $\otimes$-functors

$$\mathsf{Norm}_K^{\varphi} \to \mathsf{Norm}_K, \quad (\mathcal{V},\varphi_{\mathcal{V}},\alpha) \mapsto (\mathcal{V},\alpha) \text{ or } (\mathcal{V},\varphi_{\mathcal{V}}(\alpha))$$

where $\varphi_{\mathcal{V}}(\alpha)$ is the splittable K-norm on $\mathcal{V}$ defined by

$$\begin{array}{rl} & (\varphi_{\mathcal{V}}(\alpha))(v) \stackrel{\mathrm{def}}{=} (\varphi^*\alpha)(\varphi_{\mathcal{V}}^{-1}(v)) \\ \text{with} & (\varphi^*\alpha)(v') \stackrel{\mathrm{def}}{=} \min\left\{\max\{|\lambda_i|\,\alpha(v_i)^q\} : \begin{array}{c} v' = \sum v_i \otimes \lambda_i \\ \lambda_i \in K,\, v_i \in V \end{array}\right\} \end{array}$$

for $v \in \mathcal{V}$ and $v' \in \varphi^*\mathcal{V} := \mathcal{V} \otimes_{K,\varphi} K$. Note that for $\alpha,\beta \in \mathbf{B}(\mathcal{V})$,

$$\mathbf{d}(\varphi_{\mathcal{V}}(\alpha),\varphi_{\mathcal{V}}(\beta)) = q \cdot \mathbf{d}(\alpha,\beta) \in \mathbb{R}^r_{\geqslant} \quad \text{and} \quad \nu(\varphi_{\mathcal{V}}(\alpha),\varphi_{\mathcal{V}}(\beta)) = q \cdot \nu(\alpha,\beta) \in \mathbb{R}.$$

5.3.3. We may then consider the following setup:

$$\begin{array}{rcl} \mathsf{A} & = & \mathsf{Rep}_k(\mathrm{Gal}_K) \\ \mathsf{C} & = & \mathsf{Norm}_K^{\varphi} \end{array} \quad \text{with} \quad \left\{ \begin{array}{rcl} \omega(\mathcal{V},\varphi_{\mathcal{V}},\alpha) & = & (\mathcal{V} \otimes_K K^s)^{\varphi_{\mathcal{V}}\otimes\varphi=\mathrm{Id}}, \\ \mathrm{rank}(V,\rho) & = & \dim_k V, \\ \deg(\mathcal{V},\varphi_{\mathcal{V}},\alpha) & = & \nu(\alpha,\varphi_{\mathcal{V}}(\alpha)). \end{array} \right.$$

These data again satisfy the assumptions of sections 4.1-4.2. For instance, if

$$f : (\mathcal{V}_1,\varphi_1,\alpha_1) \to (\mathcal{V}_2,\varphi_2,\alpha_2)$$

is a mono-epi in $\mathsf{Norm}_K^{\varphi}$, then $f : (\mathcal{V}_1,\varphi_1) \to (\mathcal{V}_2,\varphi_2)$ is an isomorphism and

$$\begin{aligned} \nu\left(\alpha_1,\varphi_1(\alpha_1)\right) &= \nu\left(f_*(\alpha_1), f_*(\varphi_1(\alpha_1))\right) \\ &= \nu(f_*(\alpha_1),\alpha_2) + \nu(\alpha_2,\varphi_2(\alpha_2)) + \nu(\varphi_2(\alpha_2),\varphi_2(f_*(\alpha_1))) \\ &= \nu(\alpha_2,\varphi_2(\alpha_2)) - (q-1)\nu(f_*(\alpha_1),\alpha_2) \end{aligned}$$

where $f_*(\alpha)(x) = \alpha \circ f^{-1}(x)$, so that $f_*(\varphi_1(\alpha_1)) = \varphi_2(f_*(\alpha_1))$, thus

$$\deg(\mathcal{V}_1,\varphi_1,\alpha_1) \leqslant \deg(\mathcal{V}_1,\varphi_1,\alpha_1)$$

with equality if and only if $f_*(\alpha_1) = \alpha_2$. We thus obtain a HN-formalism on $\mathsf{Norm}_K^{\varphi}$.

We will show that for any reductive group G over k, any faithful exact $\otimes$-functor $\mathsf{Rep}(G) \to \mathsf{Norm}_K^{\varphi}$ is good, and the pair $(\mathsf{Norm}_K^{\varphi}, \deg)$ itself is good. In particular, the corresponding HN-filtration on $\mathsf{Norm}_K^{\varphi}$ is a $\otimes$-functor

$$\mathcal{F}_{HN} : \mathsf{Norm}_K^{\varphi} \to \mathsf{F}\left(\mathsf{Rep}_k\left(\mathrm{Gal}_K\right)\right).$$

5.3.4. Since $\mathcal{O}$ is Henselian, the absolute value of K has a unique extension to K^s, which we also denote by $|-| : K^s \to \mathbb{R}_+$. The corresponding valuation ring $\mathcal{O}^s := \{x \in K^s : |x| \leqslant 1\}$ is the integral closure of $\mathcal{O}$ in K^s, and it is a strictly Henselian local ring. There is a commutative diagram of $\otimes$-functors

$$\begin{tikzcd}
\mathsf{A} = \mathsf{Rep}_k(\mathrm{Gal}_K) \arrow[r, leftrightarrow] \arrow[d, "\text{forget } \rho"] & \mathsf{Vect}_K^{\varphi} \arrow[d, "-\otimes_K K^s"] & \mathsf{Norm}_K^{\varphi} = \mathsf{C} \arrow[l] \arrow[r] \arrow[d, "-\otimes_K K^s"] & \mathsf{Norm}_K \arrow[d, "-\otimes_K K^s"] \\
\mathsf{A}^s = \mathsf{Vect}_k \arrow[r, leftrightarrow] & \mathsf{Vect}_{K^s}^{\varphi} & \mathsf{Norm}_{K^s}^{\varphi} = \mathsf{C}^s \arrow[l] \arrow[r] & \mathsf{Norm}_{K^s}
\end{tikzcd}$$

in which the horizontal functors are equivalence of categories in the first square, forget the norms in the second square, and map $(\mathcal{V}, \varphi_{\mathcal{V}}, \alpha)$ to either $(\mathcal{V}, \alpha)$ or $(\mathcal{V}, \varphi_{\mathcal{V}}(\alpha))$ in the third square. The last vertical functor maps $(\mathcal{V}, \alpha)$ to $(\mathcal{V}^s, \alpha^s)$ with

$$\mathcal{V}^s \stackrel{\text{def}}{=} \mathcal{V} \otimes_K K^s \quad \text{and} \quad \alpha^s(v) \stackrel{\text{def}}{=} \min \left\{ \max\left\{|\lambda_i|\, \alpha(v_i) : i\right\} \;\middle|\; \begin{array}{l} v = \sum v_i \otimes \lambda_i \\ v_i \in \mathcal{V},\ \lambda_i \in K^s \end{array} \right\}.$$

By [8, Lemma 132], there is an extension $(K', |-|)$ of $(K^s, |-|)$ with K' algebraically closed (in which case $\mathcal{O}' := \{x \in K' : |x| \leqslant 1\}$ is strictly Henselian) and $|K'| = \mathbb{R}$. We may then add a third row to our commutative diagram,

$$\begin{tikzcd}
\mathsf{A}^s = \mathsf{Vect}_k \arrow[r, leftrightarrow] \arrow[d, equal] & \mathsf{Vect}_{K^s}^{\varphi} \arrow[d, "-\otimes_{K^s} K'"] & \mathsf{Norm}_{K^s}^{\varphi} = \mathsf{C}^s \arrow[l] \arrow[r] \arrow[d, "-\otimes_{K^s} K'"] & \mathsf{Norm}_{K^s} \arrow[d, "-\otimes_{K^s} K'"] \\
\mathsf{A}' = \mathsf{Vect}_k \arrow[r, leftrightarrow] & \mathsf{Vect}_{K'}^{\varphi} & \mathsf{Norm}_{K'}^{\varphi} = \mathsf{C}' \arrow[l] \arrow[r] & \mathsf{Norm}_{K'}
\end{tikzcd}$$

5.3.5. Let now G be a reductive group over k and let $x : \mathsf{Rep}(G) \to \mathsf{Norm}_K^{\varphi}$ be a faithful exact k-linear $\otimes$-functor, with base change

$$x^s : \mathsf{Rep}(G) \to \mathsf{Norm}_{K^s}^{\varphi} \quad \text{and} \quad x' : \mathsf{Rep}(G) \to \mathsf{Norm}_{K'}^{\varphi}$$

and Galois representation $\omega_{G,\mathsf{A}} : \mathsf{Rep}(G) \to \mathsf{Rep}_k(\mathrm{Gal}_K)$. We denote by

$$\omega_{G,\mathsf{A}} =: (V, \rho), \quad x =: (\mathcal{V}, \varphi_{\mathcal{V}}, \alpha), \quad x^s =: (\mathcal{V}^s, \varphi_{\mathcal{V}^s}, \alpha^s) \quad \text{and} \quad x' =: (\mathcal{V}', \varphi_{\mathcal{V}'}, \alpha')$$

the components of $\omega_{G,\mathsf{A}}$, x, x^s and x'. Let τ be a faithful representation of G and

$$p : \mathbf{F}(\omega_{G,\mathsf{A}}(\tau)) \twoheadrightarrow \mathbf{F}(\omega_{G,\mathsf{A}})(\tau)$$

the projection to the image of $\mathbf{F}(\omega_{G,\mathsf{A}}) \hookrightarrow \mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$. We want to show that

$$\langle x(\tau), f\rangle \leqslant \langle x(\tau), p(f)\rangle$$

for every $f \in \mathbf{F}(\omega_{G,\mathsf{A}}(\tau))$. As in 5.2.13, this amounts to

$$\left\langle \overrightarrow{\alpha(\tau)\varphi_{\mathcal{V}(\tau)}(\alpha(\tau))}, \mathcal{F} \right\rangle \leqslant \left\langle \overrightarrow{\alpha(\tau)\varphi_{\mathcal{V}(\tau)}(\alpha(\tau))}, \mathcal{G} \right\rangle$$

for the Busemann scalar product on $\mathbf{B}(\mathcal{V}(\tau))$, where $\mathcal{F}$ and $\mathcal{G}$ are the $\varphi_{\mathcal{V}(\tau)}$-stable filtrations on $\mathcal{V}(\tau)$ corresponding to the Gal_K-stable filtrations f and $p(f)$ on $V(\tau)$. Since the CAT(0)-spaces $\mathbf{B}(\mathcal{V}(\tau)\otimes -)$ are functorial on $\mathsf{HV}(k)$, this amounts to

$$\begin{aligned} &\left\langle \overrightarrow{\alpha^s(\tau)\varphi_{\mathcal{V}^s(\tau)}(\alpha^s(\tau))}, \mathcal{F}^s \right\rangle \leqslant \left\langle \overrightarrow{\alpha^s(\tau)\varphi_{\mathcal{V}^s(\tau)}(\alpha^s(\tau))}, \mathcal{G}^s \right\rangle \\ \text{or}\quad &\left\langle \overrightarrow{\alpha'(\tau)\varphi_{\mathcal{V}'(\tau)}(\alpha'(\tau))}, \mathcal{F}' \right\rangle \leqslant \left\langle \overrightarrow{\alpha'(\tau)\varphi_{\mathcal{V}'(\tau)}(\alpha'(\tau))}, \mathcal{G}' \right\rangle \end{aligned}$$

for the Busemann scalar products on $\mathbf{B}(\mathcal{V}^s(\tau))$ or $\mathbf{B}(\mathcal{V}'(\tau))$, where $\mathcal{F}^\star$ and $\mathcal{G}^\star$ are the $\varphi_{\mathcal{V}^\star(\tau)}$-stable filtrations on $\mathcal{V}^\star(\tau) := \mathcal{V}(\tau)\otimes_K K^\star = V(\tau)\otimes_k K^\star$ base changed from $\mathcal{F}$ and $\mathcal{G}$ on $\mathcal{V}(\tau)$ or equivalently, from f and $p(f)$ on $V(\tau)$ (for $\star\in\{s,\prime\}$).

5.3.6. Since k is finite, it follows from Lang's theorem and Deligne's work on tannakian categories that the fiber functor $V:\mathsf{Rep}(G)\to\mathsf{Vect}_k$ underlying $\omega_{G,\mathsf{A}}$ is isomorphic to the standard fiber functor $\omega_{G,k}:\mathsf{Rep}(G)\to\mathsf{Vect}_k$. Without loss of generality, we may thus assume that $V=\omega_{G,k}$, in which case

$$\omega_{G,\mathsf{A}}:\mathsf{Rep}(G)\to\mathsf{Rep}_k(\mathrm{Gal}_K)$$

is induced by a morphism $\rho:\mathrm{Gal}_K\to G(k)$ with open kernel. Then

$$\omega_{G,\mathsf{A}^s}=\omega_{G,\mathsf{A}'}=\omega_{G,k},\quad \mathcal{V}=(\omega_{G,\mathsf{A}}\otimes K^s)^{\mathrm{Gal}_K}\quad\text{and}\quad \mathcal{V}^\star=\omega_{G,K^\star}$$

for $\star\in\{s,\prime\}$. Moreover, the following commutative diagram in $\mathsf{CCat(0)}$

$$\begin{tikzcd} \mathbf{F}(\omega_{G,\mathsf{A}}) \arrow[r, hook] \arrow[d, hook] & \mathbf{F}(\omega_{G,\mathsf{A}}(\tau)) \arrow[d, hook] \\ \mathbf{F}(\omega_{G,k}) \arrow[r, hook] & \mathbf{F}(\omega_{G,k}(\tau)) \end{tikzcd}$$

is $G(k)$-equivariant, thus also Gal_K-equivariant, and identifies its first row with the Gal_K-invariants of its second row. It follows that the corresponding diagram of convex projections is commutative:

$$\begin{tikzcd} \mathbf{F}(\omega_{G,\mathsf{A}})(\tau) \arrow[d, hook] & \mathbf{F}(\omega_{G,\mathsf{A}}(\tau)) \arrow[l, two heads, "p"'] \arrow[d, hook] \\ \mathbf{F}(\omega_{G,k})(\tau) & \mathbf{F}(\omega_{G,k}(\tau)) \arrow[l, two heads, "p"'] \end{tikzcd}$$

It is therefore sufficient to show that for every $f\in\mathbf{F}(V(\tau))$,

$$\left\langle \overrightarrow{\alpha^\star(\tau)\,\varphi_{\mathcal{V}^\star}(\alpha^\star)(\tau)}, f \right\rangle \leqslant \left\langle \overrightarrow{\alpha^\star(\tau)\,\varphi_{\mathcal{V}^\star}(\alpha^\star)(\tau)}, p(f) \right\rangle$$

for the Busemann scalar product on $\mathbf{B}(V(\tau)\otimes K^\star)$. Note that since

$$\varphi_{\mathcal{V}^\star}=\mathrm{Id}\otimes\varphi\quad\text{on}\quad \mathcal{V}^\star=V\otimes_k K^\star,$$

the standard $\mathcal{O}^\star$-lattice $V \otimes_k \mathcal{O}^\star$ is $\varphi_{\mathcal{V}^*}$-stable, and so is the corresponding gauge norm $\alpha_{V\otimes\mathcal{O}^\star} = \boldsymbol{\alpha}(\circ)$. The additivity of the Busemann scalar product gives

$$\begin{aligned}\left\langle \overrightarrow{\alpha^\star(\tau)\,\varphi_{\mathcal{V}^\star}(\alpha^\star)(\tau)}, f\right\rangle &= \left\langle \overrightarrow{\alpha^\star(\tau)\,\boldsymbol{\alpha}(\circ)(\tau)}, f\right\rangle + \left\langle \overrightarrow{\boldsymbol{\alpha}(\circ)(\tau)\,\varphi_{\mathcal{V}^\star}(\alpha^\star)(\tau)}, f\right\rangle \\ &= -\left\langle \overrightarrow{\boldsymbol{\alpha}(\circ)(\tau)\,\alpha^\star(\tau)}, f\right\rangle + \left\langle \overrightarrow{\varphi_{\mathcal{V}^\star}(\boldsymbol{\alpha}(\circ))(\tau)\,\varphi_{\mathcal{V}^\star}(\alpha^\star)(\tau)}, f\right\rangle \\ &= (q-1)\cdot\left\langle \overrightarrow{\boldsymbol{\alpha}(\circ)(\tau)\,\alpha^\star(\tau)}, f\right\rangle\end{aligned}$$

and similarly for $p(f)$ – using the formulas of section 5.2.13 and 5.3.2. For $\star = {}'$, we also know that $\alpha' \in \mathsf{Norm}^{\otimes}_{K'}(\omega_{G,K'})$ belongs to $\mathbf{B}(\omega_G, K')$ by lemma 5.4, thus

$$\left\langle \overrightarrow{\boldsymbol{\alpha}(\circ)(\tau)\,\alpha'(\tau)}, f\right\rangle \leqslant \left\langle \overrightarrow{\boldsymbol{\alpha}(\circ)(\tau)\,\alpha'(\tau)}, p(f)\right\rangle$$

by proposition 5.6, which indeed applies since $k = \mathbb{F}_q$ is perfect.

5.3.7. We have shown that any faithful exact $\otimes$-functor $x : \mathsf{Rep}(G) \to \mathsf{Norm}^{\varphi}_K$ is good. Starting with a pair of objects $(\mathcal{V}_i, \varphi_i, \alpha_i)$ in $\mathsf{Norm}^{\varphi}_K$ (for $i \in \{1,2\}$), with Galois representations $\rho_i : \mathrm{Gal}_K \to GL(V_i)$, set $G := GL(V_1) \times GL(V_2)$ and $\rho := (\rho_1, \rho_2)$. Then $\rho : \mathrm{Gal}_K \to G(k)$ induces an exact and faithful $\otimes$-functor

$$\mathsf{Rep}(G) \to \mathsf{Rep}_k(\mathrm{Gal}_K)$$

with corresponding étale φ-module $(\mathcal{V}, \varphi_{\mathcal{V}}) : \mathsf{Rep}(G) \to \mathsf{Vect}^{\varphi}_K$ given by

$$\mathcal{V}(\tau) = (\omega_{G,k}(\tau) \otimes K^s)^{\mathrm{Gal}_K} \quad \text{and} \quad \varphi_{\mathcal{V}(\tau)} = \mathrm{Id} \otimes \varphi|_{\mathcal{V}(\tau)}.$$

In particular, $(\mathcal{V}, \varphi_{\mathcal{V}})(\tau_i') = (\mathcal{V}_i, \varphi_i)$ where $\tau_1' := \tau_1 \boxtimes 1$ and $\tau_2' := 1 \boxtimes \tau_2$ for the tautological representation τ_i of $GL(V_i)$ on V_i. We have to show that the splittable K-norms α_1 and α_2 also extend to $\alpha \in \mathsf{Norm}^{\otimes}_K(\mathcal{V})$. Since $\mathcal{V}^s = \mathcal{V} \otimes_K K^s \simeq \omega_{G,K^s}$, the base changed norms α_i^s on $\mathcal{V}_i^s = \mathcal{V}_i \otimes_K K^s$ plainly extend to a unique K^s-norm

$$\alpha^s = (\alpha_1^s, \alpha_2^s) \quad \text{in} \quad \begin{array}{ccccc} \mathbf{B}(\mathcal{V}_1^s) \times \mathbf{B}(\mathcal{V}_2^s) & \simeq & \mathbf{B}^e(G, K^s) & \simeq & \mathbf{B}(\omega_G, K^s) \\ & \subset & \mathsf{Norm}^{\otimes}_{K^s}(\omega_{G,K^s}) & \simeq & \mathsf{Norm}^{\otimes}_{K^s}(\mathcal{V}^s) \end{array}$$

on $\mathcal{V}^s : \mathsf{Rep}(G) \to \mathsf{Vect}_{K^s}$. For every $\tau \in \mathsf{Rep}(G)$, we may then define

$$\alpha(\tau) : \mathcal{V}(\tau) \to \mathbb{R}_+, \quad \alpha(\tau) \overset{\text{def}}{=} \alpha^s(\tau)|_{\mathcal{V}(\tau)}.$$

Plainly, $\alpha(\tau)$ is a K-norm on $\mathcal{V}(\tau)$ and $\alpha(\tau_i') = \alpha_i^s|_{\mathcal{V}_i} = \alpha_i$ on $\mathcal{V}(\tau_i') = \mathcal{V}_i$, which is a splittable K-norm on $\mathcal{V}_i$. Since τ_1' and τ_2' are $\otimes$-generators of the tannakian category $\mathsf{Rep}(G)$, it follows that $\alpha(\tau)$ is a splittable K-norm for every $\tau \in \mathsf{Rep}(G)$. Then $\alpha : \mathsf{Rep}(G) \to \mathsf{Norm}_K$ indeed belongs to $\mathsf{Norm}^{\otimes}_K(\mathcal{V})$, thus

$$(\mathcal{V}, \varphi_{\mathcal{V}}, \alpha) : \mathsf{Rep}(G) \to \mathsf{Norm}^{\varphi}_K$$

is a faithful exact $\otimes$-functor with $(\mathcal{V}, \varphi_{\mathcal{V}}, \alpha)(\tau_i') = (\mathcal{V}_i, \varphi_i, \alpha_i)$ for $i \in \{1,2\}$. Since it is good, the pair $(\mathsf{Norm}^{\varphi}_K, \deg)$ is indeed itself good.

5.3.8. *A variant.* We may also consider the quasi-abelian k-linear $\otimes$-category $\mathsf{Bun}_{\mathcal{O}}^{\varphi}$ of pairs $(L, \varphi_{\mathcal{V}})$ where L is a finite free $\mathcal{O}$-module and $\varphi_{\mathcal{V}} : \varphi^* \mathcal{V} \to \mathcal{V}$ is a Frobenius on $\mathcal{V} := L \otimes K$, with the obvious morphisms and tensor products. The functor

$$\mathsf{Bun}_{\mathcal{O}}^{\varphi} \to \mathsf{Norm}_{K}^{\varphi}, \qquad (L, \varphi_{\mathcal{V}}) \mapsto (\mathcal{V}, \varphi_{\mathcal{V}}, \alpha_L)$$

is a fully faithful exact k-linear $\otimes$-functor, whose essential image is stable under strict subobjects and quotients. It is thus also compatible with the corresponding HN-formalism. In particular, the HN-filtration is a $\otimes$-functor

$$\mathcal{F}_{HN} : \mathsf{Bun}_{\mathcal{O}}^{\varphi} \to \mathsf{F}(\mathsf{Rep}_k(\mathrm{Gal}_K)).$$

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