ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 27, nº 1 (2020), p. 83-123.

<http://ambp.centre-mersenne.org/item?id=AMBP_2020__27_1_83_0>

Publication éditée par le laboratoire de mathématiques Blaise Pascal
de l'université Clermont Auvergne, UMR 6620 du CNRS
Clermont-Ferrand — France

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http://www.centre-mersenne.org/

Annales mathématiques Blaise Pascal **27**, 83-123 (2020)

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Sunrose T. Shrestha

Résumé

Square-tiled surfaces can be classified by their number of squares and their cylinder diagrams (also called realizable separatrix diagrams). For the case of n squares and two cone points with angle 4π each, we set up and parametrize the classification into four diagrams. Our main result is to provide formulae for enumeration of square-tiled surfaces in these four diagrams, completing the detailed count for genus two. The formulae are in terms of various well-studied arithmetic functions, enabling us to give asymptotics for each diagram. Interestingly, two of the four cylinder diagrams occur with asymptotic density 1/4, but the other diagrams occur with different (and irrational) densities.

1. Introduction

The main result of this paper is enumeration of the number of primitive (connected) square-tiled translation surfaces in the stratum $\mathcal{H}(1,1)$ by their cylinder diagrams.

Recall that a square-tiled translation surface is a closed orientable surface built out of unit-area axis-parallel Euclidean squares glued along edges via translations. Square-tiled surfaces of genus > 1 are ramified covers (with branching over exactly one point) of the standard square torus. The principal stratum $\mathcal{H}(1,1)$ contains genus two translation surfaces with two simple cone points. (The only other possibility is a single cone point with more angle excess.) A *primitive* square-tiled surface is one that covers the standard torus with no other square-tiled surface as an intermediate cover.

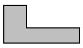
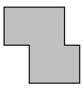
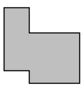
Every square-tiled surface is built out of horizontal square-tiled cylinders. We can define *cylinder diagrams* (ribbon graphs with a pairing on the boundary components) which keep track of the number of cylinders and the gluing patterns along their sides. In Section 3 we identify the four cylinder diagrams (named A, B, C and D in this paper), for $\mathcal{H}(1,1)$. A square-tiled surface with n squares will be referred to as an *n-square surface*. Next, we state our main result.

Theorem 1.1. *Let $A(n), B(n), C(n)$ and $D(n)$ count the number of primitive n-square surfaces in $\mathcal{H}(1,1)$ with cylinder diagram A, B, C and D, respectively. Let $E(n)$ be the total number of primitive n-square surfaces in $\mathcal{H}(1,1)$. Then*

$$E(n) = \frac{1}{6}(n-2)(n-3)J_2(n)$$

Keywords: Square-tiled surface, translation surface, counting, primitive.
2020 *Mathematics Subject Classification*: 00X99.

and this breaks down as

Diagram	Formula	Density
	$A(n) = \frac{1}{24}n(n-6)J_2(n) + \frac{1}{2}nJ_1(n)$	$\frac{1}{4}E(n)$
	$B(n) = \big((\mu \cdot \sigma_2) * (\sigma_1 \Delta \sigma_2)\big)(n)$ $-\frac{1}{24}(2n+9)(n-2)J_2(n) - \frac{1}{2}nJ_1(n)$	$\left(\frac{\zeta(2)\zeta(3)}{2\zeta(5)} - \frac{1}{2}\right)E(n)$
	$C(n) = \frac{1}{24}(n-2)(n-3)J_2(n)$	$\frac{1}{4}E(n)$
	$D(n) = \frac{1}{6}n(n-1)J_2(n) - \big((\mu \cdot \sigma_2) * (\sigma_1 \Delta \sigma_2)\big)(n)$	$\left(1 - \frac{\zeta(2)\zeta(3)}{2\zeta(5)}\right)E(n)$

Furthermore, while surfaces with diagram A and diagram C are each asymptotic to one-quarter of the total, the surfaces with other diagrams have unequal (and irrational) densitites.

$$A(n)/E(n) \to 0.25\ ; \qquad B(n)/E(n) \to 0.453\ldots$$
$$C(n)/E(n) = 0.25\ ; \qquad D(n)/E(n) \to 0.047\ldots$$

The statement of our main theorem uses standard notation for arithmetic functions: J_1 and J_2 are Jordan totient functions; μ is the Möbius function; σ_1 and σ_2 are divisor functions; and ζ is the Riemann zeta function. The symbol $*$ denotes the Dirichlet convolution, and Δ is additive convolution. For more detailed definitions, see Appendix A in [17].

Since $\frac{6}{\pi^2}n^2 < J_2(n) \leq n^2$, the number of primitive n-square surfaces in $\mathcal{H}(1,1)$ grows at least as fast as $\frac{1}{\pi^2}n^4$. The total count $E(n)$ was already known, and can be found in the work of Bloch–Okounkov [2] and Dijkgraaf [4]. The novelty in our result is that we get a more detailed count, by cylinder diagrams, that allows us to obtain the individual asymptotic densities as well.

Figure 1.1 shows the share of $E(n)$ by cylinder diagram for $4 \leq n \leq 101$. We see erratic but steady convergence in the direction of the asymptotics from Theorem 1.1.

The enumeration by cylinder diagram of primitive n-square surfaces in $\mathcal{H}(2)$ was done in unpublished work of Zmiaikou [19]. Let $H(n)$ be the total number of primitive square-tiled surfaces in $\mathcal{H}(2)$. There are two cylinder diagrams for surfaces in $\mathcal{H}(2)$, which we can denote by F and G, and the number of primitive n-square surfaces with

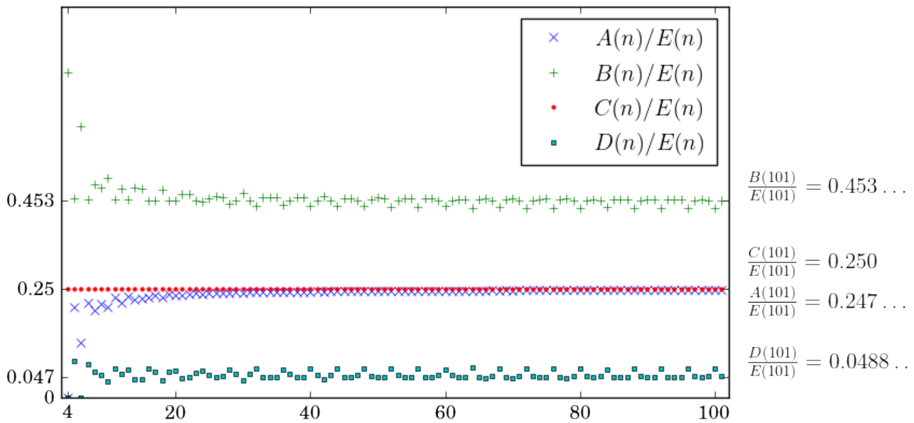


FIGURE 1.1. Scatter plot for $\frac{A(n)}{E(n)}, \frac{B(n)}{E(n)}, \frac{C(n)}{E(n)}, \frac{D(n)}{E(n)}$ for $4 \le n \le 101$

each diagram is then given as follows.

$$F(n) = \frac{1}{6}nJ_2(n) - \frac{1}{2}nJ_1(n) \sim \frac{4}{9}H(n);$$
$$G(n) = \frac{5}{24}nJ_2(n) + \frac{1}{2}nJ_1(n) - \frac{3}{4}J_2(n) \sim \frac{5}{9}H(n).$$

with $H(n) = \frac{3}{8}(n-2)J_2(n)$. The main theorem of this paper complements Zmiaikou's work, completing the enumeration of primitive square-tiled surfaces of genus 2 by cylinder diagram.

1.1. Relationship to other results

This work fits into a significant body of literature on enumeration of square-tiled surfaces. Several papers focus on the classification by orbits in the $SL_2(\mathbb{Z})$ action on square-tiled surfaces. Combined work of Hubert–Lelièvre [8] and McMullen [15] shows that for $n \ge 3$, there are either one or two $SL_2(\mathbb{Z})$ orbits for primitive n-square surfaces in $\mathcal{H}(2)$. Subsequently, Lelièvre–Royer [12] obtained formulae for enumerating these orbit-wise. They also prove that the generating functions for these countings are quasimodular forms. Dijkgraaf [4] gave generating functions for the number of n-sheeted covers of genus g of the square torus with simple ramification over distinct points, and Bloch–Okounkov [2] studied this problem for arbitrary ramification.

Square-tiled surfaces (not necessarily primitive) were also counted by their cylinder diagrams by Zorich [20] who applied the counts to compute the Masur–Veech volume of certain small genus strata. Eskin–Okounkov [6] generalized Zorich's work and used such counts to obtain formulae for the Masur–Veech volume of all strata. More recently Delecroix–Goujard–Zograf–Zorich [3] refined this result, and studied the absolute and relative contributions of square-tiled surfaces with fixed number of cylinders in their cylinder diagrams to the Masur–Veech volume of the ambient strata. They got a general formulae (albeit, not closed) for any number of cylinders and any strata. Moreover, they obtained closed sharp upper and lower bounds of the absolute contributions of 1-cylinder surfaces to the volume of any stratum and in the cases where the stratum is either $\mathcal{H}(2g-2)$ or $\mathcal{H}(1, 1, \ldots, 1)$ they obtained closed exact formulae. In the same paper, they also proved that square-tiled surfaces with a fixed cylinder diagram equidistribute in the ambient stratum.

Eskin–Masur–Schmoll [5] on the other hand used counts of primitive genus two square-tiled surfaces to obtain the asymptotics for the number of closed orbits for billiards in a square table with a barrier.

1.2. **Proof strategy and structure of paper**

The paper is organized as follows. The background in Section 2 covers generalities on square-tiled surfaces and introduces the *monodromy group* $\mathrm{Mon}(S) \leq S_n$ as a key tool that records the structure of gluings of the sides of S in terms of permutations of squares. For the remaining sections, we have streamlined the presentation by creating appendices with detailed but routine calculations. These appendices are included in the Arxiv version [17] of this paper. In Section 3 we give statements of how the four cylinder diagrams in $\mathcal{H}(1, 1)$ will be parametrized. (Full proofs that support these parametrizations are found in the Appendix C.) In Section 4 we state number theoretic criteria on the parameters obtained in Section 3 which characterize primitivity. (Full proofs appear in Appendix D.) In Section 5 we manipulate the sums that appear from the number theoretic criteria to deduce the counting formulae in the main theorem. (Full work showing simplifications of intermediate sums appears in Appendix B.) Finally, in Section 6 we complete the proof of the main theorem by computing asymptotic densities for each cylinder diagram. (Appendix A contains background on some arithmetic functions, and number theoretic identities used during the enumerations.)

The main idea in the proof is to take advantage of the key fact that a connected n-square surface is primitive if and only if the associated monodromy group, $\mathrm{Mon}(S)$, satisfies an algebraic condition also called primitivity, which is described in terms of the orbits of its action on the n squares. (See Section 2.) The number theoretic conditions result directly from this algebraic characterization, and much of the rest of the work is in cleverly handling the sums involving arithmetic functions.

Our methods can be extended directly to enumeration of primitive square-tiled surfaces in certain strata of higher genus, with asymptotic proportions by cylinder diagrams. However, the complexity becomes forbidding. Even $\mathcal{H}(4)$, which is the smallest stratum in genus 3, already has 22 different cylinder diagrams (as shown by S. Lelièvre in [11] as an Appendix to [14].) A more tractable starting point might be to consider the hyperelliptic component of $\mathcal{H}(4)$, which has just 5 cylinder diagrams. On the other hand, there are clear limitations on the generality of this counting method: it is known that in $\mathcal{H}(1,1,1,1)$, the principal stratum of genus 3, the lattice of absolute periods does not pick up primitivity, so new ideas would be needed.

1.3. Acknowledgements

We are very grateful for Moon Duchin for suggesting this problem and advising us throughout the project. We are also very grateful to Samuel Lelièvre for many insightful conversations, comments, and suggestions. We also acknowledge David Zmiaikou, Robert Lemke-Oliver and Frank Thorne for helpful conversations. Finally, we thank the referee for the numerous corrections and improvements they suggested.

2. Background

2.1. Square-tiled Surfaces

We begin this section by defining square-tiled surfaces more rigorously.

Definition 2.1 (Square-tiled surface)**.** A *square-tiled surface* is a closed orientable surface obtained from the union of finitely many Euclidean axis parallel unit area squares $\{\Delta_1, \ldots, \Delta_n\}$ such that

- the embedding of the squares in $\mathbb{R}^2$ is fixed only up to translation,
- after orienting the boundary of every square counterclockwise, for every $1 \leq j \leq n$, for every oriented side s_j of Δ_j, there exists a $1 \leq k \leq n$, and an oriented side s_k of Δ_k so that s_j and s_k are parallel, and of opposite orientation. The sides s_j and s_k are glued together by a parallel translation.

Remarks 2.2. A few key things that follow from the definition:

(1) The orientations of the glued edges s_j and s_k are opposite so that as one moves along the glued side, Δ_j appears to the left and Δ_k appears to the right (or vice versa).

(2) The total angle around a vertex is $2\pi c$ for some positive integer c. When $c > 1$, we call the point a cone point.

(3) Since the squares are embedded in $\mathbb{R}^2$ up to translation, we distinguish between two squares if one is obtained from the other by a nontrivial rotation. However, two squares are "cut, parallel transport, and paste" equivalent. Hence, square-tiled surfaces come with a well defined vertical direction.

Considering the squares as embedded in $\mathbb{C}$, one can give a square-tiled surface a complex structure as well. Moreover, since the gluing of the sides is by translation, the transition functions are translations:

$$z \to z' + C$$

Hence, the 1-form dz on $\mathbb{C}$ gives rise to a holomorphic 1-form ω on S so that locally, $\omega = dz$ around non-cone points. This is well defined since at a chart where the coordinate function is z', ω takes the form $\omega = dz'$. Around the cone points, up to a change of coordinates, the coordinate functions are z^{k+1} so that ω takes the form $\omega = z^k dz$. Hence, the cone points are zeros of ω, and if the angle around the cone point is $2\pi(k+1)$, then the degree of the zero is k.

Given the angles around each cone point, and the number of cone points, one can also recover the genus of the square-tiled surface. Note that since the squares are Euclidean, square-tiled surfaces are flat everywhere except at the cone points. Hence, the classical Gauss-Bonnet theorem takes a relatively simple form to give:

$$\int \kappa \,\mathrm{d}A = 2\pi\chi(S) \implies -2\pi \sum_{i=1}^{m} k_i = 2\pi(2-2g) \implies \sum_{i=1}^{m} k_i = 2g-2$$

where κ is Gaussian curvature, $\chi(S)$ is the Euler characteristic of S, and the sum is over the cone points with angles $2\pi(k_i + 1)$.

More generally, if we allow arbitrary Euclidean polygons in definition 2.1, we get *translation surfaces* which are a class of surfaces of which square-tiled surfaces are a particular subset.

For translation surfaces, let $\alpha = (k_1, \ldots, k_m)$ be the integer vector that records the angle data of the cone points so that there are m cone points and the cone point angles are $2\pi(k_i + 1)$ for $k_i \geq 1$. Since the genus of the surface is recovered using this data, the space of genus g translation surfaces is stratified with surfaces sharing the same data $\alpha = (k_1, \ldots, k_m)$ for the various integer partitions of $2g - 2$. These are called *strata* and are denoted $\mathcal{H}(\alpha)$.

Let $\mathbb{T} = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ be the standard torus. Given a square-tiled surface S (with a holomorphic 1-form ω), we know that the cone points are in the integer lattice, and hence we get a map,

$$\pi : S \to \mathbb{T}$$

by

$$p \to \int_{P_1}^{p} \omega \mod \mathbb{Z} + i\mathbb{Z}$$

where $\{P_1, \ldots, P_m\}$ is the set of cone points of S. π is holomorphic and onto, and hence it is a ramified covering where the ramification points are exactly the zeros of ω (or the cone points) which project to $0 \in \mathbb{T}$.

Given an element ρ in the relative homology group $H_1(S, \{P_1, \ldots, P_m\}; \mathbb{Z})$ of a square-tiled surface S, we call $\int_\rho \omega$ a *period*. Since the cone points are in the integer lattice, all periods are in $\mathbb{Z} + i\mathbb{Z} \simeq \mathbb{Z}^2$.

Since square-tiled surfaces are cut, parallel transport and paste equivalent, their representation in the plane is not unique. In particular, square-tiled surfaces can be represented by parallelograms as well. We call such representations *unfolded representations*. See Figure 2.1 for a basic example.

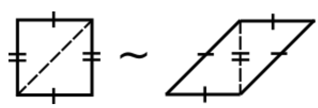


FIGURE 2.1. An unfolded representation for the standard torus

Next we define some geometrical objects of interest on these surfaces.

Definition 2.3 (Saddle connection)**.** A *saddle connection* in a translation surface S is a curve $\gamma : [0, 1] \to S$ such that $\gamma(0)$ and $\gamma(1)$ are cone points, but $\gamma(s)$ is not a cone point for any $0 < s < 1$.

Definition 2.4 (Holonomy vector)**.** The *holonomy vector* associated to a saddle connection γ in a translation surface is the relative period $v = \int_\gamma \omega$ viewed as a vector in $\mathbb{R}^2$.

Geometrically, a holonomy vector of a saddle connection γ records the Euclidean horizontal and vertical displacement of a saddle connection γ. If (v_1, v_2) is a holonomy vector of a saddle connection γ, then v_1 will be called the *horizontal holonomy* of γ and v_2 will be called the *vertical holonomy* of γ.

Definition 2.5 (Lattice of periods)**.** The *lattice of periods* of a square-tiled surface S, denoted as $\mathrm{Per}(S)$, is the rank 2 sublattice of $\mathbb{Z}^2$ holonomy vectors of S.

Definition 2.6 (Lattice of absolute periods)**.** The *lattice of absolute periods* of a square-tiled surface S, denoted as $\mathrm{AbsPer}(S)$, is the rank 2 sublattice of $\mathbb{Z}^2$ generated by the holonomy vectors of closed saddle connections of S.

Note that $\mathrm{AbsPer}(S) \subset \mathrm{Per}(S)$.

We say that a square-tiled surface S covers S' if the following diagram commutes for a ramified covering π. S will be called a *proper ramified covering* of S' if π and h' have covering degree > 1.

$$\begin{tikzcd} S \arrow[rr, "\pi"] \arrow[dr, "h"'] & & S' \arrow[dl, "h'"] \\ & \mathbb{T} & \end{tikzcd}$$

This motivates the following definition:

Definition 2.7 (Primitive square-tiled surface)**.** We call a square-tiled surface *primitive* it is not a proper ramified covering of any other square-tiled surface.

2.2. The Monodromy Group

Given an n-square surface S, first fix a labelling of the squares by $\{1, \dots, n\}$. Square-tiled surfaces, come with a well defined vertical direction, and the squares used to make the surface are axis parallel. Hence, for any square, there is a well defined notion of top, right, bottom and left neighboring squares. We associate two permutations, σ and τ, to S defined by

$$\sigma(i) = j \iff \text{right side of square } i \text{ is glued to the left side of square } j$$
$$\tau(i) = j \iff \text{top side of square } i \text{ is glued to the bottom side of square } j$$

The permutations σ and τ describe completely how to glue the squares to form S. σ is referred to as the *right permutation* associated to S and τ is referred to as the *top permutation* associated to S.

If the labelling on S is changed by a permutation $\gamma \in S_n$ (the symmetric group on n objects) so that square i is now labelled $\gamma(i)$, then one checks that the associated right and top permutations we get for the newly labelled S, are $\gamma\sigma\gamma^{-1}$ and $\gamma\tau\gamma^{-1}$.

Hence, given an unlabelled square-tiled surface S we can obtain a simultaneous conjugacy class of a pair of permutations in S_n as described above. One checks that the converse is true: given a simultaneous conjugacy class in $S_n \times S_n$, one can associate uniquely, a (possibly disconnected) square-tiled surface.

Notationally, given a pair of permutations (σ, τ) we will denote $S(\sigma, \tau)$ as the square-tiled surface that has σ and τ as its right and top permutations. Note that fixing the conjugacy class representative σ, fixes a labelling of S, so that $S(\sigma, \tau)$ comes with a labelling.

The commutator $[\sigma, \tau]$ is also of interest, since its cycle type defines the topological type of S. We will state the following known propositions, to this effect.

Proposition 2.8. *Let $\sigma, \tau \in S_n$ such that $[\sigma, \tau]$ is a product of disjoint nontrivial cycles $u_1, \ldots, u_m$ of lengths $(k_i + 1), \ldots, (k_m + 1)$. Consider $S(\sigma, \tau)$ endowed with a labelling prescribed by σ. Then, for labelled squares $x \neq y$, their lower left corners are identified if and only if and each $x, y \in \operatorname{supp}(u_j)$ for some j.*

Recall that the support of a cycle $u = (a_1 \ldots a_j)$, denoted $\operatorname{supp}(u)$, is the set $\{a_1, \ldots, a_j\}$ of elements that are nontrivially moved by u. Knowing Proposition 2.8, it follows that the cycle type of $[\sigma, \tau]$ determines the stratum:

Proposition 2.9. *$S(\sigma, \tau) \in \mathcal{H}(k_1, \ldots, k_m)$ for σ, τ as in Proposition 2.8.*

The group $\langle \sigma, \tau \rangle$ is also of interest, and has a name.

Definition 2.10 (Monodromy Group)**.** Let S be a labelled square-tiled surface with $\sigma, \tau \in S_n$ as the top and right permutations. We define the *monodromy group*, denoted $\operatorname{Mon}(S)$, as the subgroup of S_n generated by σ and τ (upto conjugacy).

Note that relabelling S results in an isomorphic monodromy group. Note also that S is connected if and only if $\operatorname{Mon}(S)$ acts transitively on the set $\{1, \ldots, n\}$ with the natural permutation action.

Next we define the notion of primitivity for a subgroup of S_n.

Definition 2.11 (Blocks)**.** A non-empty subset $\Delta \in \{1, \ldots, n\}$ is a *block* for a subgroup $H \subset S_n$ if for all $h \in H$, either $h(\Delta) = \Delta$ or $h(\Delta) \cap \Delta = \emptyset$.

Note that singletons and the whole set $\{1, \ldots n\}$ are always blocks. These will be called *trivial blocks*.

Definition 2.12 (Primitive subgroup)**.** A subgroup $H \subset S_n$ is *primitive* if H has no nontrivial blocks.

We then have the following proposition, that bridges the two notions of primitivity we have seen so far:

Proposition 2.13 ([18])**.** *A connected n-square surface S is primitive if and only if* $\operatorname{Mon}(S) \subset S_n$ *is primitive.*

We will use this proposition in Section 4 to characterize primitivity of square-tiled surfaces in $\mathcal{H}(1, 1)$ in terms of geometric parameters that define them.

3. **Cylinder Diagrams**

In this section, we study the geometry of square-tiled surfaces by studying their system of horizontal saddle connections. Let S be a square-tiled surface and consider a graph Γ on S with the vertex set as the cone points of S, and horizontal saddle connections as the edge set. Since every square-tiled surface has a complete cylinder decomposition in the horizontal direction, the complement $S \setminus \Gamma$ is a collection of horizontal cylinders of S.

Since S is an orientable surface, we take an orientation on S and endow the horizontal foliation of S with a compatible orientation. This orientation induces an orientation on the edges of Γ, and hence we obtain an oriented graph. Moreover, given any vertex v of Γ, as S is oriented, we get a cyclic order on the edges incident to v. Since S is oriented, the orientation of the edges incident to v alternate between orientation towards and away from v as we move counterclockwise.

Taking an ϵ neighborhood of the edges of Γ, we obtain a (not necessarily connected) *ribbon graph* $R(\Gamma)$ which is a collection of oriented strips glued as per the cyclic ordering on the vertices of Γ. $R(\Gamma)$ is an orientable surface with boundary. There are two orientations that are induced on the boundary components of $R(\Gamma)$. The first orientation is the canonical orientation on the boundary components coming from the orientation of $R(\Gamma)$ as an oriented surface. The second orientation is induced by the oriented edges of Γ. Note that these orientations do not necessarily match on all boundary components. We say that a boundary component is *positively oriented* if the two notions give the same orientation, and *negatively oriented* if the two notions do not match.

The complement, $S \setminus R(\Gamma)$ is then a union of flat cylinders, each of whose two boundaries are glued to boundary components of $R(\Gamma)$, one positively oriented and one negatively oriented. Hence, the boundary components of $R(\Gamma)$ decompose into pairs so that each of the components in a pair bound the same cylinder in S, and have opposite signs of orientation. This motivates the following definition:

Definition 3.1 (Separatrix Diagram)**.** A *separatrix diagram* is a pair $(R(\Gamma), P)$, where Γ is a finite directed graph and P is a pairing on the boundary components of the associated ribbon graph $R(\Gamma)$ such that

(1) edges incident to each vertex are cyclically ordered, with orientations alternating between to and from the vertex.

(2) the boundary components of $R(\Gamma)$, in each pair defined by P, are of opposite orientation.

Note that from the process described above, to each square-tiled surface one can associate a separatrix diagram. Conversely, from each separatrix diagram, one obtains a closed orientable surface by gluing in topological cylinders between the paired boundary components.

However, in order for the resulting surface to be a square-tiled surface, we first assign real variables representing lengths to the edges of Γ. Then each boundary component of $R(\Gamma)$ has as its length, the sum of the lengths of saddle connections that run parallel to it. To obtain a square-tiled surface from a separatrix diagram $(R(\Gamma), P)$, in each pair of boundary components determined by P, the lengths of the boundary components in the pair must be equal (so that one can glue in a metric cylinder with these boundary components). Imposing this condition, we obtain a system of linear equations with variables as the lengths of saddle connections. Hence, a square-tiled surface is obtained if and only if there exists a solution with positive integer lengths.

Definition 3.2 (Cylinder Diagram)**.** A cylinder diagram is a realizable separatrix diagram.

For more details, see [21].

We next state a previously known classification of the cylinder diagrams of surfaces in $\mathcal{H}(1,1)$, the proof of which we recreate in Appendix C [17]. The proof is a standard procedure of enumerating graphs on surfaces. We refer the reader to [10] for more details on such graphs.

Proposition 3.3. *There are 4 distinct cylinder diagrams in* $\mathcal{H}(1,1)$.

Using Proposition 3.3 we make Figure 3.3 that show the cylinder diagrams in $\mathcal{H}(1,1)$ and prototypical surfaces arising from them and the parameters associated to these surfaces. The gluings are indicated by the dotted lines, and if dotted lines are missing then the gluing is by the obvious opposite side horizontal or vertical translation. We parametrize each of these prototypes by the lengths and heights of the cylinders, the lengths of the horizontal saddle connections, and the amount of shear (twist) on the cylinders. In all of the parametrization, p, q, r are heights of cylinders, j, k, l, m are lengths of horizontal saddle connections, and α, β, γ are shears in the cylinders.

As shown in the Figure 3.3:

- Cylinder diagram A is parametrized by (p, j, k, l, m, α).
- Cylinder diagram B is parametrized by $(p, q, k, l, m, \alpha, \beta)$ where α is the shear in the longer cylinder and β the shear in the shorter cylinder.
- Cylinder diagram C is parametrized by $(p, q, k, l, m, \alpha, \beta)$ where α is the shear in the cylinder of width p and β is the shear in the cylinder of width q.
- Cylinder diagram D is parametrized by $(p, q, r, k, l, \alpha, \beta, \gamma)$ where α, β, γ are the shears in the cylinders with width p, q, r respectively.

We note that these parameters are not unique as stated. The non-uniqueness of parameters stems from cut and paste equivalence. For instance, given a surface with cylinder diagram D, if the bottom cylinder is longer than the top cylinder, (i.e. $k > l$ in the parametrization), one can interchange them via a cut and paste move so that the shorter cylinder is at the bottom. In other words, for cylinder diagram D, parameters $(p, q, r, k, l, \alpha, \beta, \gamma)$ and $(r, q, p, l, k, \gamma, \beta, \alpha)$ define the same surface. In order to use them to count the surfaces, we first need a unique set of parameters for each cylinder diagram. The following propositions give unique parametrizations of the these surfaces. We detail the proof for diagram B (Proposition 3.5) and refer the reader to Appendix C [17] for details on the others.

Proposition 3.4 (Uniqueness of cylinder diagram A parameters)**.** *Let* $(p, j, k, l, m, \alpha) \in \mathbb{N}^6$ *such that* $p(j + k + l + m) = n$*. The set of such tuples,* $\Sigma_A = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$*, where*

$$\begin{aligned} S_1 &:= \{(p, j, k, l, m, \alpha) \mid j < k, l, m, 0 \le \alpha < n/p\} \\ S_2 &:= \{(p, j, k, l, m, \alpha) \mid j = l < k \le m, 0 \le \alpha < n/p\} \\ S_3 &:= \{(p, j, k, l, m, \alpha) \mid j = k < l, m, 0 \le \alpha < n/p\} \\ S_4 &:= \{(p, j, k, l, m, \alpha) \mid j = k = l < m, 0 \le \alpha < n/p\} \\ S_5 &:= \{(p, j, k, l, m, \alpha) \mid j = k = l = m, 0 \le \alpha < n/(2p)\} \end{aligned}$$

uniquely parametrizes n-square surfaces in $\mathcal{H}(1, 1)$ *with cylinder diagram A.*

Proposition 3.5 (Uniqueness of cylinder diagram B parameters)**.** *The set*

$$\Sigma_B := \{(p, q, k, l, m, \alpha, \beta) \in \mathbb{N}^7 \mid 0 \le \alpha < k + l + m, 0 \le \beta < m, p(k + l + m) + qm = n\}$$

uniquely parametrizes n-square surfaces in $\mathcal{H}(1, 1)$ *with cylinder diagram B.*

Proof. Let $(p, q, k, l, m, \alpha, \beta)$ and $(p', q', k', l', m', \alpha', \beta') \in \Sigma_B$ parametrize the same surface. First, $m = m'$ and $k + l + m = k' + l' + m'$ since the first represents the unique length of horizontal closed saddle connections and the second represents the length of the longer horizontal cylinder. Hence, $k + l = k' + l'$ as well. Likewise, $p = p'$ and $q = q'$ as these represent the heights of the two cylinders.

Assume $\beta' > \beta$. Now, define S_0 by shearing the shorter cylinder of S by $-\beta$. The parameters for S_0 then are $(p, q, k, l, m, \alpha, 0)$ and $(p', q', k', l', m', \alpha', \beta' - \beta) = (p, q, k', l', m, \alpha', \beta' - \beta)$. Since there exists only one cone point each in the boundary curves of the shorter cylinder, it is clear that no non-trivial shear will be equal to a zero shear. Hence, $\beta' = \beta$. Similarly, assuming without loss of generality $\alpha' > \alpha$, obtain S_{00} from S_0 by shearing the longer cylinder by $-\alpha$ and get parameters $(p, q, k, l, m, 0, 0)$ and $(p, q, k', l', m, \alpha' - \alpha, 0)$ for S_{00}. Note S_{00} has a non-singular vertical simple closed curve of length $p + q$, crossing the core curves of both the horizontal cylinders exactly once. From

Figure 3.1 the shear in the longer cylinder must be zero, so that $\alpha' - \alpha = 0 \implies \alpha = \alpha'$. Now considering the length of the horizontal saddle connections we get that either:

(i) $k = k'$ and $l = l'$ or

(ii) $k = l'$ and $l = k'$.

We will show that the case (ii) leads to a special case of (i).

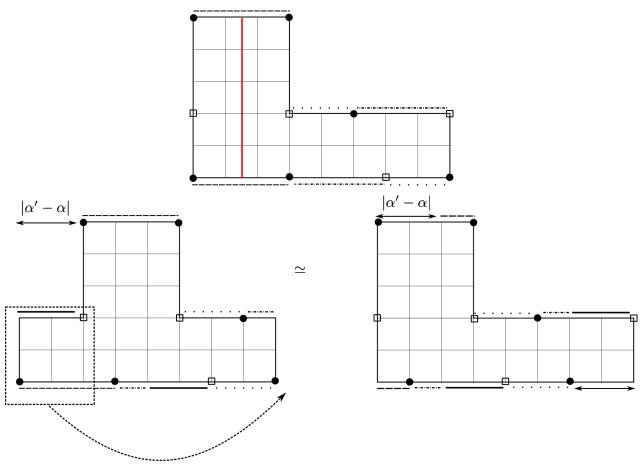


Figure 3.1. Diagram B argument for uniqueness of shear parameters. Top: S_{00} with a non-singular vertical simple closed curve of length $p + q$, crossing the core curves of both the horizontal cylinders exactly once. Bottom: S_{00} with a nontrivial $\alpha - \alpha'$ shear on the longer cylinder which will not allow the vertical $p + q$ to exist.

Consider the set of the 4 shortest non-closed saddle connections in S_{00} with vertical holonomy $-p$, transverse to the core curve of the longer cylinder and non-zero horizontal holonomy. These saddle connections have holonomy vectors in the set

$$\{(m, -p), (k + l, -p), (-k, -l), (l - k, -p)\}$$

according to one parametrization, and

$$\{(m, -p), (k' + l', -p), (-k' - l', (l' - k', -p)\}$$

according to the other parametrization. Assume $k = l'$ and $l = k'$. Then, equality of these sets implies that $l' - k' = l - k \implies k - k' = k' - k \implies k = k'$. See Figure 3.2. $\square$

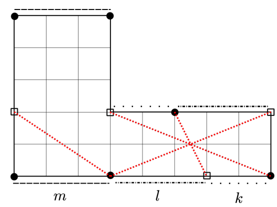


FIGURE 3.2. Special short curves with vertical holonomy $-p$ in S_{00}

Proposition 3.6 (Uniqueness of cylinder diagram C parameters)**.** *Let* $(p, q, k, l, m, \alpha, \beta) \in \mathbb{N}^7$ *such that* $p(k + l) + q(l + m) = n$, $0 \le \alpha < k + l$ *and* $0 \le \beta < l + m$. *The set of such tuples* $\Sigma_C = S_1 \cup S_2 \cup S_3$ *where*

$$S_1 := \{(p, q, k, l, m, \alpha, \beta) \mid k < m\} \qquad S_2 := \{(p, q, k, l, m, \alpha, \beta) \mid k = m, p < q\}$$
$$S_3 := \{\{(p, q, k, l, m, \alpha, \beta) \mid k = m, p = q, \alpha \le \beta\}$$

uniquely parametrizes n-square surfaces in $\mathcal{H}(1, 1)$ *with cylinder diagram C.*

Proposition 3.7 (Uniqueness of cylinder diagram D parameters)**.** *Let* $(p, q, r, k, l, \alpha, \beta, \gamma) \in \mathbb{N}^8$ *such that* $(p + q)k + (r + q)l = n$, $0 \le \alpha < k, 0 \le \beta < l + k$ *and* $0 \le \gamma < l$. *The set of such tuples* $\Sigma_D = S_1 \cup S_2 \cup S_3$ *where*

$$S_1 := \{(p, q, r, k, l, \alpha, \beta, \gamma) \mid k < l\} \qquad S_2 := \{(p, q, r, k, l, \alpha, \beta, \gamma) \in \mathbb{N}^8 \mid k = l, p \le r\}$$
$$S_3 := \{(p, q, r, k, l, \alpha, \beta, \gamma) \in \mathbb{N}^8 \mid k = l, p = r, \alpha \le \gamma\}$$

uniquely parametrizes n-square surfaces in $\mathcal{H}(1, 1)$ *with cylinder diagram D.*

4. **Primitivity criteria**

In this section we will develop a characterization of primitivity for square-tiled surfaces in $\mathcal{H}(1, 1)$ according to the parameters used to describe their cylinder diagrams.

Notation. $m \wedge n$ will denote the greatest common divisor of the integers m and n.

We start with a lemma from [19] by Zmiaikou, characterizing when certain integer vectors generate $\mathbb{Z}^2$.

Lemma 4.1. *For* $k, l, m, \alpha, \beta, p, q \in \mathbb{Z}$, *the vectors* $(k, 0), (l, 0), (\alpha, p), (\beta, q)$ *generate the lattice* $\mathbb{Z}^2$ *if and only if* $p \wedge q = 1$ *and* $k \wedge l \wedge (p\beta - q\alpha) = 1$

Next, we obtain a sufficient condition for primitivity of a square-tiled surface in $\mathcal{H}(1, 1)$ which we will use to get sufficient number theoretic conditions for primitivity in each cylinder diagram.

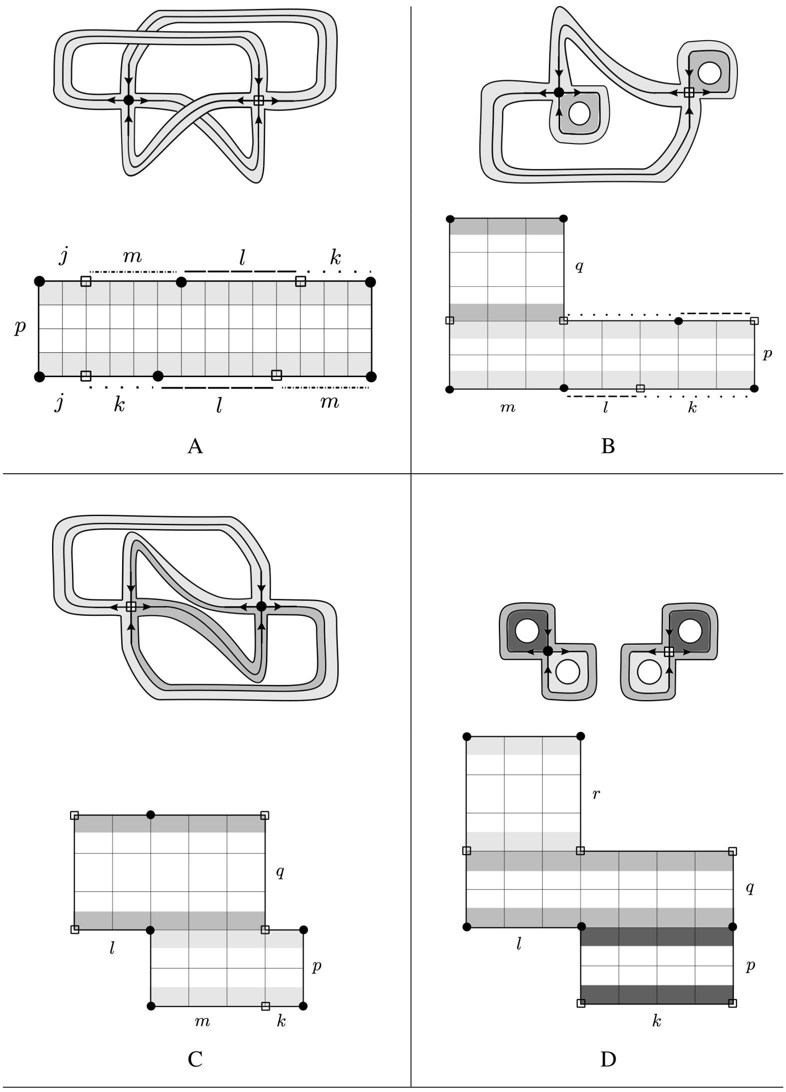


FIGURE 3.3. The 4 distinct cylinder diagrams for square-tiled surfaces of $\mathcal{H}(1, 1)$, and the prototypical square tiled surfaces (with parameters) that have those cylinder diagrams. Regions with the same shading have boundaries that are paired. The shears on the cylinders are omitted to avoid clutter of the pictures.

Lemma 4.2. *A square-tiled surface S in $\mathcal{H}(1,1)$ with* $\mathrm{AbsPer}(S) = \mathbb{Z}^2$ *is primitive.*

Proof. Assume S covers another square-tiled surface T. Since S is of genus 2, T has to be of genus 1. If $(x, y) \in \mathbb{Z}^2$ is in $\mathrm{AbsPer}(S)$, then, $(x/m, y/m) \in \mathrm{AbsPer}(T)$ for some $m \in \mathbb{N}$. So, $\mathrm{AbsPer}(S) = \mathbb{Z}^2$ implies that $\mathrm{AbsPer}(T) = \mathbb{Z}^2$ as well. Hence, T is a genus 1 surface with $\mathbb{Z}^2$ as its lattice of absolute periods, which implies that T is the standard torus $\mathbb{T}$. Therefore, as $\mathbb{T}$ is the only square-tiled surface covered by S, we conclude that S is primitive. ◻

We next state necessary and sufficient conditions for primitivity in each of the four cylinder diagrams, in terms of their parameters. Note that the parameters are defined in terms of the lengths of certain saddle connections bounding the horizontal cylinders. Hence, in geometrical terms, the following primitivity conditions are simply relations between lengths of certain saddle connections that provide an obstruction for the existence of a nontrivial square-tiled surface that is being branch covered. For instance, any surface with cylinder diagram B and parameters $(2, 2, k, l, m, \alpha, \beta)$ will cover the surface with cylinder diagram B and parameters $(1, 1, k, l, m, \alpha, \beta)$, as the height of the cylinders can be scaled down uniformly to get a surface with essentially the same properties but less area.

Lemma 4.3 (Primitivity criterion by cylinder diagram)**.** *Let $S \in \mathcal{H}(1,1)$ be an n-square surface. Then*

(1) *if S has cylinder diagram A, and is parametrized by (p, j, k, l, m, α), it is primitive if and only if*

$$p = 1 \qquad \text{and} \qquad (j + k) \wedge (k + l) \wedge n = 1.$$

(2) *if S has cylinder diagram B, and is parametrized by $(p, q, k, l, m, \alpha, \beta)$, it is primitive if and only if*

$$p \wedge q = 1 \qquad \text{and} \qquad (k + l) \wedge m \wedge (p\beta - q\alpha + (p + q)l) = 1.$$

(3) *if S has cylinder diagram C, and is parametrized by $(p, q, k, l, m, \alpha, \beta)$, it is primitive if and only if*

$$p \wedge q = 1 \qquad \text{and} \qquad (k + l) \wedge (l + m) \wedge (p\beta - q\alpha) = 1.$$

(4) *if S has cylinder diagram D, and is parametrized by $(p, q, r, k, l, \alpha, \beta, \gamma)$, it is primitive if and only if*

$$(p + q) \wedge (r + q) = 1 \qquad \text{and} \qquad k \wedge l \wedge ((p - r)\beta + (p + q)\gamma - (r + q)\alpha) = 1.$$

The general strategy to prove necessity of the number theoretic conditions is to first assume they are not satisfied, then show that this implies the monodromy group is not primitive, and hence the surface is not primitive. We carry out the proof for diagram C surfaces, but note that the arguments for the rest of the diagrams follow similarly.

Proof of 4.3(3). ($\Rightarrow$). Let $S = S(p, q, k, l, m, \alpha, \beta)$ be a surface of diagram C in $\mathcal{H}(1, 1)$. Assume $p \wedge q = 1$ and $(k + l) \wedge (l + m) \wedge (p\beta - q\alpha) = 1$ are not satisfied. By Lemma 4.1, we know $(\alpha, p), (\beta, q), (k + l, 0), (l + m, 0)$ generate some lattice L that does not coincide with $\mathbb{Z}^2$. We will then show that $G = \langle \sigma, \tau \rangle$ is not primitive where σ and τ are the right and top permutations associated to S.

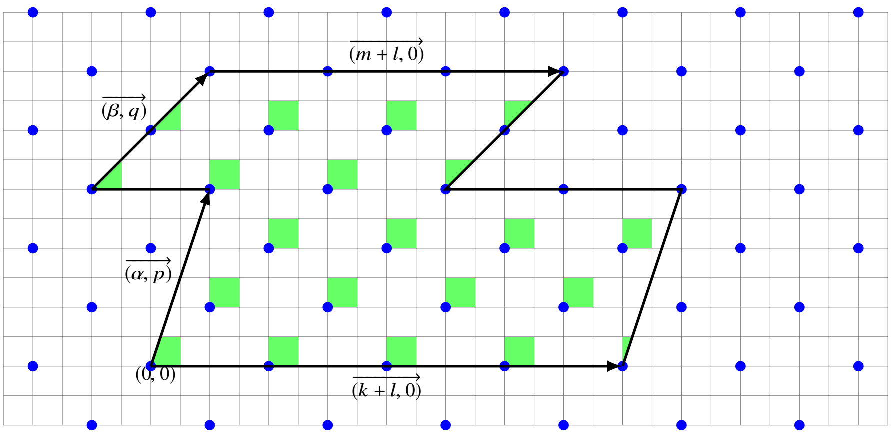


Figure 4.1. The marked points belong to the lattice L generated by the vectors with coordinates (α, p), (β, q), $(k + l, 0)$ and $(m + l, 0)$. The colored squares of the surface form a nontrivial block for the group G.

Take an unfolded representation of S as shown in Figure 4.1, and color the squares in the surface which have their lower left vertex contained in the lattice L. We will show that the set of labels of the colored squares is a block for $G = \langle \sigma, \tau \rangle$. We call this set Δ.

Consider an arbitrary square of the surface with lower left vertex (s, t). Under σ and τ this vertex is sent to:

$$(s + 1, t) - \epsilon_1(l + m, 0) - \epsilon_2(k + l, 0) \tag{4.1}$$

$$(s, t + 1) - \epsilon_3(\alpha + \beta, p + q) - \epsilon_4(\alpha, p) - \epsilon_5(\beta, q) \tag{4.2}$$

with $\epsilon_i \in \{0, 1\}$

Now, consider the action of an element $g \in G = \langle \sigma, \tau \rangle$ on the lower left vertices of the colored squares. Given an element of G note that up to a linear combination of $\epsilon_1(l + m, 0)$, $\epsilon_2(k+l, 0)$, $\epsilon_3(\alpha+\beta, p+q)$, $\epsilon_4(\alpha, p)$, $\epsilon_5(\beta, q)$, all the points in the lattice are translated by the same vector v. But, $\epsilon_1(l + m, 0), \epsilon_2(k + l, 0), \epsilon_3(\alpha + \beta, p + q), \epsilon_4(\alpha, p), \epsilon_5(\beta, q) \in L$.

Hence, either $g(\Delta) = \Delta$ or $g(\Delta) \cap \Delta = \emptyset$. This implies Δ is a block for G.

($\Leftarrow$). Assume now that $p \wedge q = 1$ and $(k+l) \wedge (l+m) \wedge (p\beta - q\alpha) = 1$ By Lemma 4.1 this means that $\langle (\alpha, p), (\beta, q), (k+l, 0), (l+m, 0) \rangle = \mathbb{Z}^2$. But $(\alpha, p), (\beta, q), (k+l, 0), (l+m, 0) \in \mathrm{AbsPer}(s)$. Hence, $\mathrm{AbsPer}(S) = \mathbb{Z}^2$ and S is primitive by Lemma 4.2. □

We next present an alternate parametrization of the primitive n-square surfaces with cylinder diagram A. Even though the previous parametrization, in terms of the lengths of the horizontal saddle connections and heights and shears of the cylinders, is in lieu with the parametrization of the other cylinder diagrams, this new parametrization lends itself better to enumeration. To do this new parametrization, we set up a bijection between the previous parametrization in Proposition 3.4 under the primitivity conditions imposed by Lemma 4.3 and the new set of parameters. We detail this bijection in Appendix D of [17].

Lemma 4.4. *The set of primitive n-square surfaces of $\mathcal{H}(1,1)$ with cylinder diagram A is parametrized uniquely by the set*

$$\Omega := \{(x, y, z, t) \in \mathbb{N}^4 \mid 1 \leq x < y < z < t \leq n, (z - x) \wedge (t - y) \wedge n = 1\}$$

Putting together the various cylinder wise primitivity criterion obtained in Lemma 4.3 we get a converse of Lemma 4.2. Then noting that $\mathrm{AbsPer}(S) = \mathrm{Per}(S)$ for square-tiled surfaces in $\mathcal{H}(2)$, we recover the combined statement, which is stated in [5] as Lemma 4.1.

Proposition 4.5. *A genus two square-tiled surface S is primitive if and only if* $\mathrm{AbsPer}(S) = \mathbb{Z}^2$.

We note that an analogous statement for genus 3 is not true, and we provide an example of a square-tiled surface with $\mathrm{AbsPer}(S) = \mathbb{Z}^2$ but is not primitive, in Figure 4.2.

5. Enumeration of Primitive Square-tiled Surfaces

Now that we have number theoretic conditions ensuring primitivity of the square-tiled surfaces with different cylinder diagrams, we can count primitive square-tiled surfaces. We begin with the following lemma about enumerating integers that are relatively prime to d in a given interval of length d with integer endpoints:

Lemma 5.1. *Let $\beta \in \mathbb{Z}$. For any positive integer d, the number of integers in the interval $[\beta, \beta + d)$ that are relatively prime to d is given by $\phi(d)$.*

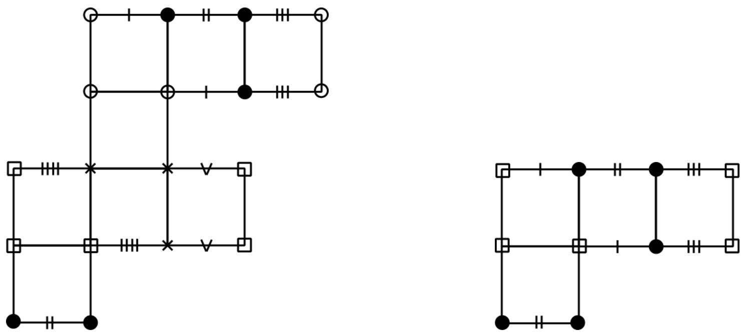


FIGURE 4.2. (i) A genus 3 non-primitive square-tiled surface S with $\mathrm{AbsPer}(S) = \mathbb{Z}^2$. (ii) A genus 2 square-tiled surface that S covers.

Proof. Given an integer in $[\beta, \beta + d)$, note that it is relatively prime to d if and only if its residue class mod d is a unit in $\mathbb{Z}/d\mathbb{Z}$. Since each residue class mod d has one and only representative in $[\beta, \beta + d)$, the number of relatively prime integers to d in $[\beta, \beta + d)$ is the number of units in $\mathbb{Z}/d\mathbb{Z}$, which is $\phi(d)$. □

Next, we prove a generalization of Lemma 8 presented in [19] by Zmiaikou that will be used to count the contribution of the shear parameters to our enumeration.

Lemma 5.2. *Let $p, q, k, l \in \mathbb{N}$ such that $p \wedge q = 1$ and $\beta_1, \beta_2 \in \mathbb{Z}$. The number of distinct pairs $(\alpha, \gamma) \in \mathbb{Z}^2$ such that*

$$\beta_1 \leq \alpha < k + \beta_1, \quad \beta_2 \leq \gamma < l + \beta_2 \quad \text{and} \quad k \wedge l \wedge (p\gamma - q\alpha) = 1$$

is equal to $kl \cdot \frac{\phi(k \wedge l)}{k \wedge l}$.

Proof. Let $d := k \wedge l$. Let (α, γ) satisfy the condition that $d \wedge (p\gamma - q\alpha) = 1$

Assume now that you have $\alpha', \gamma' \in \mathbb{Z}$ such that $d|(\alpha' - \alpha)$ and $d|(\gamma' - \gamma)$. Then,

$$d \wedge (p\gamma' - q\alpha') = d \wedge (p(\gamma' - \gamma) - q(\alpha' - \alpha) + p\gamma - q\alpha) = d \wedge (p\gamma - q\alpha) = 1$$

So, to count the number of solutions (α, γ) in the box $B := [\beta_1, k+\beta_1) \times [\beta_2, l+\beta_2)$, it suffices to count the number of solutions in the size $d \times d$ smaller box $K := [\beta_1, \beta_1 + d) \times [\beta_2, \beta_2 + d)$ then multiply the number of such solutions by $\frac{kl}{d^2}$, the number of copies of K required to tile B, to obtain the total number of solutions in B.

Now, we find the number of distinct pairs $(\alpha, \gamma) \in K$ such that $d \wedge (p\gamma - q\alpha) = 1$.

Since $p \wedge q = 1$, there exists $a, b \in \mathbb{Z}$ such that $A := \left[\begin{smallmatrix} -q & p \\ a & b \end{smallmatrix}\right] \in SL_2(\mathbb{Z})$. Call a point $(x, y) \in \mathbb{Z}^2$ *d-prime* if $x \wedge d = 1$. Then, counting $(\alpha, \gamma) \in K$ such that $d \wedge (p\gamma - q\alpha) = 1$

is equivalent to counting d-prime points in $A(K)$, since

$$A \cdot \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} -q & p \\ a & b \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} p\gamma - q\alpha \\ a\alpha + b\gamma \end{bmatrix} \in A(K)$$

By Lemma 5.1, we know there are $\phi(d)$ integers in $[\beta_1, \beta_1 + d)$ that are relatively prime to d. Hence, there are $d\phi(d)$ d-prime points in K.

Next, we show that the number of d-prime pairs in K and $A(K)$ is the same. So, first we argue $T_1(K)$ has $d\phi(d)$ d-prime pairs where $T_1 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

For each integer $k \in [\beta_2, \beta_2 + d)$ (which are the y-coordinates of integer points in $T_1(K)$) the set of integer points of $T_1(K)$ with y-coordinate k, have x-coordinates in the interval $[\beta_1 + k - \beta_2, \beta_1 + k - \beta_2 + d)$. By Lemma 5.1, each such interval has $\phi(d)$ d-prime integers implying there are $\phi(d)$ d-prime pairs in $T_1(K)$ with y-coordinate k. Ranging over the possible k, we conclude $T_1(K)$ has $d\phi(d)$ d-prime pairs.

Using a similar argument, we conclude that $T_2(K)$ has $d\phi(d)$ d-prime pairs for $T_2 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. As T_1 and T_2 generate $SL_2(\mathbb{Z})$, we see that $M(K)$ for any $M \in SL_2(\mathbb{Z})$ has $d\phi(d)$ d-prime points and in particular, $A(K)$ has $d\phi(d)$ d-prime points. Equivalently, there exists $d\phi(d)$ distinct integer (α, γ) points in K such that $k \wedge l \wedge (p\gamma - q\alpha) = 1$ Hence, there are $\frac{kl}{d^2} d\phi(d) = kl \cdot \frac{\phi(k \wedge l)}{k \wedge l}$ such points in B. $\square$

Notation. Since the function $\frac{\phi(m)}{m}$ appears frequently in our computations, we define $\phi'(m) := \frac{\phi(m)}{m}$

5.1. Enumeration for Cylinder Diagram A

In this section we count the number of primitive n-square surfaces in $\mathcal{H}(1,1)$ with cylinder diagram A. For various notation used in this section as well as the subsequent sections regarding enumeration for other cylinder diagrams, we refer the reader to Appendix A.

Proposition 5.3. *The number of primitive n-square surfaces in $\mathcal{H}(1,1)$ with cylinder diagram A is given by*

$$A(n) = \frac{1}{2} n J_1(n) + \left(\frac{1}{24} n^2 - \frac{1}{4} n \right) J_2(n)$$

Before we prove this, we need the following Lemma which counts the number of positive integer quadruples under certain number theoretic conditions.

Lemma 5.4. *Let $n > 3$. For $d|n$, the number of quadruples $(x, y, z, t) \in \mathbb{N}^4$ such that*

$$x < y < z < t \in [1, n] \qquad \textit{and} \qquad d|(z - x) \ \textit{and} \ d|(t - y)$$

is given by

$$\binom{d}{2}\binom{n/d}{2} + 3\binom{d}{2}\binom{n/d}{3} + d^2\binom{n/d}{4}$$

Proof. First we write

$$x = \alpha_1 d + r_1 \qquad y = \alpha_2 d + r_2 \qquad z = \alpha_3 d + r_3 \qquad t = \alpha_4 d + r_4$$

Where $\alpha_i \in \{0, \ldots, \frac{n}{d} - 1\}$ and $r_i \in \{1, \ldots, d\}$. Since $d|z - x$ and $d|t - y$, we get that $r_1 = r_3$ and $r_2 = r_4$. Then as $x < y < z < t \in [1, n]$, there can be 5 different cases:

Cases	α_i choices	r_i choices	Total choices
$\alpha_1 = \alpha_2 < \alpha_3 = \alpha_4; r_1 = r_3 < r_2 = r_4$	$\binom{n/d}{2}$	$\binom{d}{2}$	$\binom{n/d}{2} \cdot \binom{d}{2}$
$\alpha_1 = \alpha_2 < \alpha_3 < \alpha_4; r_1 = r_3 < r_2 = r_4$	$\binom{n/d}{3}$	$\binom{d}{2}$	$\binom{n/d}{3} \cdot \binom{d}{2}$
$\alpha_1 < \alpha_2 = \alpha_3 < \alpha_4; r_1 = r_3 > r_2 = r_4$	$\binom{n/d}{3}$	$\binom{d}{2}$	$\binom{n/d}{3} \cdot \binom{d}{2}$
$\alpha_1 < \alpha_2 < \alpha_3 = \alpha_4; r_1 = r_3 < r_2 = r_4$	$\binom{n/d}{3}$	$\binom{d}{2}$	$\binom{n/d}{3} \cdot \binom{d}{2}$
$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4; r_1 = r_3; r_2 = r_4$	$\binom{n/d}{4}$	d^2	$\binom{n/d}{4} \cdot d^2$

So in total we have

$$\binom{d}{2}\binom{n/d}{2} + 3\binom{d}{2}\binom{n/d}{3} + d^2\binom{n/d}{4}$$

choices for x, y, z, t with the given restrictions. □

We are now ready to give the proof of Proposition 5.3.

Proof of Proposition 5.3. By Lemma 4.4, to count the number of primitive n-square surfaces in $\mathcal{H}(1, 1)$ with cylinder diagram A, it suffices to count integers,

$$x < y < z < t \in [1, n] \qquad \text{such that} \qquad (z - x) \wedge (t - y) \wedge n = 1.$$

To enumerate the number of ways we can pick these numbers, we start with the $\binom{n}{4}$ quadruples $(x, y, z, t) \in \mathbb{N}^4$ such that $x < y < z < t$. We begin by subtracting the quadruples such that $z - x$ and $t - y$ are simultaneously divisible by a prime divisor of n. We then add the number of quadruples such that $z - x$ and $t - y$ are simultaneously divisible by two distinct primes divisors of n. Then we subtract the ones such that $z - x$ and $t - y$ are simultaneously divisible by three distinct prime divisors of n and so on as per the inclusion exclusion principle. Then, $A(n)$ is given by,

$$
\begin{aligned}
&= \#\{(x,y,z,t) \in \mathbb{N}^4 \mid x < y < z < t \in [1,n], n \wedge (z-x) \wedge (t-y) = 1\} \\
&= \binom{n}{4} - \sum_{p \mid n, p \text{ prime}} \#\{(x,y,z,t) \in \mathbb{N}^4 \mid x < y < z < t \in [1,n], n \wedge (z-x) \wedge (t-y) = 1\} \\
&\quad + \sum_{\substack{p_1 \mid n, p_2 \mid n \\ p_1 \neq p_2 \text{ primes}}} \#\{(x,y,z,t) \in \mathbb{N}^4 \mid x < y < z < t \in [1,n], p_1, p_2 \text{ divide } (z-x)\&(t-y)\} \\
&\quad \vdots \\
&= \sum_{d \mid n} \mu(d) \#\{(x,y,z,t) \in \mathbb{N}^4 \mid x < y < z < t \in [1,n], d \text{ divides } (z-x)\&(t-y)\}
\end{aligned}
$$

By Lemma 5.4 this sum is equal to

$$
\begin{aligned}
&\sum_{d \mid n} \mu(d) \left(\binom{d}{2}\binom{n/d}{2} + 3\binom{d}{2}\binom{n/d}{3} + d^2\binom{n/d}{4} \right) \\
&\qquad = \left(-\frac{1}{24}n^2 - \frac{1}{4}n\right) \sum_{d \mid n} \mu(d) + \left(\frac{1}{2}n^2\right) \sum_{d \mid n} \mu(d)\frac{1}{d} + \left(\frac{1}{24}n^4 - \frac{1}{4}n^3\right) \sum_{d \mid n} \mu(d)\frac{1}{d^2}
\end{aligned}
$$

Using Propositions A.5 and A.6, the sum simplifies to,

$$
\begin{aligned}
&\left(\frac{1}{2}n\right) n \prod_{p \mid n} \left(1 - p^{-1}\right) + \left(\frac{1}{24}n^2 - \frac{1}{4}n\right) n^2 \prod_{p \mid n} \left(1 - p^{-2}\right) \\
&\qquad\qquad = \frac{1}{2}nJ_1(n) + \left(\frac{1}{24}n^2 - \frac{1}{4}n\right) J_2(n) \qquad \square
\end{aligned}
$$

5.2. **Enumeration for Cylinder Diagram B**

In this section we count the number of primitive n-square surfaces in $\mathcal{H}(1,1)$ with cylinder diagram B.

Proposition 5.5. *The number of primitive n-square surfaces in $\mathcal{H}(1,1)$ with cylinder diagram B is given by,*

$$
B(n) = \big((\mu \cdot \sigma_2) * (\sigma_1 \Delta \sigma_2)\big)(n) - \left(\frac{1}{12}n^2 + \frac{5}{24}n - \frac{3}{4}\right) J_2(n) - \frac{1}{2}nJ_1(n)
$$

We first count the contribution of the shear parameters to our count, in the following lemma:

Lemma 5.6. *Let $p, q, k, l, m \in \mathbb{N}$ and $p \wedge q = 1$. The number of $(\alpha, \beta) \in \mathbb{Z}^2$ such that*

$$0 \leq \alpha < k + l + m, \ \ 0 \leq \beta < m, \quad \text{and} \quad (k + l) \wedge m \wedge (p\beta - q\alpha + (p + q)l) = 1$$

is given by,

$$(k + l + m)m \cdot \phi'((k + l) \wedge m)$$

Proof. Rewrite $p\beta - q\alpha + (p + q)l = p(\beta + l) - q(\alpha - l)$. Set $\beta' = \beta + l$ and $\alpha' = \alpha - l$. Then, we want to find $(\alpha', \beta') \in \mathbb{Z}^2$ such that

$$-l \leq \alpha' < (k + l + m) - l, \ \ l \leq \beta' < m + l, \quad \text{and} \quad (k + l + m) \wedge m \wedge (p\beta' - q\alpha') = 1$$

Applying now, Lemma 5.2 with $\beta_1 = -l$ and $\beta_2 = l$, we get that the number of required (α, β) is $(k + l + m)m \cdot \phi'((k + l + m) \wedge m)$ □

We are now ready the prove Proposition 5.5

Proof of Proposition 5.5. First note that we need to count the parameters stated in Proposition 3.5 under the conditions stated in Lemma 4.3. Since Lemma 5.6 gives the contribution of the shear parameters, we must evaluate the sum

$$\sum_{\substack{p,q,k,l,m\in\mathbb{N} \\ p(k+l+m)+qm=n \\ p\wedge q=1}} (k + l + m)m \cdot \phi'((k + l + m) \wedge m)$$

$$= \sum_{\substack{p,q,k',l,m\in\mathbb{N} \\ pk'+qm=n \\ p\wedge q=1}} \sum_{\substack{k,l\in\mathbb{N} \\ k+l=k'-m}} k'm \cdot \phi'(k' \wedge m) = \sum_{\substack{p,q,k',l,m\in\mathbb{N} \\ k'>m \\ pk'+qm=n \\ p\wedge q=1}} k'm \cdot \phi'(k' \wedge m)(k' - m - 1)$$

after a reparametrization $k' = k + l + m$ which implies $k + l = k' - m$ and that $k' - m > 0$. Then, renaming k' as k,

$$B(n) = \sum_{\substack{p,q,k,m\in\mathbb{N} \\ k>m \\ pk+qm=n \\ p\wedge q=1}} km(k - m) \cdot \phi'(k \wedge m) - \sum_{\substack{p,q,k,m\in\mathbb{N} \\ k>m \\ pk+qm=n \\ p\wedge q=1}} km \cdot \phi'(k \wedge m)$$

$$= \sum_{\substack{p,q,l,m\in\mathbb{N} \\ q>p \\ pl+qm=n \\ p\wedge q=1}} (l + m)ml \cdot \phi'(l \wedge m) - \sum_{\substack{p,q,k,m\in\mathbb{N} \\ k>m \\ pk+qm=n \\ p\wedge q=1}} km \cdot \phi'(k \wedge m)$$

where the final equality uses the reparametrization $k = l + m$ for the first term.

We will simplify the first summation term in detail in Lemma 5.7 and obtain $((\mu \cdot \sigma_2) * \sigma_1 \Delta \sigma_2)(n) - \frac{1}{12} n^2 J_2(n)$. The second term is simplified similarly (proof in Lemma B.2 of [17]) to obtain $\frac{5}{24} n J_2(n) + \frac{1}{2} n J_1(n) - \frac{3}{4} J_2(n)$. Finally, putting the two together,

$$B(n) = \big((\mu \cdot \sigma_2) * (\sigma_1 \Delta \sigma_2)\big)(n) - \left(\frac{1}{12} n^2 + \frac{5}{24} n - \frac{3}{4}\right) J_2(n) - \frac{1}{2} n J_1(n). \qquad \square$$

The first summation term in $B(n)$ in the proof above is simplified in the following Lemma:

Lemma 5.7.

$$X(n) := \sum_{\substack{q,p,k,l \in \mathbb{N} \\ p>q \\ pk+ql=n \\ p \wedge q = 1}} (k+l)kl \cdot \phi'(k \wedge l) = ((\mu \cdot \sigma_2) * \sigma_1 \Delta \sigma_2)(n) - \frac{1}{12} n^2 J_2(n)$$

Proof. Dropping the GCD condition, define,

$$\widetilde{X}(n) = \sum_{\substack{q,p,k,l \in \mathbb{N} \\ p>q \\ pk+ql=n}} (k+l)kl \cdot \phi'(k \wedge l)$$

Then, writing the sum over the divisors of n, we get,

$$\widetilde{X}(n) = \sum_{d|n} \sum_{\substack{q,p,k,l \in \mathbb{N} \\ p>q \\ pk+ql=n \\ p \wedge q = d}} (k+l)kl \cdot \phi'(k \wedge l) = \sum_{d|n} \sum_{\substack{q,p,k,l \in \mathbb{N} \\ p>q \\ pk+ql=n/d \\ p \wedge q = 1}} (k+l)kl \cdot \phi'(k \wedge l) = \sum_{d|n} X\left(\frac{n}{d}\right)$$

By the Möbius inversion formula, see Propoosition A.7 from the Appendix, this gives us, $X \equiv \mu * \widetilde{X}$

Now applying symmetry between p and q, and between k and l, we get,

$$\begin{aligned}
\widetilde{X}(n) &= \sum_{\substack{q,p,k,l \in \mathbb{N} \\ p>q \\ pk+ql=n}} (k+l)kl \cdot \phi'(k \wedge l) \\
&= \frac{1}{2} \sum_{\substack{q,p,k,l \in \mathbb{N} \\ pk+ql=n}} (k+l)kl \cdot \phi'(k \wedge l) - \frac{1}{2} \sum_{\substack{p,k,l \in \mathbb{N} \\ p(k+l)=n}} (k+l)kl \cdot \phi'(k \wedge l) \\
&= \sum_{\substack{q,p,k,l \in \mathbb{N} \\ pk+ql=n}} kl^2 \cdot \phi'(k \wedge l) - \sum_{\substack{p,k,l \in \mathbb{N} \\ p(k+l)=n}} kl^2 \cdot \phi'(k \wedge l)
\end{aligned}$$

Next define

$$\widetilde{X}_1(n) := \sum_{\substack{q,p,k,l\in\mathbb{N}\\ pk+ql=n}} kl^2 \cdot \phi'(k\wedge l) \quad \text{and} \quad \widetilde{X}_2(n) := \sum_{\substack{p,k,l\in\mathbb{N}\\ p(k+l)=n}} kl^2 \cdot \phi'(k\wedge l)$$

Again, summing over divisors and simplifying,

$$\widetilde{X}_1(n) = \sum_{d|n} \frac{\phi(d)}{d} d^3 \sum_{\substack{q,p,k,l\in\mathbb{N}\\ pk+ql=n/d\\ k\wedge l=1}} kl^2 = \sum_{d|n} d^2\phi(d) X_{11}\left(\frac{n}{d}\right) \quad \text{with} \quad X_{11}(n) := \sum_{\substack{p,q,k,l,\in\mathbb{N}\\ pk+ql=n\\ k\wedge l=1}} kl^2,$$

so that $\widetilde{X}_1 \equiv \mathrm{Id}_2 \cdot\phi * X_{11}$. Similarly, considering the sum X_{11} without the gcd condition on k and l

$$\widetilde{X}_{11}(n) := \sum_{\substack{p,q,k,l\in\mathbb{N}\\ pk+ql=n}} kl^2 = \sum_{d|n} d^3 X_{11}\left(\frac{n}{d}\right) \quad \text{so that} \quad \widetilde{X}_{11} \equiv \mathrm{Id}_3 * X_{11} \implies X_{11} \equiv \mathrm{Id}_3 \cdot\mu * \widetilde{X}_{11}$$

by Proposition A.8, and $\widetilde{X}_1 \equiv \mathrm{Id}_2 \cdot\phi * \mathrm{Id}_3 \cdot\mu * \widetilde{X}_{11}$ with

$$\widetilde{X}_{11}(n) = \sum_{\substack{p,q,k,l\in\mathbb{N}\\ pk+ql=n}} kl^2 = \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n}} \sum_{k|\alpha} k \sum_{l|\beta} l^2 = \sum_{\alpha=1}^{n-1} \sigma_1(\alpha)\sigma_2(n-\alpha) = (\sigma_1\Delta\sigma_2)(n)$$

Exactly analogously, we get $\widetilde{X}_2 \equiv \mathrm{Id}_2 \cdot\phi * \mathrm{Id}_3 \cdot\mu * \widetilde{X}_{21}$ where

$$\widetilde{X}_{21}(n) := \sum_{\substack{p,k,l\in\mathbb{N}\\ p(k+l)=n}} kl^2 = \sum_{\alpha|n} \sum_{l=1}^{\alpha-1} l^2(\alpha-l) = \sum_{\alpha|n} \left(\frac{\alpha^4}{12} - \frac{\alpha^2}{12}\right) = \left(\frac{1}{12}\sigma_4 - \frac{1}{12}\sigma_2\right)(n)$$

Hence,

$$X \equiv \mu * \widetilde{X} \equiv \mu * (\widetilde{X}_1 - \widetilde{X}_2) \equiv \mu * \mathrm{Id}_2 \cdot\phi * \mathrm{Id}_3 \cdot\mu * (\widetilde{X}_{11} - \widetilde{X}_{21})$$

Replacing $\widetilde{X}_{11}$ and $\widetilde{X}_{21}$ and using Proposition A.9 parts (iii), (i), (ii),

$$\begin{aligned} X &\equiv \mu * \mathrm{Id}_2 \cdot(\phi * \mathrm{Id}_1 \cdot\mu) * (\sigma_1\Delta\sigma_2 - \frac{1}{12}\sigma_4 + \frac{1}{12}\sigma_2) \\ &\equiv \mathrm{Id}_2 \cdot\mu * (\mu * (\sigma_1\Delta\sigma_2) - \frac{1}{12}\mathrm{Id}_4 + \frac{1}{12}\mathrm{Id}_2) \\ &\equiv \mathrm{Id}_2 \cdot\mu * \mu * (\sigma_1\Delta\sigma_2) - \frac{1}{12}\mathrm{Id}_2 \cdot(\mu * \mathrm{Id}_2) + \frac{1}{12}\mathrm{Id}_2 \cdot(\mu * 1) \\ &\equiv (\mu\cdot\sigma_2) * (\sigma_1\Delta\sigma_2) - \frac{1}{12}\mathrm{Id}_2 \cdot J_2 \end{aligned}$$

□

5.3. **Enumeration for Cylinder Diagram C**

In this section we count the number of primitive n-square surfaces in $\mathcal{H}(1,1)$ with cylinder diagram C.

Proposition 5.8. *The number of primitive n-square surfaces in* $\mathcal{H}(1,1)$ *with cylinder diagram C is given by,*

$$C(n) = \frac{1}{24}(n-2)(n-3)J_2(n)$$

Proof. We first note that we need to count the parameters stated in Proposition 3.6 under the conditions stated in Lemma 4.3. Hence, using Lemma 5.2 with k and l as $k+l$ and $l+m$ and $\beta_i = 0$, we obtain the contribution of the shear parameters. Then to count the number of primitive n-square surfaces in $\mathcal{H}(1,1)$ with cylinder diagram C, we must evaluate the sum:

$$\sum_{\substack{p,q,k,l,m\in\mathbb{N}\\ k<m\\ p(k+l)+q(l+m)=n\\ p\wedge q=1}} (k+l)(l+m)\phi'((k+l)\wedge(l+m)) + \sum_{\substack{p,q,k,l\in\mathbb{N}\\ p<q\\ (p+q)(k+l)=n\\ p\wedge q=1}} (k+l)\phi(k+l) + \frac{1}{2}\sum_{\substack{k,l\in\mathbb{N}\\ 2(k+l)=n}} (k+l)\phi(k+l)$$

Let $C_1(n), C_2(n), C_3(n)$ be the first, second and third summation terms in the above expression. We start with $C_1(n)$. Using symmetry between k and m, we have that

$$\begin{aligned} &C_1(n)\\ &:= \sum_{\substack{p,q,k,l,m\in\mathbb{N}\\ k<m\\ p(k+l)+q(l+m)=n\\ p\wedge q=1}} (k+l)(l+m)\phi'((k+l)\wedge(l+m))\\ &= \frac{1}{2}\sum_{\substack{p,q,k,l,m\in\mathbb{N}\\ p(k+l)+q(l+m)=n\\ p\wedge q=1}} (k+l)(l+m)\phi'((k+l)\wedge(l+m)) - \frac{1}{2}\sum_{\substack{p,q,k,l\in\mathbb{N}\\ (p+q)(l+k)=n\\ p\wedge q=1}} (k+l)\phi(k+l) \end{aligned}$$

Next, due to symmetry between p and q,

$$C_2(n) + C_3(n) = \sum_{\substack{p,q,k,l\in\mathbb{N} \\ p<q \\ (p+q)(k+l)=n \\ p\wedge q=1}} (k+l)\phi(k+l) + \frac{1}{2}\sum_{\substack{k,l\in\mathbb{N} \\ 2(k+l)=n}} (k+l)\phi(k+l)$$
$$= \frac{1}{2}\sum_{\substack{p,q,k,l\in\mathbb{N} \\ (p+q)(l+k)=n \\ p\wedge q=1}} (k+l)\phi(k+l)$$

Hence,

$$C(n) = C_1(n) + C_2(n) + C_3(n) = \frac{1}{2}\sum_{\substack{p,q,k,l,m\in\mathbb{N} \\ p(k+l)+q(l+m)=n \\ p\wedge q=1}} (k+l)(l+m)\phi'((k+l)\wedge(l+m))$$

Now, we write $C \equiv \frac{1}{2}(\mu * \widetilde{C})$ where

$$\widetilde{C}(n)$$
$$:= \sum_{\substack{p,q,k,l,m\in\mathbb{N} \\ p(k+l)+q(l+m)=n}} (k+l)(l+m)\phi'((k+l)\wedge(l+m))$$
$$= \sum_{\substack{p,q,k',m'\in\mathbb{N} \\ k'>m' \\ pk'+qm'=n}} \sum_{l=1}^{m'-1} k'm'\phi'(k'\wedge m') + \sum_{\substack{p,q,k',m'\in\mathbb{N} \\ k'<m' \\ pk'+qm'=n}} \sum_{l=1}^{k'-1} k'm'\phi'(k'\wedge m') + \sum_{\substack{p,q,k'\in\mathbb{N} \\ (p+q)k'=n}} \sum_{l=1}^{k'-1} k'\phi(k')$$
$$= \sum_{\substack{p,q,k,m\in\mathbb{N} \\ k>m \\ pk+qm=n}} km(m-1)\phi'(k\wedge m) + \sum_{\substack{p,q,k,m\in\mathbb{N} \\ k<m \\ pk+qm=n}} km(k-1)\phi'(k\wedge m) + \sum_{\substack{p,q,k\in\mathbb{N} \\ (p+q)k=n}} k(k-1)\phi(k)$$

where the second equality follows from a reparametrization of $k' = k + l$ and $m' = m + l$ and in the third equality we rename k' as k and m' as m. Due to symmetry between p and q the first two summation terms are equal, so we continue simplifying to get,

$$\widetilde{C}(n)$$
$$= 2\sum_{\substack{p,q,k,m\in\mathbb{N} \\ k>m \\ pk+qm=n}} km^2\cdot\phi'(k\wedge m) - 2\sum_{\substack{p,q,k,m\in\mathbb{N} \\ k>m \\ pk+qm=n}} km\cdot\phi'(k\wedge m) + \sum_{\substack{p,q,k\in\mathbb{N} \\ (p+q)k=n}} (k^2-k)\phi(k)$$

$$
\begin{aligned}
&= 2 \sum_{\substack{p,q,k,m\in\mathbb{N}\\ k>m\\ pk+qm=n}} km^2 \cdot \phi'(k\wedge m) - 2\left(\frac{1}{2}\sum_{\substack{p,q,k,m\in\mathbb{N}\\ pk+qm=n}} km\cdot\phi'(k\wedge m) - \frac{1}{2}\sum_{\substack{p,q,k\in\mathbb{N}\\ (p+q)k=n}} k\phi(k)\right)\\
&\quad + \sum_{\substack{p,q,k\in\mathbb{N}\\ (p+q)k=n}} k^2\phi(k) - \sum_{\substack{p,q,k\in\mathbb{N}\\ (p+q)k=n}} k\phi(k)\\
&= 2 \sum_{\substack{p,q,k,m\in\mathbb{N}\\ k>m\\ pk+qm=n}} km^2 \cdot \phi'(k\wedge m) + \sum_{\substack{p,q,k\in\mathbb{N}\\ (p+q)k=n}} k^2\phi(k) - \sum_{\substack{p,q,k,m\in\mathbb{N}\\ pk+qm=n}} km\cdot\phi'(k\wedge m) \qquad (5.1)
\end{aligned}
$$

Here, the second equality follows from symmetry between k and m in the second term of the first equality.

The first summation term in (5.1) is simplified in Lemma 5.9 to

$$
\left((\mathrm{Id}_2\cdot\mu) * \left(\frac{1}{24}\sigma_4 + \frac{1}{2}\sigma_3 - \frac{1}{24}(12\,\mathrm{Id}_1+1)\sigma_2\right)\right)(n).
$$

The second summation term simplifies as:

$$
\begin{aligned}
\sum_{\substack{p,q,k\in\mathbb{N}\\ (p+q)k=n}} k^2\phi(k) &= \sum_{k|n} k^2\phi(k)\left(\frac{n}{k}-1\right) = \sum_{k|n} k^2\phi(k)\frac{n}{k} - \sum_{k|n} k^2\phi(k)\\
&= (\mathrm{Id}_2\cdot\phi * \mathrm{Id}_1 - \mathrm{Id}_2\cdot\phi * 1)(n)
\end{aligned}
$$

The third summation term similarly simplifies to $\big((\mathrm{Id}_1\cdot\mu) * \big(\frac{5}{12}\sigma_3 + \frac{1}{12}\sigma_1 - \frac{1}{2}\,\mathrm{Id}_1\,\sigma_1\big)\big)(n)$ and the proof of this simplification is in Lemma B.4 of [17]. We omit the proof here since it is similar to Lemma 5.7. Putting all of the terms together and simplifying using the identities in Proposition A.9 (i),

$$
\begin{aligned}
C &\equiv \frac{1}{2}(\mu * \widetilde{C})\\
&\equiv \frac{1}{2}\mu * \left(2(\mathrm{Id}_2\cdot\mu) * \left(\frac{1}{24}\sigma_4 + \frac{1}{2}\sigma_3 - \frac{1}{24}(12\,\mathrm{Id}_1+1)\sigma_2\right) + \mathrm{Id}_2\cdot\phi * \mathrm{Id}_1 - \mathrm{Id}_2\cdot\phi * 1\right.\\
&\quad \left. - (\mathrm{Id}_1\cdot\mu) * \left(\frac{5}{12}\sigma_3 + \frac{1}{12}\sigma_1 - \frac{1}{2}\,\mathrm{Id}_1\,\sigma_1\right)\right)\\
&\equiv (\mathrm{Id}_2\cdot\mu) * \left(\frac{1}{24}\,\mathrm{Id}_4 + \frac{1}{2}\,\mathrm{Id}_3 - \frac{1}{2}(\mathrm{Id}_1\cdot\sigma_2 * \mu) - \frac{1}{24}\,\mathrm{Id}_2\right)\\
&\quad + \frac{1}{2}\left(\mu * \mathrm{Id}_2\cdot\phi * \mathrm{Id}_1 - \mu * \mathrm{Id}_2\cdot\phi * 1\right) - (\mathrm{Id}_1\cdot\mu) * \left(\frac{5}{24}\,\mathrm{Id}_3 + \frac{1}{24}\,\mathrm{Id}_1 - \frac{1}{4}(\mathrm{Id}_1\cdot\sigma_1 * \mu)\right)
\end{aligned}
$$

Finally, using Proposition A.9 again to notice that

$$(\mathrm{Id}_2 \cdot \mu) * (\mathrm{Id}_2 \cdot \sigma_2) * \mu \equiv \sigma_2 \cdot \phi; \qquad \mu * (\mathrm{Id}_2 \cdot J_1) * \mathrm{Id}_1 \equiv \sigma_2 \cdot J_2;$$
$$\text{and} \qquad (\mathrm{Id}_1 \cdot \mu) * (\mathrm{Id}_1 \cdot \sigma_1) * \mu \equiv J_2$$

we obtain, $C(n) = \frac{1}{24}(n-2)(n-3)J_2(n)$. □

The simplification of the first summation term in (5.1) can be proved in the following way, using the technique of Dirichlet series:

Lemma 5.9.

$$U(n) := \sum_{\substack{p,q,k,l\in\mathbb{N}\\ k>l\\ pk+ql=n}} kl^2 \cdot \phi'(k\wedge l) = \left((\mathrm{Id}_2 \cdot \mu) * \left(\frac{1}{24}\sigma_4 + \frac{1}{2}\sigma_3 - \frac{1}{24}(12\,\mathrm{Id}_1 + 1)\sigma_2\right)\right)(n)$$

Proof. We first rewrite the sum over the divisors of n, as in Lemma 5.7, and obtain,

$$U(n) := \sum_{\substack{p,q,k,l\in\mathbb{N}\\ k>l\\ pk+ql=n}} kl^2 \cdot \phi'(k\wedge l) = \sum_{d|n} d^2\phi(d)U_1\left(\frac{n}{d}\right) \quad \text{where} \quad U_1(n) := \sum_{\substack{p,q,k,l\in\mathbb{N}\\ k>l\\ pk+ql=n\\ k\wedge l=1}} kl^2$$

Then let

$$\widetilde{U}_1(n) := \sum_{\substack{p,q,k,l\in\mathbb{N}\\ k>l\\ pk+ql=n}} kl^2 \quad \text{so that} \quad \widetilde{U}_1 = \mathrm{Id}_3 * U_1 \implies U_1 = \mathrm{Id}_3 \cdot \mu * \widetilde{U}_1$$

To find $\widetilde{U}_1$, consider first the series,

$$R(s) := \sum_{p,q,k,l\in\mathbb{N}} \frac{kl^2}{(pk+ql)^s} \tag{5.2}$$

for s large enough that the series converges. We rewrite the series by breaking it into parts where $k = l > 0$, $k > l$ and $l > k$ to get,

$$R(s) = \sum_{k,p,q\in\mathbb{N}} \frac{k^3}{((p+q)k)^s} + \sum_{p,q,k,l\in\mathbb{N}} \frac{(k+l)l^2}{(p(k+l)+ql)^s} + \sum_{k,l,p,q\in\mathbb{N}} \frac{k(k+l)^2}{(pk+q(k+l))^s} \tag{5.3}$$

We also break up (5.2) in another way by when $p = q$, $p > q$ and $p < q$ to get,

$$R(s) = \sum_{k,l,p\in\mathbb{N}} \frac{kl^2}{(p(k+l))^s} + \sum_{k,l,p,q\in\mathbb{N}} \frac{kl^2}{((p+q)k+ql)^s} + \sum_{p,q,k,l\in\mathbb{N}} \frac{kl^s}{(pk+(p+q)l)^s} \tag{5.4}$$

Next, consider the expansion,

$$\sum_{k,l,p,q\in\mathbb{N}} \frac{k(k+l)^2}{(pk+q(k+l)^s} = \sum_{k,l,p,q\in\mathbb{N}} \frac{k^3}{(pk+q(k+l))^s} + 2\sum_{k,l,p,q\in\mathbb{N}} \frac{k^2 l}{(pk+q(k+l))^s} + \sum_{k,l,p,q\in\mathbb{N}} \frac{kl^2}{(pk+q(k+l))^s}$$

and note that

$$\sum_{k,l,p,q\in\mathbb{N}} \frac{k^2 l}{(pk+q(k+l))^s} = \sum_{p,q,k,l\in\mathbb{N}} \frac{kl^2}{(pk+(p+q)l)^s}$$

Hence, equating (5.3) and (5.4) and simplifying, we obtain,

$$\sum_{p,q,k,l\in\mathbb{N}} \frac{(k+l)l^2}{(p(k+l)+ql)^s} = \sum_{k,l,p\in\mathbb{N}} \frac{kl^2}{(p(k+l))^s} - \sum_{k,p,q\in\mathbb{N}} \frac{k^3}{((p+q)k)^s} - \sum_{k,l,p,q\in\mathbb{N}} \frac{k^3}{(pk+q(k+l))^s} - \sum_{k,l,p,q\in\mathbb{N}} \frac{k^2 l}{(pk+q(k+l))^s} \tag{5.5}$$

Next, note that,

$$\sum_{k,l,p,q\in\mathbb{N}} \frac{k^3}{(pk+q(k+l))^s} + \sum_{k,l,p,q\in\mathbb{N}} \frac{k^2 l}{(pk+q(k+l))^s} = \sum_{p,q,k,l\in\mathbb{N}} \frac{(k+l)l^2}{(p(k+l)+ql)^s}$$

so that (5.5) becomes,

$$2\sum_{p,q,k,l\in\mathbb{N}} \frac{(k+l)l^2}{(p(k+l)+ql)^s} = \sum_{k,l,p\in\mathbb{N}} \frac{kl^2}{(p(k+l))^s} - \sum_{k,p,q\in\mathbb{N}} \frac{k^3}{((p+q)k)^s} \tag{5.6}$$

We simplify the first term on the right hand side as,

$$\sum_{k,l,p\in\mathbb{N}} \frac{kl^2}{(p(k+l))^s} = \sum_{n>0} \frac{1}{n^s} \sum_{d|n} \sum_{k=1}^{d} k^2(d-k) = \sum_{n>0} \frac{1}{n^s} \sum_{d|n} \left(\frac{d^4}{12} - \frac{d^2}{12}\right) = \sum_{n} \frac{\sigma_4(n)-\sigma_2(n)}{12n^s}$$

Then the second term as,

$$\sum_{k,p,q\in\mathbb{N}} \frac{k^3}{((p+q)k)^s} = \sum_{n>0} \frac{1}{n^s} \sum_{d|n} d^3\left(\frac{n}{d}-1\right) = \sum_{n>0} \frac{n\sigma_2(n)-\sigma_3(n)}{n^s}$$

So, (5.6) then becomes,

$$\sum_{p,q,k,l\in\mathbb{N}} \frac{(k+l)l^2}{(p(k+l)+ql)^s} = \sum_{n>0}\frac{1}{n^s} \sum_{\substack{p,q,k,l\in\mathbb{N}\\ p(k+l)+ql=n}} (k+l)l^2$$
$$= \frac{1}{2}\sum_{n} \frac{\sigma_4(n)-\sigma_2(n)}{12n^s} - \frac{1}{2}\sum_{n>0}\frac{n\sigma_2(n)-\sigma_3(n)}{n^s}$$

By uniqueness of Dirichlet series (Proposition A.12), we see that

$$\widetilde{U}_1(n) = \sum_{\substack{p,q,k,l\in\mathbb{N}\\ k>l\\ pk+ql=n}} kl^2 = \sum_{\substack{p,q,k,l\in\mathbb{N}\\ p(k+l)+ql=n}} (k+l)l^2 = \frac{1}{24}\sigma_4(n) + \frac{1}{2}\sigma_3(n) - \frac{1}{24}(12n+1)\sigma_2(n)$$

Hence, using Proposition A.9 (iii),

$$U \equiv \mathrm{Id}_2\cdot\phi * U_1 \equiv (\mathrm{Id}_2\cdot\phi) * (\mathrm{Id}_3\cdot\mu) * \left(\frac{1}{24}\sigma_4 + \frac{1}{2}\sigma_3 - \frac{1}{24}(12\,\mathrm{Id}_1+1)\sigma_2\right)$$
$$\equiv \mathrm{Id}_2(\phi * \mathrm{Id}_1\cdot\mu) * \left(\frac{1}{24}\sigma_4 + \frac{1}{2}\sigma_3 - \frac{1}{24}(12\,\mathrm{Id}_1+1)\sigma_2\right)$$
$$\equiv (\mathrm{Id}_2\cdot\mu) * \left(\frac{1}{24}\sigma_4 + \frac{1}{2}\sigma_3 - \frac{1}{24}(12\,\mathrm{Id}_1+1)\sigma_2\right) \qquad \square$$

5.4. **Enumeration for Cylinder Diagram D**

In this section we count the number of primitive n-square surfaces in $\mathcal{H}(1,1)$ with cylinder diagram D.

Proposition 5.10. *The number of primitive n-square surfaces in $\mathcal{H}(1,1)$ with cylinder diagram D is given by,*

$$D(n) = \left(\frac{1}{6}n^2 - \frac{1}{6}n\right) J_2(n) - \big((\mu\cdot\sigma_2) * (\sigma_1\Delta\sigma_2)\big)(n)$$

We first count the contribution of the shear parameters to our count, in the following lemma:

Lemma 5.11. *Let $p,q,r,k,l \in \mathbb{N}$ such that $(p+q)\wedge(r+q) = 1$. The number of $(\alpha,\beta,\gamma)\in\mathbb{Z}^3$ such that*

$$0\le\alpha<k, 0\le\beta<k+l, 0\le\gamma<l \quad \textit{and} \quad k\wedge l\wedge((p+q)(\beta+\gamma)-(r+q)(\alpha+\beta)) = 1$$

is given by,

$$(k+l)kl\cdot\phi'(k\wedge l)$$

Proof. Applying Lemma 5.2, given $p, q, r, k, l \in \mathbb{N}$ such that $(p+q) \wedge (r+q) = 1$ and a fixed $0 \le \beta < k + l$, the number of (α, γ) pairs satisfying

$$0 \le \alpha < k, 0 \le \gamma < l \quad \text{and} \quad k \wedge l \wedge ((p+q)(\beta+\gamma) - (r+q)(\alpha+\beta)) = 1$$

is given by $kl \cdot \phi'(k \wedge l)$. Then, since there are $k + l$ possible β values, this implies the Lemma. □

We are now ready to prove Proposition 5.10

Proof of Proposition 5.10. First note that we need to count the parameters from Proposition 3.7 under the conditions imposed by Lemma 4.3. Since Lemma 5.11 counts the contribution of the shear parameters, we must evaluate the sum

$$D(n) = \sum_{\substack{p,q,r,k,l \in \mathbb{N} \\ k<l, \\ (p+q)k+(r+q)l=n \\ (p+q)\wedge(q+r)=1}} (k+l)kl \cdot \phi'(k \wedge l) + \sum_{\substack{p,q,r,k \in \mathbb{N} \\ p<r \\ (p+r+2q)k=n \\ (p+q)\wedge(q+r)=1}} 2k^2 \phi(k)$$

Let the first term

$$\begin{aligned} D_1(n) &:= \sum_{\substack{p,q,r,k,l \in \mathbb{N} \\ k<l, \\ (p+q)k+(r+q)l=n \\ (p+q)\wedge(q+r)=1}} (k+l)kl \cdot \phi'(k \wedge l) \\ &= \frac{1}{2} \sum_{\substack{p,q,r,k,l \in \mathbb{N} \\ (p+q)k+(r+q)l=n \\ (p+q)\wedge(q+r)=1}} (k+l)kl \cdot \phi'(k \wedge l) - \frac{1}{2} \sum_{\substack{p,q,r,k \in \mathbb{N} \\ (p+r+2q)k=n \\ (p+q)\wedge(q+r)=1}} 2k^2 \phi(k) \end{aligned}$$

by symmetry between k and l. Similarly, the symmetry between p and r in the second term gives,

$$\begin{aligned} D_2(n) = \sum_{\substack{p,q,r,k \in \mathbb{N} \\ p<r \\ (p+r+2q)k=n \\ (p+q)\wedge(q+r)=1}} 2k^2 \phi(k) &= \frac{1}{2} \sum_{\substack{p,q,r,k \in \mathbb{N} \\ (p+r+2q)k=n \\ (p+q)\wedge(q+r)=1}} 2k^2 \phi(k) - \frac{1}{2} \sum_{\substack{p,q,r,k \in \mathbb{N} \\ p=r \\ (p+r+2q)k=n \\ (p+q)\wedge(q+r)=1}} 2k^2 \phi(k) \\ &= \sum_{\substack{p,q,r,k \in \mathbb{N} \\ (p+r+2q)k=n \\ (p+q)\wedge(q+r)=1}} k^2 \phi(k) \end{aligned}$$

where the $p = r$ summand vanishes since in this case there do not exist any q satisfying $(p+q) \wedge (q+r) = 1$.

Together,

$$D(n) = D_1(n)+D_2(n) = \frac{1}{2} \sum_{\substack{p,q,r,k,l\in\mathbb{N}\\(p+q)k+(r+q)l=n\\(p+q)\wedge(q+r)=1}} (k+l)kl\cdot\phi'(k\wedge l) = \frac{1}{2} \sum_{\substack{p,q,r,k,l\in\mathbb{N}\\p>q,r>q\\pk+rl=n\\p\wedge r=1}} (k+l)kl\cdot\phi'(k\wedge l)$$

Rewriting the sum, $D(n)$ becomes,

$$\begin{aligned}
D(n) &= \frac{1}{2} \sum_{\substack{q,r,k,l\in\mathbb{N}\\r>q\\qk+rl=n\\q\wedge r=1}} (k+l)kl\cdot\phi'(k\wedge l)(q-1) + \frac{1}{2} \sum_{\substack{q,p,k,l\in\mathbb{N}\\p>q\\pk+ql=n\\q\wedge p=1}} (k+l)kl\cdot\phi'(k\wedge l)(q-1)\\
&= \frac{1}{2}\cdot 2 \sum_{\substack{q,p,k,l\in\mathbb{N}\\p>q\\pk+ql=n\\q\wedge p=1}} (k+l)kl\cdot\phi'(k\wedge l)(q-1)\\
&= \sum_{\substack{q,p,k,l\in\mathbb{N}\\p>q\\pk+ql=n\\q\wedge p=1}} (k+l)kl\cdot\phi'(k\wedge l)q - \sum_{\substack{q,p,k,l\in\mathbb{N}\\p>q\\pk+ql=n\\q\wedge p=1}} (k+l)kl\cdot\phi'(k\wedge l)
\end{aligned}$$

The first term simplifies to $\left(\frac{1}{12}n^2 - \frac{1}{6}n\right)J_2(n)$. For the proof of this simplification we refer the reader to Lemma B.5 of [17]. The simplication uses the technique of Dirichlet series, very similar to Lemma 5.9. The second summation term has already been simplified in Lemma 5.7 to obtain,

$$D(n) = \left(\frac{1}{6}n^2 - \frac{1}{6}n\right)J_2(n) - \big((\mu\cdot\sigma_2) * (\sigma_1\Delta\sigma_2)\big)(n) \qquad \square$$

5.5. **The total primitive count in $\mathcal{H}(1,1)$**

We can now put together the counts for different cylinder diagrams and get the total count of primitive n-square surfaces in $\mathcal{H}(1,1)$ as $E(n) := A(n) + B(n) + C(n) + D(n)$ to obtain the following theorem.

Theorem 5.12. *The number of primitive n-square surfaces in $\mathcal{H}(1,1)$ is given by,*

$$E(n) := \frac{1}{6}(n-2)(n-3)J_2(n)$$

6. **Proportion of the different cylinder diagrams**

In this section we look at the proportion of surfaces with the different cylinder diagrams as $n \to \infty$.

We will need the following result of Ingham [9] (see also [13] for a simpler proof and second order terms) to get the first order asymptotic for $\sigma_2 \Delta \sigma_1$.

Theorem 6.1. *For any complex numbers x and y with positive real parts,*

$$(\sigma_x \Delta \sigma_y)(n) = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)} \frac{\zeta(x+1)\zeta(y+1)}{\zeta(x+y+2)} \sigma_{x+y+1}(n) + O(n^{x+y+\alpha})$$

where

$$\alpha = \begin{cases} 0 & \text{if } \Re(x) > 1, \Re(y) > 1 \\ 1 - x + \epsilon & \text{if } \Re(x) \le 1, \Re(y) \ge 1 \\ 1 - y + \epsilon & \text{if } \Re(x) \ge 1, \Re(y) \le 1 \\ 1 - \frac{xy}{x+y-xy} + \epsilon & \text{if } \Re(x) < 1, \Re(y) < 1 \end{cases}$$

Γ is the Gamma function, and ζ is the Riemann-zeta function.

Then, we finish the proof of the main result with the following theorem.

Theorem 6.2. *The asymptotic densities of the various cylinder diagrams is given by the following limits.*

(1) $\lim_{n\to\infty} \frac{A(n)}{E(n)} = \frac{1}{4}$

(2) $\lim_{n\to\infty} \frac{B(n)}{E(n)} = \frac{\zeta(2)\zeta(3)}{2\zeta(5)} - \frac{1}{2} \approx 0.453$

(3) $\lim_{n\to\infty} \frac{C(n)}{E(n)} = \frac{1}{4}$

(4) $\lim_{n\to\infty} \frac{D(n)}{E(n)} = 1 - \frac{\zeta(2)\zeta(3)}{2\zeta(5)} \approx 0.047$

Proof. (1). We have,

$$\lim_{n\to\infty} \frac{A(n)}{E(n)} = \lim_{n\to\infty} \frac{\frac{1}{2} n J_1(n) + (\frac{1}{24} n^2 - \frac{1}{4} n) J_2(n)}{\frac{1}{6}(n^2 - 5n + 6) J_2(n)} = \lim_{n\to\infty} \left(\frac{\frac{J_1(n)}{2nJ_2(n)}}{\frac{1}{6} - \frac{5}{6n} + \frac{1}{n^2}} + \frac{\frac{1}{24} - \frac{1}{4n}}{\frac{1}{6} - \frac{5}{6n} + \frac{1}{n^2}} \right)$$

But

$$\frac{J_1(n)}{nJ_2(n)} = \frac{n \prod_{p|n} (1 - p^{-1})}{n^3 \prod_{p|n} (1 - p^{-2})} < \frac{1}{n^2} \to 0$$

as $n \to \infty$.

Hence,

$$\lim_{n\to\infty}\frac{A(n)}{E(n)}=\lim_{n\to\infty}\left(\frac{\frac{J_1(n)}{2nJ_2(n)}}{\frac{1}{6}-\frac{5}{6n}+\frac{1}{n^2}}+\frac{\frac{1}{24}-\frac{1}{4n}}{\frac{1}{6}-\frac{5}{6n}+\frac{1}{n^2}}\right)=\frac{1}{4}.$$

(2). Note that, by Theorem 6.1, we have,

$$(\sigma_1\Delta\sigma_2)(n)=\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)}\frac{\zeta(2)\zeta(3)}{\zeta(5)}\sigma_4(n)+O(n^{3+\epsilon})=\frac{\zeta(2)\zeta(3)}{12\zeta(5)}\sigma_4(n)+O(n^{3+\epsilon})$$

Hence,

$$\begin{aligned}((\mathrm{Id}_2\cdot\mu)*\mu*(\sigma_1\Delta\sigma_2))(n)&=\frac{\zeta(2)\zeta(3)}{12\zeta(5)}((\mathrm{Id}_2\cdot\mu)*\mu*\sigma_4)(n)+((\mathrm{Id}_2\cdot\mu)*\mu*f)(n)\\&=\frac{\zeta(2)\zeta(3)}{12\zeta(5)}(\mathrm{Id}_2\cdot J_2)(n)+((\sigma_2\cdot\mu)*f)(n)\end{aligned}$$

where $f\in O(n^{3+\epsilon})$. Since $\sigma_2\cdot\mu\in O(n^3)$, we have that, for large enough n,

$$|(\sigma_2\cdot\mu*f)(n)|\le\sum_{d|n}|(\sigma_2\cdot\mu)(d)f(n/d)|\le K\sum_{d|n}d^3\frac{n^{3+\epsilon}}{d^{3+\epsilon}}=Kn^{3+\epsilon}\sum_{d|n}d^{-\epsilon}$$

for some constant $K>0$ and $0<\epsilon<1$. However, $\sum_{d|n}d^{-\epsilon}\le\sum_{d|n}1=\sigma_0(n)$, and noting that $\sigma_0(n)\in O(n^{\epsilon_2})$ for any $\epsilon_2>0$, we obtain

$$|(\sigma_2\cdot\mu*f)(n)|\le Mn^{3+\delta}\quad\text{for some } M>0 \text{ and } 0<\delta<1 \text{ and } n \text{ large enough.}$$

Moreover,

$$n^4\frac{6}{\pi^2}=n^4\prod_{p\text{ prime}}(1-p^{-2})<n^4\prod_{p|n}(1-p^{-2})=n^2J_2(n)$$

so that

$$\left|\frac{((\sigma_2\cdot\mu)*f)(n)}{n^2J_2(n)}\right|\le\frac{Mn^{3+\delta}}{\frac{6}{\pi^2}n^4}\to 0\text{ as }n\to\infty$$

Hence,

$$\lim_{n\to\infty}\frac{(\mathrm{Id}_2\cdot\mu*\mu*\sigma_1\Delta\sigma_2)(n)}{\frac{1}{6}n^2J_2(n)}=\lim_{n\to\infty}\left(\frac{\frac{\zeta(2)\zeta(3)}{12\zeta(5)}n^2J_2(n)}{\frac{1}{6}n^2J_2(n)}+\frac{((\sigma_2\cdot\mu)*f)(n)}{\frac{1}{6}n^2J_2(n)}\right)=\frac{\zeta(2)\zeta(3)}{2\zeta(5)}$$

And finally,

$$\lim_{n\to\infty}\frac{B(n)}{E(n)} = \lim_{n\to\infty}\frac{(\mathrm{Id}_2\cdot\mu*\mu*\sigma_1\Delta\sigma_2)(n)-(\frac{1}{12}n^2+\frac{5}{24}n-\frac{3}{4})J_2(n)-\frac{1}{2}nJ_1(n)}{\frac{1}{6}(n^2-5n+6)J_2(n)}$$
$$= \lim_{n\to\infty}\frac{(\mu\cdot\sigma_2*\sigma_1\Delta\sigma_2)(n)}{\frac{1}{6}n^2J_2(n)}-\frac{1}{2}$$
$$= \frac{\zeta(2)\zeta(3)}{2\zeta(5)}-\frac{1}{2}\approx 0.453$$

(3). We have,

$$\lim_{n\to\infty}\frac{C(n)}{E(n)} = \lim_{n\to\infty}\frac{\frac{1}{24}(n^2-5n+6)J_2(n)}{\frac{1}{6}(n^2-5n+6)J_2(n)} = \frac{1}{4}$$

(4). Since we have already calculated the asymptotic proportions of the other three diagrams, we get,

$$\lim_{n\to\infty}\frac{D(n)}{E(n)} = \lim_{n\to\infty}\frac{E(n)-A(n)-B(n)-C(n)}{E(n)} = 1-\frac{1}{4}-\frac{\zeta(2)\zeta(3)}{2\zeta(5)}+\frac{1}{2}-\frac{1}{4}$$
$$= 1-\frac{\zeta(2)\zeta(3)}{2\zeta(5)}\approx 0.047 \qquad \square$$

This completes the proof of our main result.

We note that analogous versions of Propositions 5.3, 5.5, 5.8, and 5.10 for the unrestricted (not necessarily primitive) cases of STSs in $\mathcal{H}(1,1)$ can be obtained by routinely using similar enumeration techniques on the uniqueness parameters presented in Propositions 3.4, 3.5, 3.6 and 3.7. These can then be used to get analogous versions of Theorems 5.12 and 6.2 as well. However, we have no apriori reason to expect the individual densities of Theorem 6.2 to remain the same if we lift the primitivity restriction, and in fact, brute force experimental observations suggest that they are different.

Appendix A. Arithmetic Functions

In this section we will recall some basic definition and facts about arithmetic functions that we use throughout our calculations. Most of the content of this section can be found in [7, Chapters 16, 17] and [1, Chapter 2]

Definition A.1. An *arithmetic function* is a function $f:\mathbb{N}\to\mathbb{C}$.

We will use the following different operations on arithmetic functions. For all $n\geq 1$ and arithmetic functions f and g.

(i) $(f+g)(n)=f(n)+g(n)$ is the *sum* of f and g

(ii) $(f \cdot g)(n) = f(n)g(n)$ is the *product* of f and g

(iii) $(f/g)(n) = f(n)/g(n)$ is the *quotient* of f and g

(iv) $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$ is the *Dirichlet convolution* of f and g

(v) $(f\Delta g)(n) = \sum_{k=1}^{n-1} f(k)g(n-k)$ is the *additive convolution* of f and g.

Note that all of the above operations, except the quotient (iii), are commutative. The Dirichlet convolution, (iv) is associative. We will use the convention that $f \cdot g * h$ means $(f \cdot g) * h$.

Definition A.2. An arithmetic function f is *multiplicative* if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$ such that $m \wedge n = 1$.

It follows that $f(1) = 1$ for all multiplicative functions f.

Definition A.3. An arithmetic function f is *completely multiplicative* if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

Completely multiplicative functions distribute over the Dirichlet product as stated by the following proposition.

Proposition A.4. *Let f, g, h be arithmetic functions and let f be completely multiplicative. Then,*

$$f \cdot (g * h) \equiv (f \cdot g) * (f \cdot h)$$

The following are some well-studied multiplicative functions that we use in our computations:

(i) $1(n) = 1$ is the *constant* 1 function.

(ii) $\varepsilon(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else} \end{cases}$ is the *Dirichlet convolution identitity* since $f * \varepsilon \equiv \varepsilon * f \equiv f$ for any arithmetic function f such that $f(1) = 1$.

(iii) $\mathrm{Id}_k(n) = n^k$ will denote the *power* function of order k.

(iv) $\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } k \text{ is the number of distinct primes that divide } n \end{cases}$ is the *Möbius* function

(v) $\sigma_k(n) = \sum_{d|n} d^k$ is the *divisor function* of order k

(vi) $\phi(n) = n \prod_{p|n,\, p \text{ prime}} (1 - p^{-1})$ is the *Euler totient* function

(vii) $J_k(n) = n^k \prod_{p|n,\, p \text{ prime}} \left(1 - \frac{1}{p^k}\right)$ is the *Jordan totient* function of order k

Next we state some identities that we use in our calculations.

Proposition A.5. *Let f be a multiplicative arithmetic function. Then, for any $n \in \mathbb{N}$,*

$$\sum_{d|n} \mu(d) f(d) = \prod_{p|n} (1 - f(p))$$

Proposition A.6. *For $n > 1$, $\sum_{d|n} \mu(d) = 0$, or equivalently, $\mu * 1 \equiv \varepsilon$.*

Proposition A.7 (Möbius inversion formula)**.** *f and g are arithmetic functions satisfying $g \equiv 1 * f$ if and only if $f \equiv \mu * g$.*

Proposition A.8. *Let f, g, h be arithmetic functions such that $f \equiv g * h$. Then, $h \equiv (\mu \cdot g) * f$.*

The above stated propositions can be used to produce various identities relating the different multiplicative functions presented. We use the following identities in our calculations.

Proposition A.9. *For $k \geq 1$ we have the following relation between some of the arithmetic functions.*

(i) $\mu * \sigma_k \equiv \mathrm{Id}_k$

(ii) $\mu * \mathrm{Id}_k \equiv J_k$

(iii) $\phi * \mathrm{Id}_1 \cdot \mu \equiv \mu$

Arithmetic functions can also be related to each other via additive convolution. The following proposition is due to S. Ramanujan. The proof, along with other similar identities involving σ_3 and σ_5 can be found in [16]

Proposition A.10.

$$\sum_{\substack{p,q,k,l \in \mathbb{N} \\ pq+kl=n}} kl = (\sigma_1 \Delta \sigma_1)(n) = \frac{5}{12}\sigma_3(n) + \frac{1}{12}\sigma_1(n) - \frac{1}{2} n\sigma_1(n)$$

Next, we recall Dirichlet series which are central objects in the proof of Lemma 5.9.

Definition A.11. A Dirichlet series is a formal series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$$

where the $\{\alpha_n\}_{i=1}^{\infty}$ is a sequence of complex numbers and s is a complex number.

Dirichlet series do not always converge. However, when they do, their coefficients satisfy the following uniqueness property:

Proposition A.12 (Uniqueness of Dirichlet Series)**.** *Let*

$$F(s) = \sum_{n>0} \frac{\alpha_n}{n^s} \quad \textit{and} \quad G(s) = \sum_{n>0} \frac{\beta_n}{n^s}$$

be Dirichlet series such that $F(s)$ and $G(s)$ converge for all $s > s_0$, $s_0 \neq \infty$ and that $F(s) = G(s)$ for all $s > s_0$. Then, $\alpha_n = \beta_n$ for all n.

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