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Pradeep Boggarapu & Sundaram Thangavelu

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Mixed norm estimates for the Riesz transforms associated to Dunkl harmonic oscillators

Pradeep Boggarapu
Sundaram Thangavelu

Abstract

In this paper we study weighted mixed norm estimates for Riesz transforms associated to Dunkl harmonic oscillators. The idea is to show that the required inequalities are equivalent to certain vector valued inequalities for operator defined in terms of Laguerre expansions. In certain cases the main result can be deduced from the corresponding result for Hermite Riesz transforms.

1. **Introduction**

Let G be a Coxeter group (finite reflection group) associated to a root system R in $\mathbb{R}^d, d \geq 2$. We use the notation $\langle ., . \rangle$ for the standard inner product on $\mathbb{R}^d$. Let κ be a multiplicity function which is assumed to be non-negative and let

$$h_\kappa(x) = \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{\kappa(\nu)}$$

where R_+ is the set of all positive roots in R. Let T_j, $j = 1, 2, \ldots, d$ be the difference-differential operators defined by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\nu \in R_+} \kappa(\nu) \nu_j \frac{f(x) - f(\sigma_\nu x)}{\langle \nu, x \rangle}.$$

where σ_ν is the reflection defined by ν. The Dunkl Laplacian Δ_κ is then defined to be the operator

$$\Delta_\kappa = \sum_{j=1}^{d} T_j^2$$

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which can be explicitly calculated, see Theorem 4.4.9 in Dunkl-Xu [7]. The Dunkl harmonic oscillator is then defined by

$$H_{d,\kappa} = -\Delta_\kappa + |x|^2$$

which reduces to the Hermite operator $H_d = -\Delta + |x|^2$ when $\kappa = 0$.

Our aim in this paper is to study the L^p mapping properties of Riesz transforms associated to the Dunkl harmonic oscillator. The spectral theory of the operator $H_{d,\kappa}$ has been developed by Rösler in [16]. The eigenfunctions of $H_{d,\kappa}$ are called the generalised Hermite functions and denoted by Φ_μ^κ, $\mu \in \mathbb{N}^d$. It has been proved that they form an orthonormal basis for $L^2(\mathbb{R}^d, h_\kappa^2 dx)$. In analogy with the Riesz transforms associated to the Hermite operator, one can define the Riesz transforms $R_j^\kappa,\ R_j^{\kappa *},\ j = 1, 2, \ldots, d$ by

$$R_j^\kappa = \Big(T_j + x_j\Big) H_{d,\kappa}^{-\frac{1}{2}}, \quad R_j^{\kappa *} = \Big(-T_j + x_j\Big) H_{d,\kappa}^{-\frac{1}{2}}.$$

Note that the operators R_j^κ and $R_j^{\kappa *}$ are densely defined i.e., they are defined on the subspace V consisting of finite linear combinations of the generalised Hermite functions Φ_α^κ. In the particular case of $G = \mathbb{Z}_2^d$ treated in [14] the authors have shown that the L^2 norm of $(T_j + x_j)\Phi_\alpha^\kappa$ behaves like $(2|\alpha| + d + 2\gamma)^{1/2}$ where $\gamma = \sum_{\nu \in R^+} \kappa(\nu)$. Since Φ_α^κ are eigenfunctions of $H_{d,\kappa}$ with eigenvalues $(2|\alpha| + d + 2\gamma)$ the operator $H_{d,\kappa}^{-\frac{1}{2}}$ defined by spectral theorem satisfies $H_{d,\kappa}^{-\frac{1}{2}} \Phi_\alpha^\kappa = (2|\alpha| + d + 2\gamma)^{-1/2} \Phi_\alpha^\kappa$. From these two facts, it is clear that the Riesz transforms defined on V satisfy the inequalities

$$\|R_j^\kappa f\|_2 \le C\|f\|_2, \quad \|R_j^{\kappa *} f\|_2 \le C\|f\|_2$$

for all $f \in V$. Consequently, they extend to L^2 as bounded linear operators. In [1] a very cute argument based on the fact that

$$H_{d,\kappa} = \frac{1}{2} \sum_{j=1}^{d} \left((T_j + x_j)(-T_j + x_j) + (-T_j + x_j)(T_j + x_j)\right)$$

is used to show that the L^2 boundedness on V holds for any reflection group G. We make use of these definitions and results in the sequel.

If it can be shown that R_j^κ and $R_j^{\kappa *}$ satisfy the inequalities

$$\|R_j^\kappa f\|_p \le C\|f\|_p, \quad \|R_j^{\kappa *} f\|_p \le C\|f\|_p$$

for any $1 < p < \infty$ whenever $f \in V$ then by density arguments they can be extended to the whole of $L^p(\mathbb{R}^d, h_\kappa^2 dx)$, $1 < p < \infty$ as bounded operators. This was proved in [14] by Nowak and Stempak in the particular case when $G = \mathbb{Z}_2^d$. For general Coxeter groups the boundedness properties of the Riesz transforms are proved by Amri in [1]. We refer to these two papers for details and further information on Riesz transforms associated to the Dunkl harmonic oscillator. Weighted norm inequalities or mixed norm inequalities are not known for these Riesz transforms. In this paper our main goal is to establish certain weighted mixed norm estimates for these operators.

For $\alpha \geq -\frac{1}{2}$, let $A_p^\alpha(\mathbb{R}^+)$ be the Muckenhoupt's class of A_p-weights on $\mathbb{R}^+$ associated to the doubling measure $d\mu_\alpha(t) = t^{2\alpha+1}dt$. Let $d\sigma$ be the surface measure on unit sphere $\mathbf{S}^{d-1}$ and let w be a positive function on $\mathbb{R}^+$. We denote by $L^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ the space of all measurable functions f on $\mathbb{R}^d$ for which

$$\int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} |f(r\omega)|^2 h_\kappa^2(\omega) d\sigma(\omega)\Big)^{\frac{p}{2}} w(r) r^{d+2\gamma-1} dr < \infty.$$

The $p-$th root of the above quantity is a norm with respect to which the space becomes a Banach space. For $1 < p < \infty$ the dual of the Banach space $L^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ is nothing but the space $L^{p',2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ where p' is the index conjugate to p. This follows from a general theorem proved in [4] since we can think of the space $L^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ as an L^p space on $\mathbb{R}^+$ of functions taking values in the Hilbert space $L^2(\mathbf{S}^{d-1}, h_\kappa^2(\omega)d\sigma(\omega))$ taken with respect to the measure $w(r)r^{d+2\gamma-1}dr$. Since $L^2(\mathbf{S}^{d-1}, h_\kappa^2(\omega)d\sigma(\omega))$ is a separable Hilbert space, it can be identified with the sequence space $l^2(\mathbb{N})$ and hence a simple independent proof also can be given for the fact about the dual. We denote by $L_G^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ the subspace of G-invariant functions in $L^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$.

Let V_G stand for the set of all G-invariant functions in V. To see that V_G is a nontrivial subspace of V we proceed as follows. Given a function f on $\mathbb{R}^d$ we define the G-invariant function $f^\#$ by averaging over G. Thus

$$f^\#(x) = \frac{1}{|G|} \sum_{g \in G} f(gx)$$

where $|G|$ stands for the cardinality of G. We claim that V_G is precisely the set of all $f^{\#}$ where f runs through V. Indeed, it is obvious that for any G-invariant $f \in V$ we have $f = f^{\#}$. On the other hand, if $f \in V$ then $f^{\#}$ is G-invariant and $f^{\#}$ belongs to V. The latter can be easily seen as follows: Since $f \in V$, it is of the form

$$f(x) = \sum_{\alpha \in F} c_\alpha \Phi_\alpha^\kappa(x),$$

where F is a finite subset of $\mathbb{N}^d$. Since $H_{d,\kappa}$ is G-invariant and $H_{d,\kappa}\Phi_\alpha^\kappa = (2|\alpha| + d + 2\gamma)\Phi_\alpha^\kappa$ (see Section 2.2 below) it follows that

$$H_{d,\kappa}(\Phi_\alpha^\kappa)^{\#} = (2|\alpha| + d + 2\gamma)(\Phi_\alpha^\kappa)^{\#}.$$

Note that $H_{d,\kappa}$ is a self-adjoint operator with discrete spectrum. Moreover, each eigenspace is finite dimensional and $\{\Phi_\alpha^\kappa : \alpha \in \mathbb{N}^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d, h_\kappa^2 dx)$ consisting of eigenfunctions of $H_{d,\kappa}$. Consequently, $(\Phi_\alpha^\kappa)^{\#}$ which is an eigenfunction of $H_{d,\kappa}$ can be written as $\sum_{|\beta|=|\alpha|} a_\beta \Phi_\beta^\kappa$. This shows that $f^{\#} = \sum_{\alpha \in F} c_\alpha (\Phi_\alpha^\kappa)^{\#}$ belongs to V. This proves our claim.

Also note that the density of V in $L^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ implies the density of V_G in $L_G^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ which is an immediate consequence of Minkowski's inequality since the measures given by $h_\kappa^2(\omega)d\sigma(\omega)$ and $w(r)r^{d+2\gamma-1}dr$ are G- invariant. In Subsection 2.4 we will show that V is dense in $L^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ for all $w \in A_p^{\frac{d}{2}+\gamma-1}(\mathbb{R}^+), 1 < p < \infty$. Thus, R_j^κ and $R_j^{\kappa*}$ are well defined on the dense subspace V_G.

Theorem 1.1. *Let $d \geq 2,\ 1 < p < \infty$. Then for $j = 1, 2, \cdots, d$ the Riesz transforms R_j^κ and $R_j^{\kappa*}$ initially defined on V_G satisfy the estimates*

$$\begin{aligned}\int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} |R_j^\kappa f(r\omega)|^2 h_\kappa^2(\omega) d\sigma(\omega) \Big)^{\frac{p}{2}} w(r) r^{d+2\gamma-1} dr \\ \leq C_j(w,p,\kappa) \int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} |f(r\omega)|^2 h_\kappa^2(\omega) d\sigma(\omega) \Big)^{\frac{p}{2}} w(r) r^{d+2\gamma-1} dr\end{aligned}$$

for all $f \in V_G$, $w \in A_p^{\frac{d}{2}+\gamma-1}(\mathbb{R}^+)$. Consequently R_j^κ and $R_j^{\kappa*}$ can be extended as bounded operators from $L_G^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ into $L^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$.

The proof of this theorem is based on the fact that on radial functions the Dunkl harmonic oscillator $H_{d,\kappa}$ coincides with the Hermite operator $H_{d+2\gamma}$, when 2γ is an integer. More generally, using an analogue of Funk-Hecke formula for h-harmonics we can show that the mixed norm estimates for the Riesz transforms R_j^κ are equivalent to a vector valued inequality for a sequence of Laguerre Riesz transforms. When 2γ is an integer these inequalities can be deduced from the weighted norm inequalities satisfied by Hermite Riesz transforms. In the general case when 2γ is not an integer, we can appeal to a recent result of Ciaurri and Roncal [5].

The plan of the paper is as follows. In Section 2 we collect some facts from the spectral theory of Dunkl harmonic oscillators. Especially, we need an analogue of Mehler's formula for the generalised Hermite functions. We also collect some basic facts about h-harmonics which are analogues of spherical harmonics on $\mathbf{S}^{d-1}$. The most important result is an analogue of Funk-Hecke formula for h-harmonics. In Section 3 we consider the vector $\mathcal{R}^\kappa f = (R_1^\kappa, \cdots, R_d^\kappa)$ of Riesz transforms and show that mixed norm inequalities for $|\mathcal{R}^\kappa f| = \left(\sum_{j=1}^d |R_j^\kappa f|^2\right)^{\frac{1}{2}}$ can be reduced to vector valued inequalities for operators related to Laguerre expansions. In Section 4 we prove the required inequalities by considering the vector of Hermite Riesz transforms.

Though we have considered only the Riesz transforms in this paper, we can also treat multipliers (e.g. Bochner-Riesz means) for the Dunkl harmonic oscillator. Using the known results for the Hermite operator, we can prove an analogue of Theorem 1.1 for multipliers associated to Dunkl harmonic oscillator.

2. **Preliminaries**

2.1. **Coxeter groups and Dunkl operators:**

We assume that the reader is familiar with the notion of finite reflection groups associated to root systems. Given a root system R we define the reflection σ_ν, $\nu \in R$ by

$$\sigma_\nu x = x - 2\,\frac{\langle \nu, x\rangle}{|\nu|^2}\nu.$$

Recall that $\langle \nu, x\rangle$ is the inner product on $\mathbb{R}^d$. These reflections σ_ν, $\nu \in R$ generate a finite group which is called a Coxeter group. A function κ defined on R is called a multiplicity function if it is G invariant. We assume that our multiplicity function κ is non negative. The Dunkl operators T_j defined by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\nu \in R_+} \kappa(\nu)\nu_j \frac{f(x) - f(\sigma_\nu x)}{\langle \nu, x\rangle}.$$

form a commuting family of operators. There exists a kernel $E_\kappa(x,\ \xi)$ which is a joint eigenfunction for all T_j:

$$T_j E_\kappa(x,\ \xi) = \xi_j E_\kappa(x,\ \xi).$$

This is the analogue of the exponential $e^{\langle x,\ \xi\rangle}$ and Dunkl transform is defined in terms of $E_\kappa(ix,\ \xi)$. For all these facts we refer to Dunkl [6] and Dunkl-Xu [7]. The weight function associated to R and κ is defined by

$$h_\kappa^2(x) = \prod_{\nu \in R_+} |\langle x, \nu\rangle|^{2\kappa(\nu)}.$$

Recall that $\gamma = \sum_{\nu \in R_+} \kappa(\nu)$ and the multiplicity function $\kappa(\nu)$ is always assumed to be non-negative. We consider L^p spaces defined with respect to the measure $h_\kappa^2(x)dx$. Note that $h_\kappa^2(x)$ is homogeneous of degree 2γ.

2.2. **Generalised Hermite functions:**

In [16] Rösler has studied generalised Hermite polynomials associated to Coxeter groups. She has shown that there exists an orthonormal basis Φ_α^κ, $\alpha \in \mathbb{N}^d$ for $L^2(\mathbb{R}^d, h_\kappa^2(x)dx)$ consisting of functions for which $\Phi_\alpha^\kappa(x)e^{\frac{1}{2}|x|^2}$ are polynomials. Moreover, they are eigenfunctions of the Dunkl harmonic oscillator:

$$\Big(-\Delta_\kappa + |x|^2\Big)\Phi_\alpha^\kappa = (2|\alpha| + d + 2\gamma)\Phi_\alpha^\kappa.$$

They are also eigenfunctions of the Dunkl transform. For our purpose, the most important result is the generating function identity or the Mehler's formula for the generalised Hermite functions. For $0 < r < 1$, one has

$$\sum_{\alpha \in \mathbb{N}^d} \Phi_\alpha^\kappa(x)\Phi_\alpha^\kappa(y) r^{|\alpha|} = c_d(1-r^2)^{-\frac{d}{2}-\gamma} e^{-\frac{1}{2}\left(\frac{1+r^2}{1-r^2}\right)(|x|^2+|y|^2)} E_\kappa\Big(\frac{2rx}{1-r^2},\ y\Big)$$

see Theorem 3.12 in [16]. By taking $r = e^{-2t}$, $t > 0$ we see that the kernel of the heat semigroup generated by $-\Delta_\kappa + |x|^2$ is given by

$$K_t(x, y) = c_{d,\gamma}(\sinh 2t)^{-\frac{d}{2}-\gamma} e^{-\frac{1}{2}(\coth 2t)(|x|^2+|y|^2)} E_\kappa\Big(\frac{x}{\sinh 2t},\ y\Big). \quad (2.1)$$

We will make use of this kernel in the study of Riesz transforms.

Recall that the subspace V defined in the introduction is the algebraic span of the generalised Hermite functions Φ_α^κ, $\alpha \in \mathbb{N}^d$. As every Φ_α^κ is a Schwartz function it follows that elements of V are also of Schwartz class. It is known that V is dense in $L^p(\mathbb{R}^d, h_\kappa^2(x)dx)$, $1 \leq p < \infty$. Indeed, in [21] the authors have shown that Bochner-Riesz means $S_R^\delta f$, for large enough δ, converge to f in the norm as $R \to \infty$ as long as $1 \leq p < \infty$. Since $S_R^\delta f \in V$ for any $f \in L^p(\mathbb{R}^d, h_\kappa^2(x)dx)$ it follows that V is dense in $L^p(\mathbb{R}^d, h_\kappa^2(x)dx)$. The same thing can be proved using the fact that the heat semigroup, $e^{-tH_{d,\kappa}}$ generated by $H_{d,\kappa}$ is strongly continuous in each of $L^p(\mathbb{R}^d, h_\kappa^2(x)dx)$, $1 \leq p < \infty$. We also need to know the density of V in certain weighted L^p spaces. This will be addressed in Subsection 2.4 below.

2.3. **h-harmonics and Funk-Hecke formula:**

The best reference for this section is Chapter 5 of [7]. For the space $L^2(\mathbf{S}^{d-1}, h_\kappa^2(\omega)d\sigma(\omega))$ there exists an orthonormal basis consisting of h-harmonics. These are analogues of spherical harmonics and defined using Δ_κ in place Δ. A homogeneous polynomial $P(x)$ is said to be a solid h-harmonic if $\Delta_\kappa P(x) = 0$. Restrictions of such solid harmonics to $\mathbf{S}^{d-1}$ are called spherical h-harmonics. The space $L^2(\mathbf{S}^{d-1}, h_\kappa^2 d\sigma)$ is the orthogonal direct sum of the finite dimensional spaces $\mathcal{H}_m^d$ consisting of h-harmonics of degree m. We can choose an orthonormal basis $Y_{m,j}^h$, $j = 1, 2, \ldots, d(m)$, $d(m) = dim(\mathcal{H}_m^h)$ so that the collection $\{Y_{m,j}^h : \ j = 1, 2, \ldots, d(m),\ m = 0, 1, 2, \ldots\}$ is an orthonormal basis for $L^2(\mathbf{S}^{d-1}, h_\kappa^2 d\sigma)$.

In order to state the Funk-Hecke formula we need to recall the intertwining operator. It has been proved that there is an operator V_κ satisfying $T_j V_\kappa = V_\kappa \frac{\partial}{\partial x_j}$. The explicit form of V_κ is not known, except in a couple of simple cases, but it is a useful operator. In particular, the Dunkl kernel is given by $E_\kappa(x,\ \xi) = V_\kappa e^{\langle \cdot,\ \xi \rangle}(x)$. The operator V_κ also intertwines h-harmonics (see Proposition 5.2.8 of [7]).

The classical Funk-Hecke formula for spherical harmonics states the following. For any continuous function f on $[-1,\ 1]$ and a spherical harmonic Y_m of degree m, one has the formula

$$\int_{\mathbf{S}^{d-1}} f(\langle x',\ y'\rangle) Y_m(y') d\sigma(y') = \lambda_m(f) Y_m(x')$$

where $\lambda_m(f)$ is a constant defined by

$$\lambda_m(f) = \frac{B(\frac{d-1}{2}, \frac{1}{2})^{-1}}{C_m^{\frac{d}{2}-1}(1)} \int_{-1}^{1} f(t) C_m^{\frac{d}{2}-1}(t)(1-t^2)^{\frac{d-3}{2}} dt.$$

Here C_m^{λ} stand for ultraspherical polynomials of type λ and $B(r, s)$ stands for the beta function. A similar formula is true for h-harmonics (see Theorem 5.3.4 in [7]);

$$\int_{\mathbf{S}^{d-1}} V_\kappa f(x',\ \cdot)(y') Y_m^h(y') h_\kappa^2(y') d\sigma(y') = \lambda_m(f) Y_m^h(x')$$

where

$$\lambda_m(f) = \frac{B(\frac{d-1}{2}+\gamma, \frac{1}{2})^{-1}}{C_m^{\frac{d}{2}-1+\gamma}(1)} \int_{-1}^{1} f(t) C_m^{\frac{d}{2}-1+\gamma}(t)(1-t^2)^{\frac{d-3}{2}+\gamma} dt.$$

Let $J_\delta(z)$ stand for Bessel function of type $\delta > -1$ and define $\mathcal{I}_\delta(z) = e^{-i\frac{\pi}{2}\delta} J_\delta(iz)$. If we take $f(t) = e^{itz}$ in the above we get

$$\frac{B(\frac{d-1}{2}+\gamma, \frac{1}{2})^{-1}}{C_m^{\frac{d}{2}-1+\gamma}(1)} \int_{-1}^{1} e^{itz} C_m^{\frac{d}{2}-1+\gamma}(t)(1-t^2)^{\frac{d-3}{2}+\gamma} dt = c_{d,\gamma} \frac{J_{\frac{d}{2}+\gamma+m-1}(z)}{z^{\frac{d}{2}+\gamma-1}}$$

(see page 204-205 in [3]). By taking $f(t) = e^{t|x|\,|y|}$ and making use of the above formula we get

$$\int_{\mathbf{S}^{d-1}} E_\kappa(x,\ y) Y_m^h(y') h_\kappa^2(y') d\sigma(y') = c_{d,\gamma} \frac{\mathcal{I}_{\frac{d}{2}+\gamma+m-1}(|x|\,|y|)}{(|x|\,|y|)^{\frac{d}{2}+\gamma-1}} Y_m^h(x').$$

In view of this and (2.1) we have

$$\int_{\mathbf{S}^{d-1}} K_t(rx', sy')Y_m^h(y')h_\kappa^2(y')d\sigma(y')$$
$$= c_{d,\gamma}(\sinh 2t)^{-1}e^{-\frac{1}{2}(\coth 2t)(r^2+s^2)}\frac{\mathcal{I}_{\frac{d}{2}+\gamma+m-1}\big(\frac{rs}{\sinh 2t}\big)}{(rs)^{\frac{d}{2}+\gamma-1}}Y_m^h(x'). \quad (2.2)$$

We will make use of this formula in calculating the action of $e^{-tH_{d,\kappa}}$ on functions of the form $g(r)Y_m^h(x')$.

2.4. The density of V:

In this subsection we take up the issue of proving the density of V, defined in the introduction, in the spaces $L^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$ for $1 < p < \infty$ and $w \in A_p^{d/2+\gamma-1}(\mathbb{R}^+)$. In order to do this we will make use of the Laguerre connection. For each $\delta \geq -\frac{1}{2}$ we consider the Laguerre differential operator

$$L_\delta = -\frac{d^2}{dr^2} + r^2 - \frac{2\delta+1}{r}\frac{d}{dr}$$

whose normalised eigenfunctions are given by

$$\psi_k^\delta(r) = \Big(\frac{2\Gamma(k+1)}{\Gamma(k+\delta+1)}\Big)^{\frac{1}{2}} L_k^\delta(r^2)e^{-\frac{1}{2}r^2}$$

where $L_k^\delta(r)$ are Laguerre polynomials of type δ. These functions form an orthonormal basis for $L^2(\mathbb{R}^+, d\mu_\delta)$, where $d\mu_\delta(r) = r^{2\delta+1}dr$. The operator L_δ generates the semigroup $T_t^\delta = e^{-tL_\delta}$ whose kernel is given by

$$K_t^\delta(r,s) = \sum_{k=0}^{\infty} e^{-(4k+2\delta+2)t}\psi_k^\delta(r)\psi_k^\delta(s). \quad (2.3)$$

The generating function identity ((1.1.47) in [20]) for Laguerre functions gives the explicit expression

$$K_t^\delta(r,s) = (\sinh 2t)^{-1}e^{-\frac{1}{2}(\coth 2t)(r^2+s^2)}(rs)^{-\delta}\mathcal{I}_\delta\Big(\frac{rs}{\sinh 2t}\Big) \quad (2.4)$$

where $\mathcal{I}_\delta(z) = e^{-i\frac{\pi}{2}\delta}J_\delta(iz)$ is the modified Bessel function.

The Dunkl-Hermite semigroup $e^{-tH_{d,\kappa}}$ generated by the operator $H_{d,\kappa}$ is an integral operator given by

$$e^{-tH_{d,\kappa}}f(x) = \int_{\mathbb{R}^d} f(y)K_t(x,y)h_\kappa^2(y)dy$$

where $K_t(x,y)$ is the kernel defined in (2.1). The relation between this semigroup and the Laguerre semigroups $T_t^\delta = e^{-tL_\delta}$ is given by the following proposition. In what follows, $Y_{m,j}^h$, $j = 1,2,\cdots,d(m)$, $m = 0,1,2,\cdots$ stands for the orthonormal basis for $L^2(\mathbf{S}^{d-1}, h_\kappa^2(\omega)d\sigma(\omega))$ described in Subsection 2.3.

Proposition 2.1. *For any Schwartz class function f on $\mathbb{R}^d$ let*

$$\widetilde{f}_{m,j}(r) = r^{-m}\int_{\mathbf{S}^{d-1}} f(r\omega)Y_{m,j}^h(\omega)h_\kappa^2(\omega)d\sigma(\omega).$$

Then we have the relation

$$\int_{\mathbf{S}^{d-1}} e^{-tH_{d,\kappa}}f(r\omega)Y_{m,j}^h(\omega)h_\kappa^2(\omega)d\sigma(\omega) = c_{d,\gamma}\, r^m\Big(T_t^{d/2+m+\gamma-1}\widetilde{f}_{m,j}\Big)(r).$$

The proof of this proposition is immediate from the expressions (2.1) and (2.4) for the kernels of $e^{-tH_{d,\kappa}}$ and $T_t^{d/2+m+\gamma-1}$ and the Funk-Hecke formula.

We make use of the following lemma in order to prove that V is a dense subspace of $L^{p,2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$. That V is a subspace follows immediately from the lemma as every member of V being a finite linear combination of Φ_α^κ is a Schwartz class function.

Lemma 2.2. *Let $1 \le p < \infty$ and f be a Schwartz class function on $\mathbb{R}^d$. Then*

$$\int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} |f(r\omega)|^2 h_\kappa^2(\omega)d\sigma(\omega)\Big)^{\frac{p}{2}} w(r)r^{d+2\gamma-1}dr < \infty$$

whenever $w \in A_p^{d/2+\gamma-1}(\mathbb{R}^+)$.

Proof. First we observe that if f is a Schwartz class function on $\mathbb{R}^d$, then the function $f_0(r) := \Big(\int_{\mathbf{S}^{d-1}} |f(r\omega)|^2 h_\kappa^2(\omega)d\sigma\omega\Big)^{\frac{1}{2}}$ is a continuous function on $\mathbb{R}^+$ and for every positive integer N there exists $C_N > 0$ such that

$f_0(r) \le C_N(1+r)^{-N}$ for all $r \in \mathbb{R}^+$. To prove the lemma, it is enough to prove that

$$\int_0^\infty (f_0(r))^p w(r) r^{d+2\gamma-1} dr < \infty.$$

Let $\delta = d/2 + \gamma - 1$ and the above integral can be written as

$$\int_0^\infty (f_0(r))^p w(r) r^{2\delta+1} dr = \left(\int_0^1 + \int_1^\infty\right) ((f_0(r))^p w(r) r^{2\delta+1} dr).$$

The first integral on the left hand side of the above is finite as f_0 is continuous and w is locally integrable. And the second integral can be written as

$$\sum_{j=1}^\infty \int_{2^{j-1} \le r < 2^j} (f_0(r))^p w(r) r^{2\delta+1} dr$$

which can be bounded by

$$\sum_{j=1}^\infty (f_0(r_j))^p \int_0^{2^j} w(r) r^{2\delta+1} dr$$

where $r_j \in [2^{j-1}, 2^j]$ are the points at which f_0 attains maximum on $[2^{j-1}, 2^j]$. Such r_j's exist in the closed interval $[2^{j-1}, 2^j]$, since f_0 is continuous. The A_p-weight condition on w implies $\int_0^R w(r) r^{2\delta+1} dr \le C R^{2p(\delta+1)}$ for $R > 0$, see page 252, Eqn.16 in [12]. Choose a positive integer N such that $N > 2(\delta+1)$. Finally we see that

$$\begin{aligned} \int_1^\infty (f_0(r))^p w(r) r^{2\delta+1} dr &\le \sum_{j=1}^\infty ((1+r_j)^N f_0(r_j))^p (1+r_j)^{-Np}\, 2^{2pj(\delta+1)} \\ &\le C \sum_{j=1}^\infty 2^{-jNp} 2^{2pj(\delta+1)} \\ &\le C \sum_{j=1}^\infty 2^{-jp(N-2(\delta+1))} < \infty. \end{aligned}$$

The second inequality in the above is due to the facts that $(1+r)^N f_0(r)$ is bounded on $\mathbb{R}^+$ and $1 + r_j \ge 2^{j-1}$. This proves the lemma. □

We are now in a position to prove the density of V in the mixed norm space $L^{p,2}(\mathbb{R}^d, w(r) r^{d+2\gamma-1} h_\kappa^2(\omega) d\sigma(\omega) dr)$ for $1 < p < \infty$, $w \in A_p^{d/2+\gamma-1}(\mathbb{R}^+)$. If V is not dense in $L^{p,2}(\mathbb{R}^d, w(r) r^{d+2\gamma-1} h_\kappa^2(\omega) d\sigma(\omega) dr)$, by duality there exists a nontrivial function $f \in L^{p',2}(\mathbb{R}^d, w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)$ $d\sigma(\omega)dr)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) such that

$$\int_{\mathbb{R}^d} f(y)\Phi_\alpha^\kappa(y)w(|y|)h_\kappa^2(y)dy = 0 \tag{2.5}$$

for all $\alpha \in \mathbb{N}^d$. Since $w \in A_p^{\frac{d}{2}+\gamma-1}(\mathbb{R}^+)$ if and only if $w^{1-p'} \in A_{p'}^{\frac{d}{2}+\gamma-1}(\mathbb{R}^+)$ it follows that the function g defined by $g(y) = f(y)w(|y|)$ belongs to $L^{p',2}(\mathbb{R}^d, w^{1-p'}(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$. Since the heat kernel $K_t(x,y)$ is a Schwartz function, it follows from Lemma 2.2 that $e^{-tH_{d,\kappa}}g$ is well defined. Moreover, by Mehler's formula

$$e^{-tH_{d,\kappa}}g(x) = \sum_{\alpha\in\mathbb{N}^d} e^{-(2|\alpha|+d+2\gamma)t}\left(\int_{\mathbb{R}^d} g(y)\Phi_\alpha^\kappa(y)h_\kappa^2(y)dy\right)\Phi_\alpha^\kappa(x).$$

Consequently, $e^{-tH_{d,\kappa}}g = 0$ for all $t > 0$ in view of (2.5). In view of Proposition 2.1 it follows that for any $m = 0, 1, 2, \ldots$, $j = 1, 2, \cdots, d(m)$

$$(T_t^{d/2+m+\gamma-1}\widetilde{g}_{m,j})(r) = 0.$$

Hence we only need to conclude that the above implies $\widetilde{g}_{m,j} = 0$ for all m and j which leads to a contradiction.

But this follows from the theory of Laguerre semigroups. Indeed, what we have is

$$\int_0^\infty (rs)^m K_t^{d/2+m+\gamma-1}(r,s)f_{m,j}(s)w(s)s^{d+2\gamma-1}ds = 0.$$

Here $w \in A_p^{d/2+\gamma-1}(\mathbb{R}^+)$ and $f_{m,j} \in L^{p'}(\mathbb{R}^+, w(s)d\mu_{d/2+\gamma-1}(s))$. Once again, the above can be rewritten as

$$\int_0^\infty (rs)^m K_t^{d/2+m+\gamma-1}(r,s)g_{m,j}(s)s^{d+2\gamma-1}ds = 0$$

for all $t > 0$. Note that the function $g_{m,j}(s) = f_{m,j}(s)w(s)$ belongs to $L^{p'}(\mathbb{R}^+, w^{1-p'}(s)d\mu_{d/2+\gamma-1}(s))$ with $w^{1-p'} \in A_{p'}^{d/2+\gamma-1}(\mathbb{R}^+)$. Invoking the fact that the modified Laguerre semigroup $\widetilde{T}_t^{d/2+m+\gamma-1}$ defined by

$$\widetilde{T}_t^{d/2+m+\gamma-1}h(r) = \int_0^\infty (rs)^m K_t^{d/2+m+\gamma-1}(r,s)h(s)s^{d+2\gamma-1}ds$$

is strongly continuous on $L^{p'}(\mathbb{R}^+, u(s)s^{d+2\gamma-1}ds)$ for any $u \in A_{p'}^{\frac{d}{2}+\gamma-1}(\mathbb{R}^+)$ we conclude that $g_{m,j} = 0$ for all m and j.

Finally, we briefly indicate how the strong continuity of $\widetilde{T}_t^{d/2+m+\gamma-1}$ can be proved. It is almost trivial to prove that the kernel of this semigroup satisfies the estimates stated in Proposition 3.4 of Ciaurri-Roncal [5]. Actually, we need not care about the uniformity in m. These estimates in turn can be used to prove that $\widetilde{T}_t^{d/2+m+\gamma-1} f$ is dominated by the maximal function $M_{d/2+\gamma-1}f$ adapted to the space of homogeneous type $(\mathbb{R}^+, d\mu_{d/2+\gamma-1})$. As this maximal function is known to be bounded on $L^p(\mathbb{R}^+, wd\mu_{d/2+\gamma-1})$, $w \in A_p^{d/2+\gamma-1}(\mathbb{R}^+)$, see e.g. Duoandikoetxea [8], we conclude that $\widetilde{T}_t^{d/2+m+\gamma-1}$ is strongly continuous on $L^p(\mathbb{R}^+, wd\mu_{d/2+\gamma-1})$, $w \in A_p^{d/2+\gamma-1}(\mathbb{R}^+)$, $1 < p < \infty$. This completes the proof.

3. Riesz transforms for the Dunkl harmonic Oscillator

3.1. Preliminaries on Riesz transforms:

As in the case of Hermite operator which corresponds to the case $\kappa = 0$, we define the Riesz transforms R_j^κ , $j = 1, 2, \ldots, d$ associated to the Dunkl harmonic oscillator $H_{d,\kappa}$ by

$$R_j^\kappa f = (T_j + x_j)H_{d,\kappa}^{-\frac{1}{2}} f$$

where $H_{d,\kappa}^{-\frac{1}{2}}$ is defined by spectral theorem. More precisely,

$$H_{d,\kappa}^{-\frac{1}{2}} f = \sum_\alpha (2|\alpha| + 2\gamma + d)^{-\frac{1}{2}}(f, \Phi_\alpha^\kappa)\Phi_\alpha^\kappa$$

where Φ_α^κ are the generalised Hermite functions and $(f, g) = \int_{\mathbb{R}^d} f(x)g(x) h_\kappa^2(x)dx$. We can also define $R_j^{\kappa *}$ by as in the Hermite case. It is easy to see that R_j^κ are bounded on $L^2(\mathbb{R}^d, h_\kappa^2(x)dx)$, see Proposition 2.1 in [1] . In the same paper, Amri has proved that R_j^κ are singular integral operators whose kernels satisfy a modified Calderón-Zygmund condition and hence by a theorem of Amri and Sifi [2] they are all bounded on $L^p(\mathbb{R}^d, h_\kappa^2(x)dx)$, $1 < p < \infty$.

In the case of Hermite operator, the Riesz transforms satisfy weighted norm estimates. More precisely, if $w \in A_p(\mathbb{R}^d)$, then R_j^0 are bounded on $L^p(\mathbb{R}^d, wdx)$, $1 < p < \infty$. This has been proved by Stempak and Torrea [19] and it follows from the fact that the kernels of R_j^0 satisfy standard Calderón-Zygmund conditions. In the present situation we do not have weighted inequalities for the Riesz transforms R_j^κ. Later we will show that the weighted inequalities for R_j^0 can be used to prove mixed norm inequalities for the Hermite Riesz transforms which will then be used to prove similar results for R_j^κ.

Assume that 2γ is an integer. Then the action of $H_{d,\kappa}$ on radial functions coincides with that of $H_{d+2\gamma}$ on radial functions. More generally, let $f(x) = g(r)Y^h(\omega)$, $r = |x|, \omega \in \mathbf{S}^{d-1}$ where Y^h is h-harmonic of degree m. Then Mehler's formula for the generalised Hermite functions along with Funk-Hecke formula yields the result

$$e^{-tH_{d,\kappa}} f(x) = ce^{-tH_{d+2\gamma+2m}} g(|x|) Y^h(\omega)$$

where on the right hand side g is considered as a radial function on $\mathbb{R}^{d+2\gamma+2m}$. It is also possible to write $e^{-tH_{d+2\gamma+2m}} g(|x|)$ in terms of Laguerre semigroup. We will make use of these observations in the proof of our main result.

3.2. **More on h-harmonics:**

As indicated in the previous subsection, we plan to expand the given function f on $\mathbb{R}^d$ in terms of h-harmonics. In order to find out the action of Riesz transforms on individual terms which are of the form $g(|x|)Y_m^h(\omega)$ we need formulas for the action of T_j on such terms. More generally we let $\nabla^\kappa = (T_1, T_2, \cdots, T_d)$ stand for the Dunkl gradient which is the sum of the gradient $\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right)$ and E^κ where

$$E^\kappa f(x) = \sum_{\nu \in R^+} \kappa(\nu) \, \frac{f(x) - f(\sigma_\nu x)}{\langle x, \nu \rangle} \, \nu.$$

Let ∇_0 be the spherical part of ∇. Then the Dunkl gradient is written as

$$\nabla^\kappa = \omega \frac{\partial}{\partial r} + \frac{1}{r} \nabla_0^\kappa$$

with $\nabla_0^\kappa = \nabla_0 + E_0^\kappa$ standing for the spherical part of the Dunkl gradient, where $E_0^\kappa f(\omega) = \sum_{\nu\in R^+} \kappa(\nu)\, \frac{f(\omega)-f(\sigma_\nu \omega)}{\langle \omega,\nu\rangle}\, \nu$ for functions f defined on $\mathbf{S}^{d-1}$.

For $\xi \in \mathbb{R}^d$, let T_ξ stand for the Dunkl derivative given by

$$T_\xi f = \partial_\xi f + \sum_{\nu\in R^+} \kappa(\nu)\langle \nu, \xi\rangle\, \frac{f(x)-f(\sigma_\nu x)}{\langle x,\nu\rangle}.$$

If one of f and g is G-invariant then

$$T_\xi(fg) = f\; T_\xi g + (T_\xi f)\; g$$

remains true. Moreover, we also know that

$$\int_{\mathbb{R}^d} T_\xi f(x) g(x) h_\kappa^2(x) dx = -\int_{\mathbb{R}^d} f(x) T_\xi g(x) h_\kappa^2(x) dx.$$

In view of this we get

$$\int_{\mathbb{R}^d} \langle \nabla^\kappa f(x), \nabla^\kappa g(x)\rangle h_\kappa^2(x) dx = -\int_{\mathbb{R}^d} \Delta_\kappa f(x) g(x) h_\kappa^2(x) dx$$

We will make use of these properties in the following calculation.

We begin with some simple observations. When f is a radial function we have

$$\nabla^\kappa(fg) = f\; \nabla^\kappa g + g\; \nabla^\kappa f$$

and consequently

$$\nabla^\kappa(fg)(r\omega) = f(r)\; \nabla^\kappa g(r\omega) + g(r\omega)\; \frac{\partial f}{\partial r}\; \omega. \tag{3.1}$$

Let Y_m be a homogeneous polynomial of degree m on $\mathbb{R}^d$. Then

$$\sum_{j=1}^d (\nabla_0)_j(\omega_j Y_m(\omega)) = (d-1) Y_m(\omega) \tag{3.2}$$

where $(\nabla_0)_j$ stand for the j^{th} component of ∇_0. To see this, consider

$$\begin{aligned} \sum_{j=1}^d \frac{\partial}{\partial x_j}(x_j Y_m(x)) &= d Y_m(x) + \sum_{j=1}^d x_j \frac{\partial}{\partial x_j} Y_m(x) \\ &= (m+d) Y_m(x) \end{aligned}$$

in view of Euler's formula. On the other hand

$$\sum_{j=1}^{d}\frac{\partial}{\partial x_j}(x_jY_m(x))=\sum_{j=1}^{d}\frac{\partial}{\partial x_j}(r^{m+1}\omega_jY_m(\omega)).$$

Since $\frac{\partial}{\partial x_j}=\omega_j\frac{\partial}{\partial r}+\frac{1}{r}(\nabla_0)_j$ it follows that

$$\sum_{j=1}^{d}\frac{\partial}{\partial x_j}(x_jY_m(x))=(m+1)Y_m(x)+\sum_{j=1}^{d}r^m(\nabla_0)_j(\omega_jY_m(\omega)).$$

Comparing this with the earlier expression we get the assertion.

Proposition 3.1. *Let Y_n and Y_m be h-harmonic polynomials of degree n and m respectively. Then we have the following identities. Let $\rho^\kappa Y_n(\omega)=\sum_{\nu\in R^+}\kappa(\nu)Y_n(\sigma_\nu\omega)$.*

(1) $\langle\nabla_0^\kappa Y_n(\omega),\omega\rangle=\gamma Y_n(\omega)-\rho^\kappa Y_n(\omega)$

(2) $\langle\nabla^\kappa Y_n(x),\omega\rangle=r^{n-1}((n+\gamma)Y_n(\omega)-\rho^\kappa Y_n(\omega))$

(3) $\sum_{j=1}^{d}(\nabla_0^\kappa)_j(\omega_jY_n(\omega))=(d+\gamma-1)Y_n(\omega)+\rho^\kappa Y_n(\omega)$

(4) $\int_{\mathbf{S}^{d-1}}\langle\nabla_0^\kappa Y_n(\omega),\nabla_0^\kappa Y_m(\omega)\rangle h_\kappa^2(\omega)d\sigma(\omega)=0$ *if* $n\neq m$.

Proof. (1) follows from the definition of $\nabla_0^\kappa=\nabla_0+E_0^\kappa$ and the fact that $\langle\nabla_0Y_n(\omega),\omega\rangle=0$ for any homogeneous polynomial, see Lemma 2.2 in [15]. (2) follows from (1) since

$$\nabla^\kappa Y_n(x)=nr^{n-1}Y_n(\omega)\omega+r^{n-1}\nabla_0^\kappa Y_n(\omega).\tag{3.3}$$

To prove (3) use the definition of $\nabla_0^\kappa=\nabla_0+E_0^\kappa$;

$$\sum_{j=1}^{d}(\nabla_0^\kappa)_j(\omega_jY_n(\omega))=\sum_{j=1}^{d}(\nabla_0)_j(\omega_jY_n(\omega))+\sum_{j=1}^{d}(E_0^\kappa)_j(\omega_jY_n(\omega))$$

and

$$\begin{aligned}\sum_{j=1}^{d}(E_0^\kappa)_j(\omega_jY_n(\omega)) &= \sum_{j=1}^{d}\sum_{\nu\in R^+}\kappa(\nu)\frac{\omega_jY_n(\omega)-(\sigma_\nu\omega)_jY_n(\sigma_\nu\omega)}{\langle\nu,\omega\rangle}\nu_j\\ &= \gamma Y_n(\omega)-\sum_{\nu\in R^+}\kappa(\nu)\frac{Y_n(\sigma_\nu\omega)\langle\sigma_\nu\omega,\nu\rangle}{\langle\nu,\omega\rangle}\end{aligned}$$

Since $\langle \sigma_\nu \omega, \nu \rangle = \langle \omega, \sigma_\nu \nu \rangle = -\langle \omega, \nu \rangle$ we get (3) in view of (3.2) and the definition of ρ^κ.

Finally, in order to prove (4) we evaluate the integral

$$\int_{\mathbb{R}^d} \langle \nabla^\kappa f(x), \nabla^\kappa g(x) \rangle h_\kappa^2(x) dx$$

where $f(x) = e^{-\frac{1}{2}|x|^2} Y_n(x)$ and $g(x) = e^{-\frac{1}{2}|x|^2} Y_m(x)$ in two different ways. As we have already observed, the above integral is equal to

$$-\int_{\mathbb{R}^d} \Delta_\kappa f(x) g(x) h_\kappa^2(x) dx.$$

The Dunkl Laplacian decomposes as (see Dunkl-Xu [7])

$$\begin{aligned} \Delta_\kappa &= \frac{\partial^2}{\partial r^2} + \frac{2\lambda_\kappa + 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\kappa,0} \\ &= p(\partial_r) + \frac{1}{r^2} \Delta_{\kappa,0} \end{aligned}$$

where $\lambda_\kappa = \gamma + \frac{d-2}{2}$ and $p(\partial_r) = \frac{\partial^2}{\partial r^2} + \frac{2\lambda_\kappa+1}{r} \frac{\partial}{\partial r}$. Thus

$$\Delta_\kappa f(x) = p(\partial_r)(r^n e^{-\frac{1}{2}r^2}) Y_n(\omega) + r^{n-2} e^{-\frac{1}{2}r^2} \Delta_{\kappa,0} Y_n(\omega).$$

Since h-harmonics are eigenfunctions of the spherical part $\Delta_{\kappa,0}$ we have

$$\Delta_{\kappa,0} Y_n(\omega) = -n(n + 2\lambda_\kappa) Y_n(\omega)$$

and consequently,

$$\Delta_\kappa f(x) = p(\partial_r)(r^n e^{-\frac{1}{2}r^2}) Y_n(\omega) - n(n + \lambda_\kappa) r^{n-2} e^{-\frac{1}{2}r^2} Y_n(\omega).$$

Clearly, integrating the above against $g(x) = e^{-\frac{1}{2}r^2} Y_m(\omega)$ produces 0 whenever $m \neq n$.

We will now evaluate the same integral using the expression $\nabla^\kappa = \omega \frac{\partial}{\partial r} + \frac{1}{r}(\nabla_0 + E_0^\kappa)$. Note that

$$\nabla^\kappa f(x) = (nr^{n-1} - r^{n+1}) e^{-\frac{1}{2}r^2} Y_n(\omega)\omega + r^{n-1} e^{-\frac{1}{2}r^2} \nabla_0^\kappa Y_n(\omega)$$

with a similar expression for $\nabla^\kappa g(x)$. Thus $\langle \nabla^\kappa f(x), \nabla^\kappa g(x) \rangle$ involves terms of the form $Y_n(\omega) Y_m(\omega)$, $Y_n(\omega) \langle \omega, \nabla_0^\kappa Y_m(\omega) \rangle$, $Y_m(\omega) \langle \omega, \nabla_0^\kappa Y_n(\omega) \rangle$ and $\langle \nabla_0^\kappa Y_n(\omega), \nabla_0^\kappa Y_m(\omega) \rangle$. Hence the proposition will be proved if we show that

$$\int_{\mathbf{S}^{d-1}} Y_n(\omega) \langle \omega, \nabla_0^\kappa Y_m(\omega) \rangle h_\kappa^2(\omega) d\sigma(\omega) = 0$$

whenever $n \neq m$. In view of (1) of the proposition it suffices to show that

$$\int_{\mathbf{S}^{d-1}} Y_n(\omega)(\sum_{\nu \in R^+} \kappa(\nu) Y_m(\sigma_\nu \omega)) h_\kappa^2(\omega) d\sigma(\omega) = 0.$$

But this is obvious, since the space $\mathcal{H}_n^h$ is invariant under the action of the orthogonal group. This completes the proof of (4). □

If $Y_{m,j}$ and $Y_{m,k}$ are h-harmonics of the same degree which are orthogonal to each other, then we cannot claim that

$$\int_{\mathbf{S}^{d-1}} \langle \nabla_0^\kappa Y_{m,j}(\omega), \nabla_0^\kappa Y_{m,k}(\omega) \rangle h_\kappa^2(\omega) d\sigma(\omega) = 0.$$

This is clear from the above proof. However, if we assume that $Y_{m,j}$ and $Y_{m,k}$ are both G-invariant, then the orthogonally relation holds.

Proposition 3.2. *Let $Y_{m,j}$ and $Y_{m,k}$ be h-harmonics of degree m which are G-invariant. Then*

$$\int_{\mathbf{S}^{d-1}} \langle \nabla_0^\kappa Y_{m,j}(\omega), \nabla_0^\kappa Y_{m,k}(\omega) \rangle h_\kappa^2(\omega) d\sigma(\omega)$$
$$= \lambda_d(m, \gamma) \int_{\mathbf{S}^{d-1}} Y_{m,j}(\omega) Y_{m,k}(\omega) h_\kappa^2(\omega) d\sigma(\omega) \quad (3.4)$$

where $\lambda_d(m, \gamma) = m(m + 2\lambda_\kappa)$, with $\lambda_\kappa = \gamma + \frac{d-2}{2}$.

Proof. Proceeding as in the proof of Proposition 3.1 and noting that $\langle \nabla_0^\kappa Y_{m,j}, \omega \rangle = 0$ in view of (1) and the G-invariance we get

$$\int_{\mathbf{S}^{d-1}} \langle \nabla_0^\kappa Y_{m,j}(\omega), \nabla_0^\kappa Y_{m,k}(\omega) \rangle h_\kappa^2(\omega) d\sigma(\omega) = 0$$

whenever $Y_{m,j}$ is orthogonal to $Y_{m,k}$. When they are not orthogonal, the constant $\lambda_d(m, \gamma)$ is given by

$$\lambda_d(m, \gamma) = \frac{A_d(m, \gamma) - B_d(m, \gamma) - C_d(m, \gamma)}{D_d(m, \gamma)}$$

where

$$A_d(m, \gamma) = m(m + 2\lambda_\kappa) \int_0^\infty e^{-r^2} r^{d+2\gamma+2m-3} dr,$$
$$B_d(m, \gamma) = \int_0^\infty p(\partial_r)(r^m e^{-\frac{1}{2}r^2}) e^{-r^2} r^{d+2\gamma+m-1} dr,$$
$$C_d(m, \gamma) = \int_0^\infty (m r^{m-1} - r^{m+1})^2 e^{-r^2} r^{d+2\gamma-1} dr$$

and

$$D_d(m,\gamma) = \int_0^\infty e^{-r^2} r^{d+2\gamma+2m-3} dr.$$

Simplifying we obtain the expression for $\lambda_d(m,\gamma)$. $\square$

3.3. The vector of Riesz transforms:

In this subsection we consider the vector of Riesz transforms $\mathcal{R}f = (R_1^\kappa f, \cdots, R_d^\kappa f)$ and show that for G-invariant functions, the mixed norm estimates for $\langle \mathcal{R}f, \mathcal{R}f\rangle^{\frac{1}{2}}$ can be reduced to certain vector valued inequalities.

Let $L^2_G(\mathbf{S}^{d-1}, h_\kappa^2(\omega)d\sigma)$ stand for the subspace of $L^2(\mathbf{S}^{d-1}, h_\kappa^2(\omega)d\sigma)$ consisting of G-invariant functions. Each space $\mathcal{H}_m^h$ can be decomposed into the subspace $(\mathcal{H}_m^h)^G$ consisting of G-invariant h-harmonics in $\mathcal{H}_m^h$ and its orthogonal complement. We choose an orthonormal basis $Y_{m,j}^h$; $j = 1, 2, \ldots, d_1(m)$, $d_1(m) \leq d(m)$ for $(\mathcal{H}_m^h)^G$ and then augment it with an orthonormal basis $Y_{m,j}^h, d_1(m) < j \leq d(m)$ for the orthogonal complement. Thus, we get an orthonormal basis $\{Y_{m,j}^h : 1 \leq j \leq d(m), m \in \mathbb{N}\}$ for $L^2(\mathbf{S}^{d-1}, h_\kappa^2(\omega)d\sigma)$ such that for each m, $Y_{m,j}^h$, $1 \leq j \leq d_1(m)$ are G-invariant. It is easy to see that $\{Y_{m,j}^h : 1 \leq j \leq d_1(m), m \in \mathbb{N}\}$ is an orthonormal basis for $L^2_G(\mathbf{S}^{d-1}, h_\kappa^2(\omega)d\sigma)$. Indeed, if f is G-invariant and orthogonal to all $Y_{m,j}^h$, $1 \leq j \leq d_1(m)$, $m \in \mathbb{N}$ then for any $Y_{m,k}^h$, $k > d_1(m)$ we have

$$\begin{aligned}
\gamma \int_{\mathbf{S}^{d-1}} & f(\omega) Y_{m,k}^h(\omega) h_\kappa(\omega) d\sigma(\omega) \\
&= \sum_{\nu\in R^+} \kappa(\nu) \int_{\mathbf{S}^{d-1}} f(\sigma_\nu \omega) Y_{m,k}^h(\omega) h_\kappa^2(\omega) d\sigma(\omega) \\
&= \int_{\mathbf{S}^{d-1}} f(\omega) \Big(\sum_{\nu\in R^+} \kappa(\nu) Y_{m,k}^h(\sigma_\nu\omega) \Big) h_\kappa^2(\omega) d\sigma(\omega).
\end{aligned}$$

As $\sum_{\nu\in R^+} \kappa(\nu) Y_{m,k}^h(\sigma_\nu\omega)$ is G-invariant h-harmonic of degree m, it can be written as $\sum_{j=1}^{d_1(m)} c_{k,j} Y_{m,j}^h$ and consequently f is orthogonal to $Y_{m,k}^h$. Let $L^2_G(\mathbb{R}^d, h_\kappa^2(x)dx)$ stand for the subspace of $L^2(\mathbb{R}^d, h_\kappa^2(x)dx)$ consisting of G-invariant functions. Thus if $f \in L_G^{p,2}(\mathbb{R}^d, r^{d+2\gamma-1} h_\kappa^2(\omega) d\sigma dr) \cap$

$L^2_G(\mathbb{R}^d, h^2_\kappa(x)dx)$ then we have the expansion

$$f(r\omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{d_1(m)} f_{m,j}(r) Y^h_{m,j}(\omega)$$

where $f_{m,j}(r) = \int_{\mathbf{S}^{d-1}} f(r\omega) Y^h_{m,j}(\omega) h^2_\kappa(\omega) d\sigma(\omega)$. Note that V_G is a subspace of $L^{p,2}_G(\mathbb{R}^d, r^{d+2\gamma-1} h^2_\kappa(\omega) d\sigma(\omega) dr) \cap L^2_G(\mathbb{R}^d, h^2_\kappa(x)dx)$.

If we let $F = (-\Delta_\kappa + |x|^2)^{-\frac{1}{2}} f$, then F is also G-invariant and hence

$$F(r\omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{d_1(m)} F_{m,j}(r) Y^h_{m,j}(\omega).$$

This expansion is justified since the operator $(-\Delta_\kappa + |x|^2)^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{R}^d, h^2_\kappa(x)dx)$ and it takes G-invariant functions into G-invariant functions. We remark that V_G is also invariant under $(-\Delta_\kappa + |x|^2)^{-\frac{1}{2}}$. This can be easily seen as follows: Since the kernel $K_t(x, y)$ of the semigroup $e^{-tH_{d,\kappa}}$ satisfies $K_t(gx, gy) = K_t(x, y)$, $g \in G$, $e^{-tH_{d,\kappa}}$ preserves G-invariant functions. Consequently, $(-\Delta_\kappa + |x|^2)^{-\frac{1}{2}} f$ is G-invariant whenever f is.

We are now ready to prove the following.

Proposition 3.3. *Let $d \geq 2$ and $1 < p < \infty$. For functions f in the space $L^{p,2}_G(\mathbb{R}^d, r^{d+2\gamma-1} h^2_\kappa(\omega) d\sigma(\omega) dr) \cap L^2_G(\mathbb{R}^d, h^2_\kappa(x)dx)$, we have*

$$\int_{\mathbf{S}^{d-1}} \langle \mathcal{R} f(r\omega), \mathcal{R} f(r\omega) \rangle h^2_\kappa(\omega) d\sigma(\omega) = A_1(r)^2 + A_2(r)^2$$

where

$$A_1(r)^2 = \sum_{m=0}^{\infty} \sum_{j=1}^{d_1(m)} \left| \Big(\frac{\partial}{\partial r} + r \Big) F_{m,j}(r) \right|^2$$

and

$$A_2(r)^2 = \sum_{m=0}^{\infty} \sum_{j=1}^{d_1(m)} \frac{\lambda_d(m, \gamma)}{r^2} |F_{m,j}(r)|^2.$$

Proof. As $\mathcal{R} f = (\nabla^\kappa + x)(-\Delta_\kappa + |x|^2)^{-\frac{1}{2}} f$ we see that

$$\mathcal{R} f(r\omega) = (\omega \frac{\partial}{\partial r} + r\omega + \frac{1}{r} \nabla^\kappa_0) F(r\omega).$$

Now

$$(\omega\frac{\partial}{\partial r}+r\omega+\frac{1}{r}\nabla_0^\kappa)(F_{m,j}(r)Y_{m,j}^h)(\omega)$$
$$=(\frac{\partial}{\partial r}+r)F_{m,j}(r)Y_{m,j}^h(\omega)\omega+\frac{1}{r}F_{m,j}(r)\nabla_0^\kappa Y_{m,j}^h(\omega),$$

and consequently

$$\mathcal{R}f(r\omega)=\sum_{m=0}^{\infty}\sum_{j=1}^{d_1(m)}\Big(\frac{\partial}{\partial r}+r\Big)F_{m,j}(r)Y_{m,j}^h(\omega)\omega+\frac{1}{r}F_{m,j}(r)\nabla_0^\kappa Y_{m,j}^h(\omega).$$

As $Y_{m,j}^h$'s are G-invariant we can make use of Proposition 3.2. Also, note that $\langle\omega,\nabla_0^\kappa Y_{m,j}^h(\omega)\rangle=0$. Therefore, integrating out $\langle\mathcal{R}f(r\omega),\mathcal{R}f(r\omega)\rangle$ over $\mathbf{S}^{d-1}$ and making use of the orthogonality relations we get the proposition. □

3.4. **The Laguerre connection and a proof of Theorem 1.1:**

In view of the above proposition, Theorem 1.1 will be proved once we show that

$$\int_0^\infty A_i(r)^p w(r)r^{d+2\gamma-1}dr$$
$$\leq C\int_0^\infty\Big(\sum_{m=0}^{\infty}\sum_{j=1}^{d_1(m)}|f_{m,j}(r)|^2\Big)^{\frac{p}{2}}w(r)r^{d+2\gamma-1}dr$$

for $i=1,2$ for all $w\in A_p^{\frac{n}{2}+\gamma-1}(\mathbb{R}^+)$. Actually, we get

$$\int_0^\infty\Big(\int_{\mathbf{S}^{d-1}}\langle\mathcal{R}f(r\omega),\mathcal{R}f(r\omega)\rangle h_\kappa^2(\omega)d\sigma(\omega)\Big)^{\frac{p}{2}}w(r)r^{d+2\gamma-1}dr$$
$$\leq c\int_0^\infty\Big(\int_{\mathbf{S}^{d-1}}|f(r\omega)|^2h_\kappa^2(\omega)d\sigma(\omega)\Big)^{\frac{p}{2}}w(r)r^{d+2\gamma-1}dr$$

for all G-invariant functions f in $L^{p,2}(\mathbb{R}^d,w(r)r^{d+2\gamma-1}h_\kappa^2(\omega)d\sigma(\omega)dr)$. We now show that the above inequalities for $A_i,\ i=1,2$ can be interpreted as certain vector valued inequalities for Laguerre Riesz transforms.

For each $\delta\geq-\frac{1}{2}$ the Laguerre differential operator L_δ has been introduced in Subsection 2.4. The Laguerre functions ψ_k^δ are eigenfunctions of

L_δ and the semigroup generated by L_δ is denoted by e^{-tL_δ} or T_t^δ. Using spectral theory we can define $L_\delta^{-\frac{1}{2}}$ which is also given by the integral

$$L_\delta^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-tL_\delta} t^{-\frac{1}{2}} dt.$$

The operators $R^\delta = \left(\frac{\partial}{\partial r} + r\right) L_\delta^{-\frac{1}{2}}$ are called Laguerre Riesz transforms and they have been studied in [13]. It is known that they are bounded on $L^p(\mathbb{R}^+, d\mu_\delta)$, $1 < p < \infty$. Recently Ciaurri and Roncal [5] have proved the following vector inequality.

Theorem 3.4 (Ciaurri-Roncal)**.** *Let $\delta \geq -\frac{1}{2}$ and $1 < p < \infty$. Then*

$$\int_0^\infty \Big(\sum_{m=0}^\infty r^{2m} |R^{\delta+m} \tilde{f}_m(r)|^2 \Big)^{\frac{p}{2}} w(r) d\mu_\delta(r) \\ \leq C \int_0^\infty \Big(\sum_{m=0}^\infty |f_m(r)|^2 \Big)^{\frac{p}{2}} w(r) d\mu_\delta(r)$$

for all $w \in A_p^\delta(\mathbb{R}^+)$. Here $\tilde{f}_m(r) = r^{-m} f_m(r)$.

We only need to prove the above inequality when the right hand side is finite. If (f_m) is a sequence with this property then each function f_m belongs to $L^2(\mathbb{R}^+, w(r)d\mu_\delta)$ which will then imply that $\tilde{f}_m \in L^2(\mathbb{R}^+, w(r) d\mu_{\delta+m})$ so that $R^{\delta+m}\tilde{f}_m$ are well defined. A similar remark applies to $L_{\delta+m}^{-\frac{1}{2}} \tilde{f}_m$ which appears in the next theorem. Actually it is enough to prove the inequality when the sequence (f_m) is finite with a constant C independent of the number of terms in the sequence. In the same paper [5] they have also proved the following inequality.

Theorem 3.5 (Ciaurri-Roncal)**.** *Let $\delta \geq -\frac{1}{2}$ and $1 < p < \infty$. Then*

$$\int_0^\infty \Big(\sum_{m=0}^\infty m^2 r^{2m-2} |L_{\delta+m}^{-\frac{1}{2}} \tilde{f}_m(r)|^2 \Big)^{\frac{p}{2}} w(r) d\mu_\delta(r) \\ \leq C \int_0^\infty \Big(\sum_{m=0}^\infty |f_m(r)|^2 \Big)^{\frac{p}{2}} w(r) d\mu_\delta(r)$$

for all $w \in A_p^\delta(\mathbb{R}^+)$. Here $\tilde{f}_m(r) = r^{-m} f_m(r)$.

We claim that the required inequalities for A_1 and A_2 can be deduced from the above two theorems. Recall that $F_{m,j}$ is defined as

$$F_{m,j}(r) = \int_{\mathbf{S}^{d-1}} (-\Delta_\kappa + |x|^2)^{-\frac{1}{2}} f(r\omega) Y^h_{m,j}(\omega) h^2_\kappa(\omega) d\sigma(\omega)$$

which can be expressed in terms of the semigroup $e^{-tH_{d,\kappa}}$ as follows:

$$\begin{aligned} F_{m,j}(r) &= \frac{1}{\sqrt{\pi}} \int_{\mathbf{S}^{d-1}} \Big(\int_0^\infty e^{-tH_{d,\kappa}} f(r\omega) t^{-\frac{1}{2}} dt \Big) Y^h_{m,j}(\omega) h^2_\kappa(\omega) d\sigma(\omega) \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} e^{-tH_{d,\kappa}} f(r\omega) Y^h_{m,j}(\omega) h^2_\kappa(\omega) d\sigma(\omega) \Big) t^{-\frac{1}{2}} dt. \end{aligned}$$

Use Proposition 2.1 stated in the Subsection 2.4 to conclude that

$$F_{m,j}(r) = c_{d,\gamma} r^m L^{-\frac{1}{2}}_{\frac{d}{2}+\gamma+m-1} \tilde{f}_{m,j}(r).$$

Consequently,

$$\Big(\frac{\partial}{\partial r} + r\Big) F_{m,j}(r) = c_{d,\gamma} r^m R^{\frac{d}{2}+\gamma+m-1} \tilde{f}_{m,j}(r) + c_{d,\gamma} \frac{m}{r} F_{m,j}(r).$$

From these expressions for $F_{m,j}$ and $(\frac{\partial}{\partial r} + r) F_{m,j}(r)$ it is clear that the weighted inequalities for A_1 and A_2 follow from Theorem 3.4 and 3.5.

In the next section we give a simple proof Theorem 3.4 and 3.5 when 2γ is an integer.

4. **Riesz transforms for the Hermite operator**

4.1. **Hermite operator in spherical coordinates:**

The Hermite operator $H = -\Delta + |x|^2$ admits a family of eigenfunctions viz., the Hermite functions Φ_α, $\alpha \in \mathbb{N}^d$ which forms an orthonormal basis for $L^2(\mathbb{R}^d)$. On the other hand there is another family of orthonormal basis given by

$$\tilde{\varphi}_{m,j,l}(x) = \Big(\frac{2\Gamma(j+1)}{\Gamma(m-j+\frac{d}{2})} \Big)^{\frac{1}{2}} L_j^{\frac{d}{2}-1+m-2j}(|x|^2) Y_{m-2j,l}(x) e^{-\frac{1}{2}|x|^2}$$

where $m \geq 0$, $j = 0, 1, \ldots, [\frac{m}{2}]$, $l = 1, 2, \ldots, d(m-2j)$, $Y_{m-2j,l}(x)$ are solid spherical harmonics and L_k^δ are Laguerre polynomials of type δ. The

Hermite operator in spherical coordinates takes the form

$$H = -\frac{\partial^2}{\partial r^2} - \frac{d-1}{r}\frac{\partial}{\partial r} + r^2 - \frac{1}{r^2}\Delta_0$$

where Δ_0 is the spherical Laplacian on $\mathbf{S}^{d-1}$. It can be shown that $H = A^*A + d$, where

$$A = \Big(\frac{\partial}{\partial r} + r\Big)\omega + \frac{1}{r}\nabla_0$$

where ∇_0 is the spherical part of the gradient and

$$A^* = -\Big(\frac{\partial}{\partial r} - r\Big)\omega - \frac{1}{r}(div)_0$$

where $(div)_0$ is the spherical part of the divergence. It is therefore natural to look at the vector valued Riesz transform $AH^{-\frac{1}{2}}f$. The natural space suitable for studying this is the mixed norm space $L^{p,2}(\mathbb{R}^d, w(r)r^{d-1}drd\sigma)$ consisting of functions for which

$$\int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} |f(r\omega)|^2 d\sigma(\omega)\Big)^{\frac{p}{2}} w(r)r^{d-1}dr < \infty.$$

In [5] Ciaurri and Roncal have proved the following theorem.

Theorem 4.1 (Ciaurri-Roncal)**.** *Let $d \geq 2$, $1 < p < \infty$ and $w \in A_p^{\frac{d}{2}-1}(\mathbb{R}^+)$. Then*

$$\|\langle AH^{-\frac{1}{2}}f,\ AH^{-\frac{1}{2}}f\rangle^{\frac{1}{2}}\|_{L^{p,2}(\mathbb{R}^d, wr^{d-1}drd\sigma)} \leq C\|f\|_{L^{p,2}(\mathbb{R}^d, wr^{d-1}drd\sigma)} \quad (4.1)$$

for all f in the algebraic span of Hermite functions with a constant C independent of f. Consequently, the above inequality remains valid for all $f \in L^{p,2}(\mathbb{R}^d, wr^{d-1}drd\sigma)$.

For the Hermite operator we also have the standard Riesz transforms $R_j = A_jH^{-\frac{1}{2}}$ studied by several authors in the literature, see [20] and [19]. It is well known that R_j are Calderón-Zygmund singular integral operators and hence satisfy the weighted norm inequalities

$$\Big(\int_{\mathbb{R}^d} |R_jf(x)|^p w(x)dx\Big)^{\frac{1}{p}} \leq C\Big(\int_{\mathbb{R}^d} |f(x)|^p w(x)dx\Big)^{\frac{1}{p}}$$

for every $w \in A_p(\mathbb{R}^d)$, $1 < p < \infty$. This has been proved by Stempak and Torrea in [19]. We will give an easy proof of the above theorem of Ciaurri and Roncal based on the connection between $AH^{-\frac{1}{2}}$ and the vector $Rf = (R_1f, \cdots, R_df)$.

Theorem 4.2. *Let $d \geq 2$ and $1 < p < \infty$. Then the inequality* (4.1) *for $AH^{-\frac{1}{2}}$ stated in the previous theorem holds for all finite linear combination of Hermite functions if and only if*

$$\|\Big(\sum_{j=1}^{d} |R_j f(x)|^2\Big)^{\frac{1}{2}}\|_{L^{p,2}(\mathbb{R}^d, wr^{d-1}drd\sigma)} \leq C\|f\|_{L^{p,2}(\mathbb{R}^d, wr^{d-1}drd\sigma)}. \tag{4.2}$$

for all such functions.

The proof of this theorem is easy. We have already observed in the previous section that the mixed norm estimates for the Riesz transforms $R_j f$, (which corresponds to $\kappa = 0$ of Theorem 1.1) is equivalent to the weighted norm inequalities for $A_1(r)$ and $A_2(r)$ appearing in Proposition 3.3. Our claim is substantiated by comparing this with the proof of Theorem 2.1 in [5]. The terms they call $O_1(f)$ and $O_2(f)$ are precisely our terms $A_1(r)$ and $A_2(r)$ respectively.

4.2. **A simple proof of Theorem 4.1:**

We now give a simple proof of mixed norm estimates (4.2) for the (standard) Riesz transforms associated to the Hermite operator, which implies Theorem 4.1. When 2γ is an integer it also implies the weighted norm inequalities for $A_1(r)$ and $A_2(r)$ and hence we get another proof of Theorem 1.1 without using the result of Ciaurri and Roncal [5].

We will be following an idea of Rubio de Francia. This method described briefly in [17] is based on an extension of a theorem of Marcinkiewicsz and Zygmund as expounded in Herz and Riviere [11]. Indeed, we make use of the following lemma which can be found in [11]

Lemma 4.3 (Herz-Riviere)**.** *Let (G, μ) and (H, ν) be arbitrary measure spaces and $T : L^p(G) \to L^p(G)$ a bounded linear operator. Then if $p \leq q \leq 2$ or $p \geq q \geq 2$, there exists a bounded linear operator $\tilde{T} : L^p(G; L^q(H)) \to L^p(G; L^q(H))$ with $\|\tilde{T}\| \leq \|T\|$ such that for $g \in L^p(G; L^q(H))$ of the form $g(x, \xi) = f(\xi)u(x)$ where $f \in L^p(G)$ and $u \in L^q(H)$ we have*

$$(\tilde{T}g)(\xi, x) = (Tf)(\xi)u(x).$$

The idea of Rubio de Francia is as follows (we are indebted to Gustavo Garrigos for bringing this to our attention). Suppose $T : L^p(\mathbb{R}^d, dx) \to L^p(\mathbb{R}^d, dx)$ is a bounded linear operator. Then by the lemma of Herz and

Riviere, it has an extension $\tilde{T}$ to $\mathcal{H}$ valued functions on $\mathbb{R}^d$ where $\mathcal{H}$ is the Hilbert space $L^2(K)$, $K = SO(d)$. Moreover, the extension satisfies $(\tilde{T}\tilde{f})(x,k) = Tg(x)h(k)$ if $\tilde{f}(x,k) = g(x)h(k)$, $x \in \mathbb{R}^d$, $k \in SO(d)$. Given $f \in L^p(\mathbb{R}^d, dx)$ consider $\tilde{f}(x,k) = f(kx)$. Then $\int_{\mathbb{R}^d}(\int_K |\tilde{f}(x,k)|^2 dk)^{\frac{p}{2}} dx$ can be calculated as follows. If $x = r\omega$, $\omega \in \mathbf{S}^{d-1}$, $\tilde{f}(x,k) = f(rk\omega)$ and hence

$$\int_K |\tilde{f}(x,k)|^2 dk = \int_{K_\omega} \Big(\int_{K/K_\omega} |f(rk\omega)|^2 d\mu \Big) d\nu \tag{4.3}$$

where $K_\omega = \{k \in K : k\omega = \omega\}$ is the isotropy subgroup of K, $d\nu$ is the Haar measure on K_ω and $d\mu$ is the K_ω invariant measure on K/K_ω which can be identified with $\mathbf{S}^{d-1}$. Hence

$$\int_K |\tilde{f}(x,k)|^2 dk = c \int_{\mathbf{S}^{d-1}} |f(r\omega)|^2 d\sigma(\omega). \tag{4.4}$$

Therefore,

$$\int_{\mathbb{R}^d} \Big(\int_K |\tilde{f}(x,k)|^2 dk \Big)^{\frac{p}{2}} dx = c' \int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} |f(r\omega)|^2 d\sigma(\omega) \Big)^{\frac{p}{2}} r^{d-1} dr. \tag{4.5}$$

Let us define $\rho(k)f(x) = f(kx)$ so that $\tilde{f}(x,k) = \rho(k)f(x)$. If T commutes with rotation i.e. $T\rho(k) = \rho(k)T$ then

$$\begin{aligned} \tilde{T}\tilde{f}(x,k) &= T(\rho(k)f)(x) \\ &= \rho(k)(Tf)(x) \\ &= (Tf)(kx). \end{aligned}$$

The boundedness of $\tilde{T}$ on $L^p(\mathbb{R}^d, \mathcal{H})$ gives

$$\int_{\mathbb{R}^d} \Big(\int_K |Tf(kx)|^2 dk \Big)^{\frac{p}{2}} dx \leq C \int_{\mathbb{R}^d} \Big(\int_K |f(kx)|^2 dk \Big)^{\frac{p}{2}} dx \tag{4.6}$$

which translates into the mixed norm estimate for T.

Given a unit vector $u \in \mathbf{S}^{d-1}$ let us consider the operator $T_u f = \sum_{j=1}^d u_j R_j f(x)$ where $R_j = A_j H^{-\frac{1}{2}}$ are the Hermite Riesz transforms. This operator T_u is not rotation invariant but has a nice transformation property under the action of $SO(d)$. Indeed,

$$T_u f(x) = (x \cdot u + u \cdot \nabla) H^{-\frac{1}{2}} f(x)$$

and as $H^{-\frac{1}{2}}$ commutes with $\rho(k)$ it follows that

$$T_u\rho(k)f = \rho(k)T_{ku}f \quad \text{or} \quad T_{k^{-1}u}\rho(k)f = \rho(k)T_u f.$$

This leads us to

$$T_u f(kx) = \sum_{j=1}^{d}(k^{-1}u)_j R_j(\rho(k)f)(x).$$

We make use of this in proving the mixed norm estimate (4.2) .

The operator R_j are singular integral operators and hence bounded on $L^p(\mathbb{R}^d, wdx)$ for any weight function $w \in A_p(\mathbb{R}^d)$, $1 < p < \infty$. By the lemma of Herz and Riviere, R_j extends as a bounded operator $\widetilde{R}_j$ on $L^p(\mathbb{R}^d, \mathcal{H}; wdx)$ where $\mathcal{H} = L^2(\mathbf{S}^{d-1})$ and $\widetilde{R}_j(\rho(k)f)(x) = R_j(\rho(k)f)(x)$. When w is radial, it can be easily checked that

$$\begin{aligned}\|\rho(k)f(x)\|^p_{L^p(\mathbb{R}^d,\mathcal{H};\, wdx)} &= \int_{\mathbb{R}^d}\Big(\int_K |\rho(k)f(x)|^2 dk\Big)^{\frac{p}{2}} w(x)dx \qquad (4.7)\\ &= c\int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} |f(r\omega)|^2 d\sigma(\omega)\Big)^{\frac{p}{2}} w(r) r^{d-1} dr.\end{aligned}$$

Moreover, by the result of Duoandikoetxea et al (Theorem 3.2 in [9]), a radial weight w belongs to $A_p(\mathbb{R}^d)$ if and only if $w(r) \in A_p^{\frac{d}{2}-1}(\mathbb{R}^+)$. From the identity

$$\begin{aligned}T_u f(kx) &= \sum_{j=1}^{d}(k^{-1}u)_j R_j(\rho(k)f)(x)\\ &= \sum_{j=1}^{d}(k^{-1}u)_j \widetilde{R}_j(\rho(k)f)(x)\end{aligned}$$

we obtain

$$\begin{aligned}\|T_u f(kx)\|_{L^p(\mathbb{R}^d,\mathcal{H};\, wdx)} &\le C\sum_{j=1}^{d}\|\widetilde{R}_j(\rho(k)f)(x)\|_{L^p(\mathbb{R}^d,\mathcal{H};\, wdx)}\\ &\le C\sum_{j=1}^{d}\|\rho(k)f(x)\|_{L^p(\mathbb{R}^d,\mathcal{H};\, wdx)}\end{aligned}$$

which translates into the required inequality (4.2) by (4.7) and taking u to be coordinate vectors.

4.3. **Higher order Riesz transforms:**

In this section we show that Theorem 1.1 remains true for higher order Riesz transforms associated to the Hermite operator H_d. As explained in Sanjay-Thangavelu [18], operators of the form $R_P f = G(P)H^{-(\frac{m+n}{2})}$ where P is a solid bigraded harmonic of total degree $(m+n)$ and $G(P)$ is the Weyl correspondence of P, are natural analogues of higher order Riesz transforms. When

$$P(z) = \sum_{|\alpha|=m, |\beta|=n} c_{\alpha,\beta} z^\alpha \bar{z}^\beta$$

is a solid harmonic, Geller [10] has shown that

$$G(P) = \sum_{|\alpha|=m, |\beta|=n} c_{\alpha,\beta} A^\alpha A^{*\beta}$$

where $A = (A_1, \cdots, A_d)$, $A^* = (A_1^*, \cdots, A_d^*)$. In particular when $P(z) = z^\alpha$ (resp. $\bar{z}^\alpha$), $G(P) = A^\alpha$ (resp. $A^{*\alpha}$). The Riesz transforms $G(P)H^{-(\frac{m+n}{2})}$ have been studied in [18]. There, by using a transference result of Mauceri it has been shown that $G(P)H^{-(\frac{m+n}{2})}$ are all bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$.

However, we can also directly prove the boundedness of $G(P)H^{-(\frac{m+n}{2})}$. In fact,

$$G(P)H^{-(\frac{m+n}{2})} = \frac{1}{\Gamma(\frac{m+n}{2})} \int_0^\infty G(P) e^{-tH} t^{\frac{m+n}{2}-1} dt$$

and hence the kernel $K_P(x,y)$ of $R_P := G(P)H^{-(\frac{m+n}{2})}$ is given by

$$K_P(x,y) = \frac{1}{\Gamma(\frac{m+n}{2})} \int_0^\infty G(P) K_t(x,y) t^{\frac{m+n}{2}-1} dt$$

where K_t is the kernel of e^{-tH} which is explicitly known. Though it is tedious, it is not difficult to show that K_P is a Calderón-Zygmund kernel (see Stempak-Torrea [19] for the case $m+n=1$). Hence the Riesz transforms R_P are bounded on $L^p(\mathbb{R}^d, w dx)$, $1 < p < \infty$, $w \in A_p(\mathbb{R}^d)$. Using this we can prove

Theorem 4.4. *Let P be a solid harmonic of bidegree (m,n), $1<p<\infty$ and $w\in A_p^{\frac{d}{2}-1}(\mathbb{R}^+)$. Then there exists $C>0$ such that*

$$\int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} |R_P f(r\omega)|^2 d\sigma(\omega)\Big)^{\frac{p}{2}} w(r) r^{d-1} dr \\ \leq C \int_0^\infty \Big(\int_{\mathbf{S}^{d-1}} |f(r\omega)|^2 d\sigma(\omega)\Big)^{\frac{p}{2}} w(r) r^{d-1} dr \quad (4.8)$$

for all $f\in L^{p,2}(\mathbb{R}^d, w(r)r^{d-1}drd\sigma(\omega))$.

The proof is similar to that of Theorem 4.1. Consider $\mathcal{H}_{m,n}$ the space of all bigraded spherical harmonics of bidegree (m,n). If $Y\in\mathcal{H}_{m,n}$ then $P(z)=|z|^{m+n}Y(z')$, $z=|z|z'$ is a solid harmonic. The group $U(d)$ acts on $\mathcal{H}_{m,n}$ and we have an irreducible unitary representation, denoted by $R(\sigma)$ supported by $\mathcal{H}_{m,n}$. We choose an orthonormal basis Y_j, $j=1,2,\ldots,d(m,n)$ and let P_j stand for the corresponding solid harmonics. Consider the operator T which takes $L^p(\mathbb{C}^d)$ into $L^p(\mathbb{C}^d,\mathcal{H}_{m,n})$ given by the prescription

$$Tf(z,\zeta)=\sum_{j=1}^{d(m,n)} R_{P_j}f(z)Y_j(\zeta).$$

This operator has a very nice transformation property. Let $\rho(\sigma)f(z)=f(\sigma^{-1}z)$ stand for the action of $U(d)$ on functions on $\mathbb{C}^d$.

Lemma 4.5. *For any $\sigma\in U(d)$ we have*

$$Tf(z,\sigma^{-1}\zeta)=\sum_{j=1}^{d(m,n)} \rho(\sigma)R_{P_j}\rho(\sigma^{-1})f(z)Y_j(\zeta).$$

This lemma has been essentially proved in [18], see the proof of Theorem 1.4. Once we have the above Lemma we can easily prove Theorem 4.4. Indeed, from the lemma we have

$$Tf(\sigma z,\zeta)=\sum_{j=1}^{d(m,n)} R_{P_j}\rho(\sigma^{-1})f(z)Y_j(\sigma\zeta).$$

With the same notation as in the proof of Theorem 4.1, the above reads as

$$\rho(\sigma^{-1})Tf(z,\zeta)=\sum_{j=1}^{d(m,n)} \tilde{R}_{P_j}\tilde{f}(z,\sigma^{-1})Y_j(\sigma\zeta).$$

where we keep $\zeta \in \mathbf{S}^{2d-1}$ fixed. Then by similar calculations, using the Lemma of Herz-Riviere we can obtain the desired inequality for $T(\cdot, \zeta)$ and hence for any $R_{P_j} f$. This completes the proof of Theorem 4.4.

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Pradeep Boggarapu
Department of Mathematics
Indian Institute of Science
Bangalore-560012 (India)
pradeep@math.iisc.ernet.in

Sundaram Thangavelu
Department of Mathematics
Indian Institute of Science
Bangalore-560012 (India)
veluma@math.iisc.ernet.in