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Lisette Jager

Abstract

The construction, in [1], of a pseudodifferential calculus analogous to the Weyl calculus, in an infinite dimensional setting, required the introduction of convenient symbol classes. The symbols are functions defined on an infinite dimensional Hilbert space H, which compels us to investigate, in particular, their stochastic extensions to a Wiener space linked with H.

In this article, we define a new class of symbols, which applications in quantum electrodynamics render necessary. We proceed with the study of the first classes too. The results will be used to establish, later on, the properties that a pseudodifferential calculus is expected to satisfy. They reveal as well links between functions defined on the Hilbert space (the Cameron–Martin space) and their stochastic extensions.

More precisely, we prove here that the symbols of both classes and the terms of their Taylor expansions admit stochastic extensions. We define, in this infinite dimensional setting, a semigroup H_t analogous to the classical heat semigroup on a Wiener space [5, 7]. But, instead of acting on functions defined on a Wiener space, it acts on functions defined on its Cameron–Martin space, namely on functions belonging to one of our symbol classes. The heat operator commutes with a second order operator similar to the Laplacian, which is the infinitesimal generator of the semigroup. A surprising feature is that this Laplacian is continuous on the second class of symbols. This allows us, in this case, to give an expansion in powers of t of $H_t f$, or to invert it.

Extensions stochastiques de symboles et opérateur de la chaleur

Résumé

La construction d'un calcul pseudodifférentiel en dimension infinie analogue au calcul de Weyl, dans [1], a conduit à introduire des classes de symboles adaptées. Les symboles sont des fonctions définies sur un espace de Hilbert de dimension infinie H, ce qui contraint à étudier, en particulier, leur extension stochastique à un espace de Wiener construit sur H.

Dans cet article, nous définissons une nouvelle classe de symboles, nécessaire pour une application en électrodynamique quantique. Nous poursuivons aussi l'étude des premières classes de symboles. Ces résultats seront utilisés ultérieurement pour établir les propriétés qu'un calcul pseudodifférentiel est censé vérifier. Ils révèlent aussi les liens entre les fonctions définies sur l'espace de Hilbert (l'espace de Cameron–Martin) et leurs extensions stochastiques.

Plus précisément, on prouve ici que les symboles appartenant aux deux types de classes et les termes de leur développement de Taylor admettent des extensions stochastiques. On définit, en dimension infinie, un semi-groupe H_t analogue au semi-groupe de la chaleur classique sur l'espace de Wiener [5, 7]. Mais au lieu d'agir sur des fonctions définies sur un espace de Wiener, il agit sur des fonctions définies sur l'espace de Cameron–Martin, c'est-à-dire sur des fonctions appartenant à nos classes de symboles. L'opérateur de la chaleur commute avec un opérateur du second ordre analogue au Laplacien et qui est le générateur infinitésimal du semi-groupe. Un fait surprenant est que ce Laplacien est continu sur le second type de classes de symboles. Cela permet de développer H_t en termes de puissances de t et de l'inverser.

Keywords: stochastic extensions, heat operator, Wiener spaces, pseudodifferential calculus, symbol classes.
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1. Introduction

This article is motivated by pseudodifferential analysis in an infinite dimensional context. A central notion in the usual Weyl pseudodifferential calculus is the symbol classes. Symbols are functions defined on $\mathbb{R}^{2n}$, with which the Weyl calculus associates a linear operator, according to a classical procedure involving integrations. The space $\mathbb{R}^n$ appears both as a Hilbert space and as a measure space.

What becomes of the spaces and of the symbols in an infinite dimensional framework? On an infinite dimensional Hilbert space H, there is no measure which can replace the Lebesgue measure. One solution is to use a Wiener extension B of H, as was done in [1]. One may consult [3, 4, 5, 6, 9, 11] about Wiener spaces and the notions needed will be recalled in the article. But let us sketch here the relationship between the Hilbert space and its extension. The Hilbert space H is real and separable. Its Wiener extension B is its completion with respect to a convenient norm on H (called "measurable") and it is endowed with a Gaussian probability measure. There is no such measure on H and the H scalar product does not extend as such on B, even if B is sometimes a Hilbert space too.

This distribution of properties compels us to shift constantly from one space to the other, operation which is not necessary in the classical, finite dimensional case. The choice made in [1] was to define the symbols on the Hilbert space. To be able to integrate, one must associate a function $\widetilde{f}$, defined on the Wiener extension, with a function f, defined on the Hilbert space. This is the notion of the stochastic extension, which goes back to [3, 4, 5, 6, 9, 11]. It is studied in the present article in relation with the symbol classes. Although the motivations come from pseudodifferential analysis, even if we sometimes allude to the classical, finite dimensional theory, it will not be central in this work which is focused on the notion of stochastic extension of specific symbol classes.

In this article we principally treat two different symbol classes, one originating from a former article and one which is introduced here.

In [1], the Weyl calculus was constructed for symbols belonging to a given class, $S_m(\mathcal{B}, \varepsilon)$, recalled in Definition 2.3. Only the properties strictly necessary in view of the construction of the pseudodifferential calculus were proved there. The symbols, defined on H^2, satisfy partial differentiability conditions with respect to a fixed orthonormal basis $\mathcal{B}$, as well as estimates (formally, they satisfy the conditions required to apply the finite-dimensional Calderòn–Vaillancourt Theorem). In the present article we go further with the study of these classes. We prove that the symbols in $S_m(\mathcal{B}, \varepsilon)$ have more general stochastic extensions and are Fréchet-differentiable for sufficiently large m. We also define the analogue of the Laplace operator and relax the dependence on the basis of H (Remark 4.7), which was important in [1].

In Definition 3.5 we introduce another class of symbols, $S(Q_A)$, defined thanks to a quadratic form linked with a trace class operator A. Indeed, the classes $S_m(\mathcal{B}, \varepsilon)$ could be used in quantum electrodynamics but only under a cutoff assumption (see [2]), which the classes $S(Q_A)$ enable one to lift. Moreover, the new classes $S(Q_A)$ do not depend on a basis. The symbols admit stochastic extensions as well. Their smoothness is included in their definition.

Our principal purpose is to construct and study a semigroup of operators denoted by $(H_t)_{t\geq 0}$, similar to the heat operator, for both symbol classes. Then we state the properties which will be needed, for example, to treat the composition of operators, for both classes.

For bounded Borel functions defined on the Wiener space B itself, the heat operator is a classical notion, to which [5] is almost entirely devoted and which is still being studied ([7]). Let us recall it briefly. If $\mu_{B,t}$ is the Gaussian probability measure of (variance) parameter t on B, the heat operator is given by

$$\forall\, x \in B, \quad H_t f(x) = \int_B f(x+y)\, \mathrm{d}\mu_{B,t}(y).$$

When f is bounded and uniformly continuous on B, $H_t f$, defined on B, converges uniformly to f when t converges to 0. If, moreover, f is Lipschitz continuous on B, $H_t f$ has further differentiability properties ([5, 9]).

But our construction requires this notion for symbols f defined on the initial Hilbert space H. Stochastic extensions are then clearly necessary and we are led to set:

$$\forall\, x \in H, \quad H_t f(x) = \int_B \widetilde{f}(x+y)\, \mathrm{d}\mu_{B,t}(y),$$

where $\widetilde{f}$ is a stochastic extension of f in a certain sense and the resulting function $H_t f$ is defined on H. Since H is $\mu_{B,t}$-negligible in B, restricting $\widetilde{f}$ makes no sense, nor is $\widetilde{f}$, in general, a continuity extension of f. But one can choose a measurable norm on H, giving rise to a specific Wiener extension B_A of H, for which the stochastic extensions have topological properties. This is true for both kinds of symbol classes (Propositions 3.14 and 4.6). This allows us to use the theory of [5] and [9], classical in the frame of the Wiener space theory. The extension B_A may be different from the extension B initially chosen and is used temporarily. Of course, one checks that the integral defining $H_t f(x)$ does not depend on the chosen Wiener extension.

The main results of this article are Theorems 5.8 and 5.14, which establish, for the classes $S_m(\mathcal{B}, \varepsilon)$ and for the classes $S(Q_A)$, the existence of a Laplacian commuting with the heat operator and which is its infinitesimal operator. We insist on the fact that, in the classes $S(Q_A)$, the Laplace operator is continuous, which allows one to inverse the heat operator.

Let us stress, for instance, the following expansion for $f \in S(Q_A)$, in terms of powers of the (bounded) Laplace operator:

$$H_t f = f + \sum_{k=1}^{N} \frac{t^k}{k!} \left(\frac{1}{2}\Delta\right)^k f + t^{N+1} R_N(t),$$

the remainder R_N satisfying estimates independent of t. This kind of results is important for the composition of symbols and problems purely linked to pseudodifferential analysis, like the relationship between Wick, Anti-Wick and Weyl symbols.

Section 2 recalls the indispensable notions about Wiener space and measure. Then it gives the vital definitions and results about the Weyl calculus in an infinite dimensional setting. Section 3 is devoted to various stochastic extensions. It recalls and states more precisely the results about stochastic extensions for the classes $S_m(\mathcal{B}, \varepsilon)$ of [1]. It proves similar results for the classes $S(Q_A)$, which are defined at this point. It also treats the case of products of scalar products in view of the polynomial terms in the Taylor expansions of the symbols. This brings up the alternative definition of the Weyl calculus, as a quadratic form, which enables us to use unbounded symbols. In Section 4 we prove the Fréchet-differentiability of the symbols in the classes $S_m(\mathcal{B}, \varepsilon)$. Then we extend stochastically the Taylor's expansions of symbols of both classes. The polynomial terms do not belong to the symbol classes since they are unbounded. Still, they admit stochastic extensions (Proposition 4.9) (and give rise, in a certain sense, to pseudodifferential operators [1]). This section gives tools to define the heat operator. Section 5 defines the heat operator H_t for functions initially defined on the Hilbert space and which are impossible to integrate on the Wiener space without an extension. We establish the semigroup property for both classes, together with useful properties of H_t : infinitesimal generator, commutation.

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2. The Weyl calculus on a Wiener space

The construction of the Wiener space may be found in [3, 4, 6, 9]. The Weyl calculus on a Wiener space has been developed in [1]. We just recall here the notions which are necessary to read the present article.

The abstract Wiener space (H, B) is a couple where H is a real, separable, infinite dimensional Hilbert space and B is a Banach space containing H as a dense subspace. The space B is called a Wiener extension of H. One denotes by $\langle \cdot \rangle$ (or sometimes $\cdot$) and $|\cdot|$ the scalar product and the norm on H and by $\|\cdot\|$ the norm on B.

In what follows, one identifies H with its dual space, so that $B' \subset H \subset B$, each space being a dense subspace of the following one. One denotes by $\mathcal{F}(X)$ the set of all finite dimensional subspaces of a vector space X. If $E \in \mathcal{F}(H)$, π_E is the orthogonal projection of H onto E. If $E \in \mathcal{F}(B')$, one denotes by P_E a generalization of the projection, where the scalar products are replaced by the B', B duality.

It is impossible to extend to H itself the Gaussian measure which is naturally defined on its finite dimensional subspaces. Nevertheless, if the norm $\|\cdot\|$ of B has a property called measurability (see [9, Definition 4.4, Chapter 1] or [3]), one can construct a Gaussian measure on the Borel σ-algebra of B. Let

$$d\mu_{\mathbb{R}^n,h}(x) = (2\pi h)^{-n/2} e^{-\frac{1}{2h}\sum_{i=1}^n x_i^2}\, d\lambda(x_1,\dots,x_n)$$

be the Gaussian measure with variance $h > 0$ on $\mathbb{R}^n$. A cylinder of B is a set of the form

$$C = \{x \in B : (y_1(x),\dots,y_n(x)) \in A\}, \tag{2.1}$$

where n is a positive integer, $y_1,\dots,y_n$ are elements of B' and A is a Borel set of $\mathbb{R}^n$. If $(y_1,\dots,y_n)$ is orthonormal with respect to the scalar product of H, one sets

$$\mu_{B,h}(C) = \int_A d\mu_{\mathbb{R}^n,h}(x). \tag{2.2}$$

The parameter h represents the variance of the Gaussian measure and can also be considered as a semiclassical parameter in the Weyl calculus. One can prove that this measure extends as a probability measure, still denoted by $\mu_{B,h}$, on the σ-algebra generated by the cylinders of B, which is the Borel σ-algebra of B (the same definition, but starting from cylinders of H, yields a pseudomeasure which is not σ-additive). One may then integrate. The subscript p will often denote the L^p norm in B or B^2, for the measure with variance h.

If $E \in \mathcal{F}(B')$ has dimension n, one can identify E and $\mathbb{R}^n$ by choosing a basis, orthonormal with respect to the scalar product of H and thus define a measure $\mu_{E,h}$ on E. For every function $\varphi \in L^1(E,\mu_{E,h})$, the transfer theorem gives

$$\int_B \varphi \circ P_E(x)\, d\mu_{B,h}(x) = \int_E \varphi(u)\, d\mu_{E,h}(u). \tag{2.3}$$

If y is an element of B', it can be considered as a random variable on B. If y is not zero one sees, using (2.2), that, for every Borel set A of $\mathbb{R}$,

$$\mu_{B,h}(y \in A) = \int_A e^{-\frac{v^2}{2h|y|^2}} (2\pi h|y|^2)^{-1/2}\, dv,$$

which means that y has the normal distribution $\mathcal{N}(0,\sigma^2 = h|y|^2)$ [9]. Up to the factor $\sqrt{h}$, this is an isometry from $(B', |\cdot|)$ in $L^2(B,\mu_{B,h})$. It can be extended as an isometry from H in $L^2(B,\mu_{B,h})$ and one denotes by ℓ_a the image of an element a of H. If $a \in B'$, $\ell_a = a$

is a linear application but if $a \in H$, ℓ_a is only defined $\mu_{B,t}$- almost everywhere and is therefore not necessarily linear. However, $\ell_a(-x) = -\ell_a(x)$ and $\ell_a(x+y) = \ell_a(x) + a \cdot y$ for $y \in H$.

If $E \in \mathcal{F}(H)$ has an orthonormal basis $(e_1, \dots, e_n)$, one sets, for $x \in B$,

$$\widetilde{\pi}_E(x) = \sum_{j=1}^{n} \ell_{u_j}(x) u_j, \tag{2.4}$$

in keeping with the projection. Then, for all $a \in H$, $a \cdot \widetilde{\pi}_E(x) = \ell_{\pi_E(a)}(x)$. The functions ℓ_a satisfy the following identities, recalled in [1].

$$\begin{aligned} &\forall\ (u,v) \in H^2, && \int_B e^{\ell_u(x)+i\ell_v(x)}\, \mathrm{d}\mu_{B,h}(x) = e^{\frac{h}{2}(|u|^2-|v|^2+2iu\cdot v)}. \\ &\forall\ a \in H,\ \forall\ p \in [1,+\infty[, && \int_B |\ell_a(x)|^p\, \mathrm{d}\mu_{B,h}(x) = \frac{(2h)^{p/2}}{\sqrt{\pi}} |a|^p\, \Gamma\left(\frac{p+1}{2}\right). \end{aligned} \tag{2.5}$$

Setting

$$K(p) = 2^{1/2}\pi^{-1/2p}\left(\Gamma\left(\frac{p+1}{2}\right)\right)^{1/p}, \tag{2.6}$$

one can write that $\|\ell_a\|_{L^p(B,\mu_{B,h})} = K(p)h^{1/2}|a|$. Notice that $K(2) = 1$. One sees, too, that for all a and b in H,

$$\int_B e^{\ell_b(u)} |\ell_a(u)|^p\, \mathrm{d}\mu_{B,h}(u) = e^{h\frac{|b|^2}{2}} \int_{\mathbb{R}} |\sqrt{h}|a|v + ha\cdot b|^p\, \mathrm{d}\mu_{\mathbb{R},1}(v). \tag{2.7}$$

Let us recall the Wick Theorem:

Theorem 2.1. *Let $u_1, \dots u_{2p}$ be vectors of H ($p \geq 1$). Let $h > 0$. Then one has*

$$\int_B \ell_{u_1}(x) \dots \ell_{u_{2p}}(x)\, \mathrm{d}\mu_{B,h}(x) = h^p \sum_{(\varphi,\psi)\in S_p} \prod_{j=1}^{p} \langle u_{\varphi(j)}, u_{\psi(j)} \rangle \tag{2.8}$$

where S_p is the set of all couples (φ, ψ) of one to one maps from $\{1, \dots, p\}$ into $\{1, \dots, 2p\}$ such that:

(1) *For all $j \leq p$, $\varphi(j) < \psi(j)$.*

(2) *The sequence $(\varphi(j))_{(1\leq j\leq p)}$ is an increasing sequence.*

The measure $\mu_{B,h}$ transforms, under translation of a vector $a \in H$, into another measure which is absolutely continuous with respect to the former one. More precisely, for all $g \in L^1(B, \mu_{B,h})$, one has, for all a in H:

$$\int_B g(x)\, \mathrm{d}\mu_{B,h}(x) = e^{-\frac{1}{2h}|a|^2} \int_B g(x+a) e^{-\frac{1}{h}\ell_a(x)}\, \mathrm{d}\mu_{B,h}(x). \tag{2.9}$$

If a belongs to $B \setminus H$ or if h changes, both measures are mutually singular.

The Weyl calculus on the Wiener space has been constructed in two different ways. One of the constructions is rather similar to the finite dimensional definition of a calculus, in that it relies on classes of symbols which satisfy differentiability conditions and it yields operators which are bounded on a L^2 space. We will work in this frame most of the time. We do not have, though, an integral definition of $\mathrm{Op}(f)u$, neither on H nor on B. The symbols are functions defined on H^2 by Definition 2.3. It is possible (and necessary) to extend them to functions defined on B^2 according to the definition below. This notion is inspired by the theory of Wiener spaces (see [3, 4, 6, 9, 11]), where the sequences converge in probability and not necessarily in a L^p space.

Definition 2.2. Let (H, B) be an abstract Wiener space and h, a positive number. A function f admits a stochastic extension $\widetilde{f}$ in $L^p(B, \mu_{B,h})$ $(1 \leq p < \infty)$ if, for every increasing sequence (E_n) in $\mathcal{F}(H)$, whose union is dense in H, the functions $f \circ \widetilde{\pi}_{E_n}$ are in $L^p(B, \mu_{B,h})$ and if the sequence $f \circ \widetilde{\pi}_{E_n}$ converges in $L^p(B, \mu_{B,h})$ to $\widetilde{f}$.

One defines likewise the stochastic extension of a function on H^2 to a function on B^2.

One can check, for example thanks to (2.5), that ℓ_a is the stochastic extension of the scalar product with a and that $\widetilde{\pi}_E$ is the stochastic extension of π_E in L^p. The stochastic extension can be obtained in a more topological manner (see [9, Chapter 1, Section 6]). There exists a result about extensions of holomorphic functions ([1, Theorem 8.8]), obtained by martingale methods, which proves a property announced by [8].

The symbol classes used in [1] and which we recall below share derivability properties and estimates with the classes of the finite dimensional Calderón–Vaillancourt Theorem:

Definition 2.3. Let (H, B) be an abstract Wiener space, let $\mathcal{B} = (e_j)_{(j \in \Gamma)}$ be a Hilbert basis of H, indexed by a countable set Γ, with $e_j \in B'$ for all j. Set $u_j = (e_j, 0)$ and $v_j = (0, e_j)$ $(j \in \Gamma)$. A multiindex is a map (α, β) from Γ into $\mathbb{N} \times \mathbb{N}$ such that $\alpha_j = \beta_j = 0$ except for a finite number of indices. Let M be a nonnegative real number, m a nonnegative integer and $\varepsilon = (\varepsilon_j)_{(j \in \Gamma)}$ a family of nonnegative real numbers. One denotes by $S_m(\mathcal{B}, M, \varepsilon)$ the set of bounded continuous functions $F : H^2 \to \mathbf{C}$ satisfying the following condition. For every multiindex (α, β) of depth m (that is to say such that $\max_{j \in \Gamma}(\alpha_j, \beta_j) \leq m$), the derivative in the inequality below is well defined, continuous on H^2 and satisfies, for every (x, ξ) in H^2

$$\left| \left[\prod_{j \in \Gamma} \partial_{u_j}^{\alpha_j} \partial_{v_j}^{\beta_j} \right] F(x, \xi) \right| \leq M \prod_{j \in \Gamma} \varepsilon_j^{\alpha_j + \beta_j} . \tag{2.10}$$

One recalls the following very useful property, stated in the proof of Proposition 4.14 of [1]. If ε is square summable, every F in $S_1(\mathcal{B}, M, \varepsilon)$ verifies a Lipschitz condition:

$$\forall\ (X, V) \in H^2,\ |F(X+V) - F(X)| \leq M|V|\sqrt{2}\left[\sum_{j\in\Gamma} \varepsilon_j^2\right]^{1/2}. \tag{2.11}$$

It is more convenient to represent classes of symbols as vector spaces.

Definition 2.4. Let ε be a sequence of positive real numbers and let $m \in \mathbb{N}$. One sets $S_m(\mathcal{B}, \varepsilon) = \bigcup_{M\geq 0} S_m(\mathcal{B}, M, \varepsilon)$. For $F \in S_m(\mathcal{B}, \varepsilon)$ one sets $\|F\|_{m,\varepsilon} = \inf\{M \geq 0 : F \in S_m(\mathcal{B}, M, \varepsilon)\}$.

Notice that $S_m(\mathcal{B}, \varepsilon)$, equipped with $\|\cdot\|_{m,\varepsilon}$, is a Banach space. Setting $S^\infty(\mathcal{B}, \varepsilon) = \bigcap_{m=0}^\infty S_m(\mathcal{B}, \varepsilon)$, one can, classically, define a distance by $\mathrm{d}(F, G) = \sum_{m=0}^\infty 2^{-m} \frac{\|F-G\|_{m,\varepsilon}}{1+\|F-G\|_{m,\varepsilon}}$. Then $(S^\infty(\mathcal{B}, \varepsilon), d)$ is complete.

An alternative construction of the Weyl calculus uses an analogue of the Wigner function in order to associate a quadratic form with a function $\widetilde{F}$ defined, this time, on B^2. This quadratic form is applied to cylindrical functions, depending on a finite number of variables. Let us only recall that this construction requires of $\widetilde{F}$ to belong to $L^1(B^2, \mu_{B^2,h/2})$ and to be such that there exists a nonnegative integer m such that

$$N_m(\widetilde{F}) = \sup_{Y\in H^2} \frac{\|\widetilde{F}(\cdot + Y)\|_{L^1(B^2,\mu_{B^2,h/2})}}{(1+|Y|)^m} < +\infty. \tag{2.12}$$

This norm is finite if the function $\widetilde{F}$ is bounded or if it is a polynomial expression of degree m with respect to functions $(x, \xi) \to \ell_a(x) + \ell_b(\xi)$, with a and b in H, as we shall see in Subsection 3.3.

These approaches complement one another. The approach relying on symbol classes enables us to work on L^2 spaces on B, but the symbol has to be bounded, the other one allows us to use unbounded symbols, but the domain of the quadratic forms contains only cylindrical functions. Both definitions coincide under certain conditions ([1, Theorem 1.4]).

3. Stochastic extensions

3.1. Stochastic extensions of symbols in $S_m(\mathcal{B}, \varepsilon)$

We first generalize a proposition stated in [1, Proposition 8.4] in the case when $p = 1$. Moreover, we show that the stochastic extension does not depend on h, which is not necessarily the case since the measures are mutually orthogonal for different variance parameters h. This result is important to integrate with respect to the variance parameter, as in Section 3.4.

Proposition 3.1. *Let F be a function in $S_1(\mathcal{B}, \varepsilon)$, with respect to a Hilbert basis $\mathcal{B} = (e_j)_{(j\in\Gamma)}$, where the sequence $(\varepsilon_j)_{(j\in\Gamma)}$ is summable. Then, for every positive h and every $q \in [1, +\infty[$, F admits a stochastic extension in $L^q(B^2, \mu_{B^2,h})$.*

Moreover, there exists a function $\widetilde{F}$ which is the stochastic extension of F in $L^q(B^2, \mu_{B^2,h})$ for all $h \in]0, \infty[$ and $q \in [1, +\infty[$.

For any $E \in \mathcal{F}(H^2)$, we then have the inequality : $\forall\ (h, q) \in]0, +\infty[\times [1, +\infty[$,

$$\|F \circ \widetilde{\pi}_E - \widetilde{F}\|_{L^q} \le \|F\|_{1,\varepsilon} K(q) h^{1/2} \sum_{j=1}^{\infty} \varepsilon_j (|u_j - \pi_E(u_j)| + |v_j - \pi_E(v_j)|). \quad (3.1)$$

Proof. Let (E_n) be an increasing sequence of $\mathcal{F}(H^2)$, whose union is dense in H^2. For all m and n such that $m < n$, let S_{mn} be the orthogonal complement of E_m in E_n. We can state an inequality analogous to the inequality (120) of [1]:

$$\|F \circ \widetilde{\pi}_{E_m} - F \circ \widetilde{\pi}_{E_n}\|_{L^q} \le \|F\|_{1,\varepsilon} K(q) h^{1/2} \sum_{j=1}^{\infty} \varepsilon_j (|\pi_{S_{mn}}(u_j)| + |\pi_{S_{mn}}(v_j)|). \quad (3.2)$$

Indeed, one just needs to replace L^1 by L^q in the original proof, since the only changes take place in the explicit L^q norms of the ℓ_a functions appearing there. This inequality proves that $F \circ \widetilde{\pi}_{E_m}$ is a Cauchy sequence in $L^q(B^2, \mu_{B^2,h})$ and one can verify that the limit does not depend on the sequence (E_n).

We first construct a representative of the stochastic extension common to all $(h, q) \in]0, h_0] \times [1, q_0]$, for a given finite q_0. Let (E_n) be an increasing sequence of elements of $\mathcal{F}(B')$. The right term of (3.2) is smaller than an expression $C(m, n)$ which depends only on h_0, q_0. Then, there exist an increasing sequence $(n_i)_i$ satisfying $C(n_{i+1}, n_i) < 2^{-i-1}$ and a sequence of functions $(F_N)_N$ defined by

$$F_N := F \circ \widetilde{\pi}_{E_{n_1}} + \sum_{j=1}^{N} \left(F \circ \widetilde{\pi}_{E_{n_{j+1}}} - F \circ \widetilde{\pi}_{E_{n_j}} \right) \text{ on } B^2,$$

exactly as in the classical proof of the Riesz–Fisher Theorem. The functions F_N are defined everywhere on B^2 and independent of (h, q). The limit $\widetilde{F}$ of this sequence is the representative we are looking for, it takes finite values on a subset of B^2 whose $\mu_{B^2,h}$-measure is 1 for all $h \le h_0$. Inequality (3.1) is a consequence of (3.2), with $E_m = E$ and n growing to infinity.

We now lift the restriction on q_0. Denote by $\widetilde{F}_2$ (resp. $\widetilde{F}_n$) the stochastic expansion valid for $h \in]0, h_0]$ and $q \in [1, 2]$ (resp. $q \in [1, n]$). Let (E_s) be an increasing sequence of $\mathcal{F}(H^2)$, whose union is dense in H^2. One then has, for $q \le 2$,

$$\lim_{s\to\infty} \|F \circ \widetilde{\pi}_{E_s} - \widetilde{F}_2\|_{q,h} = 0, \quad \lim_{s\to\infty} \|F \circ \widetilde{\pi}_{E_s} - \widetilde{F}_n\|_{q,h} = 0.$$

Consequently $\widetilde{F}_2 = \widetilde{F}_n$ $\mu_{B^2,h}$, almost everywhere. It follows that $\widetilde{F}_2 \in L^q(B^2, \mu_{B^2,h})$ for $q \le n$ and that the convergence is true. To obtain the inequality one similarly replaces $\widetilde{F}_n$ by $\widetilde{F}_2$.

We lift the restriction on h_0 in the same way. Consider $\widetilde{F}_{h_0}, \widetilde{F}_{h_0'}$ the extensions corresponding to different values of h_0 and an increasing sequence (E_n) as before. Then $\|F \circ \widetilde{\pi}_{E_s} - \widetilde{F}_{h_0}\|_{1,h}$ and $\|F \circ \widetilde{\pi}_{E_s} - \widetilde{F}_{h_0'}\|_{1,h}$ converge to 0 for $h \le \min(h_0, h_0')$, which proves that $\widetilde{F}_{h_0} = \widetilde{F}_{h_0'}$, $\mu_{B^2,h}$ almost everywhere. □

Corollary 3.2. *If $F \in S_1(\mathcal{B}, \varepsilon)$ where ε is summable, then for all $h > 0$ and all $p \in [1, +\infty[$,*

$$|\widetilde{F}| \le \|F\|_{1,\varepsilon} \quad \mu_{B^2,h}\text{-a.s.}, \qquad \|\widetilde{F}\|_{L^p(B^2,\mu_{B^2,h})} \le \|F\|_{1,\varepsilon}.$$

Let $\mathcal{P}$ be the operator which associates, with a function in $S_1(\mathcal{B}, \varepsilon)$, its stochastic extension in $L^p(B^2, \mu_{B^2,t})$. This linear operator is thus bounded, with norm smaller than 1.

Proof. For every increasing sequence $(E_n)_n$ of $\mathcal{F}(H^2)$, whose union is dense in H^2, the sequence $(F \circ \widetilde{\pi}_{E_n})_n$ converges to $\widetilde{F}$ in $L^p(B^2, \mu_{B^2,t})$. Since $|F \circ \widetilde{\pi}_{E_n}|$ is smaller than $\|F\|_{1,\varepsilon}$ $\mu_{B^2,h}$, almost everywhere on B^2 (on the domain where $\widetilde{\pi}_{E_n}$ is defined or on B^2 if $E_n \subset B'$), so is $|\widetilde{F}|$. Moreover, $\|F \circ \widetilde{\pi}_{E_n}\|_{L^p} \le \|F\|_{1,\varepsilon}$ and letting n grow to infinity yields $\|\widetilde{F}\|_{L^p} \le \|F\|_{1,\varepsilon}$. □

We now state a result about translations, denoting by $\tau_Y F$ the function defined by $\tau_Y F(X) = F(X+Y)$ (with $X, Y \in H$ or H^2).

Lemma 3.3. *Let F be a globally Lipschitz continuous function on H, admitting a stochastic extension $\widetilde{F}$ in L^p for every $p \in [1, +\infty[$ and $h > 0$. If $Y \in H$, then $\tau_Y F$ admits $\tau_Y \widetilde{F}$ as a stochastic extension in $L^p(B, \mu_{B,h})$ for $h > 0$ and $p \in [1, +\infty[$.*

Proof. Let (E_j) be an increasing sequence of $\mathcal{F}(H)$, whose union is dense in H. If we denote by an index p the $L^p(B, \mu_{B,h})$ norm, we obtain that

$$\|\tau_Y \widetilde{F} - (\tau_Y F) \circ \widetilde{\pi}_{E_j}\|_p \le \|\tau_Y \widetilde{F} - \tau_Y (F \circ \widetilde{\pi}_{E_j})\|_p + \|\tau_Y (F \circ \widetilde{\pi}_{E_j}) - (\tau_Y F) \circ \widetilde{\pi}_{E_j}\|_p.$$

For all $p' > p$, the inequality

$$\begin{aligned}
\|\tau_Y \widetilde{F} - \tau_Y (F \circ \widetilde{\pi}_{E_j})\|_p &= \left(\int_{B^2} |\widetilde{F} - F \circ \widetilde{\pi}_{E_j}|^p(X) e^{\frac{1}{h}\ell_Y(X)}\, \mathrm{d}\mu_{B^2,h}(X) e^{-\frac{1}{2h}|Y|^2} \right)^{1/p} \\
&\le \|\widetilde{F} - F \circ \widetilde{\pi}_{E_j}\|_{p'} e^{\frac{|Y|^2}{2h(p'-p)}}
\end{aligned}$$

holds true, thanks to the translation change of variables (2.9), to Hölder's inequality and to the formula (2.5). The first term then goes to 0 as $j \to \infty$. So does the second one. Indeed, since F is globally Lipschitz, one has

$$\begin{aligned}|F(\widetilde{\pi}_{E_j}(X+Y)) - F(\widetilde{\pi}_{E_j}(X)+Y)| &= |F(\widetilde{\pi}_{E_j}(X)+\pi_{E_j}(Y)) - F(\widetilde{\pi}_{E_j}(X)+Y)| \\ &\leq C|\pi_{E_j}(Y) - Y|_H,\end{aligned}$$

which implies that $\|\tau_Y(F \circ \widetilde{\pi}_{E_j}) - (\tau_Y F) \circ \widetilde{\pi}_{E_j}\|_p \leq C|\pi_{E_j}(Y) - Y|_H$ and it converges to 0 when $j \to \infty$. □

Corollary 3.4. *If $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 1$ and ε summable, if $\widetilde{F}$ is the stochastic extension in the L^p given above, if $Y \in H^2$, then $\tau_Y F$ admits $\tau_Y \widetilde{F}$ as a stochastic extension in the $L^p(B^2, \mu_{B^2,h})$ for $h > 0$ and $p \in [1, +\infty[$.*

This result is a consequence of the lemma, thanks to Proposition 3.1 for the stochastic extensions and to (2.11) for the Lipschitz condition. The fact that the functions are defined on H or H^2 does not change anything.

3.2. Symbol classes defined thanks to a quadratic form

We now define a new class of symbols.

Definition 3.5. Let A be a linear, selfadjoint, nonnegative, trace class application on a Hilbert space H. For all $x \in H$ one sets $Q_A(x) = \langle Ax, x\rangle$. Let $S(Q_A)$ be the class of all functions $f \in C^\infty(H)$ such that there exists $C(f) > 0$ satisfying:

$$\begin{aligned}&\forall\, x \in H,\ |f(x)| \leq C(f),\ \forall\, m \in \mathbb{N}^*,\ \forall\, x \in H,\ \forall\, (U_1, \dots, U_m) \in H^m, \\ &\qquad |(\mathrm{d}^m f)(x)(U_1, \dots, U_m)| \leq C(f) \prod_{j=1}^{m} Q_A(U_j)^{1/2}. \qquad (3.3)\end{aligned}$$

The smallest constant $C(f)$ such that (3.3) holds is denoted by $\|f\|_{Q_A}$.

Notice that $S(Q_A)$, equipped with the norm $\|\cdot\|_{Q_A}$, is a Banach space. One can also check that, if A and B satisfy the conditions of Definition 3.5, a product of functions belonging to $S(Q_A)$, $S(Q_B)$ is in $S(Q_{2(A+B)})$ with

$$\|fg\|_{Q_{2(A+B)}} \leq \|f\|_{Q_A} \|g\|_{Q_B}. \qquad (3.4)$$

Moreover, if A is as above but defined on H^2, the class $S(Q_A)$ is included in a class $S_\infty(\mathcal{B}, \varepsilon)$ for any orthonormal basis $\mathcal{B} = (e_j)$ of H, with $\varepsilon_j = \max(Q_A(e_j, 0)^{1/2}, Q_A(0, e_j)^{1/2})$. Since the sequence ε is only square summable, the existence results for the stochastic extensions must be obtained otherwise.

Remark 3.6. One may think of a less restrictive class, of functions f satisfying the inequality (3.3) with constants $C_m(f)$ depending on the order m. This space is a Fréchet space and some of the results below are still valid in this frame (for example Propositions 3.9 and 3.11 for stochastic extensions, inequality 5.8, with a constant $C_{k+1}(f)$ depending on the order). Here we use the (smaller) class of Definition 3.3, for which the reverse heat equation will be solvable. In the finite dimensional case, this class corresponds to some analytic class with $|\partial^\alpha f(x)| \leq C(f)R^{|\alpha|}$ (and $R = 1$) and is therefore better than real-analytic.

Lemma 3.7. *For $E \in \mathcal{F}(H)$ and $h > 0$, $y \mapsto Q_A(\widetilde{\pi}_E(y))^{1/2}$ belongs to $L^p(B, \mu_{B,h})$ for all $p \in [1, +\infty[$. More precisely, if (u_j) is a Hilbert basis of H whose vectors are eigenvectors of A and if one denotes by λ_j the corresponding eigenvalues (which can be equal to* 0*), one obtains*

$$\|Q_A^{1/2} \circ \widetilde{\pi}_E\|_{L^p(B,\mu_{B,h})} \leq C(p) \left(\sum_0^\infty \lambda_j |\pi_E(u_j)|^{\max(p,2)} \right)^{1/\max(p,2)} h^{1/2},$$

with $K(p)$ defined by (2.6) *and*

$$C(p) = K(p) \left(\sum_0^\infty \lambda_j \right)^{1/2 - \frac{1}{p}} \textit{ for } p > 2, \ C(p) = 1 \textit{ for } p \leq 2. \tag{3.5}$$

Proof. By decomposing A on its eigenvector basis, one obtains that

$$Q_A(\widetilde{\pi}_E(y)) = \sum_{j=0}^\infty \lambda_j (u_j \cdot \widetilde{\pi}_E(y))^2 = \sum_{j=0}^\infty \lambda_j (\ell_{\pi_E(u_j)})^2,$$

using the properties of $\widetilde{\pi}$. For $p = 2$ it suffices to integrate this equality and to use (2.5). For $p > 2$, one uses Jensen's inequality for a probability measure on $\mathbb{N}$. Set $S = \sum_0^\infty \lambda_j$. One then has

$$Q_A(\widetilde{\pi}_E(y))^{\frac{p}{2}} = \left(\sum_{j=0}^\infty \frac{\lambda_j}{S} S(\ell_{\pi_E(u_j)})^2 \right)^{\frac{p}{2}} \leq \sum_{j=0}^\infty \frac{\lambda_j}{S} (S(\ell_{\pi_E(u_j)})^2)^{\frac{p}{2}},$$

and it remains to integrate. Finally, for $p \in [1, 2[$, one applies Hölder's inequality. □

Remark 3.8. One can give an upper bound for $\|Q_A^{1/2} \circ \widetilde{\pi}_E\|_{L^p(B,\mu_{B,h})}$, which does not depend on E:

$$\left\|Q_A^{1/2} \circ \widetilde{\pi}_E\right\|_{L^p(B,\mu_{B,h})} \leq C(p) \left(\sum_0^\infty \lambda_j \right)^{1/\max(p,2)} h^{1/2}.$$

One can prove the following result.

Proposition 3.9. *Let $h > 0$ and let $p \in [1, +\infty[$. Every function f belonging to $S(Q_A)$ admits a stochastic extension $\widetilde{f}$ in $L^p(B, \mu_{B,h})$. The function $\widetilde{f}$ is bounded $\mu_{B,h}$, almost everywhere by $\|f\|_{Q_A}$. Moreover, for all $E \in \mathcal{F}(H)$,*

$$\|f \circ \widetilde{\pi}_E - \widetilde{f}(x)\|_{L^p(B,\mu_{B,h})} \leq C(p)h^{1/2}\|f\|_{Q_A}\left(\sum_{j\geq 0}\lambda_j|\pi_E(u_j) - u_j|^{\max(p,2)}\right)^{1/\max(p,2)},$$

with the notations of Lemma 3.7.

Proof. Let (E_n) be an increasing sequence of $\mathcal{F}(H)$, whose union is dense in H. Let f be in $S(Q_A)$. Let m and n be such that $m < n$. Let S_{mn} be an orthogonal complement of E_m in E_n. Then

$$f(\widetilde{\pi}_{E_n}(x)) - f(\widetilde{\pi}_{E_m}(x)) = \int_0^1 (\mathrm{d}f)(\widetilde{\pi}_{E_m}(x) + \theta\widetilde{\pi}_{S_{mn}}(x))(\widetilde{\pi}_{S_{mn}}(x))\,\mathrm{d}\theta.$$

Hence

$$|f(\widetilde{\pi}_{E_n}(x)) - f(\widetilde{\pi}_{E_m}(x))| \leq \|f\|_{Q_A}\int_0^1 Q_A(\widetilde{\pi}_{S_{mn}}(x))^{1/2}\,\mathrm{d}\theta.$$

This implies that

$$\|f \circ \widetilde{\pi}_{E_n} - f \circ \widetilde{\pi}_{E_m}\|_{L^p(B,\mu_{B,h})} \leq \|f\|_{Q_A}\left\|Q_A^{1/2} \circ \widetilde{\pi}_{S_{mn}}\right\|_{L^p(B,\mu_{B,h})}.$$

Using the preceding Lemma 3.7, one gets that

$$\|f \circ \widetilde{\pi}_{E_n} - f \circ \widetilde{\pi}_{E_m}\|_{L^p(B,\mu_{B,h})} \leq C(p)h^{1/2}\|f\|_{Q_A}\left(\sum_{j\geq 0}\lambda_j|\pi_{S_{mn}}(u_j)|^{\max(p,2)}\right)^{1/\max(p,2)}.$$

The right term converges to 0 when m grows to infinity, according to the dominated convergence Theorem. Indeed, for all j, $|\pi_{S_{mn}}(u_j)|$ converges to 0 when m grows to infinity, $|\pi_{S_{mn}}(u_j)|^{\max(p,2)} \leq 1$ and the series $\sum\lambda_j$ converges. The sequence $(f(\widetilde{\pi}_{E_n}))_n$ is therefore a Cauchy sequence in $L^p(B, \mu_{B,h})$. One can verify that its limit, in $L^p(B, \mu_{B,h})$, does not depend on the sequence (E_n). Since the function $|f \circ \widetilde{\pi}_{E_n}|$ is almost everywhere smaller than $\|f\|_{Q_A}$, so is its limit. To get the final inequality, one takes $E = E_m$ in one of the above inequalities and lets n converge to infinity. □

Remark 3.10. This result holds true for the class of Remark 3.6, with $\max(C_0(f), C_1(f))$ instead of $\|f\|_{Q_A}$ in the estimates.

Proposition 3.11. *Let $h > 0$, $p \in [1, +\infty[$. Let k be a positive integer and let x be a fixed point in H. Set $S = \sum\lambda_j$. The function $y \mapsto \mathrm{d}^k f(x) \cdot y^k$ defined on H admits a stochastic* *extension in $L^p(B, \mu_{B,h})$. Moreover, for all $E \in \mathcal{F}(H)$, denoting by $\mathcal{P}$ the passage to a stochastic extension,*

$$\|\mathrm{d}^k f(x) \cdot \widetilde{\pi}_E(y)^k - \mathcal{P}(y \mapsto \mathrm{d}^k f(x) \cdot y^k)\|_p \\ \leq k\|f\|_{Q_A} C(pk)^k S^{\frac{k-1}{\alpha_{pk}}} h^{\frac{k}{2}} \left(\sum \lambda_s |\pi_E(u_s) - u_s|^{\alpha_{pk}}\right)^{\frac{1}{\alpha_{pk}}},$$

where the subscript p indicates the $L^p(B, \mu_{B,h})$ norm.

Proof. We still use the notations of Lemma 3.7 and set $\alpha_p = \max(p, 2)$. Let $E, F \in \mathcal{F}(H)$ with $E \subset F$. For all $y \in B$, one has the telescopic summation

$$\mathrm{d}^k f(x) \cdot \widetilde{\pi}_E(y)^k - \mathrm{d}^k f(x) \cdot \widetilde{\pi}_F(y)^k \\ = \sum_{j=1}^{k} \mathrm{d}^k f(x)(\widetilde{\pi}_E(y)^{j-1}, \widetilde{\pi}_E(y) - \widetilde{\pi}_F(y), \widetilde{\pi}_F(y)^{k-j}).$$

Using Definition 3.5, one deduces that

$$|\mathrm{d}^k f(x) \cdot \widetilde{\pi}_E(y)^k - \mathrm{d}^k f(x) \cdot \widetilde{\pi}_F(y)^k| \\ \leq \sum_{j=1}^{k} \|f\|_{Q_A} Q_A^{\frac{j-1}{2}}(\widetilde{\pi}_E(y)) Q_A^{\frac{k-j}{2}}(\widetilde{\pi}_F(y)) Q_A^{1/2}(\widetilde{\pi}_E(y) - \widetilde{\pi}_F(y)).$$

Integrating, using Hölder's inequality and Remark 3.8, one obtains that

$$\|\mathrm{d}^k f(x) \cdot \widetilde{\pi}_E(y)^k - \mathrm{d}^k f(x) \cdot \widetilde{\pi}_F(y)^k\|_p \\ \leq \sum_{j=1}^{k} \|f\|_{Q_A} \left\|Q_A^{1/2} \circ \widetilde{\pi}_E\right\|_{pk}^{j-1} \left\|Q_A^{1/2} \circ \widetilde{\pi}_F\right\|_{pk}^{k-j} \left\|Q_A^{1/2} \circ (\widetilde{\pi}_E - \widetilde{\pi}_F)\right\|_{pk} \\ \leq k\|f\|_{Q_A} (C(pk) S^{\frac{1}{\alpha_{pk}}} h^{1/2})^{k-1} C(pk) h^{1/2} \left(\sum \lambda_s |(\pi_E - \pi_F)(u_s)|^{\alpha_{pk}}\right)^{\frac{1}{\alpha_{pk}}}.$$

Then one proceeds as in the preceding proposition, replacing E and F by the terms of an increasing sequence of $\mathcal{F}(H)$ whose union is dense in H and whose first term is E. □

Remark 3.12. This result holds with the class defined by Remark 3.6, with $C_k(f)$ instead of $\|f\|_{Q_A}$.

A consequence of Lemma 3.7 is the following result, which partly generalizes Proposition 8.7 of [1]:

Corollary 3.13. *Let $h > 0$ and $p \in [1, +\infty[$. The function $Q_A^{1/2}$ admits a stochastic extension in $L^p(B, \mu_{B,h})$.*

Proof. As in the proof of Proposition 3.9, one introduces an increasing sequence (E_n) of $\mathcal{F}(H)$. Let S_{mn} be an orthogonal complement of E_m in E_n if $m \leq n$. Lemma 3.7 implies the inequality

$$\|Q_A^{1/2} \circ \widetilde{\pi}_{S_{mn}}\|_{L^p(B,\mu_{B,h})} \leq C(p) \left(\sum_0^\infty \lambda_j |\pi_{S_{mn}}(u_j)|^{\max(p,2)} \right)^{1/\max(p,2)} h^{1/2},$$

which proves that $(Q_A^{1/2} \circ \widetilde{\pi}_{E_n})_n$ is a Cauchy sequence in $L^p(B, \mu_{B,h})$. □

The point of the following result is that it enables us to use another Wiener space associated with H than the space B initially chosen and to use uniform continuity. We may then use the results of the classical Wiener-space theory ([5, 7]) and restrict stochastic extensions to the (dense but negligible) subspace H, as in Proposition 5.3 below.

Proposition 3.14. *Let A be a linear, selfadjoint, nonnegative, trace class application in a Hilbert space H. There exists a measurable norm (see* [9, Definition 4.4] *or* [3]*), $\|\cdot\|_{A,n}$ on H, and hence a completion B_A of H with respect to this norm, such that the following property is satisfied: if f belongs to the class $S(Q_A)$, then f is uniformly continuous on H with respect to the norm $\|\cdot\|_{A,n}$.*

The function f admits a uniformly continuous extension f_A on B_A. The stochastic extension $\widetilde{f}$ of f in the sense of Proposition 3.9 (on $B = B_A$) is equal to f_A $\mu_{B_A,h}$-a.e.

Proof. If A is a one to one map, one sets $\|x\|_{A,n} = \langle Ax, x\rangle^{1/2} = Q_A(x)^{1/2}$. If not, if $(e_s)_{s\in\mathbb{N}}$ is an orthonormal basis of $\mathrm{Ker}(A)$, one can add to A the operator C defined, for example, by $Cx = \sum_s e^{-s}\langle x, e_s\rangle e_s$. The operator $A + C$ is selfadjoint, nonnegative, trace class and it is one to one. One then sets $\|x\|_{A,n} = \langle (A+C)x, x\rangle^{1/2} = Q_{A+C}(x)^{1/2}$. It follows from Theorem 3 in [3] that $\|\cdot\|_{A,n}$ is a measurable seminorm. It is a norm since A (or $A + C$) is one to one. Taylor's formula gives, in both cases, the inequality

$$|f(y) - f(x)| \leq \|f\|_{Q_A} Q_A(y - x)^{1/2} \leq \|f\|_{Q_A} \|x - y\|_{A,n},$$

which in turn implies the uniform continuity. The topological extension f_A of f is then uniformly continuous on B_A. According to Theorem 6.3 in [9, Chapter 1], f_A and $\widetilde{f}$ coincide almost everywhere. □

3.3. Stochastic extension of products

This part begins with stochastic extensions of general products of functions defined on H. The results are then applied to products of scalar products, which appear in the Taylor expansions of Section 4.

The following lemma is a straightforward consequence of the general Hölder's inequality, applied to a telescopic decomposition of $\prod_{i=1}^N f_i - \prod_{i=1}^N g_i$:

$$\prod_{i=1}^N f_i - \prod_{i=1}^N g_i = \sum_{k=1}^N \left(\prod_{i=1}^{k-1} f_i \prod_{i=k+1}^N g_i \right) (f_k - g_k).$$

Lemma 3.15. *Let $(\Omega, \mathcal{T}, m)$ be a measure space. Let $N \geq 2$ be an integer. For $i \leq N$, let f_i, g_i be functions on Ω with values in $\mathbb{R}$ such that, for all $p \in [1, +\infty[$, $f_i \in L^p(\Omega, \mathcal{T}, m)$, $g_i \in L^p(\Omega, \mathcal{T}, m)$. For all $p \in [1, +\infty[$, set $M_p = \max_{1 \leq i \leq N}(\|g_i\|_p, \|f_i\|_p)$. Then for all $p \in [1, +\infty[$,*

$$\left\| \prod_{i=1}^N f_i - \prod_{i=1}^N g_i \right\|_p \leq (M_{pN})^{N-1} \sum_{k=1}^N \|f_k - g_k\|_{pN}. \tag{3.6}$$

More precisely, one has

$$\left\| \prod_{i=1}^N f_i - \prod_{i=1}^N g_i \right\|_p \leq \sum_{k=1}^N \left(\prod_{i=1}^{k-1} \|f_i\|_{pN} \prod_{i=k+1}^N \|g_i\|_{pN} \right) \|(f_k - g_k)\|_{pN}. \tag{3.7}$$

Corollary 3.16. *Let $F_1, \dots, F_N$ N be functions defined on H^2 and admitting stochastic extensions $\widetilde{F_1}, \dots, \widetilde{F_N}$ in $L^p(B^2, \mu_{B^2,h})$ for all $p \in [1, +\infty[$. Then $\prod_{i=1}^N F_i$ admits $\prod_{i=1}^N \widetilde{F}_i$ as a stochastic extension in $L^p(B^2, \mu_{B^2,h})$ for all $p \in [1, +\infty[$.*

Proof. Let (E_n) be an increasing sequence of $\mathcal{F}(H^2)$, whose union is dense in H^2. According to (3.6),

$$\begin{aligned} &\left\| \prod_{i=1}^N F_i \circ \widetilde{\pi}_{E_n} - \prod_{i=1}^N \widetilde{F}_i \right\|_p \\ &\qquad \leq \left(\sup_{n \in \mathbb{N}, i \leq N} (\|F_i \circ \widetilde{\pi}_{E_n}\|_{Np}, \|\widetilde{F}_i\|_{Np}) \right)^{N-1} \sum_{i=1}^N \|F_i \circ \widetilde{\pi}_{E_n} - \widetilde{F}_i\|_{Np}, \end{aligned} \tag{3.8}$$

which gives the result. □

Let $a_1, \dots, a_n$ be vectors of H. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multiindex such that $\alpha_i > 0$ for every i. One defines the function a^α on H by

$$a^\alpha(x) = \prod_{i=1}^n \langle a_i, x \rangle^{\alpha_i}.$$

Proposition 3.17. *For $h > 0$ and $p \in [1, +\infty[$, the function a^α admits the function $\prod_{i=1}^{n} \ell_{a_i}^{\alpha_i}$ as a stochastic extension in $L^p(B, \mu_{B,h})$. Moreover, for all $E \in \mathcal{F}(H)$,*

$$\left\| a^\alpha \circ \widetilde{\pi_E} - \prod_{i=1}^{n} \ell_{a_i}^{\alpha_i} \right\|_p \leq K(p|\alpha|)^{|\alpha|} h^{|\alpha|/2} \left(\max_{1 \leq i \leq n} |a_i| \right)^{|\alpha|-1} \sum_{i=1}^{n} \alpha_i |\pi_E(a_i) - a_i|.$$

Proof. First prove the result for a multiindex of length one, that is to say, for a single scalar product $\varphi_b : x \mapsto \langle x, b \rangle$. Let $(E_j)_j$ be an increasing sequence of $\mathcal{F}(H)$ such that $\overline{\bigcup E_j} = H$. Using $\varphi_b \circ \widetilde{\pi_{E_j}} = \ell_{\pi_{E_j}(b)}$ and (2.5), one obtains that, for a finite p

$$\|\varphi_b \circ \widetilde{\pi_{E_j}} - \ell_b\|_p = \|\ell_{\pi_{E_j}(b)-b}\|_p = K(p) h^{1/2} |\pi_{E_j}(b) - b|.$$

The result is thus valid for $x \mapsto \langle x, b \rangle$.

For a general multiindex α, one starts by applying Lemma 3.15 stated in the appendix to the $|\alpha|$ functions appearing in the products $a^\alpha \circ \widetilde{\pi}_E$ and $\prod_{i=1}^{n} \ell_{a_i}^{\alpha_i}$. One then notices that

$$\|\ell_{\pi_E(a_i)}\|_{p|\alpha|} = K(p|\alpha|) h^{1/2} |\pi_E(a_i)| \leq K(p|\alpha|) h^{1/2} |a_i|.$$

It remains to replace E by the sequence $(E_j)_j$ to obtain the stochastic extension. □

Besides, one can define a pseudodifferential operator whose symbol is a product of ℓ_a functions. This operator is defined as a quadratic form, as in the end of Section 2. For example, if $|\alpha| = 1$, we recover the Segal fields for symbols of the kind $(x, \xi) \mapsto \ell_a(x) + \ell_b(\xi)$ defined on B^2 ([1, Proposition 8.10]). This relies on the following result:

Proposition 3.18. *Let $a_1, \ldots, a_n, b_1, \ldots, b_p$ belong to H, let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_p$ be positive integers, set $m = \max(|\alpha|, |\beta|)$. The function $\widetilde{F} \colon (x, \xi) \mapsto \prod_{i=1}^{n} \ell_{a_i}^{\alpha_i}(x) \prod_{i=1}^{p} \ell_{b_i}^{\beta_i}(\xi)$ has a finite norm N_m defined by* (2.12)*. More precisely,*

$$N_m(\widetilde{F}) \leq \max\left(1, \sqrt{\frac{h}{2}}\right)^{|\alpha|+|\beta|} \prod_1^n |a_j|^{\alpha_j} \prod_1^p |b_i|^{\beta_i} \times \cdots$$
$$\times \prod_1^n \left(\int_{\mathbb{R}} (1 + |v|)^{n\alpha_j} \, \mathrm{d}\mu_{\mathbb{R},1}(v) \right)^{\frac{1}{n}} \prod_1^p \left(\int_{\mathbb{R}} (1 + |v|)^{p\beta_i} \, \mathrm{d}\mu_{\mathbb{R},1}(v) \right)^{\frac{1}{p}}$$

Proof. Recall that

$$N_m(\widetilde{F}) = \sup_{Y \in H^2} \frac{\|\widetilde{F}(\,\cdot\, + Y)\|_{L^1(B^2, \mu_{B^2, \frac{h}{2}})}}{(1 + |Y|)^m}.$$

By the change of variables formula (2.9) one obtains

$$\int_{B^2} |\widetilde{F}(X + Y)| \, \mathrm{d}\mu_{B^2, \frac{h}{2}}(X) \leq A \times B,$$

with

$$A = e^{-\frac{1}{h}|y|^2} \int_B \prod_{j=1}^{n} |\ell_{a_j}(x)|^{\alpha_j} e^{\frac{2}{h}\ell_y(x)}\, \mathrm{d}\mu_{B,\frac{h}{2}}(x),$$

$$B = e^{-\frac{1}{h}|\eta|^2} \int_B \prod_{j=1}^{p} |\ell_{b_j}(\xi)|^{\beta_j} e^{\frac{2}{h}\ell_\eta(\xi)}\, \mathrm{d}\mu_{B,\frac{h}{2}}(\xi).$$

Hölder's inequality yields

$$A \leq e^{-\frac{1}{h}|y|^2} \prod_{j=1}^{n} \left(\int_B |\ell_{a_j}(x)|^{n\alpha_j} e^{\frac{2}{h}\ell_y(x)}\, \mathrm{d}\mu_{B,\frac{h}{2}}(x) \right)^{1/n}.$$

According to (2.7),

$$A \leq e^{-\frac{1}{h}|y|^2} \prod_{j=1}^{n} \left(e^{|y|^2/h} \int_{\mathbb{R}} |\sqrt{\frac{h}{2}}|a_j|v + < y, a_j > |^{n\alpha_j}\, \mathrm{d}\mu_{\mathbb{R},1}(v) \right)^{1/n}.$$

One can factor $|a_j|$ and, since $\sqrt{\frac{h}{2}}|v| + |y|$ is smaller than $\max\left(1, \sqrt{\frac{h}{2}}\right)(1+|v|)(1+|y|)$, one gets

$$A \leq \max\left(1, \sqrt{\frac{h}{2}}\right)^{|\alpha|} \prod_{1}^{n} |a_j|^{\alpha_j} (1+|y|)^{|\alpha|} \prod_{1}^{n} \left(\int_{\mathbb{R}} (1+|v|)^{n\alpha_j}\, \mathrm{d}\mu_{\mathbb{R},1}(v) \right)^{1/n}.$$

One treats the factor B similarly, which gives the desired result. □

3.4. Stochastic extension in an integral

This section is devoted to integrals of symbols of the Calderón–Vaillancourt type (Definition 2.3).

By differentiating, one sees that the function defined by the integral below belongs to the same class as the integrand. This implies the existence of a stochastic extension by Proposition 3.1, but does not give a precise form:

Proposition 3.19. *Let $G \in S_m(\mathcal{B}, \varepsilon)$, $f \in L^1([0,1], \mathbb{R})$ and $X \in H^2$. For all $Y \in H^2$, set $T_X G(Y) = \int_0^1 f(s) G(X + sY)\, \mathrm{d}s$. Then $T_X G \in S_m(\mathcal{B}, \varepsilon)$ and*

$$\|T_X G\|_{m,\varepsilon} \leq \int_0^1 |f(s)|\, \mathrm{d}s\, \|G\|_{m,\varepsilon}.$$

A precise form for the stochastic extension is given by the following result, which uses the existence of extensions which do not depend on h, provided it remains bounded:

Proposition 3.20. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function, let G be in $S_1(\mathcal{B}, \varepsilon)$, with ε summable and let us fix $X \in H^2$. For all $q \in [1, +\infty[$ and all $h \in]0,1]$, the function*

$$Y \mapsto \int_0^1 f(s) G(X + sY)\, \mathrm{d}s,$$

defined on H^2, admits, as a stochastic extension in $L^q(B^2, \mu_{B^2,h})$, the function

$$Y \mapsto \int_0^1 f(s) \widetilde{G}(X + sY)\, \mathrm{d}s,$$

defined on B^2, where $\widetilde{G}$ is the stochastic extension of G for all $L^r(B^2, \mu_{B^2,h})$, $(r,h) \in [1, +\infty[\times]0,1]$.

Moreover, if $E \in \mathcal{F}(H^2)$, one has the inequality

$$\begin{aligned} &\left\| \int_0^1 f(s) \left(G(X + s\widetilde{\pi_E}(\,\cdot\,)) - \widetilde{G}(X + s\,\cdot\,) \right) \mathrm{d}s \right\|_{L^q} \\ &\quad \leq \|G\|_{1,\varepsilon} \int_0^1 |f(s)|\, \mathrm{d}s \left(\sqrt{2 \sum_{\Gamma} \varepsilon_j^2} |X - \pi_E(X)| \right. \\ &\qquad\qquad \left. + \sqrt{2h(q+2)} C e^{\frac{|X|^2}{2h}} \sum_0^{\infty} \varepsilon_j (|u_j - \pi_E(u_j)| + |v_j - \pi_E(v_j)|) \right), \end{aligned}$$

where the constant C does not depend on the parameters.

Proof. One checks that the functions $(s, Y) \mapsto X + sY$ and $(s, Y) \mapsto X + s\widetilde{\pi_E}(Y)$ are measurable. Define a function U_E on B^2 by setting $U_E(Y) = \int_0^1 f(s) g(s, Y)$, with $g(s, Y) = (G(X + s\widetilde{\pi_E}(Y)) - \widetilde{G}(X + sY))\, \mathrm{d}s$.

One notices that

$$\int_0^1 |f(s)| \left(\int_{B^2} |g(s, Y)|^q \, \mathrm{d}\mu_{B^2,h}(Y) \right)^{1/q} \mathrm{d}s < \infty.$$

This implies that $U_E \in L^q(B^2, \mu_{B^2,h})$ and that

$$\|U_E\|_{L^q(B^2, \mu_{B^2,h})} \leq \int_0^1 |f(s)| \left(\int_{B^2} |g(s, Y)|^q \, \mathrm{d}\mu_{B^2,h}(Y) \right)^{1/q} \mathrm{d}s.$$

Indeed, this is straightforward if $q = 1$. If $q > 1$, one introduces a function of $L^{q'}(B^2, \mu_{B^2,h})$ (with $q^{-1} + q'^{-1} = 1$) to prove that U_E belongs to the dual space of $L^{q'}$.

This proves that

$$\|U_E\|_{L^q(B^2,\mu_{B^2,h})} \leq \int_0^1 |f(s)| \, \|G(X + s\widetilde{\pi_E}(\,\cdot\,)) - G(\widetilde{\pi_E}(X + s\,\cdot\,))\|_{L^q} \,\mathrm{d}s \\ + \int_0^1 |f(s)| \, \|G(\widetilde{\pi_E}(X + s\,\cdot\,)) - \widetilde{G}(X + s\,\cdot\,)\|_{L^q} \,\mathrm{d}s.$$

Formula (2.11) proves that the first term is smaller than

$$\int_{B^2} |G(X + s\widetilde{\pi_E}(Y)) - G(\widetilde{\pi_E}(X + sY))|^q \,\mathrm{d}\mu_{B^2,h} \\ \leq \int_{B^2} \left(\|G\|_{1,\varepsilon} \sqrt{2 \sum \varepsilon_j^2} |X - \pi_E(X)| \right)^q \,\mathrm{d}\mu_{B^2,h}.$$

For the second term, successive change of variables give

$$\int_{B^2} |G(\widetilde{\pi_E}(X + sY)) - \widetilde{G}(X + sY)|^q \,\mathrm{d}\mu_{B^2,h}(Y) \\ = \int_{B^2} |G(\widetilde{\pi_E}(X + Z)) - \widetilde{G}(X + Z)|^q \,\mathrm{d}\mu_{B^2,s^2h}(Z) \\ = \int_{B^2} |G(\widetilde{\pi_E}(Z)) - \widetilde{G}(Z)|^q e^{-\frac{|X|^2}{2s^2h}} e^{\frac{1}{s^2h}\ell_X(Z)} \,\mathrm{d}\mu_{B^2,s^2h}(Z).$$

One then applies Hölder's inequality to the last term, raising $|G(\widetilde{\pi_E}(Z)) - \widetilde{G}(Z)|^q$ to the power q'/q with $q' = q + \frac{1}{s^2}$. This gives

$$\|G(\widetilde{\pi_E}(X + s\,\cdot\,)) - \widetilde{G}(X + s\,\cdot\,)\|_{L^q(\mu_{B^2,h})} \leq e^{\frac{|X|^2}{2h}} \|G \circ \widetilde{\pi_E} - \widetilde{G}\|_{L^{q+\frac{1}{s^2}}(\mu_{B^2,s^2h})}.$$

Using (3.1), one obtains

$$\int_0^1 |f(s)| \|G(\widetilde{\pi_E}(X + s\,\cdot\,)) - \widetilde{G}(X + s\,\cdot\,)\|_{L^q(B^2,\mu_{B^2,h})} \,\mathrm{d}s \\ \leq \int_0^1 |f(s)| e^{\frac{|X|^2}{2h}} \|G\|_{1,\varepsilon} \sqrt{2hs^2} \left(\pi^{-1/2} \Gamma\left(\frac{q + s^{-2} + 1}{2} \right) \right)^{\frac{1}{q+s^{-2}}} \\ \times \sum_0^\infty \varepsilon_j (|u_j - \pi_E(u_j)| + |v_j - \pi_E(v_j)|) \,\mathrm{d}s.$$

For large values of $|z|$ and $|\arg(z)| < \pi$, one has, by Stirling's formula,

$$\Gamma(z) = z^{-1/2} e^{z(\ln(z)-1)} \sqrt{2\pi}\,(1 + O(z^{-1})).$$

It follows that, for a constant C which is independent of the parameters,

$$\left(\pi^{-1/2}\Gamma\left(\frac{q+s^{-2}+1}{2}\right)\right)^{\frac{1}{q+s^{-2}}} \leq C(q+s^{-2}+1)^{1/2},$$

which gives the estimate for the second term. □

The following result is useful in the following Section 4 to treat the remainder in a Taylor formula.

Corollary 3.21. *Let f be continuous on $[0,1]$. Let $G \in S_1(\mathcal{B},\varepsilon)$, with ε summable. Let $(a_1,\dots,a_n,X) \in H^2$ and let (p,h) be in $[1,+\infty[\,\times\mathbb{R}^{+*}$. The function*

$$Y \in H^2 \mapsto \left(\prod_{i=1}^{k}\langle a_i, Y\rangle\right)\int_0^1 f(s)G(X+sY)\,\mathrm{d}s$$

admits, as a stochastic extension in $L^p(B^2,\mu_{B^2,h})$, the function

$$Y \in B^2 \mapsto \left(\prod_{i=1}^{k}\ell_{a_i}(Y)\right)\int_0^1 f(s)\widetilde{G}(X+sY)\,\mathrm{d}s,$$

where $\widetilde{G}$ is the stochastic extension of G valid for all $h' \in]0,+\infty[$ and all finite p. Moreover, there exists a constant K depending on p,k,h but not on the a_i, G, X, f, E or ε such that, for all $E \in \mathcal{F}(H^2)$,

$$\begin{aligned}\Bigg\|\prod_{i=1}^{k}\langle a_i,\widetilde{\pi}_E(Y)\rangle\int_0^1 f(s)G(X+s\widetilde{\pi}_E(Y))\,\mathrm{d}s &- \prod_{i=1}^{k}\ell_{a_i}(Y)\int_0^1 f(s)\widetilde{G}(X+sY)\,\mathrm{d}s\Bigg\|_p \\ \leq K\int_0^1 |f(s)|\,\mathrm{d}s\|G\|_{1,\varepsilon}A^{k-1}\times\Bigg(&\sum_{i=1}^{k}|\pi_E(a_i)-a_i| + A\|\varepsilon\|_2|\pi_E(X)-X| \\ &+ Ae^{\frac{|X|^2}{2h}}\sum_{j=0}^{\infty}\varepsilon_j(|\pi_E(u_j)-u_j|+|\pi_E(v_j)-v_j|)\Bigg)\end{aligned}$$

where $A = \max_{1\leq i\leq k}(|a_i|)$ and the norm is in $L^p(B^2,\mu_{B^2,h})$.

Proof. The existence of the stochastic extension for the product is a consequence of Corollary 3.16. To prove the inequality, one uses (3.7) to establish that the left hand side is smaller than

$$\prod_{i=1}^{k}\|\ell_{a_i}\|\times\left\|\int_0^1 f(s)G(X+s\widetilde{\pi}_E(Y))\,\mathrm{d}s-\int_0^1 f(s)\widetilde{G}(X+sY)\,\mathrm{d}s\right\|$$
$$+\sum_{i=1}^{k}\left(\prod_{j=1}^{i-1}\|\ell_{\pi_E(a_j)}\|\prod_{j=i+1}^{k}\|\ell_{a_j}\|\right)\|\ell_{\pi_E(a_i)}-\ell_{a_i}\|\times\left\|\int_0^1 f(s)G(X+s\widetilde{\pi}_E(Y))\,\mathrm{d}s\right\|,$$

the norm in the second term being the $L^{p(k+1)}(B^2,\mu_{B^2,h})$-norm. But $\|\ell_a\| = K(p(k+1))h^{1/2}|a|$, according to (2.5) and (2.6). An upper bound is, consequently, $(K(p(k+1))h^{1/2})^k$ multiplied by the following expression:

$$\left(\prod_{i=1}^{k}|a_j|\right)\left\|\int_0^1 f(s)G(X+s\widetilde{\pi}_E(Y))\,\mathrm{d}s-\int_0^1 f(s)\widetilde{G}(X+sY)\,\mathrm{d}s\right\|$$
$$+\sum_{i=1}^{k}|\pi_E(a_i)-a_i|\left(\prod_{1\le j\le k,j\neq i}|a_j|\right)\times\left\|\int_0^1 f(s)G(X+s\widetilde{\pi}_E(Y))\,\mathrm{d}s\right\|$$

One concludes by noticing that $|G|$ is smaller than $\|G\|_{1,\varepsilon}$, (thanks to Proposition 3.20) and that the $|a_i|$ are smaller than A. □

4. Taylor expansions

4.1. Differentiability of the symbols in $S_m(\mathcal{B},\varepsilon)$

By definition, the symbols in the Calderón–Vaillancourt classes admit partial derivatives, but no differentiability assumption is made. This section gives results about the Fréchet differentiablity of such symbols. Notice that one loses an order of differentiability (Remark 4.3)

As a consequence, Proposition 4.6 states that one may also work with another, more suitable Wiener extension of H, in which one can use continuity results from the general theory.

In this subsection, the sequence ε is supposed to be square summable when orthogonality can be used. Summability is needed in the other cases, for example for the existence of stochastic extensions.

The following straightforward lemma lists useful properties of the S_m classes:

Lemma 4.1. *Let $F \in S_m(\mathcal{B}, \varepsilon)$, with ε square summable.*

- *If $m \geq k \geq 1$ and if α, β are two multiindices of depth k (such that $\max_{j\in\Gamma}(\alpha_j, \beta_j) \leq k$), then $\partial_u^\alpha \partial_v^\beta F \in S_{m-k}(\mathcal{B}, \varepsilon)$ and*
$$\|\partial_u^\alpha \partial_v^\beta F\|_{m-k,\varepsilon} \leq \|F\|_{m,\varepsilon} \prod_{j\in\Gamma} \varepsilon_j^{\alpha_j+\beta_j}.$$
- *If $m \geq 2$, one defines $\Delta_{\mathcal{B}}$ by*
$$\Delta_{\mathcal{B}} F = \left(\sum_{j\in\Gamma} \left(\frac{\partial}{\partial u_j}\right)^2 + \left(\frac{\partial}{\partial v_j}\right)^2\right) F.$$
It satisfies $\Delta_{\mathcal{B}} F \in S_{m-2}(\mathcal{B}, \varepsilon)$, with $\|\Delta_{\mathcal{B}} F\|_{m-2,\varepsilon} \leq 2 \sum_j \varepsilon_j^2 \|F\|_{m,\varepsilon}$.
- *If $G \in S_m(\mathcal{B}, \delta)$ with δ square summable too, then $FG \in S_m(\mathcal{B}, \varepsilon + \delta)$ with $\|FG\|_{m,\varepsilon+\delta} \leq \|F\|_{m,\varepsilon} \|G\|_{m,\delta}$.*

One can prove that, under certain conditions, the Laplace operator does not depend on the chosen basis (see Remark 4.7 below).

Proposition 4.2. *If $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$ and ε square summable, then F is Fréchet differentiable on H^2 and*
$$DF(X) \cdot Y = \sum_{j\in\Gamma} \langle Y, u_j\rangle \frac{\partial F}{\partial u_j}(X) + \langle Y, v_j\rangle \frac{\partial F}{\partial v_j}(X).$$
Moreover, for all X and Y in H^2,
$$|F(X+Y) - F(X) - DF(X) \cdot Y| \leq \|F\|_{m,\varepsilon} \sum_{j\in\Gamma} \varepsilon_j^2 (1 + 2\sqrt{2})\ |Y|^2.$$

Proof. Let $X, Y \in H^2$. Since Γ is countable, we may order it and replace it formally by $\mathbb{N}$. Let P_N be the orthogonal projection onto $\mathrm{Vect}(u_i, v_i, i \leq N)$ if $N \geq 0$, $P_{-1} = 0$ and $P_{N,1/2}$ the orthogonal projection onto $\mathrm{Vect}(u_i, v_j, i \leq N+1, j \leq N)$. By approaching $P(X+Y)$ by $P(X + P_N(Y))$ one obtains
$$\begin{aligned} F(X + P_N(Y)) - F(X) = \sum_{j=0}^{N} & F(X + P_j(Y)) - F(X + P_{j-1,1/2}(Y)) \\ & + F(X + P_{j-1,1/2}(Y)) - F(X + P_{j-1}(Y)). \end{aligned}$$

Taylor's formula gives, for example for the part of the j-th term concerned with v_j,

$$
\begin{aligned}
F(X + \langle Y, v_j\rangle v_j + P_{j-1,1/2}(Y)) - F(X + P_{j-1,1/2}(Y)) \\
&= \langle Y, v_j\rangle \frac{\partial F}{\partial v_j}(X + P_{j-1,1/2}(Y)) \\
&\qquad + \langle Y, v_j\rangle^2 \int_0^1 (1-s)\frac{\partial^2 F}{\partial v_j^2}(X + P_{j-1,1/2}(Y) + s\langle Y, v_j\rangle v_j)\,\mathrm{d}s \\
&= \langle Y, v_j\rangle \frac{\partial F}{\partial v_j}(X) + \langle Y, v_j\rangle \left(\frac{\partial F}{\partial v_j}(X + P_{j-1,1/2}(Y)) - \frac{\partial F}{\partial v_j}(X)\right) \\
&\qquad + \langle Y, v_j\rangle^2 \int_0^1 (1-s)\frac{\partial^2 F}{\partial v_j^2}(X + P_{j-1,1/2}(Y) + s\langle Y, v_j\rangle v_j)\,\mathrm{d}s.
\end{aligned}
$$

The first term gives the expression of the differential and it is the general term of a convergent series (apply Cauchy–Schwarz inequality). Since $\frac{\partial F}{\partial v_j}$ is in $S_{m-1}(\mathcal{B}, \varepsilon)$ with $\left\|\frac{\partial F}{\partial v_j}\right\|_{m-1,\varepsilon} \le \varepsilon_j \|F\|_{m,\varepsilon}$, one can use (2.11) to treat the second term. It then yields a convergent series too, its sum being smaller than Cste. $|Y|^2$. The integral term can be estimated thanks to the estimates on the second derivatives and the sum of the corresponding terms is of order 2 in $|Y|$. Since F and its derivatives are bounded by $\|F\|_{m,\varepsilon}$ and powers of ε independently of X and Y, the remainder can be bounded as is asserted in the theorem, with a constant C independent of X, Y, $\|F\|_{m,\varepsilon}$ and ε. One may take $C = (1 + 2\sqrt{2})$. □

Remark 4.3. Since there are infinitely many terms, we need a precise bound for the remainder in Taylor's formula, which explains the loss of one order of differentiability.

Differentiating term by term and using the continuity of the extension operator $\mathcal{P}$ (Corollary 3.2) we get the following results:

Proposition 4.4. *Let $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \ge 2$ and ε square summable. Then, for all $Y \in H^2$, $X \mapsto DF(X) \cdot Y$ is in $S_{m-1}(\mathcal{B}, \varepsilon)$, with $\|X \mapsto DF(X) \cdot Y\|_{m-1,\varepsilon} \le 2\|F\|_{m,\varepsilon}|Y|\sqrt{\sum_{j\in\Gamma}\varepsilon_j^2}$.*

If, moreover, ε is summable, the application $X \mapsto DF(X) \cdot Y$ from H^2 in $\mathbb{R}$ admits a stochastic extension in $L^p(B^2, \mu_{B^2,t})$, which is the application

$$
\sum_\Gamma \langle Y, u_j\rangle \mathcal{P}\left(\frac{\partial F}{\partial u_j}\right) + \langle Y, v_j\rangle \mathcal{P}\left(\frac{\partial F}{\partial v_j}\right). \tag{4.1}
$$

The summability of ε in the second part is needed to ensure the existence of the stochastic extension.

Let $F \in S_m(\mathcal{B}, \varepsilon)$ with ε square summable. For $k \in \{1, \dots, m\}$ and $X \in H^2$, let us denote by $\Phi_k(X)$ on $(H^2)^k$ the k-linear symmetric continuous form defined by

$$\Phi_k(X)(Y_1, \dots, Y_k) = \sum_{\substack{J \in \Gamma^k, \\ \delta \in \{0,1\}^k}} \left(\prod_{s=1}^{k} \langle Y_s, w_{j_s}^{\delta_s} \rangle \right) \frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}}(X), \tag{4.2}$$

for all $(X, Y_1, \dots, Y_k) \in (H^2)^{k+1}$: with $w_j^0 = u_j$, $w_j^1 = v_j$, $J = (j_1, \dots, j_k)$, $\delta = (\delta_1, \dots, \delta_k)$. We notice that

$$|\Phi_k(X)(Y_1, \dots, Y_k)| \leq 2^k \|F\|_{m,\varepsilon} \, \|\varepsilon\|_2^k \prod_{s=1}^{k} |Y_s| \tag{4.3}$$

From now on, for the sake of brevity, we shall write J, δ instead of $(j_1, \dots, j_k)$, $(\delta_1, \dots, \delta_k)$.

Proposition 4.5. *Let $F \in S_m(\mathcal{B}, M, \varepsilon)$ with ε square summable. Then F is C^{m-1} on H^2 and, for all $k \in \{1, \dots, m-1\}$ and all $X \in H^2$,*

$$D^k F(X) = \Phi_k(X).$$

The inequality (4.3) *is satisfied. Finally, for* $0 \leq k \leq m-2$, *one has*

$$\|D^k F(X+Z) - D^k F(X) - D^{k+1} F(X)(\,\cdot\,, Z)\| \leq 2^k \|F\|_{m,\varepsilon} \|\varepsilon\|_2^{k+2} (1 + 2\sqrt{2}) |Z|^2,$$

where the norm is the norm of k-linear continuous applications on H^2.

Proof. Propositions 4.2 and 4.4 give the result for $m = 2$, except for the fact that F is C^1. This can be proved by applying (2.11) to the partial derivatives of F. For a general m, one uses induction. ◻

This allows one to state Taylor's formula to the order k for $F \in S_m(\mathcal{B}, \varepsilon)$, with ε square summable and $m \geq k+1$. For $X, Y \in H^2$,

$$\begin{aligned} F(X+Y) &= F(X) + \sum_{i=1}^{k-1} \frac{1}{i!} D^i F(X) \cdot Y^i + \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} D^k F(X+sY) \cdot Y^k \, \mathrm{d}s \\ &= F(X) + \sum_{i=1}^{k-1} \frac{1}{i!} D^i F(X) \cdot Y^i \\ &\quad + \sum_{\substack{J \in \Gamma^k, \\ \delta \in \{0,1\}^k}} \left(\prod_{r=1}^{k} \langle w_{j_r}^{\delta_r}, Y \rangle \right) \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} \frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}}(X+sY) \, \mathrm{d}s, \end{aligned} \tag{4.4}$$

exchanging the sums to get the last equality.

In the following Subsection 4.2, one proves the existence of stochastic extensions for each of the terms appearing here, the polynomial terms as well as the remainder, under the assumption that ε is summable. Note that these extensions are series indexed by Γ.

From Taylor's Formula one deduces the following result, which allows us to construct another completion B_A of H in the case when ε is summable.

Proposition 4.6. *Let ε be a summable sequence such that $\varepsilon_j > 0$ for all $j \in \Gamma$. One defines a symmetric, definite positive and trace class operator A by setting*

$$\forall\, X \in B^2,\; AX = \sum_{j\in\Gamma} \varepsilon_j \langle X, u_j\rangle u_j + \varepsilon_j \langle X, v_j\rangle v_j.$$

Set $\|X\|_A = \langle AX, X\rangle^{1/2}$. Then $\|\cdot\|_A$ is a measurable norm on H, in the sense of [9, Definition 4.4] *or* [3]. *One denotes by B_A the completion of H for this norm.*

If $F \in S_m(\mathcal{B}, \varepsilon)$ for $m \geq 2$, then F is uniformly continuous on H^2 with respect to the norm $\|\cdot\|_A$. The function F admits a uniformly continuous extension F_A on B_A and the stochastic extension $\widetilde{F}$ of F given by Proposition 3.1 is equal to F_A $\mu_{B_A,h}$-a.e.

Proof. It follows from Theorem 3 in [3] that $\|\cdot\|_A$ is a measurable norm, since A is one to one. Since $m \geq 2$, F is C^1 on H, Taylor's Formula with to the order 1 and Definition 4.2 imply the inequality

$$\begin{aligned} |F(X) - F(Y)| &\leq \int_0^1 \sum_{j\in\{0,1\},\,\delta\in\Gamma} \left| \frac{\partial F}{\partial w_j^\delta}(X + t(Y - X))\langle Y - X, w_j^\delta\rangle \right| \mathrm{d}t \\ &\leq \sum_{j\in\Gamma} \|F\|_{m,\varepsilon}\varepsilon_j(|\langle Y - X, u_j\rangle| + |\langle Y - X, v_j\rangle|) \\ &\leq \|F\|_{m,\varepsilon}\sqrt{2}\left(\sum \varepsilon_j\right)^{1/2} \|X - Y\|_A, \end{aligned}$$

thanks to the Cauchy–Schwarz Inequality. Then F is uniformly continuous on H^2 and admits a uniformly continuous extension on B_A, F_A. According to Theorem 6.3 in [9, Chapter 1]), F_A and $\widetilde{F}$ coincide almost everywhere. □

Remark 4.7. If $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 3$ and ε summable, one can define ΔF more intrinsically. Indeed, one can state an inequality more precise than (4.3). For $k \leq 3$ one gets, for all $X, Y_1, \ldots, Y_k \in (H^2)^{k+1}$,

$$|\Phi_k(X)(Y_1, \ldots, Y_k)| \leq 2^k \|F\|_{m,\varepsilon} \left(\sum_\Gamma \varepsilon_j\right)^{k/2} \prod_{s=1}^k \langle AY_s, Y_s\rangle^{1/2},$$

reasoning as in the proof of Proposition 4.6. The function F is C^2 since $m \geq 3$ and the inequality, for $k = 2$, ensures the existence of a self-adjoint, trace class operator M_x

satisfying

$$\forall\, U, V, X \in H^2, \quad \mathrm{d}^2 F(X) \cdot (U, V) = \langle M_X U, V \rangle.$$

One then sets $\Delta F(X) = \mathrm{Tr}(M_X)$ and the expression as a sum of partial derivatives does not depend on the chosen orthonormal basis.

One can notice, too, that if ε is summable, if F belongs to $S_m(\mathcal{B}, \varepsilon)$ for all m and if there exists a constant M such that $\|F\|_{m,\varepsilon} \leq M$ for all m, then $F \in S(Q_B)$ with B defined by $B = 4(\sum_\Gamma \varepsilon_j)A$, A being as in Proposition 4.6.

4.2. Taylor's formula and stochastic extensions

In contrast to the preceding subsection, where sums like $\sum_\Gamma \varepsilon_j \langle u_j, x\rangle$ have been treated by the Cauchy–Schwarz Inequality, we must suppose here that the sequence ε is summable. The sums now have the form $\sum_\Gamma \varepsilon_j\ \ell_{u_j}$ and, since the functions ℓ_{u_j} have a L^p norm independent of j, one needs stronger assumptions on ε.

Proposition 4.8. *Let ε be summable and let $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$. Let $X \in H^2$. For all $k \leq m-1$, all $h > 0$ and $p \in [1, +\infty[$, one can write, in $L^p(B^2, \mu_{B^2,h})$:*

$$\widetilde{F}(X+Y) = F(X) + \sum_{i=1}^{k-1} \frac{1}{i!} \sum_{\substack{J \in \Gamma^i,\\ \delta \in \{0,1\}^i}} \left(\prod_{r=1}^{i} \ell_{w_{j_r}^{\delta_r}}(Y) \right) \frac{\partial^i F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_i}^{\delta_i}}(X)$$
$$+ \sum_{\substack{J \in \Gamma^k,\\ \delta \in \{0,1\}^k}} \left(\prod_{r=1}^{k} \ell_{w_{j_r}^{\delta_r}}(Y) \right) \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} \mathcal{P} \left(\frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}} \right) (X + sY)\, \mathrm{d}s. \tag{4.5}$$

Proof. Let us denote by $\widetilde{\Phi}_i(X) \cdot Y^i$ the i-th term in the first sum and by $R_k(X)$ the last one, corresponding to the remainder.

We first prove that the polynomial part of the development in (4.4) has a stochastic extension in $L^p(B^2, \mu_{B^2,h})$. For $E \in \mathcal{F}(H)$ and $\widetilde{\pi}_E$ defined by (2.4), let us study $\Phi_k(X) \cdot (\widetilde{\pi}_E(Y)^k) - \widetilde{\Phi_k(X)}(Y, \dots, Y)$. In order to show that both terms belong to $L^p(B^2, \mu_{B^2,h})$, one has to find an upper bound for each term

$$\left\| \prod_{s=1}^{k} \ell_{a_s}(Y) \right\|_{L^p} \left| \frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}}(X) \right|$$

of the sum, with $a_s = w_{j_s}^{\delta_s}$ or $\pi_E(w_{j_s}^{\delta_s})$. Proposition 3.17 gives that

$$\left\| \prod_{s=1}^{k} \langle \widetilde{\pi}_E(Y), w_{j_s}^{\delta_s} \rangle - \prod_{s=1}^{k} \ell_{w_{j_s}^{\delta_s}} \right\|_{L^p} \leq (K(pk)h^{1/2})^k \sum_{s=1}^{k} |\pi_E(w_{j_s}^{\delta_s}) - w_{j_s}^{\delta_s}|,$$

since the $w_{j_s}^{\delta_s}$ and their projections have norms smaller than 1. Therefore

$$\|\Phi_k(X)\cdot(\widetilde{\pi}_E(Y)^k) - \widetilde{\Phi_k(X)}(Y,\dots,Y)\|_{L^p(B^2,\mu_{B^2,h})}$$
$$\leq \|F\|_{m,\varepsilon}(K(pk)h^{1/2})^k \sum_{\substack{J\in\Gamma^k,\\ \delta\in\{0,1\}^k}} \prod_{s=1}^{k} \varepsilon_{j_s} \sum_{s=1}^{k} |\pi_E(w_{j_s}^{\delta_s}) - w_{j_s}^{\delta_s}|.$$

One then replaces E by E_n, where (E_n) is an increasing sequence of $\mathcal{F}(H^2)$ whose union is dense in H^2. Since the terms $|\pi_{E_n}(w_{j_s}^{\delta_s}) - w_{j_s}^{\delta_s}|$ converge to 0 and are smaller than 2, the difference converges to 0 thanks to the dominated convergence Theorem. Hence $Y \mapsto \Phi_k(X)\cdot Y^k$ admits, as a stochastic extension in $L^p(B^2,\mu_{B^2,h})$, the application $Y \mapsto \widetilde{\Phi_k(X)}\cdot Y^k$.

The remainder is the sum indexed by $J\in\Gamma^k, \delta\in\{0,1\}^k$. One applies Corollary 3.21, replacing, in the upper bound, $\int_0^1 \frac{(1-s)^{k-1}}{(k-1)!}\,\mathrm{d}s$ by $(k!)^{-1}$, $\|G\|_{1,\varepsilon}$ by $\|F\|_{m,\varepsilon}\prod_1^k \varepsilon_{j_i}$ and $A = \max(|w_{j_i}^{\delta_i}|)$, by 1. One finds

$$\sum_{J,\delta}\Bigg\|\int_0^1 \frac{(1-s)^{k-1}}{(k-1)!}\Bigg(\frac{\partial^k F}{\partial w_{j_1}^{\delta_1}\dots\partial w_{j_k}^{\delta_k}}(X+s\widetilde{\pi}_E(Y))\prod_1^k\langle\widetilde{\pi}_E(Y), w_{j_i}^{\delta_i}\rangle$$
$$-\left(\mathcal{P}\frac{\partial^k F}{\partial w_{j_1}^{\delta_1}\dots\partial w_{j_k}^{\delta_k}}\right)(X+sY)\prod_1^k \ell_{w_{j_i}^{\delta_i}}(Y)\Bigg)\,\mathrm{d}s\Bigg\|_{L^p(B^2,\mu_{B^2,h})}$$
$$\leq \frac{1}{k!}K\|F\|_{m,\varepsilon}\sum_{\substack{J\in\Gamma^k,\\ \delta\in\{0,1\}^k}}(\varepsilon_{j_1}\dots\varepsilon_{j_k})\sum_{i=1}^k|\pi_E(w_{j_i}^{\delta_i}) - w_{j_i}^{\delta_i}|$$
$$+\frac{1}{k!}K\|F\|_{m,\varepsilon}\sum_{\substack{J\in\Gamma^k,\\ \delta\in\{0,1\}^k}}(\varepsilon_{j_1}\dots\varepsilon_{j_k})$$
$$\times\left(\sqrt{\sum\varepsilon_j^2}|\pi_E(X)-X| + e^{|X|^2/2h}\sum_0^\infty \varepsilon_j(|\pi_E(u_j)-u_j| + |\pi_E(v_j)-v_j|)\right).$$

If one replaces E by E_n from an increasing sequence of $\mathcal{F}(H^2)$ whose union is dense in H^2, this converges to 0 when n converges to infinity. □

With each term of the extended Taylor expansion (4.5), one can associate a quadratic form (see [1, Definition 1.2]) thanks to the following result:

Proposition 4.9. *Let $F \in S_m(\mathcal{B}, \varepsilon)$ with ε summable and $m \geq k+1$, where k is the order of differentiation. Each of the terms of* (4.5) *has a N_s norm (cf.* (2.12)*), for a well-chosen s. Precisely*

$$N_i\left(\frac{1}{i!}\widetilde{\Phi}_i(X)\cdot Y^i\right) \leq \frac{1}{i!}\|F\|_{m,\varepsilon}\left(2\max\left(1,\sqrt{\frac{h}{2}}\right)\sum_{\Gamma}\varepsilon_j\right)^i\int_{\mathbb{R}}(1+|v|)^i\,\mathrm{d}\mu_{\mathbb{R},1}(v).$$

and

$$N_k(R_k(X)) \leq \frac{1}{k!}\|F\|_{m,\varepsilon}\left(2\max\left(1,\sqrt{\frac{h}{2}}\right)\sum_{\Gamma}\varepsilon_j\right)^k\int_{\mathbb{R}}(1+|v|)^k\,\mathrm{d}\mu_{\mathbb{R},1}(v).$$

Proof. One uses the computations of Proposition 3.18. Then

$$\left\|\prod_{r=1}^{i}\ell_{w_{jr}^{\delta_r}}(\,\cdot\,+Y)\right\|_{L^1(B^2,\mu_{B^2,\frac{h}{2}})} \leq (1+|Y|)^i\max\left(1,\sqrt{\frac{h}{2}}\right)^i\int_{\mathbb{R}}(1+|v|)^i\,\mathrm{d}\mu_{\mathbb{R},1}(v).$$

Hence

$$\begin{aligned}&\left\|\sum_{J\in\Gamma^i,\,\delta\in\{0,1\}^i}\prod_{r=1}^{i}\ell_{w_{jr}^{\delta_r}}(\,\cdot\,+Y)\frac{\partial^i F}{\partial w_{j_1}^{\delta_1}\ldots\partial w_{j_i}^{\delta_i}}(X)\right\|_{L^1(B^2,\mu_{B^2,\frac{h}{2}})}\\ &\quad\leq \sum_{J\in\Gamma^i,\,\delta\in\{0,1\}^i}\|F\|_{m,\varepsilon}\varepsilon_{j_1}\ldots\varepsilon_{j_i}\,(1+|Y|)^i\max\left(1,\sqrt{\frac{h}{2}}\right)^i\int_{\mathbb{R}}(1+|v|)^i\,\mathrm{d}\mu_{\mathbb{R},1}(v)\\ &\quad\leq \|F\|_{m,\varepsilon}\left(2\max\left(1,\sqrt{\frac{h}{2}}\right)\sum_{\Gamma}\varepsilon_j\right)^i\int_{\mathbb{R}}(1+|v|)^i\,\mathrm{d}\mu_{\mathbb{R},1}(v)(1+|Y|)^i.\end{aligned}$$

It follows that

$$N_i\left(\frac{1}{i!}\widetilde{\Phi}_i(X)\cdot Y^i\right) \leq \frac{1}{i!}\|F\|_{m,\varepsilon}\left(2\max\left(1,\sqrt{\frac{h}{2}}\right)\sum_{\Gamma}\varepsilon_j\right)^i\int_{\mathbb{R}}(1+|v|)^i\,\mathrm{d}\mu_{\mathbb{R},1}(v).$$

We treat the remainder in the same way: the sum indexed by $J\in\Gamma^k$, $\delta\in\{0,1\}^k$ contains a product of k terms ℓ and the integral, which is bounded by $\frac{1}{k!}\|F\|_{m,\varepsilon}\varepsilon_{j_1}\ldots\varepsilon_{j_k}$. Therefore the remainder has a N_k norm bounded like the polynomial terms. ◻

5. The heat operator on H

The heat operator defined below associates a function defined on a (real, separable, infinite dimensional) Hilbert space, with a function defined on the same Hilbert space. We aim at extending the notion of the heat operator, which is classical in the finite dimensional setting.

The results proved here are different from the results obtained by ([5, 9]), inasmuch as they are concerned with symbols, which are initially defined on H (or H^2) and not on B.

We first present general results and then give the features in each symbol class.

5.1. General definition

Definition 5.1. Let F be a function defined on H, admitting a stochastic extension in $L^p(B, \mu_{B,t})$ for a given $p \in [1, +\infty[$. One defines $H_t F$ on H by

$$(H_t F)(X) = \int_B \widetilde{F}(X+Y)\, \mathrm{d}\mu_{B,t}(Y) = \int_B \widetilde{F}(Y) e^{-\frac{|X|^2}{2t}} e^{\ell_X/t}\, \mathrm{d}\mu_{B,t}(Y), \tag{5.1}$$

the second identity coming from (2.9). Sometimes, H_t is denoted by $e^{\frac{t}{2}\Delta}$.

If F is defined on the product H^2, one replaces H by H^2 and B by B^2.

Remark 5.2. This definition does not depend on the stochastic extension chosen, nor on the measurable norm and on the completion of H associated with it. Indeed, the fact that a sequence $F \circ \widetilde{\pi}_{E_n}$ is a Cauchy sequence in $L^p(B, \mu_{B,h})$ is expressed by integrals on finite dimensional subspaces of H (using (2.3)) and not at all by integrals on B. Likewise, the integral of (5.1) does not depend on the integration space B, since it is a limit of integrals on finite dimensional spaces of H.

Proposition 5.3. *Let F belong to a class $S(Q_A)$ of Definition 3.5 or to a class $S_m(\mathcal{B}, \varepsilon)$, with ε summable, of Definition 2.4. The semigroup property is verified: for all positive s, t and all X in the Hilbert space,*

$$H_t(H_s F)(X) = H_{t+s} F(X).$$

Moreover, one has (according to whether $F \in S(Q_A)$ or $S_m(\mathcal{B}, \varepsilon)$),

$$\forall\, X \in H^2,\ \ |(H_t F)(X)| \leq \|F\|_{m,\varepsilon} \quad or \quad \forall\, X \in H,\ \ |(H_t F)(X)| \leq \|F\|_{Q_A}. \tag{5.2}$$

Proof. We give the proof in the case when $F \in S(Q_A)$. Let B_A be the completion of H with respect to the measurable norm $\|\cdot\|_A$ given by Proposition 3.14. The function F is uniformly continuous on H and extends continuously as a function denoted by F_A, uniformly continuous and bounded on B_A. By Theorem 6.3 of [9, Chapter 1], every stochastic extension of F in $L^p(B_A, \mu_{B_A,h})$ coincides with F_A $\mu_{B_A,h}$-a.e. Considering that the heat operator is defined by an integral on B_A, one may write that,

$$\forall\, X \in H, \quad H_t F(X) = \int_{B_A} F_A(X+Y)\, \mathrm{d}\mu_{B_A,t}(Y).$$

This formula allows one to define a function, denoted by H_tF_A, on B_A. Since F_A is uniformly continuous and bounded on B_A, H_tF_A is uniformly continuous and bounded on B_A too, by [9, Theorem 4.1, Chapter 3]. Then H_tF_A is the stochastic extension of its restriction to H, H_tF and, for all $X \in H$,

$$H_s(H_tF)(X) = \int_{B_A} H_tF_A(X+Y)\,\mathrm{d}\mu_{B_A,s}(Y) = H_{t+s}F_A(X) = H_{t+s}F(X).$$

The same proof holds on H^2 for $F \in S_m(\mathcal{B},\varepsilon)$ with ε summable and $\|\cdot\|_A$, B_A from Proposition 4.6.

The inequalities (5.2) come from the fact that F_A is bounded on B_A like F on H. □

5.2. The heat operator in the classes $S_m(\mathcal{B},\varepsilon)$

Proposition 5.4. *Let $F \in S_m(\mathcal{B},\varepsilon)$ with $m \geq 2$, ε summable. If α,β are depth 1 multiindices (such that $\max(\alpha_j,\beta_j) \leq 1$), then*

$$\partial_u^\alpha \partial_v^\beta (H_tF)(X) = H_t(\partial_u^\alpha \partial_v^\beta)(X).$$

Moreover, for $m \geq 1$, $H_tF \in S_{m-1}(\mathcal{B},\varepsilon)$, with $\|H_tF\|_{m-1,\varepsilon} \leq \|F\|_{m,\varepsilon}$. The operator H_t is continuous from $S_m(\mathcal{B},\varepsilon)$ in $S_{m-1}(\mathcal{B},\varepsilon)$.

Proof. If $m = 1$, the continuity of H_t from $S_1(\mathcal{B},\varepsilon)$ in $S_0(\mathcal{B},\varepsilon)$ comes from the inequalities (5.2). Now suppose that $m \geq 2$ and prove (first) that

$$\frac{\partial}{\partial w}(H_tF)(X) = H_t\left(\frac{\partial}{\partial w}F\right)(X)$$

with $w = u_i$ or v_i and $X \in H^2$. By Taylor's formula

$$F(X+rw) - F(X) = r\frac{\partial F}{\partial w}(X) + r^2\int_0^1 (1-s)\frac{\partial^2 F}{\partial w^2}(X+rsw)\,\mathrm{d}s. \tag{5.3}$$

According to Proposition 3.1 and its corollaries, F and $\frac{\partial F}{\partial w}$ together with their translated of a vector $Y \in H^2$ admit stochastic extensions in $L^p(B^2,\mu_{B^2,t})$ and $\widetilde{\tau_Y F} = \tau_Y\widetilde{F}$. By substraction, for all $r \in \mathbb{R}^*$, the function $G_r : X \mapsto \int_0^1 (1-s)\frac{\partial^2 F}{\partial w^2}(X+rsw)\,\mathrm{d}s$ admits a stochastic extension in $L^p(B^2,\mu_{B^2,t})$, denoted by $\widetilde{G_r}$.

For all r, $|G_r| \leq \frac{1}{2}\|F\|_{m,\varepsilon}\sup(\varepsilon_i)^2$. Hence, so does $\widetilde{G_r}$ $\mu_{B^2,t}$-a.s.

Applying (5.3) in the point $\widetilde{\pi}_{E_j}(X)$ with $X \in B^2$ and taking a limit in $L^p(B^2,\mu_{B^2,t})$, one obtains

$$\tau_{rw}\widetilde{F} - \widetilde{F} = r\mathcal{P}\left(\frac{\partial F}{\partial w}\right) + r^2\widetilde{G_r} \quad \text{in } L^p(B^2,\mu_{B^2,t}) \tag{5.4}$$

By (5.1) one gets that, for all X of H^2,

$$\frac{H_t F(X+rw) - H_t F(X)}{r} = \left(H_t \frac{\partial F}{\partial w}\right)(X) + r(H_t G_r)(X),$$

and that

$$\left|\frac{H_t F(X+rw) - H_t F(X)}{r} - \left(H_t \frac{\partial F}{\partial w}\right)(X)\right| \leq |r| \int_{B^2} |\widetilde{G_r}|(X+Y)\, \mathrm{d}\mu_{B^2,t}(Y).$$

The bound on $\widetilde{G_r}$ shows that

$$\lim_{r\to 0} \frac{(H_t F)(X+rw) - (H_t F)(X)}{r} = \left(H_t \frac{\partial F}{\partial w}\right)(X),$$

which means that $H_t F$ admits order 1 partial derivatives in the (canonical) directions u_i, v_i.

Let α, β be two depth 1 multiindices. Let $w = u_i$ (or v_i) be a coordinate, with respect to which one has not yet differentiated (that is, such that $\alpha_i = 0$ or $\beta_i = 0$). Applying the preceding reasoning to $\partial_u^\alpha \partial_v^\beta F$, we get that

$$\frac{\partial}{\partial w} H_t(\partial_u^\alpha \partial_v^\beta F)(X) = H_t\left(\frac{\partial}{\partial w}\partial_u^\alpha \partial_v^\beta F\right)(X)$$

and an induction on $|\alpha| + |\beta|$ allows us to exchange H_t and differentiations. By (5.2), one gets that

$$|\partial_u^\alpha \partial_v^\beta H_t(F)(X)| = |H_t(\partial_u^\alpha \partial_v^\beta F)(X)| \leq \|\partial_u^\alpha \partial_v^\beta F\|_{m-1,\varepsilon} \leq \varepsilon^{\alpha+\beta}\|F\|_{m,\varepsilon}.$$

If $m = 2$, the proposition is proved. Otherwise one completes the proof by induction. □

The Heat operator commutes with the Laplace operator:

Proposition 5.5. *Let ε be summable. The operator $\Delta_{\mathcal{B}}$ is continuous from $S_m(\mathcal{B}, \varepsilon)$ to $S_{m-2}(\mathcal{B}, \varepsilon)$, for $m \geq 2$. Moreover, for $m \geq 3$,*

$$\forall\ F \in S_m(\mathcal{B}, \varepsilon), \quad \Delta_{\mathcal{B}} H_t F = H_t \Delta_{\mathcal{B}} F \in S_{m-3}(\varepsilon).$$

Proof. The continuity of $\Delta_{\mathcal{B}}$ comes from Lemma 4.1, since

$$\|\Delta_{\mathcal{B}} F\|_{m-2,\varepsilon} \leq 2 \sum_{j\in\Gamma} \varepsilon_j^2 \|F\|_{m,\varepsilon}.$$

Consider that Γ is ordered and, for $n \in \mathbb{N}$, set $\Delta_n = \sum_{j\leq n} \frac{\partial^2}{\partial u_j^2} + \frac{\partial^2}{\partial v_j^2}$. One can see that $\Delta_n F$ converges to $\Delta_{\mathcal{B}} F$ in $S_{m-2}(\mathcal{B}, \varepsilon)$. Moreover, one can exchange H_t and the differentiations with respect to u_j, v_j. This fact and the continuity of the operators allow us to write

$$H_t \Delta_{\mathcal{B}} F = H_t \lim_{n\to\infty} \Delta_n F = \lim_{n\to\infty} H_t \Delta_n F = \lim_{n\to\infty} \Delta_n H_t F = \Delta_{\mathcal{B}} H_t F,$$

which completes the proof. □

Let us state a result about commutators. For $Z \in H^2$ and F a function defined on H^2, denote by $M_Z F$ the function defined by $(M_Z F)(X) = \langle Z, X\rangle F(X)$.

Proposition 5.6. *Consider $S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$ and ε square summable. For all $i \in \mathbb{N}$, one has*

$$\frac{1}{t}\left[H_t, M_{u_i}\right] = H_t \frac{\partial}{\partial u_i} \quad \textit{and} \quad \frac{1}{t}\left[H_t, M_{v_i}\right] = H_t \frac{\partial}{\partial v_i} \quad \textit{on} \quad S_m(\mathcal{B}, \varepsilon).$$

Proof. Let $F \in S_m(\mathcal{B}, \varepsilon)$. Notice that $\ell_Z \widetilde{F}$ is a stochastic extension of $M_Z F$ in $L^p(B^2, \mu_{B^2,t})$ for all p in $[1, +\infty[$, by Corollary 3.16. According to Theorem 6.2 (Chap. 2, par. 6) of [9], for all $X \in H^2$,

$$\frac{\partial H_t F}{\partial u_i}(X) = \frac{1}{t}\int_{B^2} \widetilde{F}(X+Y)\ell_{u_i}(Y)\,\mathrm{d}\mu_{B^2,t}(Y).$$

But $\ell_{u_i}(Y) = \ell_{u_i}(Y+X) - \langle u_i, X\rangle$, since $X \in H^2$. Then

$$\begin{aligned}\frac{\partial H_t F}{\partial u_i}(X) = \frac{1}{t}\int_{B^2} \widetilde{F}(X+Y)\ell_{u_i}(Y+X)\,\mathrm{d}\mu_{B^2,t}(Y) \\ - \langle u_i, X\rangle \frac{1}{t}\int_{B^2} \widetilde{F}(X+Y)\,\mathrm{d}\mu_{B^2,t}(Y).\end{aligned}$$

This is the desired result. □

We shall use the Taylor expansions and their stochastic extensions to prove a preliminary result before stating the main result of this subsection, Theorem 5.8.

Proposition 5.7.

(1) *Let $m \geq 3$. There exists $C_m \in \mathbb{R}^+$ such that, for all $F \in S_m(\mathcal{B}, \varepsilon)$,*

$$\begin{aligned}\|H_t F - F\|_{m-3,\varepsilon} \leq C_m \|F\|_{m,\varepsilon} t. \\ \textit{if} \quad m \geq 5,\ \forall\, s > 0, \quad \|H_{t+s}F - H_s F\|_{m-4,\varepsilon} \leq C_m \|F\|_{m,\varepsilon} t.\end{aligned} \tag{5.5}$$

(2) *Let $m \geq 4$. There exists $C_m \in \mathbb{R}^+$ such that, for all $F \in S_m(\mathcal{B}, \varepsilon)$,*

$$\begin{aligned}\left\|\frac{H_t F - F}{t} - \frac{1}{2}\Delta F\right\|_{m-4,\varepsilon} \leq C_m \|F\|_{m,\varepsilon} t^{1/2}. \\ \textit{if} \quad m \geq 5,\ \forall\, s > 0, \quad \left\|\frac{H_{t+s}F - H_s F}{t} - \frac{1}{2}\Delta H_s F\right\|_{m-5,\varepsilon} \leq C_m \|F\|_{m,\varepsilon} t^{1/2}.\end{aligned} \tag{5.6}$$

Proof. Formula (4.5), integrated with respect to Y on B^2, gives, for $k \le m-1$:

$$\begin{aligned}
&\int_{B^2} \widetilde{F}(X+Y)\,\mathrm{d}\mu_{B^2,t}(Y)\\
&= F(X) + \sum_{i=1}^{k-1} \frac{1}{i!} \sum_{\substack{J\in\Gamma^i,\\ \delta\in\{0,1\}^i}} \int_{B^2} \prod_{r=1}^{i} \ell_{w_{jr}^{\delta_r}}\,\mathrm{d}\mu_{B^2,t}(Y) \frac{\partial^i F}{\partial w_{j_1}^{\delta_1}\dots\partial w_{j_i}^{\delta_i}}(X)\\
&\quad + \sum_{J,\delta} \int_{B^2} \prod_{r=1}^{k} \ell_{w_{jr}^{\delta_r}} \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} \mathcal{P}\left(\frac{\partial^k F}{\partial w_{j_1}^{\delta_1}\dots\partial w_{j_k}^{\delta_k}}\right)(X+sY)\,\mathrm{d}s\,\mathrm{d}\mu_{B^2,t}(Y).
\end{aligned}$$

We denote by R_k the last term in the preceding formula. We have seen in Subsection 4.2 that these functions admit L^1 norms, which allows us to exchange sums and integrals on B^2. Using Wick's formula, we see that odd order terms are equal to 0.

The remainder is bounded as follows:

$$\forall\ X\in H^2,\ \forall\ k\le m-1,\quad |R_k(X)| \le \frac{1}{\sqrt{\pi}k!}\|F\|_{m,\varepsilon} 2^{\frac{3k}{2}} t^{\frac{k}{2}} \Gamma\left(\frac{k+1}{2}\right)\left(\sum_{\Gamma}\varepsilon_j\right)^k.$$

Indeed, $\mathcal{P}\left(\frac{\partial^k F}{\partial w_{j_1}^{\delta_1}\dots\partial w_{j_k}^{\delta_k}}\right)$ is bounded by $\|F\|_{m,\varepsilon}\varepsilon_{j_1}\dots\varepsilon_{j_k}$. One applies Hölder's formula to the product of ℓ functions and one sums over $j_1,\dots,j_k$.

Thanks to Wick's Theorem, even order terms give the successive powers of the Laplace operator. Hence:

$$H_tF(X) = \int_{B^2}\widetilde{F}(X+Y)\,\mathrm{d}\mu_{B^2,t}(Y) = F(X) + \sum_{0<2p\le k-1} \frac{1}{p!}\frac{t^p}{2^p}\Delta^p F(X) + R_k.$$

Let us prove the point about continuity. For $k=2$ and $m=3$, since the remainder is of order t, one has:

$$\forall\ X\in H^2,\quad |H_tF(X)-F(X)| \le C_2\|F\|_{3,\varepsilon}t,$$

with $C_2 = 2(\sum\varepsilon_j)^2$. This yields the first part of (5.5) when $m=3$. To treat the general case one uses induction, working with $\partial_u^\alpha\partial_v^\beta F$, where α and β have depth 1 at most and using Proposition 5.4. To obtain the second formula of (5.5) one applies H_s to the first one (and loses one order of differentiability) and applies the semigroup property (Proposition 5.3).

Let us prove the point about differentiability. For $k=3$ and $m=4$, one has the following result since the remainder is of order $t^{3/2}$:

$$\forall\ X\in H^2,\quad \left|\frac{H_tF(X)-F(X)}{t} - \frac{1}{2}\Delta F(X)\right| \le C_3\|F\|_{4,\varepsilon}t^{1/2},$$

with $C_3 = \frac{1}{\sqrt{\pi}3!} 2^{9/2}\Gamma(2)(\sum \varepsilon_j)^3$. This gives the first part of (5.6) when $m = 4$. In the general case one uses induction and Proposition 5.4 as above. To get the second formula one applies H_s to the first one, losing one order of differentiability and uses the semigroup property. This completes the proof of Proposition 5.7 □

We now state the main result about the heat operator in S_m classes. For the sake of clarity, the two first points repeat former results of this subsection.

Theorem 5.8. *Let ε be summable.*

(1) *If $m \geq 1$, the operator H_t is continuous from $S_m(\mathcal{B}, \varepsilon)$ to $S_{m-1}(\mathcal{B}, \varepsilon)$ and for $m \geq 2$, the operator Δ is continuous from $S_m(\mathcal{B}, \varepsilon)$ to $S_{m-2}(\mathcal{B}, \varepsilon)$.*

(2) *For $m \geq 3$, H_t and Δ commute: for all $F \in S_m(\mathcal{B}, \varepsilon)$, $\Delta H_t F = H_t \Delta F \in S_{m-3}(\mathcal{B}, \varepsilon)$.*

(3) *Let $m \geq 6$ and $F \in S_m(\mathcal{B}, \varepsilon)$. The application $t \mapsto H_t F$ is C^1 from $[0, +\infty[$ in $S_{m-6}(\mathcal{B}, \varepsilon)$ and its derivative is $t \mapsto \frac{1}{2} H_t \Delta F$.*

Proof. It remains to prove the last point. Set $\varphi(t) = H_t F \in S_{m-1}(\mathcal{B}, \varepsilon)$. According to the preceding proposition, φ is differentiable on $[0, +\infty[$ and $\varphi'(t) = \frac{1}{2}\Delta H_t F = \frac{1}{2} H_t \Delta F$. But $H_t \Delta F \in S_{m-3}(\mathcal{B}, \varepsilon) \subset S_{m-6}(\mathcal{B}, \varepsilon)$. Since $\Delta F \in S_{m-2}(\mathcal{B}, \varepsilon)$, an application of point 1 (about continuity) proves that $t \mapsto H_t \Delta F$ is continuous from $[0, +\infty[$ in $S_{m-6}(\mathcal{B}, \varepsilon)$. □

Remark 5.9. It is not necessary to write $\Delta_{\mathcal{B}}$, because of Remark 4.7.

5.3. The heat operator in the classes $S(Q_A)$

This subsection is concerned with the heat operator and the Laplace operator in the frame of the classes introduced in Definition 3.5. Note, in particular, that the Laplace operator is bounded in $S(Q_A)$. This rather unusual fact allows one, for example, to invert the heat operator and, in view of pseudodifferential analysis, to recover the Anti-Wick symbol from the Wick symbol. This is due to the regularity of the classes, which are more than analytic. A similar fact is valid, in a discrete context, for finite or locally finite graphs.

The first results about H_t and Δ do not require technicalities and appear very early in this part (Propositions 5.10 and 5.11). A more complete and slightly more technical result is stated further (Theorem 5.14).

In this subsection, the operator A is self-adjoint, nonnegative and trace class.

Proposition 5.10. *Let $f \in S(Q_A)$. For all $t > 0$, the application $H_t f$ belongs to $S(Q_A)$ and $\|H_t f\|_{Q_A} \leq \|f\|_{Q_A}$.*

Let $g_{m,U}$ be the application $x \mapsto \mathrm{d}^m f(x)(U_1, \ldots, U_m)$ for $m \in \mathbb{N}^*$ and $U = (U_1, \ldots, U_m)$. Then $g_{m,U}$ belongs to $S(Q_A)$ and one has:

$$\mathrm{d}^m(H_t f)(x) \cdot (y_1, \ldots, y_m) = H_t(g_{m,y_1,\ldots,y_m})(x).$$

We denote by $\Delta f(x) = \mathrm{Tr}(\mathrm{d}^2 f(x))$ the trace of the operator M_x satisfying $\langle M_x U, V\rangle = \mathrm{d}^2 f(x)(U, V)$ for all vectors U, V of H. Its existence is ensured by the inequalities (3.3) and one can see it, too, as a sum of partial derivatives (with respect to an arbitrary orthonormal basis of H). One can state the following proposition:

Proposition 5.11. *If $f \in S(Q_A)$, then $\Delta f \in S(Q_A)$ with $\|\Delta f\|_{Q_A} \leq \mathrm{Tr}(A)\|f\|_{Q_A}$. Moreover, for all $t > 0$,*

$$\Delta(H_t f)(x) = H_t(\Delta f)(x).$$

Proof of Proposition 5.10. One checks that $g_{m,U}$ is C^∞ and that, for all integer $k \geq 1$ and all $h_1, \ldots, h_k \in H$,

$$\mathrm{d}^k g_{m,U}(x) \cdot (h_1, \ldots, h_k) = \mathrm{d}^{m+k} f(x) \cdot (h_1, \ldots, h_k, U_1, \ldots, U_m).$$

This proves that $g_{m,U} \in S(Q_A)$. Moreover, it satisfies

$$\|g_{m,U}\|_{Q_A} \leq \|f\|_{Q_A} \prod_{j=1}^{m} Q(U_j)^{1/2}.$$

We now turn to $H_t f$. It is differentiable on H and

$$\mathrm{d}(H_t f)(x) \cdot y = \int_B \mathcal{P}(u \mapsto \mathrm{d}f(u) \cdot y)(x + z)\, \mathrm{d}\mu_{B,t}(z) = (H_t g_{1,y})(x).$$

Moreover

$$|H_t f(x + y) - H_t f(x) - (H_t g_{1,y})(x)| \leq \frac{1}{2}\|f\|_{Q_A} Q_A(y).$$

By Taylor's Formula, for all $x, y \in H$,

$$\tau_y f(x) = f(x) + \mathrm{d}f(x) \cdot y + \int_0^1 (1 - s)\mathrm{d}^2 f(x + sy) \cdot y^2\, \mathrm{d}s.$$

Let $R_2(x, y)$ be the integral term. Since $\tau_y f, f$ and $x \mapsto \mathrm{d}f(x) \cdot y$ have stochastic extensions in $L^p(B, \mu_{B,t})$, so does $x \mapsto R_2(x, y)$. Hence

$$H_t f(x + y) = H_t f(x) + \int_B (\mathcal{P}(\mathrm{d}f(\,\cdot\,) \cdot y)(x + z)\, \mathrm{d}\mu_{B,t}(z) + \int_B \widetilde{R_2}(x + z)\, \mathrm{d}\mu_{B,t}(z).$$

The first integral gives a linear application with respect to y. The hypotheses on f prove its continuity and the bound on the remainder. Proposition 5.10 then follows by induction. □

Proof of Proposition 5.11. Let (e_j) be an orthonormal basis of H. One can write

$$\mathrm{Tr}(\mathrm{d}^2 f(x)) = \lim_{n\to\infty} \sum_{s=1}^{n} \mathrm{d}^2 f(x) \cdot (e_s, e_s) = \lim_{n\to\infty} \sum_{s=1}^{n} g_{2,e_s,e_s}(x),$$

with the notations of Proposition 5.10. Then the series $\sum g_{2,e_s,e_s}$ converges in $S(Q_A)$ because $\|g_{2,e_s,e_s}\|_{Q_A} \leq \|f\|_{Q_A} \langle Ae_s, e_s\rangle$ and A is trace class. Hence $\Delta f \in S(Q_A)$ with $\|\Delta f\|_{Q_A} \leq \mathrm{Tr}(A)\|f\|_{Q_A}$. Since H_t is continuous on $S(Q_A)$, one has

$$\begin{aligned}\mathrm{Tr}(\mathrm{d}^2 H_t f(x)) &= \lim_{n\to\infty} \sum_{s=1}^{n} \mathrm{d}^2 H_t f(x) \cdot (e_s, e_s) = \lim_{n\to\infty} H_t \left(\sum_{s=1}^{n} g_{2,e_s,e_s} \right)(x) \\ &= H_t \left(\sum_{s=1}^{\infty} g_{2,e_s,e_s} \right)(x) = H_t(\mathrm{Tr}(\mathrm{d}^2 f))(x).\end{aligned}$$

□

Proposition 5.12. *For all $f \in S(Q_A)$, one has*

$$\lim_{t\to 0} \left\| \frac{H_t(f) - f}{t} - \frac{1}{2}\Delta f \right\|_{Q_A} = 0.$$

Moreover, for all $s > 0$, one has

$$\lim_{t\to 0} \frac{(H_{t+s} f) - H_s f}{t} = \frac{1}{2}\mathrm{Tr}(\mathrm{d}^2 H_s f) = \frac{1}{2}\Delta H_s f = \frac{1}{2} H_s \Delta f, \tag{5.7}$$

the convergence taking place in $S(Q_A)$.

For higher orders. Let $x \in H$, let (e_n) be an arbitrary orthonormal basis of H. Let $\frac{\partial}{\partial x_j}$ be the differentiation in the direction of e_j. For all integer j set:

$$(\Delta^j) f(x) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \right)^j f(x).$$

One has for all $t > 0$,

$$H_t f(x) = f(x) + \sum_{j=1}^{N} \frac{1}{j!} \left(\frac{t}{2}\right)^j \Delta^j f(x) + \int_B \widetilde{R}_{2N+1}(y)\, \mathrm{d}\mu_{B,t}(y),$$

Proof. Let $x \in H$. First prove (5.7) for $s = 0$. For $y \in H$, Taylor's formula gives

$$f(x+y) = f(x) + \sum_{j=1}^{k} \frac{1}{j!} \mathrm{d}^j f(x) \cdot y^j + \int_0^1 \frac{(1-s)^k}{k!} \mathrm{d}^{k+1} f(x+sy) \cdot y^{k+1}\, \mathrm{d}s.$$

Denote by $R_k(y)$ the last term of the sum just above. According to Lemma 3.3, $\tau_x f$ has a stochastic extension $\widetilde{\tau_x f}$ in $L^p(B, \mu_{B,h})$, with respect to the variable y. Indeed, f admits a stochastic extension for all p and Definition 3.5 implies that it is Lipschitz continuous. By substraction, the remainder R_k also admits a stochastic extension $\widetilde{R_k}$. Let us prove that this extension is bounded as follows: $\forall\, t > 0, \forall\, p \in [1, +\infty[, \forall\, k \in \mathbb{N}^*$,

$$\|\widetilde{R_k}\|_{L^p(B,\mu_{B,t})} \leq \frac{1}{(k+1)!}\|f\|_{Q_A} C(p(k+1))^{k+1} \left(\sum_j \lambda_j\right)^{\frac{k+1}{\alpha(p(k+1))}} t^{\frac{k+1}{2}}, \tag{5.8}$$

with $C(p)$ from Lemma 3.7. Let $(E_n)_n$ be an increasing sequence of $\mathcal{F}(H)$, whose union is dense in H. Then, the norm being taken in $L^p(B, \mu_{B,t})$,

$$\begin{aligned}\|\widetilde{R_k}\|_{L^p} &\leq \|\widetilde{R_k} - R_k \circ \widetilde{\pi}_{E_n}\|_{L^p} + \|R_k \circ \widetilde{\pi}_{E_n}\|_{L^p}\\ &\leq \|\widetilde{R_k} - R_k \circ \widetilde{\pi}_{E_n}\|_{L^p} + \|f\|_{Q_A}\|Q_A^{\frac{k+1}{2}} \circ \widetilde{\pi}_{E_n}\|_{L^p}\end{aligned}$$

by definition of R_k. Remark 3.8 enables us to give an upper bound independent of n for the second term and to let n converge to infinity. This concludes the treatment of the remainder.

One can then write, extending in $L^1(B, \mu_{B,t})$, according to Proposition 3.11 :

$$\int_B \widetilde{f}(x+y)\,\mathrm{d}\mu_{B,t}(y) = f(x) + \int_B \sum_{j=1}^{k} \mathcal{P}\left(y \mapsto \frac{1}{j!}\mathrm{d}^j f(x)\cdot y^j\right) + \widetilde{R_k}(y)\,\mathrm{d}\mu_{B,t}(y)$$

where $\mathcal{P}$ represents the passage to the stochastic extension. For $j \leq k$ one uses the L^1 convergence and formula (2.3) to obtain

$$\begin{aligned}\int_B \mathcal{P}(y \mapsto \mathrm{d}^j f(x)\cdot y^j)\,\mathrm{d}\mu_{B,t}(y) &= \lim_{n\to\infty}\int_B \mathrm{d}^j f(x)\cdot \widetilde{\pi}_{E_n}(y)^j\,\mathrm{d}\mu_{B,t}(y)\\ &= \lim_{n\to\infty}\int_{E_n} \mathrm{d}^j f(x)\cdot z^j\,\mathrm{d}\mu_{E_n,t}(z),\end{aligned}$$

where $(E_n)_n$ is an increasing sequence of $\mathcal{F}(H)$, whose union is dense in H. For odd j, the terms are equal to 0. For even j, one takes an arbitrary orthonormal basis of E_n, $(e_s)_{1\leq s\leq \dim(E_n)}$, and one checks that

$$\int_{E_n} \mathrm{d}^2 f(x)\cdot z^2\,\mathrm{d}\mu_{E_n,t}(z) = \sum_{s=1}^{\dim(E_n)} t\frac{\partial^2 f}{\partial e_s^2}(x).$$

One then gets that, for any orthonormal basis of H,

$$\int_B \mathcal{P}\left(y \mapsto \mathrm{d}^2 f(x)\cdot y^2\right)\mathrm{d}\mu_{B,t}(y) = t\sum_{j\in\mathbb{N}}\frac{\partial^2 f}{\partial e_s^2}(x) = t\,\mathrm{Tr}(\mathrm{d}^2 f(x)).$$

Applying the former reasoning to $k = 3$ and using the upper bound of $\widetilde{R_3}$ in L^1 yield

$$\left|\frac{(H_t(f)(x) - f(x))}{t} - \frac{1}{2}\mathrm{Tr}(\mathrm{d}^2 f(x))\right| \leq \|f\|_{Q_A}\frac{1}{4!}C(4)^4 S^{\frac{4}{\alpha(4)}} t,$$

which holds for all $x \in H$. This proves Formula (5.7). Replacing f by $g_{m,y_1,\ldots,y_m}$ in this inequality, we obtain, thanks to Proposition 5.10,

$$\begin{aligned}\left|\frac{(\mathrm{d}^m H_t(f)(x)\cdot Y - \mathrm{d}^m f(x)\cdot Y)}{t} - \frac{1}{2}\mathrm{d}^m \operatorname{Tr}(\mathrm{d}^2 f(x))\cdot Y\right|& \\ \le \|g_{m,y_1,\ldots,y_m}\|_{Q_A}\frac{1}{4!}C(4)^4 S^{\frac{4}{\alpha(4)}}t& \\ \le \|f\|_{Q_A}\prod Q_A(y_i)^{1/2}\frac{1}{4!}C(4)^4 S^{\frac{4}{\alpha(4)}}t.&\end{aligned}$$

with $Y = (y_1,\ldots,y_m)$. One then has

$$\left\|\frac{H_t f - f}{t} - \frac{1}{2}\Delta f\right\|_{Q_A} \le \frac{1}{4!}C(4)^4 S^{\frac{4}{\alpha(4)}}t\|f\|_{Q_A},$$

which gives the convergence in $S(Q_A)$.

According to Proposition 5.10, H_s is continuous on $S(Q_A)$ and its norm is smaller than 1. The semigroup property (Proposition 5.3) gives

$$\left\|\frac{H_{t+s} f - H_s f}{t} - \frac{1}{2}H_s\Delta f\right\|_{Q_A} \le \frac{1}{4!}C(4)^4 S^{\frac{4}{\alpha(4)}}t\|f\|_{Q_A},$$

Since H_s and Δ commute, the first point is proved.

The proof of the second point is similar but one considers $k = 2N+1$ instead of stopping at $k = 3$. For even j one has

$$\int_{E_n} \mathrm{d}^j f(x)\cdot z^j \,\mathrm{d}\mu_{E_n,t}(z) = \int_{E_n} j! \sum_{\alpha\in\mathbb{N}^{\dim(E_n)},|\alpha|=j} \frac{1}{\alpha!}\frac{\partial^j f}{\partial z^\alpha} z^\alpha \,\mathrm{d}\mu_{E_n,t}(z)$$

and the terms where a coordinate of the multiindex α is odd are equal to 0. The computation of the other terms gives the result, thanks to classical equalities giving the moments of the normal law. This achieves the proof of Proposition 5.12. □

As a corollary of Propositions 5.11 and 5.12, one can state the following commutation result, which may be used, in view of pseudodifferential analysis, to prove a covariance result.

Proposition 5.13. *Let φ be linear, continuous on H and such that $\varphi^*\varphi = \varphi\varphi^* = \mathrm{Id}_H$. Let A be a linear application satisfying the hypotheses of Definition 3.5. For all $f \in S(Q_A)$, one can write*

$$\forall\, t \ge 0, \quad (H_t f)\circ\varphi = H_t(f\circ\varphi). \tag{5.9}$$

Proof. One verifies that $f\circ\varphi$ (denoted by f_φ) is in $S(Q_{\varphi^*A\varphi})$, with

$$\mathrm{d}^2 f_\varphi(x)\cdot(U,V) = \mathrm{d}^2 f(\varphi(x))\cdot(\varphi(U),\varphi(V)) = \langle \varphi^* M_{\varphi(x)}(f)\varphi U, V\rangle$$

and $\|f_\varphi\|_{Q_{\varphi^* A\varphi}} = \|f\|_{Q_A}$. (We still denote here by $\Delta f(x) = \mathrm{Tr}(\mathrm{d}^2 f(x))$ the trace of the operator M_x satisfying $\langle M_x U, V\rangle = \mathrm{d}^2 f(x)(U,V)$ for all vectors U, V in H.) Moreover, the operator $\varphi^* M_{\varphi(x)}(f)\varphi$ is trace class and has the same trace as $M_{\varphi(x)}(f)$. Thus

$$\begin{aligned}\Delta(f_\varphi(x)) = \mathrm{Tr}(\mathrm{d}^2 f_\varphi(x)) &= \mathrm{Tr}(\varphi^* M_{\varphi(x)}(f)\varphi)\\ &= \mathrm{Tr}(M_{\varphi(x)}(f)) = \mathrm{Tr}(\mathrm{d}^2 f(\varphi(x))) = (\Delta f)(\varphi(x)).\end{aligned}$$

Applying 5.12 and the above remark to $f \circ \varphi$, one gets that

$$\lim_{t\to 0} \frac{H_t(f_\varphi) - f_\varphi}{t} = \frac{1}{2}\Delta(f_\varphi) = \frac{1}{2}(\Delta f)\circ\varphi \quad \text{in} \quad S(Q_{\varphi^* A\varphi}).$$

Composing with φ^*, one obtains that

$$\lim_{t\to 0}\left(\frac{H_t(f_\varphi) - f_\varphi}{t}\right)\circ\varphi^* = \frac{1}{2}(\Delta f) = \lim_{t\to 0}\left(\frac{H_t(f - f}{t}\right) \quad \text{in} \quad S(Q_A).$$

If one denotes by T_t the operator defined on $S(Q_A)$ by $T_t f = H_t(f\circ\varphi)\circ\varphi^*$, one can verify that (T_t) is a semigroup on $S(Q_A)$. Since both semigroups (T_t) and (H_t) have the same infinitesimal generator $\frac{1}{2}\Delta$, which is continuous on $S(Q_A)$ (Proposition 5.11), they are uniformly continuous and equal ([10, Theorems 1.2 and 1.3, Chapter 1]). This achieves the proof. □

We now can state the main result of this part. For the sake of clarity, the first points repeat former results of the same part.

Theorem 5.14. *Let A be a linear application on H satisfying the hypotheses of Definition 3.5. Let $f \in S(Q_A)$.*

(1) *The function Δf belongs to $S(Q_A)$ with $\|\Delta f\|_{Q_A} \leq \mathrm{Tr}(A)\|f\|_{Q_A}$.*

(2) *For all $t \in]0,\infty[$, the application $H_t f$ belongs to $S(Q_A)$ and $\|H_t f\|_{Q_A} \leq \|f\|_{Q_A}$. Moreover, $\Delta(H_t f)(x) = H_t(\Delta f)(x)$.*

(3) *The function $t \mapsto H_t f$ is C^∞ on $]0,\infty[$ with values in $S(Q_A)$, with*
$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} H_t f = \left(\frac{1}{2}\Delta\right)^m H_t f.$$

(4) *For all $N \in \mathbb{N}^*$, one has*
$$H_t f = f + \sum_{k=1}^{N} \frac{t^k}{k!}\left(\frac{1}{2}\Delta\right)^k f + t^{N+1} R_N(t),$$
where $R_N \in S(Q_A)$ is bounded independently of $t \in [0,1]$.

(5) *As a bounded operator on* $S(Q_A)$*,*

$$H_t = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t\Delta}{2} \right)^n .$$

(6) *Consequently, the definition of* H_t *can be extended for negative values of* t*, implying that points 2, 3 and 5 are valid for* $t \in \mathbb{R}$*. In point 4, the condition can be replaced by "where* $R_N \in S(Q_A)$ *is bounded independently of* $t \in [-M, M]$ *for an arbitrary* $M > 0$*".*

Proof. The first two points come from Propositions 5.10 and 5.11. For the differentiability, according to Proposition 5.12, the result holds for $m = 1$. But then, since Δ commutes with H_t (Proposition 5.11), one concludes by induction on m.

For the fourth point, one applies one of Taylor's formulae to $t \mapsto H_t f$, which gives

$$\|R_N(t)\|_{Q_A} \leq \frac{1}{(N+1)!} \sup_{s\in[0,t]} \left\| H_s \left(\left(\frac{\Delta}{2} \right)^{N+1} f \right) \right\|_{Q_A} \leq \frac{1}{(N+1)!} \left\| \left(\frac{\Delta}{2} \right)^{N+1} f \right\|_{Q_A}$$

according to Proposition 5.10. Point 5 is a consequence of [10], since the infinitesimal generator of the semigroup is a bounded operator. The last point is a consequence of 5 and of the properties of series in a Banach space. □

References

[1] Laurent Amour, Lisette Jager, and Jean Nourrigat. On bounded pseudodifferential operators in Wiener spaces. *J. Funct. Anal.*, 269(9):2747–2812, 2015.

[2] Laurent Amour, Lisette Jager, and Jean Nourrigat. Weyl calculus in QED I. The unitary group. *J. Math. Phys.*, 58(1): article no. 013501 (24 pages), 2017.

[3] Leonard Gross. Measurable functions on Hilbert space. *Trans. Am. Math. Soc.*, 105:372–390, 1962.

[4] Leonard Gross. Abstract Wiener spaces. In *Proc. 5th Berkeley Symp. Math. Stat. Probab., Univ. Calif. 1965/66. Vol. II, Part 1: Contributions to probability theory*, pages 31–42. University of California Press, 1967.

[5] Leonard Gross. Potential theory on Hilbert space. *J. Funct. Anal.*, 1:123–181, 1967.

[6] Leonard Gross. Abstract Wiener measure and infinite dimensional potential theory. In *Lectures in modern Analysis and applications. II*, volume 140 of *Lecture Notes in Mathematics*, pages 84–116. Springer, 1970.

[7] Brian C. Hall. The heat operator in infinite dimensions. In *Infinite Dimensional Analysis in Honor of H.-H. Kuo*, volume 22 of *QP–PQ: Quantum Probability and White Noise Analysis*, pages 161–174. World Scientific, 2008.

[8] Paul Krée and Ryszard Raczka. Kernels and symbols of operators in quantum field theory. *Ann. Inst. Henri Poincaré, Phys. Théor.*, 28:41–73, 1978.

[9] Hui-Hsiung Kuo. *Gaussian measures in Banach spaces*, volume 463 of *Lecture Notes in Mathematics*. Springer, 1975.

[10] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer, 1983.

[11] Roald Ramer. On nonlinear transformations of Gaussian measures. *J. Funct. Anal.*, 15:166–187, 1974.

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