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Structure and bases of modular spaces sequences $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$

Jean-Christophe Feauveau

Abstract

The modular discriminant Δ is known to structure the sequence of modular forms of level 1 $(M_{2k}(\mathrm{SL}_2(\mathbb{Z})))_{k\in\mathbb{N}^*}$. For any positive integer N, we define a strong modular unit Δ_N of level N which enables us to structure the sequence $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$ in an identical way. We then apply this novel result to the search of bases for each of the $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$ spaces.

Structure et bases des suites d'espaces modulaires $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$

Résumé

Le discriminant modulaire Δ est connu pour structurer la famille de formes modulaires de niveau 1, $(M_{2k}(\mathrm{SL}_2(\mathbb{Z})))_{k\in\mathbb{N}^*}$. Pour tout entier N, nous définissons une unité modulaire forte de niveau N notée Δ_N, qui permet de structurer la famille $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$ de manière identique. Nous appliquerons ce résultat à la recherche de bases pour chacun des espaces $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$.

Introduction

When studying modular forms, an important result relates to the structure of the sequence $(M_{2k}(\mathrm{SL}_2(\mathbb{Z})))_{k\in\mathbb{N}^*}$ obtained using the Δ function, and the opportunity to obtain an explicit basis for each subspace [11, p. 143–144].

Such a result appears to be missing for the sequences $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$, whenever $N \geqslant 2$. We propose in this paper an explicit decomposition of modular form spaces $(M_{2k}(\Gamma_0(N)))_{(k,N)\in\mathbb{N}^{*2}}$. As the formulae providing the dimension of these spaces [2, 12] hint towards, such a reduction cannot be simple. Nevertheless, we will show that for any given level N, there exists a function Δ_N that will play for $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$ the same rôle that $\Delta = \Delta_1$ played in the study of $(M_{2k}(\mathrm{SL}_2(\mathbb{Z})))_{k\in\mathbb{N}^*}$.

More specifically, ρ_N being the weight of Δ_N, we will prove that for any fixed positive integer N and any integer k:

> *Knowing bases of* $M_{2k}(\Gamma_0(N))$ *for* $1 \leqslant k \leqslant \frac{1}{2}\rho_N + 1$ *leads to knowing bases of* $M_{2k}(\Gamma_0(N))$ *for all* k.

What is more, for any N, this result is algorithmic. It allows us to derive the Fourier series of bases $(B_{2k}(N))_{k\in\mathbb{N}^*}$ to any given accuracy level as soon as one has such series for $1 \leqslant k \leqslant \frac{1}{2}\rho_N + 1$, which SAGE for example may provide.

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First, the structure of families $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$ will be studied in Section 2 under the assumption of the existence of a strong modular unit Δ_N. This assumption will then be proven when N is prime in Section 5. Finally, this result will be generalized to any N in Section 7, on top of Sections 4 and 6 where modular units are constructed. Sections 8 and 9 will conclude.

Sections 1 and 3 are primers on two essential tools: modular forms and Dedeking function, respectively.

1. Primer on modular forms

Let $\mathcal{H} = \{\tau \in \mathbb{C} \ / \ \mathrm{Im}(\tau) > 0\}$ be the Poincaré half-plane. From now on, let τ be a complex variable belonging to $\mathcal{H}$, and we define $q = e^{2i\pi\tau}$.

For $(N, k) \in \mathbb{N}^{*2}$, let $M_{2k}(\Gamma_0(N))$ be the space of modular forms of weight $2k$ with respect to $\Gamma_0(N)$, and let $d_{2k}(N)$ be the dimension of $M_{2k}(\Gamma_0(N))$. For a primer on these spaces (definitions, theorems on cusps or cuspidal modular forms . . .), one can read [1, 2].

For $k \geqslant 2$ and $\tau \in \mathcal{H}$, the normalized Eisenstein series are defined as the following modular forms:

$$E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^{2k}} = 1 + O(q).$$

It is easy to show that $E_{2k} \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$, which ensures the non-triviality of this space. It is nevertheless the function $\Delta \in M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ that will structure the sequence $(M_{2k}(\mathrm{SL}_2(\mathbb{Z})))_{k\in\mathbb{N}^*}$:

$$\forall\ \tau \in \mathcal{H}, \quad \Delta(\tau) = q \prod_{n=1}^{+\infty} (1 - q^n)^{24} = q + O(q).$$

The Δ function is holomorphic and does not cancel on $\mathcal{H}$, but since $\lim_{\tau\to\infty} \Delta(\tau) = 0$, it vanishes at the infinite cusp.

Lastly, let us recall the well-known structural result of modular forms with respect to $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1)$:

$$\forall\ k \geqslant 6,\ M_{2k}(\Gamma_0(1)) = \mathrm{span}(E_{2k}) \oplus \Delta.M_{2k-12}(\Gamma_0(1)).$$

Indeed, the mapping $\Phi \mapsto \Phi.\Delta^{-1}$ is an isomorphism between the space of modular forms of weight $2k$ vanishing at the infinite cusp (named cuspidal modular forms) and $M_{2k-12}(\Gamma_0(1))$ [11]. It is this very result that we generalize from $N = 1$ to $N \in \mathbb{N}^*$.

2. **Structure of** $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$ **spaces**

Let us define two natural ways to generalize the function Δ, which vanishes only at the infinite cusp with respect to $\Gamma_0(1)$.

Definition 2.1. Let k and N be two positive integers, and $\Phi \in M_{2k}(\Gamma_0(N))$. The function Φ is said to be a $2k$ strong modular unit with respect to $\Gamma_0(N)$ (or equivalently "of level N") if and only if:

(i) The function Φ does vanish on $\mathcal{H}$,

(ii) The function Φ vanishes at the infinite cusp,

(iii) The function Φ does not vanish at any other cusp with respect to $\Gamma_0(N)$.

If we replace condition (iii) by

$(\widetilde{\text{iii}})$ The function Φ vanishes at all rational cusps

we are instead defining cuspidal modular forms.

Definition 2.2. An integer n is said to be the valuation of a modular form Φ if

$$\Phi(\tau) = aq^n + O(q^{n+1})$$

with $a \neq 0$ and we write $\nu(\Phi) = n$. Of particular interest is the case $a = 1$, in which case the function Φ is said to be unitary. A basis

$$\mathcal{B}_{2k}(\Gamma_0(N)) = (E^{(r)}_{2k,N})_{0\leqslant r\leqslant d_{2k}(N)-1}$$

of the space $M_{2k}(\Gamma_0(N))$ that verifies $\nu(E^{(r)}_{2k,N}) < \nu(E^{(r+1)}_{2k,N})$ for all $0 \leqslant r \leqslant d_{2k}(N) - 2$ is said to be upper triangular, or in echelon form. If the elements of $\mathcal{B}_{2k}(\Gamma_0(N))$ are also unitary, we say that this basis is unitary upper triangular.

Lemma 2.3. *For any positive integers N and k, the space $M_{2k}(\Gamma_0(N))$ has a unitary upper triangular basis. Moreover, the sequence of integers $(\nu(E_{2k,N}(r)))_{0\leqslant r\leqslant d_{2k}(N)-1}$ is independent of the choice of such a basis $(E^{(r)}_{2k,N})_{0\leqslant r\leqslant d_{2k}(N)-1}$.*

Proof. Existence comes directly from a Gaussian elimination. The result on valuations is straightforward. □

Theorem 2.4. *Let N be a positive integer such that there exists a strong modular unit of level N. Let Φ_0 be such a strong modular unit of level N and of minimal weight $2k_0$. Other strong modular units of the family $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$ are then exactly of the form $\alpha\Phi_0^n$ with $\alpha \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$.*

Proof. Let Φ be a modular unit of weight $2k$ with $k \geqslant k_0$. By Euclidean division

$$k = qk_0 + r, \quad 0 \leqslant r < k_0.$$

The inequality $\nu(\Phi) < q.\nu(\Phi_0)$ would lead to $\Phi_0^q \Phi^{-1} \in M_{-2r}(\Gamma_0(N))$. This function would then vanish at the infinite cusp and would therefore be null, which is impossible.

The inequality $q.\nu(\Phi_0) < \nu(\Phi)$ would lead to $\Phi\Phi_0^{-q} \in M_{2r}(\Gamma_0(N))$ being a strong modular unit, which would contradict the minimality of k_0.

Therefore $q.\nu(\Phi_0) = \nu(\Phi)$ and $\Phi\Phi_0^{-q}$ does not cancel on $\mathcal{H}$ nor at any cusp, which is a well-known characteristic of constant modular forms. □

The following result provides the structure of the sequence of modular forms spaces $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$ under the assumption that a strong modular unit exists (which is always the case, as will be shown later).

Theorem 2.5. *Let N be a positive integer and Φ a strong modular unit in $M_{2\ell}(\Gamma_0(N))$, with $\ell \in \mathbb{N}^*$. For $k \in \mathbb{N}^*$, let $(E_{2k,N}^{(s)})_{0\leqslant s\leqslant d_{2k}(N)-1}$ be a unitary upper triangular basis of $M_{2k}(\Gamma_0(N))$. Then for all integer $k \geqslant \ell$,*

$$M_{2k}(\Gamma_0(N)) = \Phi.M_{2k-2\ell}(\Gamma_0(N)) \oplus \operatorname{span}\big\{E_{2k,N}^{(s)} \;/\; \nu(E_{2k,N}^{(s)}) < \nu(\Phi)\big\}.$$

Therefore, if $k \in \mathbb{N}^*$ and $k = q\ell + r$ with $1 \leqslant r \leqslant \ell$,

$$M_{2k}(\Gamma_0(N)) = \Phi^q.M_{2r}(\Gamma_0(N)) \bigoplus_{n=0}^{q-1} \Phi^n \operatorname{span}\big\{E_{2k-2n\ell,N}^{(s)} \;/\; \nu(E_{2k-2n\ell,N}^{(s)}) < \nu(\Phi)\big\}.$$

Proof. Just like in the $N = 1$ case, the result stems from the isomorphism

$$\begin{aligned}\varphi : \operatorname{span}\big\{E_{2k,N}^{(s)} \;/\; \nu(E_{2k,N}^{(s)}) \geqslant \nu(\Phi)\big\} &\longrightarrow M_{2k-2\ell}(\Gamma_0(N))\\ \Psi &\longmapsto \Psi/\Phi.\end{aligned}$$ □

Our primary goal is to provide *concrete and computable results*. Theorem 2.5 does not meet these criteria until we know how to compute the elements of $\big\{E_{2k,N}^{(s)}/\nu(E_{2k,N}^{(s)}) < \nu(\Phi)\big\}$. In particular, we need to prove the existence of Φ once and for all instead of assuming it.

To construct the strong modular units, the central tool will be Dedekind η function. For clarity, we first recall the properties of this function.

3. **Primer on Dedekind η function**

Together with (Weierstrass or Jacobi) elliptic functions, the Dedekind η function is a must-have tool to construct modular functions and forms. Rademacher [10] first proposed modular functions (of weight 0) with respect to $\Gamma_0(p)$, for p prime, by constructing them on top of the η function. But it was Newman [7, 8] who first constructed a (weakly) modular function with respect to $\Gamma_0(N)$ for any N, also starting from η. More studies followed, extending these results to the modular forms with respect to $\Gamma_0(N)$ [5, 9], leading to the results used here [1, 4].

Let us define the Dedekind function, of weight $\frac{1}{2}$ [1]:

$$\forall\, \tau \in \mathcal{H}, \quad \eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^{+\infty} (1 - q^n).$$

Definition 3.1. Let N be a positive integer. We call η-quotient of level N any function of the form

$$\forall\, \tau \in \mathcal{H}, \quad \Phi(\tau) = \prod_{m|N} \eta(m\tau)^{a_m} \tag{3.1}$$

where $(a_1, \ldots, a_N)$ is a sequence of integers indexed on the divisors of N.

The relation 3.1 shows that if Φ is modular, its weight is necessarily $2k = \frac{1}{2}\sum_{m|N} a_m$ and in this case $\nu(\Phi) = \frac{1}{24}\sum_{m|N} m a_m \in \mathbb{N}^*$.

The following results are handy since they remove lots of calculations from future proofs. They are derived from the modular properties of the η function and are found in the literature under various forms. The initial sources are [7, Theorem 1], [5, Proposition 3.2.1] and finally [4, Corollary 2.3], as well as [9, Theorem 1.64] and [3, Theorem 1].

Theorem 3.2. *Let $\Phi(\tau) = \prod_{m|N} \eta(m\tau)^{a_m}$ be an η-quotient of level N. For m a divisor of N, we define $m' = N/m$. If the function Φ satisfies the following four conditions*

(i) $\prod_{m|N} m'^{a_m} \in \mathbb{Q}^2$

(ii) $\frac{1}{24}\sum_{m|N} m a_m \in \mathbb{Z}$

(iii) $\frac{1}{24}\sum_{m|N} m' a_m \in \mathbb{Z}$

(iv) $\frac{1}{2}\sum_{m|N} a_m \in 2\mathbb{N}^*$

then Φ is weakly modular with respect to $\Gamma_0(N)$ and of weight $2k = \frac{1}{2}\sum_{m|N} a_m$.

Definition 3.3. For $r = -\frac{d}{c} \in \mathbb{Q}$ with $\gcd(c, d) = 1$, the vanishing order of

$$\Phi(\tau) = \prod_{m|N} \eta(m\tau)^{a_m}$$

at the r cusp is defined by

$$\operatorname{ord}(\Phi, r) = \frac{N}{24} \sum_{m|N} \frac{\gcd(c, m)^2}{m} a_m.$$

Theorem 3.4. *The function Φ has a limit at the cusp r if and only if* $\operatorname{ord}(\Phi, r) \geqslant 0$*, and Φ vanishes at this cusp if and only if* $\operatorname{ord}(\Phi, r) > 0$*. Therefore, under assumptions* (i)*,* (ii)*,* (iii) *and* (iv) *of Theorem 3.2, if for any cusp $r = -\frac{d}{c} \in \mathbb{Q}$ we have* $\operatorname{ord}(\Phi, r) \geqslant 0$*, then $\Phi \in M_{2k}(\Gamma_0(N))$.*

As noted in [4], the behavior of Φ at the cusp $-d/c$ only depends on c. We can restrict the analysis even further: given that $\gcd(c, m) = \gcd(\gcd(c, N), m)$ for any divisor m of N, it is enough to check the condition $\operatorname{ord}(\Phi, r) \geqslant 0$ at the cusps $r = 1/c$ for the divisors c of N, $1 \leqslant c \leqslant N$.

For $1 \leqslant c \leqslant N - 1$, condition $\operatorname{ord}(\Phi, \frac{1}{c}) = 0$ indicates the non-nullity of Φ at all rational cusps. The $\operatorname{ord}(\Phi, \frac{1}{N}) > 0$ condition indicates that Φ vanishes at the infinite cusp, because $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$ are two representatives of the $\Gamma_0(N)$ class. This leads to the following result.

Theorem 3.5. *Let $\Phi(\tau) = \prod_{m|N} \eta(m\tau)^{a_m}$ be an η-quotient of level N such that:*

(i) $P(\Phi) = \prod_{m|N} m'^{a_m} \in \mathbb{Q}^2$

(ii) $\operatorname{ord}(\Phi, \infty) = \frac{1}{24} \sum_{m|N} m a_m \in \mathbb{N}^*$

(iii) $\forall\, c \in \{1, \ldots N - 1, \}, \quad \operatorname{ord}(\Phi, \frac{1}{c}) = \frac{1}{24} \sum_{m|N} \frac{\gcd(c,m)^2}{m} a_m = 0$

(iv) $W(\Phi) = \frac{1}{2} \sum_{m|N} a_m \in 2\mathbb{N}^*$

The function Φ is then a strong modular unit of level N and of weight $W(\Phi)$.

Proof. For such a function Φ, condition (ii) of Theorem 3.2 results from condition (ii) above, and condition (iii) of Theorem 3.2 is derived from condition (iii) above. The same goes for condition (iv).

Condition (ii) shows that Φ vanishes at the infinite cusp and provides the order of Φ at infinity (i.e. its valuation). Finally, condition (iii) indicates that Φ does not vanish at any cusp other than the infinite cusp. □

We will use Theorem 3.5 to construct in Section 4 a modular unit Δ_p when the level p is prime. This will give, in Section 5, a more precise and operational version of Theorem 2.5. The results obtained for p prime will be extended in Sections 6 and 7 to any level $N \geqslant 1$.

4. Strong modular units Δ_p, p prime

It is simpler to start by constructing strong modular units of minimum weight for $p = 2$ and $p = 3$, these cases being exceptions.

The $p = 2$ case

It is well known that the space $M_2(\Gamma_0(2))$ is one-dimensional and is generated by a form $E_{2,2}^{(0)}(\tau) = 1 + O(q)$. This excludes the existence of a strong modular form of weight 2 which must vanish at the infinity cusp.

Theorem 4.1. *The function*

$$\Delta_2(\tau) = \eta(2\tau)^{16}\eta(\tau)^{-8} = q \prod_{n=1}^{+\infty} \frac{(1-q^{2n})^{16}}{(1-q^n)^8}$$

belongs to $M_4(\Gamma_0(2))$. *It is a strong modular unit of level* 2 *with minimal weight.*

Proof. The function Δ_2 is an η-quotient of level $N = 2$. Its divisors are $m \in \{1, 2\}$, providing two coefficients $a_1 = -8$ and $a_2 = 16$.

The function Δ_2 is of weight $W(\Delta_2) = \frac{1}{2}(a_1 + a_2) = 4 \in 2\mathbb{N}^*$ and satisfies the other hypotheses of Theorem 3.5:

$$\prod_{m|2} m'^{a_m} = 2^{-8} \in \mathbb{Q}^2, \quad \frac{1}{24}\sum_{m|2} m a_m = 1 \in \mathbb{N}^* \quad \text{and} \quad \sum_{m|2} \frac{a_m}{m} = 0 \in 2\mathbb{N}^*. \qquad \square$$

The $p = 3$ case

The space $M_2(\Gamma_0(3))$ is also one-dimensional, generated by the modular form $E_{2,3}^{(0)}(\tau) = 1 + 12q + O(q^2)$. Hence, as in the case $p = 2$, there is no strong modular unit in this space.

The $M_4(\Gamma_0(3))$ space is 2-dimensional and does not contain any strong modular unit. Indeed, we can choose $E_{4,3}^{(0)} = [E_{2,3}^{(0)}]^2 = 1 + 24q + O(q^2)$, but we also know an element of $M_4(\Gamma_0(3))$ constructed from the Eisenstein series E_4, namely $E_4(3\tau) = 1 + 240q^3 + O(q^6)$.

We deduce from these two linearly independent modular forms that $E_{4,3}^{(1)}$ is of valuation 1, and unique if we require it to be unitary. This function could still be a strong modular unit, but by division, we would then derive $\dim(M_6(\Gamma_0(3))) = 2$ which is false, the space being of dimension 3. This leads us to the following result.

Theorem 4.2. *The function*

$$\Delta_3(\tau) = \eta(3\tau)^{18}\eta(\tau)^{-6} = q^2 \prod_{n=1}^{+\infty} \frac{(1-q^{3n})^{18}}{(1-q^n)^6}$$

belongs to $M_6(\Gamma_0(3))$. *It is the strong modular unit of level* 3 *of minimal weight.*

Proof. The function Δ_3 is an η-quotient of level $N = 3$, having as divisors $m \in \{1, 3\}$ with $a_1 = -6$ and $a_3 = 18$.

The function Δ_3 is of weight $W(\Delta_3) = \frac{1}{2}(a_1 + a_3) = 6 \in 2\mathbb{N}^*$ and satisfies the other hypotheses of Theorem 3.5:

$$\prod_{m|3} m'^{a_m} = 3^{-6} \in \mathbb{Q}^2, \quad \text{and} \quad \frac{1}{24}\sum_{m|3} m a_m = 2 \in \mathbb{N}^*.$$

Finally, for $c = 1$ and $c = 2$, we find $\sum_{m|3} \frac{\gcd(c,m)^2}{m} a_m = \sum_{m|3} \frac{a_m}{m} = 0$. □

The $p \geqslant 5$, p prime case

We can now derive a general formula for strong modular units of level $p \geqslant 5$ prime. Let us first define the function Δ_p on $\mathcal{H}$ for any prime number $p \geqslant 5$ as:

$$\Delta_p(\tau) = \eta(p\tau)^{2p}\eta(\tau)^{-2} = q^{(p^2-1)/12}\prod_{n=1}^{+\infty}\frac{(1-q^{pn})^{2p}}{(1-q^n)^2}. \tag{4.1}$$

Theorem 4.3. *The Δ_p function is a $M_{p-1}(\Gamma_0(p))$ modular unit.*

Notice that equality $\Delta_p(\tau)^{12} = \Delta(p\tau)^p\Delta(\tau)^{-1}$ indicates a modular property of Δ_p for the weight $p-1$.

Also, note that if $p \geqslant 5$ is prime, then $\frac{p^2-1}{12} \in \mathbb{N}$. More generally, if $N \geqslant 5$ is an integer such that $N \equiv 1 \pmod 6$ or $N \equiv 5 \pmod 6$, which is the case for $p \geqslant 5$ prime, then $\frac{N^2-1}{12} \in \mathbb{N}$. Indeed, if $N = 6k+1$ then $\frac{N^2-1}{12} = 3k^2 + k$ and if $N = 6k+5$ then $\frac{N^2-1}{12} = 3k^2 + k + 2$.

Proof. The function Δ_p is an η-quotient of level p, with p divisors $m \in \{1, p\}$ corresponding to the coefficient $a_1 = -2$ and $a_p = 2p$. The function Δ_p is of weight $W(\Delta_p) = p - 1 \in 2\mathbb{N}^*$ and satisfies the other hypotheses of Theorem 3.5:

$$\prod_{m|p} m'^{a_m} = p^{-2} \in \mathbb{Q}^2 \qquad \text{and} \qquad \frac{1}{24}\sum_{m|p} m a_m = \frac{p^2-1}{12} \in \mathbb{N}^*.$$

Finally, for $c \in \{1, \ldots, p-1\}$, we derive $\sum_{m|p} \frac{\gcd(c,m))^2}{m} a_m = \sum_{m|p} \frac{a_m}{m} = 0$. □

5. Structure and bases of $(M_{2k}(\Gamma_0(p)))_{k\in\mathbb{N}^*}$, p prime

Our goal is to construct a family of unitary upper triangular bases $(\mathcal{B}_{2k}(\Gamma_0(p)))_{k\in\mathbb{N}^*}$ of $(M_{2k}(\Gamma_0(p)))_{k\in\mathbb{N}^*}$, using the generic notation $\mathcal{B}_{2k}(\Gamma_0(p)) = (E_{2k,p}^{(s)})_{0\leqslant s\leqslant d_{2k}(p)-1}$.

Let us start with the special cases $p = 2$ and $p = 3$ that need to be treated separately. This also allows us to get the gist of the coming algorithm producing bases.

The $p = 2$ case

Let $E_{2,2}^{(0)}$ be the unit generator of $M_2(\Gamma_0(2))$ which is of valuation 0. It is therefore possible to choose $E_{2k,2}^{(0)} = [E_{2,2}^{(0)}]^k$ as the first vector of the unitary upper triangular basis $\mathcal{B}_{2k}(\Gamma_0(2))$. Since function Δ_2 has a weight of 4 and a valuation of 1, Theorem 2.5 gives the following result.

Corollary 5.1. *For all $k \geqslant 3$,*

$$M_{2k}(\Gamma_0(2)) = \operatorname{span}\{E_{2k,2}^{(0)}\} \oplus \Delta_2.M_{2k-4}(\Gamma_0(2))$$

Moreover, for all $k \geqslant 1$,

$$\mathcal{B}_{2k}(\Gamma_0(2)) = \left([E_{2,2}^{(0)}]^a \Delta_2^b, \text{ with } (a,b) \in \mathbb{N}^2 \text{ such as } a + 2b = k\right)$$

is a unitary upper triangular basis of $M_{2k}(\Gamma_0(2))$.

Proof. These are direct consequences of Theorem 2.5. □

Of course, a similar result stands for $N = 1$ and leads to unitary upper triangular bases structured by Δ, instead of the usual result obtained with the generators E_4 and E_6.

The $p = 3$ case

The strong modular form Δ_3 has a weight of 6, a valuation of 2. Applying Theorem 2.5 gives us a useful corollary:

Corollary 5.2. *For all $k \geqslant 4$,*

$$M_{2k}(\Gamma_0(3)) = \operatorname{span}\{E_{2k,3}^{(0)}, E_{2k,3}^{(1)}\} \oplus \Delta_3.M_{2k-6}(\Gamma_0(3)). \tag{5.1}$$

We then have a basis of $M_{2k}(\Gamma_0(3))$ for $k \geqslant 1$:

$$\begin{aligned}\mathcal{B}_{2k}(\Gamma_0(3)) = \left([E_{2,3}^{(0)}]^a.\Delta_3^b,\ (a,b) \in \mathbb{N}^2 \ / \ a + 3b = k\right)&\\ \cup \left(E_{4,3}^{(1)}.[E_{2,3}^{(0)}]^a.\Delta_3^b,\ (a,b) \in \mathbb{N}^2 \ / \ a + 3b = k - 2\right)&.\end{aligned} \tag{5.2}$$

Proof. Once again, the first equality is a direct application of Theorem 2.5. The second equality comes from a recursion.

We know that $\dim(M_2(\Gamma_0(3))) = 1, \dim(M_4(\Gamma_0(3))) = 2$ with $\nu(E_{2,3}^{(0)}) = 0, \nu(E_{4,3}^{(0)}) = 0$ and $\nu(E_{4,3}^{(1)}) = 1$. Therefore, $E_{4,3}^{(0)} = [E_{2,3}^{(0)}]^2$, and more generally, $E_{2k,3}^{(0)} = [E_{2,3}^{(0)}]^k$ can be chosen as the first element of $M_{2k}(\Gamma_0(3))$ unitary upper triangular basis.

Similarly, we can choose for any $k \geqslant 3$, $E_{2k,3}^{(1)} = E_{4,3}^{(1)}[E_{2,3}^{(0)}]^{k-2}$.

It is easy to check that relation (5.2) produces a basis for $k = 1$ and $k = 2$, and assume the result holding true to the order $k - 1 \geqslant 2$. Given the above, the relation (5.1) shows that

$$\Big([E_{2,3}^{(0)}]^k, E_{4,3}^{(1)}[E_{2,3}^{(0)}]^{k-2}\Big) \cup \Delta_3 \mathcal{B}_{2k-4}(\Gamma_0(3))$$

gives a basis $\mathcal{B}_{2k}(\Gamma_0(3))$. We can then see that

$$\begin{aligned}&\Big([E_{2,3}^{(0)}]^a.\Delta_3^b,\ (a,b) \in \mathbb{N}^2 \ /\ a + 3b = k\Big)\\ &\qquad = \Big([E_{2,3}^{(0)}]^k\Big) \cup \Delta_3 \Big([E_{2,3}^{(0)}]^a.\Delta_3^b,\ (a,b) \in \mathbb{N}^2 \ /\ a + 3b = k - 3\Big)\end{aligned}$$

and

$$\begin{aligned}&\Big(E_{4,3}^{(1)}.[E_{2,3}^{(0)}]^a.\Delta_3^b,\ (a,b) \in \mathbb{N}^2 \ /\ a + 3b = k - 2\Big)\\ &\qquad = \Big(E_{4,3}^{(1)}[E_{2,3}^{(0)}]^{k-2}\Big) \cup \Delta_3 \Big(E_{4,3}^{(1)}.[E_{2,3}^{(0)}]^a.\Delta_3^b,\ (a,b) \in \mathbb{N}^2 \ /\ a + 3b = k - 5\Big)\end{aligned}$$

which, by induction, completes the proof. □

The $p \geqslant 5$ case, p prime

Let us fix $p \geqslant 5$, p prime.

Lemma 5.3. *For all $k \in \mathbb{N}^*$,*

$$\dim(M_{2k+p-1}(\Gamma_0(p))) - \dim(M_{2k}(\Gamma_0(p))) = \nu(\Delta_p) = \frac{p^2 - 1}{12}.$$

Proof. The second equality is known. The first is in fact a special case of Theorem 7.2, valid for any N, which will be proven in Section 7. The central element of this proof is an explicit formula providing the dimension of the space $M_{2k}(\Gamma_0(N))$ as a function of k and N. See [12]. □

Moreover, we can deduce from Theorem 2.5 the following equality, for any $k \in \mathbb{N}^*$:

$$\dim(M_{2k+p-1}(\Gamma_0(p))) = \dim(M_{2k}(\Gamma_0(p))) + \operatorname{card}\left(\left\{s \ /\ \nu(E_{2k+p-1,N}^{(s)}) < \frac{p^2 - 1}{12}\right\}\right).$$

As a result $\operatorname{card}(\{s \ /\ \nu(E_{2k+p-1,N}^{(s)}) < \frac{p^2-1}{12}\}) = \frac{p^2-1}{12}$, from which we derive the following theorem.

Theorem 5.4. *For any $p \geqslant 5$ prime and any integer $k \geqslant 1$, let $(E_{2k,p}^{(s)})_{0 \leqslant s \leqslant d_{2k}(N)-1}$ be a unitary upper triangular basis of $M_{2k}(\Gamma_0(p))$. Then,*

$$\forall\, k \geqslant \frac{p+1}{2}, \quad \forall\, s \in \left\{0, \dots, \frac{p^2 - 1}{12} - 1\right\}, \quad \nu(E_{2k,p}^{(s)}) = s.$$

This result is important: it shows that the new elements appearing in $\mathcal{B}_{2k}(\Gamma_0(p))$ have regularly spaced valuations, with the remaining elements coming from $\Delta_p.\mathcal{B}_{2k-(p-1)}(\Gamma_0(p))$. We still need to characterize these new elements. Let us first prove the following result and its corollary:

Theorem 5.5. *For any integer $N \geqslant 2$, $M_2(\Gamma_0(N))$ has elements of valuation* 0.

Proof. This is a well-known result and is usually obtained thanks to Eisenstein series G_2 (see [2, p. 18] or [6]). Let us define

$$G_2(\tau) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}'_m} \frac{1}{(m\tau+n)^2} = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{+\infty} \sigma(n) q^n$$

where $\mathbb{Z}'_m = \mathbb{Z} - \{0\}$ if $\mathbb{Z}'_m$ =0 and $\mathbb{Z}'_m = \mathbb{Z}$ otherwise. Then, some calculations allow us to derive that $G_{2,N}(\tau) = G_2(\tau) - NG_2(N\tau)$ belongs to $M_2(\Gamma_0(N))$. Moreover,

$$\lim_{\tau\to+\infty} G_{2,N}(\tau) = 2(1-N)\zeta(2) \neq 0,$$

which concludes the proof. □

Corollary 5.6. *Let $N \geqslant 2$ be an integer. If $(E^{(s)}_{2k,p})_{0\leqslant s\leqslant d_{2k}(N)-1}$ is a unitary upper triangular basis of $M_{2k}(\Gamma_0(N))$, then $\nu(E^{(0)}_{2k,p}) = 0$ and we can choose $E^{(0)}_{2k,p} = [E^{(0)}_{2,p}]^k$.*

Theorem 5.5 and Corollary 5.6 enable an algorithmic construction of structured bases. Indeed, for $k \geqslant \frac{p+1}{2}$, we can choose

$$E^{(s)}_{2k,p} = E^{(s)}_{p+1,p}[E^{(0)}_{2,p}]^{k-\frac{p+1}{2}}, \quad 0 \leqslant s < \frac{p^2-1}{2}.$$

These elements are spread evenly (without jumps) and unitary in $M_{2k}(\Gamma_0(p))$. As such, they are potential candidates to be the first $\frac{p^2-1}{2}$ elements of $\mathcal{B}_{2k}(\Gamma_0(p))$. We can now give a more precise version of Theorem 2.5:

Theorem 5.7. *Let $p \geqslant 5$ be a prime number. Then for all $k \in \mathbb{N}^*$ such that $k \geqslant \frac{p-1}{2}$,*

$$M_{2k}(\Gamma_0(p)) = \Delta_p.M_{2k-(p-1)}(\Gamma_0(p)) \oplus \operatorname{span}\left\{E^{(s)}_{p+1,p}[E^{(0)}_{2,p}]^{k-\frac{p+1}{2}} \;/\; 0 \leqslant s < \frac{p^2-1}{12}\right\}.$$

Therefore, if $k \in \mathbb{N}^*$ is such that $k = q\frac{p-1}{2} + r$ with $1 \leqslant r \leqslant \frac{p-1}{2}$,

$$M_{2k}(\Gamma_0(p)) \\ = \Delta^q_p.M_{2r}(\Gamma_0(p)) \bigoplus_{n=0}^{q-1} \Delta^n_p.\operatorname{span}\left\{E^{(s)}_{p+1,p}[E^{(0)}_{2,p}]^{k-(n+1)\frac{p-1}{2}-1} \;/\; 0 \leqslant s < \frac{p^2-1}{12}\right\}.$$

In order to get a unitary upper triangular basis $\mathcal{B}_{2k}(\Gamma_0(p))$, $k \geqslant 1$, Theorem 2.5 is now operational since the knowledge of all bases is reduced to the knowledge of the finite family of bases $(\mathcal{B}_{2k}(\Gamma_0(p)))_{1\leqslant k\leqslant \frac{p+1}{2}}$.

6. Strong modular units Δ_N, $N \geqslant 1$

In Section 5, we derived structured bases of $(M_{2k}(\Gamma_0(p)))_{k\in\mathbb{N}^*}$ when p is prime. The central tool, which reduced the search of an infinity of unitary upper triangular bases to the search of a finite number of bases, was the existence of a strong modular form Δ_p. The next logical step is thus to establish the existence of strong modular forms Δ_N in the general case $N \geqslant 1$. With this in mind, the Definition 4.1 of Δ_p, for $p \geqslant 5$ prime, lead to defining the family of functions η_k:

Notation 6.1. For any $k \in \mathbb{N}^*$,

$$\forall\, \tau \in \mathcal{H},\ \ \eta_k(\tau) = \eta(k\tau)^k.$$

Additionally, the empirical search of strong modular units $(\Delta_N)_{1\leqslant N\leqslant 10}$ (of minimal weight) lead to the following notations:

Notation 6.2. Let $N \in \mathbb{N}^*$ be an integer, with $N = p_1^{r_1} \dots p_n^{r_n}$, $(r_1, \dots, r_n) \in (\mathbb{N}^*)^n$ its prime factors decomposition. Let $R = R(N) = p_1 \dots p_n$ be the radical of N. We can now define

$$\Lambda_R(\tau) = \prod_{m|R} \eta(m\tau)^{m\mu_m} = \prod_{m|R} \eta_m(\tau)^{\mu_m}$$

and

$$\Lambda_N(\tau) = \Lambda_R\left(\frac{N}{R}\tau\right) = \Lambda_R(p_1^{r_1-1} \dots p_n^{r_n-1}\tau) = \prod_{m|R} \eta_m(p_1^{r_1-1} \dots p_n^{r_n-1}\tau)^{\mu_m},$$

where μ denotes the Möbius function and $\mu_m = \mu(m)$, for $m \in \mathbb{N}^*$.

We can see that Λ_N is an η-product of level N and that the two definitions of Λ_N coincide when N is its own radical. The weight of Λ_N is given by:

$$\frac{1}{2}\sum_{d|R} d\mu_d = \frac{1}{2}\sum_{(\varepsilon_1,\dots\varepsilon_n)\in\{0,1\}^n} (-p_1)^{\varepsilon_1} \dots (-p_n)^{\varepsilon_n} = \frac{(-1)^n}{2}\prod_{i=1}^{n}(p_i - 1).$$

Table 6.1 presents the minimal strong modular units of level N for $1 \leqslant N \leqslant 10$ empirically found.

As suggested above, we will show, for $N \in \mathbb{N}^*$, that there exists $\alpha \in \mathbb{Z}^*$ such that Λ_N^α is a strong modular unit of level N. To that end, we will systematically apply Theorem 3.5

TABLE 6.1. Empirical table of minimal strong modular units of level N for $1 \leqslant N \leqslant 10$

$$\Delta_1(\tau) = \Lambda_1^{24}(\tau) = \eta(\tau)^{24} = q \prod_{i=1}^{+\infty} (1-q^i)^{24}$$

$$\Delta_2(\tau) = \Lambda_2^{-8}(\tau) = \eta(\tau)^{-8}\eta(2\tau)^{16} = q \prod_{i=1}^{+\infty} \frac{(1-q^{2i})^{16}}{(1-q^i)^8}$$

$$\Delta_3(\tau) = \Lambda_3^{-6}(\tau) = \eta(\tau)^{-6}\eta(3\tau)^{18} = q^2 \prod_{i=1}^{+\infty} \frac{(1-q^{3i})^{18}}{(1-q^i)^6}$$

$$\Delta_4(\tau) = \Lambda_4^{-4}(\tau) = \eta(2\tau)^{-4}\eta(4\tau)^{8} = q \prod_{i=1}^{+\infty} \frac{(1-q^{4i})^{8}}{(1-q^{2i})^4}$$

$$\Delta_5(\tau) = \Lambda_5^{-2}(\tau) = \eta(\tau)^{-2}\eta(5\tau)^{10} = q^2 \prod_{i=1}^{+\infty} \frac{(1-q^{5i})^{10}}{(1-q^i)^2}$$

$$\Delta_6(\tau) = \Lambda_6^{2}(\tau) = \eta(\tau)^{2}\eta(2\tau)^{-4}\eta(3\tau)^{-6}\eta(6\tau)^{12} = q^2 \prod_{i=1}^{+\infty} \frac{(1-q^{i})^{2}(1-q^{6i})^{12}}{(1-q^{2i})^4(1-q^{3i})^6}$$

$$\Delta_7(\tau) = \Lambda_7^{-2}(\tau) = \eta(\tau)^{-2}\eta(7\tau)^{14} = q^4 \prod_{i=1}^{+\infty} \frac{(1-q^{7i})^{14}}{(1-q^i)^2}$$

$$\Delta_8(\tau) = \Lambda_8^{-4}(\tau) = \eta(4\tau)^{-4}\eta(8\tau)^{8} = q^2 \prod_{i=1}^{+\infty} \frac{(1-q^{8i})^{8}}{(1-q^{4i})^4}$$

$$\Delta_9(\tau) = \Lambda_9^{-2}(\tau) = \eta(3\tau)^{-2}\eta(9\tau)^{6} = q^2 \prod_{i=1}^{+\infty} \frac{(1-q^{9i})^{6}}{(1-q^{3i})^2}$$

$$\Delta_{10}(\tau) = \Lambda_{10}^{2}(\tau) = \eta(\tau)^{2}\eta(2\tau)^{-4}\eta(5\tau)^{-10}\eta(10\tau)^{20} = q^6 \prod_{i=1}^{+\infty} \frac{(1-q^{i})^{2}(1-q^{10i})^{20}}{(1-q^{2i})^4(1-q^{5i})^{10}}$$

whose assumptions generate exceptions that should be treated separately, when $n \in \{1,2\}$ and $p \in \{2,3\}$. Let us now translate Theorem 3.5 for functions Λ_N^{α}.

Theorem 6.3. *Let $N \in \mathbb{N}^*$ be an integer, $N = p_1^{r_1} \dots p_n^{r_n}$, $(r_1, \dots, r_n) \in (\mathbb{N}^*)^n$ its prime factors decomposition, and $\alpha \in \mathbb{Z}^*$. If Λ_N^{α} satisfies the three conditions*

(1) $P(\Lambda_N^{\alpha}) = P(\Lambda_N)^{\alpha} = \prod_{i=1}^n p_i^{\delta_i} \in \mathbb{Q}^2$, *with* $\delta_i = -\alpha \prod_{1 \leqslant j \leqslant n,\ j \neq i} (1-p_j)$ *for* $1 \leqslant i \leqslant n$,

(2) $\operatorname{ord}(\Lambda_N^{\alpha}, \infty) = \alpha \operatorname{ord}(\Lambda_N, \infty) = \alpha \frac{N}{R(N)} \frac{(-1)^n}{24} \prod_{1 \leqslant i \leqslant n} (p_i^2 - 1) \in \mathbb{N}^*$,

(3) $W(\Lambda_N^\alpha) = (-1)^n \frac{\alpha}{2} \prod_{i=1}^n (p_i - 1) \in 2\mathbb{N}^*$

then Λ_N^α *is a strong modular unit of level* N *and of weight* $(-1)^n \frac{\alpha}{2} \prod_{i=1}^n (p_i - 1)$.

We can already notice that the structure of the Λ_N^α functions leads to the automatic satisfaction of hypothesis (iii) of Theorem 3.5.

Proof. Let $R = p_1 \dots p_n$ be the radical of N and $M = \frac{N}{R}$. Following the notations of Theorem 3.5, Λ_N^α is an η-product of level N with $a_m = 0$ except if $m = Md$ where $d|R$. In this case, $a_m = \alpha \mu_d d$.

(1) First, we have

$$P(\Lambda_N^\alpha) = \prod_{d|R} \left(\frac{N}{Md}\right)^{\alpha d \mu_d} = \prod_{d|R} (d')^{\alpha d \mu_d} = P(\Lambda_R^\alpha) = \prod_{i=1}^n p_i^{\delta_i}.$$

By symmetry, it is enough to study δ_1.

$$\delta_1 = -\alpha \sum_{(\varepsilon_2, \dots \varepsilon_n) \in \{0,1\}^{n-1}} (-p_2)^{\varepsilon_2} \dots (-p_n)^{\varepsilon_n} = -\alpha \prod_{i=2}^n (1 - p_i).$$

We deduce the equivalence between (i) and (1) for the functions Λ_N^α.

(2) Then,

$$\begin{aligned} \operatorname{ord}(\Lambda_N^\alpha, \infty) &= \frac{1}{24} \sum_{d|R} (Md) \alpha \mu_d d \\ &= \frac{1}{24} \alpha M \sum_{d|R} d^2 \mu_d \\ &= \frac{1}{24} \alpha M \sum_{(\varepsilon_1, \dots \varepsilon_n) \in \{0,1\}^n} (-p_1^2)^{\varepsilon_1} \dots (-p_n^2)^{\varepsilon_n} \\ &= \frac{(-1)^n}{24} \alpha M \prod_{i=1}^n (p_i^2 - 1). \end{aligned}$$

We deduce the equivalence between (ii) and (2) for the functions Λ_N^α.

(3) Let us check assumption (iii) is satisfied for all functions Λ_N^α. For $c \in \{1, \dots, N-1\}$,

$$24 \ \operatorname{ord}\left(\Lambda_N^\alpha, \frac{1}{c}\right) = \sum_{m|N} \frac{\gcd(c, m)^2}{m} a_m = \alpha \frac{R}{N} \sum_{d|R} \gcd\left(c, \frac{N}{R} d\right)^2 \mu_d.$$

We can then write $c = \widetilde{c} p_1^{s_1} \dots p_n^{s_n}$ with $\gcd(\widetilde{c}, R) = 1$, showing there exists i, with $1 \leqslant i \leqslant n$, such that $s_i < r_i$. Let us assume, for example, that $s_1 < r_1$ and define $D_0 = \{d \ / \ d | p_2 \dots p_n\}$ and $D_1 = \{p_1 d \ / \ d | p_2 \dots p_n\}$ that together form a partition of $D = \{d \ / \ d | p_1 \dots p_n\}$.

For $d = p_2^{\varepsilon_2} \dots p_n^{\varepsilon_n} \in D_0$, $(\varepsilon_2, \dots, \varepsilon_n) \in \{0, 1\}^{n-1}$, we notice that

$$\begin{aligned} \gcd\left(c, \frac{N}{R} d\right) &= \gcd(p_1^{s_1} \dots p_n^{s_n}, p_1^{r_1 - 1} p_2^{r_2 - 1 + \varepsilon_2} \dots p_n^{r_n - 1 + \varepsilon_n}) \\ &= p_1^{s_1} \gcd(p_2^{s_2} \dots p_n^{s_n}, p_2^{r_2 - 1 + \varepsilon_2} \dots p_n^{r_n - 1 + \varepsilon_n}), \end{aligned}$$

Furthermore,

$$\begin{aligned} \gcd\left(c, \frac{N}{R} p_1 d\right) &= \gcd(p_1^{s_1} \dots p_n^{s_n}, p_1^{r_1} p_2^{r_2 - 1 + \varepsilon_2} \dots p_n^{r_n - 1 + \varepsilon_n}) \\ &= p_1^{s_1} \gcd(p_2^{s_2} \dots p_n^{s_n}, p_2^{r_2 - 1 + \varepsilon_2} \dots p_n^{r_n - 1 + \varepsilon_n}). \end{aligned}$$

The two terms are therefore equal, leading to the last equality needed to finish the proof:

$$\sum_{d|R} \gcd\left(c, \frac{N}{R} d\right)^2 \mu_d = \sum_{d \in D_0} \gcd\left(c, \frac{N}{R} d\right)^2 \mu_d + \sum_{d \in D_0} \gcd\left(c, \frac{N}{R} p_1 d\right)^2 \mu_{p_1 d} = 0. \quad \square$$

The strong modular units Δ_N will be expressed using Λ_N^α functions. The following result reduces the general case to the case $N = R(N)$.

Corollary 6.4. *Let $N \in \mathbb{N}$ be an integer, $N \geqslant 2$ with $N = p_1^{r_1} \dots p_n^{r_n}$, $(r_1, \dots, r_n) \in (\mathbb{N}^*)^n$ its prime factors decomposition, R the radical of N and $\alpha \in \mathbb{Z}^*$. If the function Λ_R^α satisfies the hypotheses of Theorem 6.3, then Λ_N^α is a strong modular unit of the same weight with respect to $\Gamma_0(N)$.*

Proof. Verifying that Λ_N^α satisfies the assumptions of Theorem 6.3 is enough:

$$P(\Lambda_N^\alpha) = P(\Lambda_R^\alpha) \in \mathbb{Q}^2, \quad \operatorname{ord}(\Lambda_N^\alpha, \infty) = \frac{N}{R} \operatorname{ord}(\Lambda_R^\alpha, \infty) \in \mathbb{N}^*$$
$$\text{and} \quad W(\Lambda_N^\alpha) = W(\Lambda_R^\alpha) \in 2\mathbb{N}^*.$$

This proves that Λ_N^α is a strong modular unit of level N. $\square$

The case $N = p^r$, p prime and $r > 0$

As always, we need to separate cases $p = 2$, $p = 3$ and $p \geqslant 5$. The result is as follows:

Theorem 6.5. *The following Δ_{p^r} functions are strong modular units with respect to $\Gamma_0(p^r)$. When $p = 2$, for all $r \geqslant 2$,*

$$\Delta_2 = \Lambda_2^{-8} = \left(\frac{\eta_2}{\eta_1}\right)^8 \in M_2(\Gamma_0(2)),$$

$$\Delta_{2^r}(\tau) = \Lambda_{2^r}^{-4}(\tau) = \Delta_4(2^{r-2}\tau) = \left(\frac{\eta_2}{\eta_1}\right)^4 (2^{r-1}\tau) \in M_2(\Gamma_0(2^r)).$$

When $p = 3$, for all $r \geqslant 2$,

$$\Delta_3 = \Lambda_3^{-6} = \left(\frac{\eta_3}{\eta_1}\right)^6 \in M_2(\Gamma_0(3)),$$

$$\Delta_{3^r}(\tau) = \Lambda_{3^r}^{-2}(\tau) = \Delta_9(3^{r-2}\tau) = \left(\frac{\eta_3}{\eta_1}\right)^2 (3^{r-1}\tau) \in M_2(\Gamma_0(3^r)).$$

When $p \geqslant 5$ prime, for all $r \geqslant 1$,

$$\Delta_{p^r}(\tau) = \Lambda_{p^r}^{-2}(\tau) = \Delta_p(p^{r-1}\tau) = \left(\frac{\eta_p}{\eta_1}\right)^2 (p^{r-1}\tau) \in M_{p-1}(\Gamma_0(p^r)).$$

Proof. Let us handle the various subcases separately.

- *Subcase $N = p$*
  The case $r = 1$, that is Δ_p, p prime, has been handled in Section 4.

- *Subcase $N = p^r$, $p \geqslant 5$ prime, $r \geqslant 2$*
  Theorems 4.3 and 6.3 with its Corollary 6.4 provided the expected result.

- *Subcase $N = 2^r$*
  If $r = 2$, function $\Delta_4 = \Lambda_4^{-4}$ is an η-quotient of level $N = 4$ that satisfies the hypotheses of Theorem 6.3:

  $$P(\Lambda_4^{-4}) = 2^{-4} \in \mathbb{Q}^2, \;\; \mathrm{ord}(\Lambda_4^{-4}, \infty) = 1 \in \mathbb{N}^* \;\text{ and }\; W(\Lambda_4^{-4}) = 2 \in 2\mathbb{N}^*.$$

  Hence, Δ_4 is a 2^2-strong modular unit. If $r \geqslant 3$, Corollary 6.4 gives the result.

- *Subcase $N = 3^r$*
  This case is treated similarly to $N = 2^r$. When $r = 2$, function $\Delta_9 = \Lambda_9^{-2}$ is an η-quotient of level $N = 9$ satisfying the hypotheses of Theorem 6.3:

  $$P(\Lambda_9^{-2}) = 3^{-2} \in \mathbb{Q}^2, \;\; \mathrm{ord}(\Lambda_9^{-2}, \infty) = 2 \in \mathbb{N}^* \;\text{ and }\; W(\Lambda_9^{-2}) = 2 \in 2\mathbb{N}^*.$$

  Thus, Δ_9 is a 3^2-strong modular unit. If $r \geqslant 3$, Corollary 6.4 gives the result. □

The case $N = p_1^{r_1} p_2^{r_2}$, with p_1, p_2 distinct prime numbers and $(r_1, r_2) \in \mathbb{N}^{*2}$

The result is as follows:

Theorem 6.6. *Let $p \geqslant 3$ be a prime number and $(r_1, r_2) \in \mathbb{N}^{*2}$. Then*

$$\Delta_{2^{r_1} p^{r_2}}(\tau) = \Lambda^2_{2^{r_1} p^{r_2}}(\tau) = \left(\frac{\eta_1\, \eta_{2p}}{\eta_2\, \eta_p}\right)^2 (2^{r_1-1} p^{r_2-1}\tau) \in M_{(p-1)}(\Gamma_0(2^{r_1} p^{r_2})). \tag{6.1}$$

Let $p_1 \geqslant 3$ and $p_2 \geqslant 3$ be two distinct prime numbers and $(r_1, r_2) \in \mathbb{N}^{*2}$. Then

$$\Delta_{p_1^{r_1} p_2^{r_2}}(\tau) = \Lambda_{p_1^{r_1} p_2^{r_2}}(\tau) = \frac{\eta_1 \eta_{p_1 p_2}}{\eta_{p_1}\, \eta_{p_2}}(p_1^{r_1-1} p_2^{r_2-1}\tau) \in M_{\frac{1}{2}(p_1-1)(p_2-1)}(\Gamma_0(p_1^{r_1} p_2^{r_2})). \tag{6.2}$$

These functions are strong modular units of their corresponding modular spaces.

Proof. Given Corollary 6.4, it is sufficient to prove that Δ_{2p} and $\Delta_{p_1 p_2}$ are strong modular units for $M_{(p-1)}(\Gamma_0(2p))$ and $M_{\frac{1}{2}(p_1-1)(p_2-1)}(\Gamma_0(p_1 p_2))$ respectively. Let us focus on

$$\Delta_{2p} = \left(\frac{\eta_1\, \eta_{2p}}{\eta_2\, \eta_p}\right)^2 = \Lambda^2_{2p}$$

which is an η-quotient of level $2p$. The divisors of $2p$, namely $m \in \{1, 2, p, 2p\}$, are associated with the coefficients $a_1 = 2$, $a_2 = -4$, $a_p = -2p$, $a_{2p} = 4p$. The weight of Δ_{2p} is thus $\frac{1}{2}(a_1 + a_2 + a_p + a_{2p}) = p - 1$. This function satisfies the hypotheses of Theorem 6.3:

$$P(\Delta_{2p}) = (2^{1-p} p^{-1})^2 \in \mathbb{Q}^2, \;\; \mathrm{ord}(\Delta_{2p}, \infty) = \frac{p^2-1}{4} \in \mathbb{N}^* \;\text{ and }\; W(\Lambda^2_{2p}) = p - 1 \in 2\mathbb{N}^*.$$

As a result, Δ_{2p} is indeed a $M_{(p-1)}(\Gamma_0(2p))$ strong modular unit. It is noteworthy that the square root of Δ_{2p} does not satisfy condition (1) of Theorem 6.3.

After studying $2p$, let us replace 2 by any prime number but p; the reasoning is similar, up to one detail. When $p_1 \geqslant 3$ and $p_2 \geqslant 3$ are distinct prime numbers, the function

$$\Delta_{p_1 p_2} = \frac{\eta_1 \eta_{p_1 p_2}}{\eta_{p_1} \eta_{p_2}} = \Lambda_{p_1 p_2}$$

is an η-quotient of level $N = p_1 p_2$. The divisors of $p_1 p_2$, namely $m \in \{1, p_1, p_2, p_1 p_2\}$, correspond to the coefficients a_1 =1, $a_{p_2} = -p_1$, $a_{p_2} = -p_2$, $a_{p_1 p_2} = p_1 p_2$. The weight of $\Delta_{p_1 p_2}$ is thus $\frac{1}{2}(a_1 + a_{p_1} + a_{p_2} + a_{p_1 p_2}) = \frac{1}{2}(p_1 - 1)(p_2 - 1)$. This function satisfies the hypotheses of Theorem 6.3 because

$$P(\Lambda_{p_1 p_2}) = p_1^{1-p_2} p_2^{1-p_1} \in \mathbb{Q}^2, \quad \mathrm{ord}(\Lambda_{p_1 p_2}, \infty) = \frac{(p_1^2 - 1)(p_2^2 - 1)}{24} \in \mathbb{N}^*$$
$$\text{and} \quad W(\Lambda_{p_1 p_2}) = \frac{1}{2}(p_1 - 1)(p_2 - 1) \in 2\mathbb{N}^*$$

and as such is a strong modular unit belonging to $M_{\frac{1}{2}(p_1-1)(p_2-1)}(\Gamma_0(p_1 p_2))$. □

We can unify the previous two results by saying that

$$\Lambda^2_{p_1^{r_1} p_2^{r_2}} \in M_{(p_1-1)(p_2-1)}(\Gamma_0(p_1^{r_1} p_2^{r_2}))$$

is a strong modular unit for all prime numbers $p_1 \neq p_2$. However, the relation (6.2) allows to divide by 2 the weight of the strong modular unit selected when 2 is not one of the prime factors, which will be useful when searching for bases, for example. Additionally, relation (6.1) provides the valuation of $\Delta_{2^{r_1} p^{r_2}}$

$$\nu(\Delta_{2^{r_1} p^{r_2}}) = 2^{r_1-3} p^{r_2-1}(p^2-1),$$

while relation (6.2) provides the valuation of $\Delta_{p_1^{r_1} p_2^{r_2}}$

$$\nu(\Delta_{p_1^{r_1} p_2^{r_2}}) = p_1^{r_1-1} p_2^{r_2-1} \frac{(p_1^2-1)(p_2^2-1)}{24}.$$

These two numbers are always integers. Let us give two examples.

- For $N = 3.5 = 15$,

$$\begin{aligned}\Delta_{15}(\tau) &= \frac{\eta(\tau)\eta(15\tau)^{15}}{\eta(3\tau)^3\eta(5\tau)^5} \in M_4(\Gamma_0(15)) \\ &= q^8 \prod_{n=1}^{+\infty}(1-q^n)(1-q^{3n})^{-3}(1-q^{5n})^{-5}(1-q^{15n})^{15}.\end{aligned}$$

- For $N = 2^2.3^2 = 36$,

$$\begin{aligned}\Delta_{36}(\tau) &= \frac{\eta(6\tau)^2\eta(36\tau)^{12}}{\eta(12\tau)^4\eta(18\tau)^6} \in M_2(\Gamma_0(36)) \\ &= q^{12} \prod_{n=1}^{+\infty}(1-q^{6n})^2(1-q^{12n})^{-4}(1-q^{18n})^{-6}(1-q^{36n})^{12}.\end{aligned}$$

The general case $N = p_1^{r_1} \dots p_n^{r_n}$ with $n \geqslant 3$, $p_1, \dots, p_n$ distinct prime numbers, $(r_1, \dots, r_n) \in \mathbb{N}^{*n}$

We have the following result:

Theorem 6.7. *Let $n \geqslant 3$, $p_1, \dots, p_n$ be n distinct prime numbers, $r_1, \dots, r_n$ be positive integers, and $N = p_1^{r_1} \dots p_n^{r_n}$. Then*

$$\Delta_N(\tau) = \Lambda_N(\tau)^{(-1)^n} = \left[\prod_{m | p_1 \dots p_n} \eta_m^{\mu_m}\right]^{(-1)^n} \left(\frac{N\tau}{p_1 \dots p_n}\right).$$

is a strong modular unit with respect to $\Gamma_0(N)$ such that $\Delta_N \in M_{\frac{1}{2}(p_1-1)\dots(p_n-1)}(\Gamma_0(N))$.

Proof. Given Corollary 6.4, where $R = p_1 \dots p_n$, we need only establish that $\Delta_R = \Lambda_R^{(-1)^n}$ is a strong modular unit of level R and of weight $\frac{1}{2}(p_1 - 1)\dots(p_n - 1)$. To this end, let us check that $\Lambda_R^{(-1)^n}$ satisfies the hypotheses of Theorem 6.3. For any $i \in \{1, \dots, n\}$,

$$\delta_i = (-1)^{n+1} \prod_{1 \leqslant k \leqslant n, k \neq i} (1 - p_k)$$

is even since there is at least one prime factor other than 2 in the product. As a result,

$$P(\Lambda_R^{(-1)^n}) = \prod_{i=1}^{n} p_i^{\delta_i} \in \mathbb{Q}^2.$$

Moreover,

$$\operatorname{ord}(\Lambda_R^{(-1)^n}, \infty) = \frac{1}{24} \prod_{i=1}^{n} (p_i^2 - 1) \in \mathbb{N}^*$$

because one of the factors, calling it p, is greater than or equal to 5, making $\frac{1}{24}(p^2 - 1)$ an integer. Finally, the missing piece comes straightforwardly:

$$W(\Lambda_R^{(-1)^n}) = \frac{1}{2}(p_1 - 1)\dots(p_n - 1) \in 2\mathbb{N}^*.$$

Thus, Δ_R is a strong modular unit of level R, which finishes the proof. □

Let us give two examples. The first example with $n = 3$ is $N = 60 = 2^2.3.5$:

$$\Delta_{60}(\tau) = q^{48} \prod_{i=1}^{+\infty} \frac{(1 - q^{4i})^2 (1 - q^{6i})^3 (1 - q^{10i})^5 (1 - q^{60i})^{30}}{(1 - q^{2i})(1 - q^{12i})^6 (1 - q^{20i})^{10} (1 - q^{30i})^{15}}$$

which is a strong modular unit in $M_4(\Gamma_0(60))$. The second example is the smallest product of four distinct prime factors $N = 210 = 2.3.5.7$:

$$\Delta_{210}(\tau) = q^{1152} \prod_{i=1}^{+\infty} \frac{(1 - q^{i})(1 - q^{6i})^6 (1 - q^{10i})^{10} (1 - q^{14i})^{14}}{(1 - q^{2i})^2 (1 - q^{3i})^3 (1 - q^{5i})^5 (1 - q^{7i})^7} \times \frac{(1 - q^{15i})^{15} (1 - q^{21i})^{21} (1 - q^{35i})^{35} (1 - q^{210i})^{210}}{(1 - q^{30i})^{30} (1 - q^{42i})^{42} (1 - q^{70i})^{70} (1 - q^{105i})^{105}}$$

which is a strong modular unit in $M_{24}(\Gamma_0(210))$.

Notation 6.8. Let us call ρ_N the weight of δ_N.

Before moving on to the last piece of the proof, Table 6.2 summarizes the characteristics of Δ_N and its representations as functions of Λ_N and η.

TABLE 6.2. Summary of the characteristics of Δ_N and its representations as functions of Λ_N and η. Above, p, $p_1 \ldots$ are distinct prime numbers and r, $r_1 \ldots$ are positive integers.

N	ρ_N	$\nu(\Delta_N)$	Δ_N	$\Delta_N(\tau)$ as η – quotient
2	4	1	Λ_N^{-8}	$\eta(\tau)^{-8}\eta(2\tau)^{16}$
4	2	1	Λ_N^{-4}	$\eta(2\tau)^{-4}\eta(4\tau)^{8}$
$2^r,\ r \geqslant 2$	2	2^{r-2}	Λ_N^{-4}	$\Delta_4(2^{r-2}\tau)$
3	6	2	Λ_N^{-6}	$\eta(\tau)^{-6}\eta(3\tau)^{18}$
9	2	2	Λ_N^{-2}	$\eta(3\tau)^{-2}\eta(9\tau)^{6}$
$3^r,\ r \geqslant 2$	2	2.3^{r-2}	Λ_N^{-2}	$\Delta_9(3^{r-2}\tau)$
$p \geqslant 5$	$p-1$	$\frac{1}{12}(p^2-1)$	Λ_N^{-2}	$\eta(\tau)^{-2}\eta(p\tau)^{2p}$
$p^r,\ r \geqslant 1$	$p-1$	$\frac{1}{12}p^{r-1}(p^2-1)$	Λ_N^{-2}	$\Delta_p(p^{r-1}\tau)$
$2p$	$p-1$	$\frac{1}{4}(p^2-1)$	Λ_N^{2}	$\eta(\tau)^2\eta(2\tau)^{-4}\eta(p\tau)^{-2p}\eta(2p\tau)^{4p}$
$2^{r_1}p^{r_2}$	$p-1$	$2^{r_1-3}p^{r_2-1}(p^2-1)$	Λ_N^{2}	$\Delta_{2p}(2^{r_1-1}p^{r_2-1}\tau)$
$p_1p_2,\ p_1,p_2 \geqslant 3$	$\frac{1}{2}(p_1-1)(p_2-1)$	$\frac{1}{24}(p_1^2-1)(p_2^2-1)$	Λ_N	$\eta(\tau)\eta(p_1\tau)^{-p_1}\eta(p_2\tau)^{-p_2}\eta(p_1p_2\tau)^{p_1p_2}$
$p_1^{r_1}p_2^{r_2},\ p_1,p_2 \geqslant 3$	$\frac{1}{2}(p_1-1)(p_2-1)$	$\frac{p_1^{r_1-1}p_2^{r_2-1}}{24}(p_1^2-1)(p_2^2-1)$	Λ_N	$\Delta_{p_1p_2}(p_1^{r_1-1}p_2^{r_2-1}\tau)$
$p_1\ldots p_n,\ n \geqslant 3$	$\frac{1}{2}(p_1-1)\ldots(p_n-1)$	$\frac{1}{24}(p_1^2-1)\ldots(p_n^2-1)$	$\Lambda_N^{(-1)^n}$	$\left(\prod_{m\mid p_1\ldots p_n}\eta(m\tau)^{m\mu_m}\right)^{(-1)^n}$
$p_1^{r_1}\ldots p_n^{r_n},\ n \geqslant 3$	$\frac{1}{2}(p_1-1)\ldots(p_n-1)$	$\frac{p_1^{r_1-1}\ldots p_n^{r_n-1}}{24}(p_1^2-1)\ldots(p_n^2-1)$	$\Lambda_N^{(-1)^n}$	$\Delta_{p_1\ldots p_n}(p_1^{r_1-1}\ldots p_n^{r_n-1}\tau)$

7. **Structure and bases of** $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$**,** N **positive integer**

Let us first remind of the explicit formula for the dimension of $M_2(\Gamma_0(N))$ in the general case. Once more, we refer to [2, 6, 12].

Notation 7.1. For p prime and $N \in \mathbb{N}^*$, we call $\nu_p(N)$ the power of p in the prime factors decomposition of N. We'll need additional notations:

$$\mu_0(N) = \prod_{p|N} \left(p^{\nu_p(N)} + p^{\nu_p(N)-1}\right),$$

$$\mu_{0,2}(N) = \begin{cases} 0 & \text{if } 4 \mid N, \\ \prod_{p|N} \left(1 + \left(\frac{-4}{p}\right)\right) & \text{otherwise,} \end{cases}$$

$$\mu_{0,3}(N) = \begin{cases} 0 & \text{if } 2 \mid N \text{ or } 9 \mid N, \\ \prod_{p|N} \left(1 + \left(\frac{-3}{p}\right)\right) & \text{otherwise,} \end{cases}$$

$$c_0(N) = \sum_{d|N} \varphi(\gcd(d, N/d)),$$

$$g_0(N) = 1 + \frac{\mu_0(N)}{12} - \frac{\mu_{0,2}(N)}{4} - \frac{\mu_{0,3}(N)}{3} - \frac{c_0(N)}{2}.$$

The $M_{2k}(\Gamma_0(N))$ space is decomposable into the cuspidal subspace $S_{2k}(\Gamma_0(N))$ and the Eisenstein subspace $E_{2k}(\Gamma_0(N))$

$$M_{2k}(\Gamma_0(N)) = S_{2k}(\Gamma_0(N)) \oplus E_{2k}(\Gamma_0(N))$$

whose dimensionalities are known:

$$\dim(S_{2k}(\Gamma_0(N))) = \begin{cases} g_0(N) & \text{if } k = 1, \\ (2k-1)(g_0(N)-1) + (k-1)c_0(N) & \\ \qquad + \mu_{0,2}(N)\lfloor \frac{k}{2} \rfloor + \mu_{0,3}(N)\lfloor \frac{2k}{3} \rfloor & \text{if } k \geqslant 2, \end{cases}$$

$$\dim(E_{2k}(\Gamma_0(N))) = \begin{cases} c_0(N) - 1 & \text{if } k = 1, \\ c_0(N) & \text{if } k \geqslant 2. \end{cases}$$

hence the following result holding true for any $k \in \mathbb{N}^*$:

$$\dim(M_{2k}(\Gamma_0(N))) = (2k-1)(g_0(N)-1) + kc_0(N) + \mu_{0,2}(N)\left\lfloor \frac{k}{2} \right\rfloor + \mu_{0,3}(N)\left\lfloor \frac{2k}{3} \right\rfloor. \quad (7.1)$$

For any positive integer N, let ρ_N be the weight of Δ_N. We can now give a result generalizing Lemma 5.3, as announced.

Theorem 7.2. *For any $k \in \mathbb{N}^*$,*

$$\dim(M_{2k+\rho_N}(\Gamma_0(N))) - \dim(M_{2k}(\Gamma_0(N))) = \nu(\Delta_N). \tag{7.2}$$

We can check formula (7.2) for $N = 1$: the weight of $\Delta_1 = \Delta$ is 12 and its valuation is 1, which are the values found in the literature.

To prove this result, we could make direct use of equation (7.1) but that modus operandi would require studying several cases according to the divisibility of ρ_N by 3 and 4. A more pleasant approach follows a lemma analogous to Corollary 5.6:

Lemma 7.3. *Let N and k be integers larger or equal to 2. If $(E^{(r)}_{2k,N})_{0\leqslant r\leqslant d_{2k}(N)-1}$ is an upper triangular basis of $M_{2k}(\Gamma_0(N))$, then $\nu(E^{(1)}_{2k,N}) = 1$.*

Proof. Let us reuse some elements of the proof of Theorem 5.5. With $\mathbb{Z}'_0 = \mathbb{Z} - \{0\}$ and $\mathbb{Z}'_m = \mathbb{Z}$ if $m \neq 0$, we have

$$G_2(\tau) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}'_m} \frac{1}{(m\tau+n)^2} = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{+\infty} \sigma(n)q^n = 2\zeta(2) - 8\pi^2 q + O(q^2)$$

It is already known that

$$H_{2,N}(\tau) = \frac{1}{2(1-N)\zeta(2)}(G_2(\tau) - NG_2(N\tau)) = 1 - \frac{24}{(1-N)}q + O(q^2)$$

belongs to $M_2(\Gamma_0(N))$. Similarly, it can then be seen that

$$H_{2,N}(\tau)^2 = 1 - \frac{48}{(1-N)}q + O(q^2)$$

belongs to $M_4(\Gamma_0(N))$. On the other hand, the Eisenstein series

$$E_4(\tau) = 1 + 240q + O(q^2)$$

also belongs to the vector space $M_4(\Gamma_0(N))$, and consequently

$$E_4 - H_{2,N}^2 = \left(240 + \frac{48}{(1-N)}\right)q + O(q^2)$$

as well belongs to $M_4(\Gamma_0(N))$, with valuation 1. For any $k \geqslant 2$, using common notations,

$$(E_4 - H_{2,N}^2)[E^{(0)}_{2,N}]^{2k-4}$$

is an element of $M_{2k}(\Gamma_0(N))$ with valuation 1. The result follows. □

Proof of Theorem 7.2. Once more, let $(E^{(r)}_{2k,N})_{0\leqslant r\leqslant d_{2k}(N)-1}$ be a unitary upper triangular basis of $M_{2k}(\Gamma_0(N))$. We deduce from Lemma 7.3 the equalities

$$\nu(E^{(0)}_{4,N}) = 0 \quad \text{and} \quad \nu(E^{(1)}_{4,N}) = 1.$$

For any integers $a \geqslant 1$ and $\ell \in \{0, \ldots, a\}$, the modular form

$$[E_{2,2}^{(0)}]^{2a-2\ell} [E_{4,N}^{(1)}]^{\ell}$$

belongs to $M_{4a}(\Gamma_0(N))$ with valuation ℓ.

We now fix $a = \nu(\Delta_N)$. Therefore, any unitary upper triangular basis of $M_{4a}(\Gamma_0(N))$ will not show any jump among its first a elements, as we just showed. This property remains true for $M_{2k}(\Gamma_0(N))$ whenever $k \geqslant 2a$; to see it, we only need to multiply the first a elements of the $M_{4a}(\Gamma_0(N))$ unitary upper triangular basis by $[E_{2,2}^{(0)}]^{k-2a}$.

Then, using Theorem 2.5, for all $h \in \mathbb{N}$,

$$\begin{aligned} &M_{4a+2h+\rho_N}(\Gamma_0(N)) \\ &\qquad = \Delta_N . M_{4a+2h}(\Gamma_0(N)) \oplus \operatorname{span}\big\{E^{(s)}_{4a+2h+\rho_N,N} \;/\; \nu(E^{(s)}_{4a+2h+\rho_N,N}) < \nu(\Delta_N)\big\}. \end{aligned}$$

However, we just established that for all $h \geqslant 0$,

$$\Big\{E^{(s)}_{4a+2h+\rho_N,N} \;/\; \nu(E^{(s)}_{4a+2h+\rho_N,N}) < \nu(\Delta_N)\Big\} = \Big\{E^{(s)}_{4a+2h+\rho_N,N} \;/\; 0 \leqslant s < \nu(\Delta_N)\Big\}$$

which is of cardinal $\nu(\Delta_N)$. The relation 7.2 is therefore proven for all $k \geqslant 2a$. Finally, thanks to relation 7.1, we can notice that

$$k \longmapsto \dim(M_{2k+\rho_N}(\Gamma_0(N))) - \dim(M_{2k}(\Gamma_0(N)))$$

is periodic starting from $k = 1$, with a period of 6. Since it is constant above $2a$, it is a constant function for all $k \geqslant 1$, necessarily equal to $\nu(\Delta_N)$, as expected. □

Moreover, from Theorem 2.5, we deduce that for all $k \in \mathbb{N}$

$$\dim(M_{2k+\rho_N}(\Gamma_0(N))) = \dim(M_{2k}(\Gamma_0(N))) + \operatorname{card}(\{s \;/\; \nu(E^{(s)}_{2k+\rho_N,N}) < \nu(\Delta_N)\}).$$

Consequently, $\operatorname{card}(\{s \;/\; \nu(E^{(s)}_{2k+\rho_N,N}) < \nu(\Delta_N)\}) = \nu(\Delta_N)$ for $k \geqslant 1$ and this next result:

Theorem 7.4. *Let N be a positive integer and, for any $k \geqslant 1$, $(E^{(r)}_{2k,N})_{0 \leqslant r \leqslant d_{2k}(N)-1}$ be a unitary upper triangular basis of $M_{2k}(\Gamma_0(N))$. Then*

$$\forall\, k \geqslant \frac{\rho_N}{2} + 1, \quad \forall\, r \in \{0, \ldots, \nu(\Delta_N) - 1\}, \quad \nu(E^{(r)}_{2k,N}) = r.$$

In addition, one can choose the basis $(E^{(r)}_{2k,N})$ such that

$$\forall\, k \geqslant \frac{\rho_N}{2} + 1, \quad \forall\, r \in \{0, \ldots, \nu(\Delta_N) - 1\}, \quad E^{(r)}_{2k,N} = E^{(r)}_{\rho_N+2,N} [E^{(0)}_{2,N}]^{k-\frac{\rho_N}{2}-1}.$$

At last, the theorem to structure and construct unitary upper triangular bases takes its final form.

Theorem 7.5. *Let N be a positive integer. Then, for any $k \in \mathbb{N}$ such that $k \geqslant \rho_N/2$,*

$$M_{2k}(\Gamma_0(N)) = \Delta_N.M_{2k-\rho_N}(\Gamma_0(N)) \oplus \mathrm{span}\{E^{(s)}_{\rho_N+2,N}[E^{(0)}_{2,N}]^{k-\frac{\rho_N}{2}-1} \,/\, 0 \leqslant s < \nu(\Delta_N)\}.$$

Therefore, if $k \in \mathbb{N}^*$ and $k = q\frac{\rho_N}{2} + r$ with $1 \leqslant r \leqslant \frac{\rho_N}{2}$,

$$M_{2k}(\Gamma_0(N)) = \Delta_N^q.M_{2r}(\Gamma_0(N)) \bigoplus_{n=0}^{q-1} \Delta_N^n.\,\mathrm{span}\{E^{(s)}_{\rho_N+2,N}[E^{(0)}_{2,N}]^{k-(n+1)\frac{\rho_N}{2}-1} \,/\, 0 \leqslant s < \nu(\Delta_N)\}.$$

8. Putting theory into practice

Theorem 7.5 reveals the structure of classical modular forms spaces with respect to $\Gamma_0(N)$. To obtain unitary upper triangular bases of these spaces, it remains to determine partial bases $\mathcal{B}_{2k}(\Gamma_0(N)) = (E^{(s)}_{2k,N})_{0 \leqslant s \leqslant d_{2k}(N)-1}$, for $1 \leqslant k \leqslant \frac{\rho_N}{2}$, as well as the first elements of $\mathcal{B}_{\rho_N+2}(\Gamma_0(N))$: $(E^{(s)}_{\rho_N+2,N})_{0 \leqslant s \leqslant \nu(\Delta_N)-1}$.

This is no easy task, but many modular forms are identified in the literature; one can for example consult [4] for a broad study on the subject. We have checked that this work can be carried out, essentially thanks to Weierstrass elliptic functions, for N between 1 and 10.

Moreover the knowledge of unitary upper triangular bases $(\mathcal{B}_{2k}(\Gamma_0(N)))_{1 \leqslant k \leqslant k_0}$, for a fixed value $k_0 \leqslant \frac{\rho_N}{2}$ makes it possible to obtain many elements of $\mathcal{B}_{2k_0+2}(\Gamma_0(N))$. Noticeably, $E^{(0)}_{2,N}\mathcal{B}_{2k_0}(\Gamma_0(N)) \subset \mathcal{B}_{2k_0+2}(\Gamma_0(N))$, which greatly reduces the number of new modular forms to determine in order to obtain a unitary upper triangular basis of $M_{2k_0+2}(\Gamma_0(N))$.

Noteworthily, the computational approach can benefit directly from the results of previous sections. The knowledge of the unitary upper triangular bases for $1 \leqslant 2k \leqslant \rho_N+2$ with a precision of m terms in the development in powers of q enables one to directly obtain unitary upper triangular bases for any weight $2k > \rho_N + 2$, still with a precision of m terms.

9. Conclusion

Let us conclude this study with a few words to better put the Δ_N functions back into the context of previous works. Products of η functions have been studied by Rademacher [10] who introduced the functions $\varphi_\delta(\tau) = \eta(\delta\tau)/\eta(\tau)$ in order to establish that, if $p \geqslant 5$ was prime and r an even integer, then φ_p^r would be a weakly modular function of weight 0 with respect to $\Gamma_0(p)$. This result was extended by Newmann [7, 8] who constructed, also starting from φ_δ functions, weakly modular functions with respect to $\Gamma_0(N)$, for any N this time, and thus of weight 0.

Theorem 3.5, stating that functions Δ_N are strong modular units, was essentially proven by Ligozat [5] in his study of elliptical modular curves. From then on, mathematicians essentially looked for η-quotients in their quest for cuspidal modular forms. Perhaps therein lies the reason why the notion of strong modular units did not pan out, having been overshadowed by the highly-justified importance given to cuspidal forms that followed from Hecke's seminal work.

By introducing the Δ_N functions, we were able to clarify the structure of the sequences of modular spaces $(M_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$, and provide an effective tool to provide bases for each of these spaces. The reader will certainly appreciate that, in a similar way, the strong modular unit Δ_N also makes it possible to structure sequences of cuspidal modular spaces $(S_{2k}(\Gamma_0(N)))_{k\in\mathbb{N}^*}$, and to give explicit bases for each of these spaces.

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