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Perturbed linear rough differential equations

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Abstract

We study linear rough differential equations and we solve perturbed linear rough differential equations using the Duhamel principle. These results provide us with a key technical point to study the regularity of the differential of the Itô map in a subsequent article. Also, the notion of linear rough differential equations leads to consider multiplicative functionals with values in Banach algebras more general than tensor algebras and to consider extensions of classical results such as the Magnus and the Chen-Strichartz formula.

Équations différentielles linéaires rugueuses perturbées

Résumé

Nous étudions les équations différentielles linéaires rugueuses et résolvons des équations linéaires rugueuses perturbées à l'aide du principe de Duhamel. Ces résultats donnent un argument technique pour étudier la différentiabilité de l'application d'Itô. La notion d'équation différentielle rugueuses nous condition à considérer des fonctionnelles multiplicatives à valeurs dans des algèbres de Banach plus générales que celle des algèbres tensorielles, ainsi que des extensions de résultats classiques tels que les formules de Magnus et Chen-Strichartz.

1. Introduction

Linear Rough Differential Equations (RDE) have been considered by several authors since they are an essential tool for studying the derivative of the Itô map and its flow properties (See the bibliography in [19]). However, with the exception of the works of D. Feyel, A. de la Pradelle and G. Mokobodzki [27] and S. Aida [1], linear RDE have hardly been considered as objects as such with their own properties, excepted to control the growth of the solution as in [28, 36]. Instead, they are generally presented

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as a special case of RDE. However, the Baker-Campbell-Hausdorff-Dynkin formula was among the inspirations of the theory of rough paths [3, 9]. The algebraic view of solution of controlled differential equations through for example Chen-Fliess series [17] as well as some numerical simulation algorithms stem directly from the theory of linear controlled ordinary differential equations (See [7] for an overview and [28, 29, 43, 42, 18, 44] for the relationship between algebra and RDE). Linear equations are among the first examples given in the article [43] and the book [42] as motivation for developing rough paths theory.

The main goal of this article is then to study linear RDE in a general setting. For this, we define the notion of *p-rough resolvent*, which is an extension of the notion of *multiplicative functionals* [43, 42] taking their values in a Banach algebra. Chen series, and their rough paths extensions which are the core of the theory, are solutions to linear RDE taking their values in tensor spaces, for which more precise results could be given.

Our initial motivation for this article was to consider the perturbation of the Itô map. We then deal with a variation of constant/Duhamel principle for perturbed linear RDE. Yet we also extend the Chen-Strichartz formula, and the Magnus formula providing exponential representations of solutions.

The content of this article may be applied to bounded linear operators on an infinite dimensional Banach space. Of course, dealing with unbounded family of operators, for example for solving Stochastic Partial Differential Equations, is much more intricate and needs specific treatments. The reader is referred to the quickly growing literature on these subjects [30, 29, 21, 10, 11], ... The variation of constant principle is also linked to Volterra equations which have been studied in the rough path context by A. Deya [22, 23].

In Section 3, we define for any $p \geq 1$ the notion of *rough resolvent* which is a family of linear operators giving the solutions to the linear RDE. The idea is to solve

$$\mathsf{Y}_t = \mathsf{Id} + \int_0^t \mathsf{Y}_s \, \mathrm{d}\mathsf{A}_s$$

when $(\mathsf{A}_t)_{t\in[0,T]}$ is an operator values path of finite p-variation through the constitutive relation $\mathsf{Y}_{r,t} = \mathsf{Y}_{r,s}\mathsf{Y}_{s,t}$ for any $0 \leq s \leq r \leq t \leq T$ with $\mathsf{Y}_{s,t} = \mathsf{Y}_s^{-1}\mathsf{Y}_t$. For this, we find an approximation $\mathsf{B}_{s,t}$ of $\mathsf{Y}_{s,t}$ when $t-s$ is small enough. The *sewing lemma*, which is the technical core of the theory of rough paths, is then extended to transform $\mathsf{B}_{s,t}$ into $\mathsf{Y}_{s,t}$.

In Section 4, we provide a series expansion for such solutions when the solutions are represented in the formal algebra of power series. As a byproduct, we obtain an exponential representation of the p-rough resolvents, at least for a small time. This provides us with extensions of the Magnus [7] and Chen-Strichartz formula [53]. Although these formulae were already known for the (fractional) Brownian motion (See *e.g.* [4, 5, 12, 6]), it was not clear whether or not it could be extended easily to the case of drivers of finite p-variation as soon as $p < 1$.

In Sections 5 and 6.1, we consider perturbed linear RDE of type

$$\mathsf{Y}_t = \mathsf{Id} + \int_0^t \mathsf{Y}_s \, \mathrm{d}\mathsf{A}_s + \mathsf{B}_t$$

and we show a variation of constant/Duhamel principle. Of course, when $p \in [2, 3)$, we need to know some appropriate extensions of both $\mathsf{A}_\bullet$ and $\mathsf{B}_\bullet$.

In Section 6.2, we show that our notion is well adapted for the kind of linear equations which arise when one differentiates the Itô map with respect to its starting point. In a subsequent article [19], we use these properties to show that the Itô map is differentiable with a Lipschitz or Hölder continuous Fréchet derivative.

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2. Notations and standard results

By C, we denote a constant whose value may vary from line to line.

We set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and V denotes a Banach space over $\mathbb{K}$ with a norm $|\cdot|$ and its dual V^*.

When $\mathrm{V} = \mathrm{U} \oplus \mathrm{W}$ for two Banach spaces U and W, we denote by $\pi_{\mathrm{U}} : \mathrm{V} \to \mathrm{U}$ the projection onto U orthogonal to W.

We also denote by $\mathfrak{L}$ a Banach algebra with a norm $\|\cdot\|$ (See Section 2.2 below for a definition). The unit element of $\mathfrak{L}$ is constantly denoted by Id.

2.1. Times, multiplicative properties and control

Fix $0 \le S < T$. Let

$$\begin{aligned}\Delta_2^+(S,T) &:= \{(s,t) | S \le s \le t \le T\},\\ \Delta_2^-(S,T) &:= \{(s,t) | S \le t \le s \le T\},\\ \Delta_3^+(S,T) &:= \{(s,r,t) | S \le s \le r \le t \le T\}\\ \text{and } \Delta_3^-(S,T) &:= \{(s,r,t) | S \le t \le r \le s \le T\}.\end{aligned}$$

For $i = 2, 3$, we write $\Delta_i^\pm(S,T)$ to denote either $\Delta_i^+(S,T)$ or $\Delta_i^-(S,T)$ depending on the context. When $S = 0$, we set $\Delta_i^\pm(T) := \Delta_i^\pm(S,T)$.

For the sake of simplicity, a family $(\mathsf{A}_t)_{t\in\mathbb{T}}$ is also denoted by $\mathsf{A}_\bullet$ when this notation introduces no ambiguity.

For a family $(\mathsf{A}_t)_{t\in\mathbb{T}}$ of elements of $\mathfrak{L}$ indexed by a set $\mathbb{T}$, we write

$$\mathsf{A}_\bullet^\sharp := \sup_{t\in\mathbb{T}} \|\mathsf{A}_t\|.$$

Definition 2.1. A family $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ is said to satisfy the *right/left multiplicative property* if

$$\mathsf{A}_{s,r}\mathsf{A}_{r,t} = \mathsf{A}_{s,t} \text{ for } (s,r,t) \in \Delta_3^\pm(T). \tag{2.1}$$

By this, we mean that a family $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^+(T)}$ satisfies the right multiplicative property if $\mathsf{A}_{s,r}\mathsf{A}_{r,t} = \mathsf{A}_{s,t}$ for $(s,r,t) \in \Delta_3^+(T)$, and a family $(\mathsf{A}_{t,s})_{(s,t)\in\Delta_2^+(T)}$ satisfies the left multiplicative property if $\mathsf{A}_{t,r}\mathsf{A}_{r,s} = \mathsf{A}_{t,s}$ for $(s,r,t) \in \Delta_3^+(T)$. This notational trick will be widely used below.

Definition 2.2 (Control)**.** A *control* is a function $\omega : \Delta_2^+(T) \to \mathbb{R}_+$ which is continuous close to its diagonal and such that $\omega(s,r) + \omega(r,t) \le \omega(s,t)$ for all $(s,r,t) \in \Delta_3^+(T)$.

We extend a control ω on $[0,T]^2$ by setting $\omega(t,s) = \omega(s,t)$ for $(s,t) \in \Delta_2^+(T)$.

Given a control ω, a constant C and $p \ge 1$, we write

$$\mathsf{A}_\bullet \prec C\omega^{1/p} \text{ to mean that } \|\mathsf{A}_{s,t}\| \prec C\omega(s,t)^{1/p} \text{ for } (s,t) \in \Delta_2^\pm(T)$$

for a family $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ of elements of $\mathfrak{L}$.

Remark 2.3. Of course, since $\omega(t,t) = 0$ for $t \in [0,T]$, $\mathsf{A}_\bullet \prec C\omega^{1/p}$ implies that $\mathsf{A}_{t,t} = 0$ for any $t \in [0,T]$.

A family indexed by $t \in [0, T]$ may be transformed into a family indexed by $(s, t) \in \Delta^{\pm}(T)$ satisfying the left/right multiplicative property.

Lemma 2.4. *Given a family $(\mathsf{A}_t)_{t\in[0,T]}$ of invertible elements in $\mathfrak{L}$, we set $\mathsf{A}_{s,t} := \mathsf{A}_s^{-1}\mathsf{A}_t$ for $(s, t) \in [0, T]^2$. Thus $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^+(T)}$ satisfies the right multiplicative property and $(\mathsf{A}_{t,s})_{(s,t)\in\Delta_2^+(T)}$ satisfies the left multiplicative property.*

2.2. Associative algebras and Banach algebras

A *Banach algebra* $\mathfrak{L}$ is a unital algebra $(\mathfrak{L}, +, \cdot)$ over $\mathbb{K}$ with a unit Id which is a Banach space with a norm $\|\cdot\|$ such that $\|\mathsf{ab}\| \leq \|\mathsf{a}\| \cdot \|\mathsf{b}\|$ for any $\mathsf{a}, \mathsf{b} \in \mathfrak{L}$, $\|\lambda\mathsf{a}\| = |\lambda| \cdot \|\mathsf{a}\|$ for $\lambda \in \mathbb{K}$ and $\|\mathsf{Id}\| = 1$ (See [24]).

2.2.1. Inverse, exponential and logarithm.

Several operations on $\mathfrak{L}$ may be defined using series representations. For example, when converging (for example when $\|\mathsf{a} - \mathsf{Id}\| < 1$),

$$\mathsf{a}^{-1} := \sum_{i=0}^{+\infty} (-1)^k (\mathsf{a} - \mathsf{Id})^k \tag{2.2}$$

is a left and right inverse of a. The exponential and logarithm maps are defined by

$$\exp(\mathsf{a}) = \mathsf{Id} + \sum_{k=1}^{+\infty} \frac{1}{k!}\mathsf{a}^k \text{ and } \log(\mathsf{a}) = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k}(\mathsf{a} - \mathsf{Id})^k \text{ when } \mathsf{a} \in \exp(\mathfrak{L}). \tag{2.3}$$

In particular, a condition for the existence of the logarithm is $\|\mathsf{a} - \mathsf{Id}\| < 1$. In this case, $\|\log(\mathsf{a})\| \leq \|\mathsf{a} - \mathsf{Id}\|/(1 - \|\mathsf{a} - \mathsf{Id}\|)$.

Lemma 2.5. *For $\mathsf{a}, \mathsf{b} \in \mathfrak{L}$,*

$$\|\exp(\mathsf{a}) - \exp(\mathsf{b})\| \leq \|\mathsf{a} - \mathsf{b}\| \exp(\|\mathsf{a}\| + \|\mathsf{b}\|),$$

$$\|\log(\mathsf{a} - \mathsf{Id}) - \log(\mathsf{b} - \mathsf{Id})\| \leq \frac{\|\mathsf{a} - \mathsf{b}\|}{\|\mathsf{a}\| + \|\mathsf{b}\|} \log \frac{1}{1 - \|\mathsf{a}\| - \|\mathsf{b}\|}$$

for $\|\mathsf{a}\| + \|\mathsf{b}\| < 1$,

$$\|\mathsf{a}^{-1} - \mathsf{b}^{-1}\| \leq \frac{\|\mathsf{b} - \mathsf{a}\|}{1 - \|\mathsf{a} - \mathsf{Id}\| - \|\mathsf{b} - \mathsf{Id}\|} \text{ for } \|\mathsf{a} - \mathsf{Id}\| + \|\mathsf{b} - \mathsf{Id}\| < 1.$$

Proof. All these inequalities stem from the following result easily shown by recurrence $\|\mathsf{a}^n - \mathsf{b}^n\| \leq \|\mathsf{a} - \mathsf{b}\|(\|\mathsf{a}\| + \|\mathsf{b}\|)^{n-1}$ for any $n \geq 1$. $\square$

There are several kind of associative algebras and Banach algebra we consider in this article.

2.2.2. **Space of operators**

Let $\mathfrak{L} = L(\mathrm{V}, \mathrm{V})$ be the space of linear bounded operators on V. The norm of $\mathfrak{L}$ is $\|\mathsf{A}\| = \sup_{u \in V,\ |u|=1} |\mathsf{A}u|$.

An operator in $L(\mathrm{V}, \mathrm{V})$ will be seen as an operator acting on the right, will an operator in $L(\mathrm{V}^*, \mathrm{V}^*)$ is seen as an operator acting on the left.

Then $\mathrm{V} = \mathbb{K}^d$, we identify $L(\mathrm{V}, \mathrm{V})$ with the space of matrices $\mathcal{M}_{d\times d}(\mathbb{K})$.

2.2.3. **Space of sequences**

An element a of $\mathfrak{L}^{\mathbb{Z}^+}$ will be written $\mathsf{a} = (\mathsf{a}_0, \mathsf{a}_1, \dots,)$. Let $\pi_k : \mathfrak{L}^{\mathbb{Z}^+} \to \mathfrak{L}$ and $\pi_{\leq k} : \mathfrak{L}^{\mathbb{Z}^+} \to \mathfrak{L}^{\mathbb{Z}^+}$ be defined

$$\pi_k(\mathsf{a}) = \mathsf{a}_k \text{ and } \pi_{\leq k}(\mathsf{a}) = (\mathsf{a}_0, \dots, \mathsf{a}_k, 0, \dots).$$

Let $\mathsf{1} := (\mathsf{Id}, 0, \dots)$. We also consider $\overline{\mathfrak{L}} = \{\mathsf{a} \in \mathfrak{L}^{\mathbb{Z}^+}; \pi_0(\mathsf{a}) = \alpha\mathsf{Id}, \alpha \in \mathbb{K}\}$ and we identify $(\alpha, \mathsf{a}_1, \mathsf{a}_2, \dots)$ with $(\alpha\mathsf{Id}, \mathsf{a}_1, \mathsf{a}_2, \dots)$ for $\mathsf{a} \in \overline{\mathfrak{L}}$.

For a subspace $\mathfrak{X}$ of $\mathfrak{L}^{\mathbb{Z}^+}$, we set $\pi_{\leq k}\mathfrak{X} = \{\pi_{\leq k}(\mathsf{a}); \mathsf{a} \in \mathfrak{X}\}$.

The space $(\overline{\mathfrak{L}}, +, \boxtimes)$ is an algebra with the convolution product $\boxtimes$ defined by

$$\pi_k(\mathsf{a} \boxtimes \mathsf{b}) = \sum_{i=0}^{k} \mathsf{a}_i \mathsf{b}_{n-i} \text{ for } k = 0, 1, 2, \dots .$$

For $\mathsf{a} \in \overline{\mathfrak{L}}$, set $\|\mathsf{a}\|_{\mathrm{sum}} := \sum_{i \geq 0} \|\pi_i(\mathsf{a})\|$.

Lemma 2.6. $\mathfrak{S} = \{\mathsf{a} \in \overline{\mathfrak{L}}; \|\mathsf{a}\|_{\mathrm{sum}} < +\infty\}$*, equipped with addition* $+$ *and product* $\boxtimes$*, is a Banach algebra with unit* $\mathsf{1}$*.*

2.2.4. **Graded algebra**

An associative algebra $\mathfrak{L}$ is a *graded algebra* if it may be decomposed as $\mathfrak{L} = \mathbb{K} \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \cdots$ where the $\mathfrak{L}_i$'s are vector spaces and $ab \in \mathfrak{L}_{i+j}$ when $a \in \mathfrak{L}_i$ and $b \in \mathfrak{L}_j$ for $i, j \geq 1$. We call $\mathfrak{L}_i$ the subspace of *elements homogeneous of degree* j.

When $\mathfrak{L}$ is a graded algebra, there exists an isomorphism ϕ between $(\mathfrak{L},+)$ and $(\overline{\mathfrak{L}},+)$ by setting $\pi_k\phi(\mathsf{a})=\mathsf{a}_k$ for $\mathsf{a}=\mathsf{a}_0\mathsf{Id}+\mathsf{a}_1+\cdots$ with $\mathsf{a}_k\in\mathfrak{L}_k$. We then define a norm $\|\cdot\|_{\mathrm{sum}}$ on $\mathfrak{L}$ by setting $\|\mathsf{a}\|_{\mathrm{sum}}:=\|\phi(\mathsf{a})\|_{\mathrm{sum}}$. Note that $\|\mathsf{a}\|\leq\|\mathsf{a}\|_{\mathrm{sum}}$. We then extend naturally π_k and $\pi_{\leq k}$ to a graded algebra $\mathfrak{L}$.

On the graded algebra, we define

$$\|\mathsf{a}\|_{\mathrm{hom}}:=\sup_{i\geq 1}\{\|\pi_i(\mathsf{a})\|^{1/i}\}. \tag{2.4}$$

2.2.5. **Tensor algebra**

Given a vector space U, we construct a *tensor algebra* by

$$\mathfrak{T}(\mathrm{U})=\mathbb{K}\oplus\mathrm{U}\oplus(\mathrm{U}\otimes\mathrm{U})\oplus(\mathrm{U}\otimes\mathrm{U}\otimes\mathrm{U})\oplus\cdots$$

with the tensor product $\otimes$. This is a graded algebra.

The tensor space $(\mathrm{U})^{\otimes\ell}$ is endowed with a norm $|\cdot|$ such that $|a\otimes b|\leq|a|\cdot|b|$ for any $a\in(\mathrm{U})^{\otimes\ell'}$ and $b\in(\mathrm{U})^{\otimes(\ell-\ell')}$, $\ell'=1,\dots,\ell-1$ [24, Ex. 1.36 and 2.31]. This norm is then extended to $\mathfrak{T}(\mathrm{U})$.

For the sake of notational simplicity, for $k=1,\dots,\infty$, we write $\mathrm{T}_k(\mathrm{U})$ for the subset of elements a of $\pi_{\leq k}\mathfrak{T}(\mathrm{U})$ with $\|\mathsf{a}\|_{\mathrm{sum}}<+\infty$.

With the tensor product $\otimes_k$ defined by $\mathsf{a}\otimes_k\mathsf{b}=\pi_{\leq k}(\mathsf{a}\otimes\mathsf{b})$, $(\mathrm{T}_k(\mathrm{U}),+,\otimes_k)$ is a Banach algebra with $\|\cdot\|_{\mathrm{sum}}$. The space $\mathrm{T}_k(\mathrm{U})$ is the quotient space $\mathrm{T}_\infty(\mathrm{U})/\sim_k$ with $\mathsf{a}\sim_k\mathsf{b}$ when $\pi_{\leq k}(\mathsf{a}-\mathsf{b})=0$. To simplify the notations, when there is no ambiguity we simply write $\|\cdot\|$ for $\|\cdot\|_{\mathrm{sum}}$ and $\otimes$ for $\otimes_k$.

Remark 2.7. With $\phi:x\in\mathrm{V}\mapsto 1+x\in\mathrm{T}_1(\mathrm{V})$, we embed V into $\mathrm{T}_1(\mathrm{V})$. Since $\exp(\mathrm{V})=\{1+x\in\mathrm{T}_1(\mathrm{V});x\in\mathrm{V}\}$ forms a group, $(\mathrm{V},+)$ is isomorphic to $(\exp(\mathrm{V}),\otimes_1)$. Besides, $|x|\leq\|\phi(x)\|_{\mathrm{sum}}$ and $\|\phi(x)-1\|_{\mathrm{sum}}=0$ implies that $x=0$.

2.2.6. **Algebra of words**

Fix $d\in\mathbb{N}$, let us consider the *algebra of words* $\mathfrak{W}$ whose basis is

$$\{\emptyset\}\cup\{I=(i_1,\dots,i_k)\text{ with }(i_1,\dots,i_k)\in\{1,\dots,d\}^k,\ k\in\mathbb{N}\}.$$

The multiplication on $\mathfrak{W}$ is defined by the concatenation

$$IJ=(i_1,\dots,i_k,j_1,\dots,j_\ell)\text{ for }I=(i_1,\dots,i_k)\text{ and }J=(j_1,\dots,j_k),$$

and $\emptyset I = I\emptyset = I$. The word $\emptyset$ is the *null word*. For $I = (i_1, \dots, i_k)$, we write $|I| = k$, the *length* of I, and $|\emptyset| = 0$. The algebra of words is a *graded algebra* $\mathbb{K} \oplus \mathfrak{W}_1 \oplus \cdots$ where $\mathfrak{W}_k = \mathrm{Span}\{I; |I| = k\}$.

When U is a finite dimensional vector space with basis $\{e_1, \dots, e_d\}$, then $\mathfrak{W}$ is homomorphic to $\mathfrak{T}(\mathrm{U})$ through ϕ by setting

$$\phi(I) = e_I := e_{i_1} \otimes \cdots \otimes e_{i_k} \text{ when } I = (i_1, \dots, i_k).$$

More generally, for a finite family $\mathfrak{X} := \{\mathsf{a}^1, \dots, \mathsf{a}^d\}$ of d elements, called the *alphabet*, we write $\mathsf{a}^I := \mathsf{a}^{i_1} \cdots \mathsf{a}^{i_k}$ for $I = (i_1, \dots, i_k)$. This way, to $\mathfrak{X}$ is associated $\mathfrak{T}(\mathfrak{X}) = \mathbb{K} \oplus \mathfrak{L}_1 \oplus \cdots$ which is isomorphic to $\mathfrak{T}(\mathbb{R}^d)$ through the isomorphism defined by $\phi(\mathsf{a}^I) = e_I$ for any word I.

2.2.7. **Free Lie algebra and groups**

A *Lie algebra* $\mathfrak{l}$ is a vector space over $\mathbb{K}$ with a bilinear mapping $[\cdot, \cdot]$ satisfying $[\mathsf{a}, \mathsf{b}] = -[\mathsf{b}, \mathsf{a}]$ and the Jacobi identity $[\mathsf{a}, [\mathsf{b}, \mathsf{c}]] + [\mathsf{c}, [\mathsf{a}, \mathsf{b}]] + [\mathsf{b}, [\mathsf{c}, \mathsf{a}]] = 0$ for any $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathfrak{l}$. For $\mathsf{a}, \mathsf{b} \in \mathfrak{L}$, $[\mathsf{a}, \mathsf{b}] = \mathsf{ab} - \mathsf{ba}$ defines a Lie bracket [33].

Let $\mathfrak{X} = \{\mathsf{a}^1, \dots, \mathsf{a}^d\}$ be a finite set of d elements. A *free Lie algebra* on $\mathfrak{X}$ is a Lie algebra $\mathfrak{Lie}(\mathfrak{X})$ and a map $\imath : \mathfrak{X} \to \mathfrak{Lie}(\mathfrak{X})$ such that for any Lie algebra $\mathfrak{g}$ and any $\jmath : \mathfrak{X} \to \mathfrak{g}$ there exists a unique Lie algebra morphism $k : \mathfrak{Lie}(\mathfrak{X}) \to \mathfrak{g}$ such that $k \circ \imath = \jmath$ (See *e.g.* [9, 51]). In addition, the Lie algebra $\mathfrak{Lie}(\mathfrak{X})$ is isomorphic to the Lie algebra $\mathfrak{Lie}(\mathbb{R}^d)$ where for a d-dimensional vector space V with basis $\{e_i\}_{i=1,\dots,d}$,

$$\mathfrak{Lie}(\mathrm{V}) := \bigoplus_{k \geq 1} \mathfrak{Lie}_k(\mathrm{V}), \ \ \mathfrak{Lie}_1(\mathrm{V}) := \mathrm{V}$$
$$\text{and } \mathfrak{Lie}_{k+1}(\mathrm{V}) := \mathrm{Span}\{[\mathsf{a}, \mathsf{b}]; \mathsf{a} \in \mathrm{V}, \mathsf{b} \in \mathfrak{Lie}_k(\mathrm{V})\}$$

with $[e_i, e_j] = e_i \otimes e_j - e_j \otimes e_i$. Indeed, $\mathfrak{Lie}(\mathrm{V})$ is the smallest Lie subalgebra of $\mathfrak{T}(\mathrm{V})$ containing V.

Basically, in the theory of rough paths, there are three kinds of free Lie algebras that are under consideration:

- Tensor algebras and tensor Lie algebras, which allows one to define rough paths, following the work of K.T. Chen on iterated integrals [14].
- Lie groups of matrices, in which live the flows of some linear differential equations [3].
- Left-invariant vector fields [8].

2.2.8. **Baker-Campbell-Hausdorff-Dynkin formula**

For a and b in a Banach algebra $\mathfrak{L}$, the Baker-Campbell-Hausdorff-Dynkin (BCHD) formula states that under appropriate conditions, there exists c such that $\exp(\mathsf{a})\exp\mathsf{b} = \exp(\mathsf{c})$ with

$$\mathsf{c} := \sum_{j=1}^{+\infty}\sum_{n=1}^{j}\sum_{\substack{(H,K)\in\mathbb{W}_n\\|H|+|K|=j}} \frac{1}{H!K!(h_1+\cdots+h_n+k_1+\cdots+k_n)} D_{h,k}(\mathsf{a},\mathsf{b}) \tag{2.5}$$

with $H! = h_1!\cdots h_n!$,

$$D_{H,K}(\mathsf{a},\mathsf{b}) = \overbrace{[\mathsf{a},\cdots[\mathsf{a},}^{h_1\text{ times}}\overbrace{[\mathsf{b},\cdots[\mathsf{b},}^{k_1\text{ times}}\cdots\overbrace{[\mathsf{a},\cdots[\mathsf{a},}^{h_n\text{ times}}\overbrace{[\mathsf{b},[\cdots,\mathsf{b}}^{k_n\text{ times}}]]]\cdots]\cdots]\cdots]]\cdots],$$

and

$$\mathbb{W}_n = \left\{(I,J)\in(\mathbb{Z}^+)^n; (i_1,j_1),\dots,(i_n,j_n)\neq(0,0)\right\}.$$

When this formula holds, we write $\mathsf{a}\star\mathsf{b} := \mathsf{c}$. If $\mathfrak{L}$ is a nilpotent matrix group, then $\mathsf{a}\star\mathsf{b}$ is always defined. Otherwise, for a simply connected group, it holds for elements with norms small enough. A detailed discussion in given in [9, Sect. 5.5]. The BCHD formula stems from combinatorial considerations, so that (2.5) is formally valid in associative algebras.

Theorem 2.8 (Convergence of the BCHD formula [9, Theorem 5.54, p. 341])**.** *For* $\mathsf{a},\mathsf{b}\in\mathfrak{L}$ *with* $\max\{\|\mathsf{a}\|,\|\mathsf{b}\|\}\leq\frac{1}{4}\log(2)$, *the series in* (2.5) *is absolutely convergent. Besides, the series in* (2.5) *is totally convergent on any set* $\{(\mathsf{a},\mathsf{b}), \|\mathsf{a}\|+\|\mathsf{b}\|\leq\delta\}$, $0<\delta<\frac{1}{2}\log 2$.

There exist several alternative representations for the series giving $\mathsf{a}\star\mathsf{b}$. We refer to [9] for a detailed account on this rich topic and various proofs.

2.2.9. **Shuffle algebra and Lie elements**

The shuffle algebra is the main tool to relate Lie elements and elements in tensor algebras.

Definition 2.9 (Shuffle product and shuffle algebra)**.** For words $I = (i_1,\dots,i_{|I|})$ and $J = (j_1,\dots,j_{|J|})$, let $\mathrm{Sh}(I,J)$ be the set of words of type $K = (k_1,\dots,k_{|I|+|J|})$ such that each letter of K corresponds exactly to one letter of I or J and the order of the letters of I and the order of the letters in J is preserved.

Hence the *shuffle product* on two words is defined by linearity from $I \circledast J = \sum_{K\in\mathrm{Sh}(I,J)} K$ for words I and J in $\mathfrak{W}$, and $(\mathfrak{W}, +, \circledast)$ is the *shuffle algebra.*

Definition 2.10 (Lie element)**.** An element $x = \sum_I e_I x^I$ of $\mathfrak{T}(\mathbb{R}^d)$ is called a *Lie element* if $x_\emptyset = 0$ and for each k, the homogeneous term $\sum_{I;|I|=k} e_I x^I$ of length k belongs to $\mathfrak{Lie}_k(\mathbb{R}^d)$.

Since $\mathfrak{T}(\mathbb{R}^d)$ is a unital algebra, we define inverse, exponential and logarithm by power series using formal series given by (2.2) and (2.3). We set

$$\mathfrak{gr}(\mathbb{R}^d) = \exp(\mathfrak{Lie}(\mathbb{R}^d)) \subset \mathfrak{T}(\mathbb{R}^d).$$

Hence, if x is a Lie element, then $\exp(x) \in \mathfrak{gr}(\mathbb{R}^d)$ and such an element is called *group like*, and $(\mathfrak{gr}(\mathbb{R}^d), \otimes)$ is a group, as $\exp(x)\otimes\exp(y) = \exp(x\star y)$ with $x \star y$ given formally by (2.5).

Remark 2.11. When $\mathfrak{l}$ is a matrix Lie algebra, then $\exp(\mathfrak{l})$ is a matrix Lie group with respect to the product of matrices [3].

Following the work of K.T. Chen [15], R. Ree has proved the following result.

Theorem 2.12 (R. Ree [50, Theorem 2.5])**.** *For elements $x \in \mathfrak{T}(\mathbb{R}^d)$ of type*

$$x = 1 + \sum_{I;|I|\geq 1} e_I \alpha^I,\ \alpha^I \in \mathbb{K},$$

then $\log(x)$ is a Lie element if and only if the linear map $\phi : \mathfrak{W} \to \mathbb{K}$ defined by $\phi(\emptyset) = 1$ and $\phi(I) = \alpha^I$ is an algebra homomorphism between $(\mathfrak{T}(\mathbb{R}^d), +, \otimes)$ and $(\mathfrak{W}, +, \circledast)$, that is if and only if $\alpha^I\alpha^J = \sum_{K\in\mathrm{Sh}(I,J)} \alpha^K$ for any words I and J.

Besides, if the coefficients satisfies the shuffle relation,

$$x^{-1} = 1 + \sum_{k\geq 0} \sum_{I;|I|=k} (-1)^k \alpha^{\overleftarrow{I}} e_I$$

with $\overleftarrow{I} = (i_k, \dots, i_1)$ when $I = (i_1, \dots, i_k)$.

Example: The Heisenberg group. The Heisenberg group provides us with a simple non trivial example of a non-commutative matrix Lie group, which is nilpotent of step 2. The Heisenberg algebra is

$$\mathfrak{h} := \left\{ \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \middle| (a, b, c) \in \mathbb{R}^3 \right\}.$$

The space $\mathfrak{h}$ is the Lie algebra of the Heisenberg group

$$\mathfrak{H} := \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \middle| (a, b, c) \in \mathbb{R}^3 \right\} = \exp(\mathfrak{h}).$$

The product of 3 matrices in $\mathfrak{h}$ equals 0. Indeed, $\mathfrak{h}$ is a simply connected nilpotent Lie algebra of step 2. Let $A = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ and $C = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. Then $[A, B] := AB - BA = C$ so that the Lie algebra $\mathfrak{h}$ is generated by the two matrices A and B, although $\mathfrak{h}$ is a vector space of dimension 3.

The BCHD formula is always valid, with

$$\exp(A)\exp(B) = \exp(A \star B) \text{ with } A \star B = A + B + \frac{1}{2}[A, B] \text{ for } A, B \in \mathfrak{h}.$$

2.3. Spaces of functions

2.3.1. Chen series and Chen-Strichartz formula

The notion of *Chen series*, initiated by K.T. Chen in the 50's, provides us with a way to transform a regular path $x : [0, T] \to \mathbb{R}^d$ into an algebraic object taking its values in $\mathfrak{T}(\mathbb{R}^d)$. This is one of the core ideas of the theory of rough paths to extend this theory to paths of irregular variations.

In the next statement, formula (2.7) has been given by R. Strichartz in [53] and relies on the Baker-Campbell-Hausdorff-Dynkin formula.

Theorem 2.13 (K.T. Chen [16, 14, 15], R. Strichartz [53])**.** *Let x be a path of bounded variation with values in $\mathbb{R}^d$. Then the solutions, called* Chen series*, to*

$$\mathbf{x}_t = 1 + \int_0^t \mathbf{x}_s \otimes \mathrm{d}x_s \text{ and } \mathbf{y}_t = 1 - \int_0^t \mathrm{d}x_s \otimes \mathbf{y}_s \tag{2.6}$$

are such that for any $t\in[0,T]$, $\mathbf{x}_t=\mathbf{y}_t^{-1}$,

$$\mathbf{x}_t=1+\sum_{I;|I|\geq 1} e_I\mathbf{x}_{0,t}^I \text{ and } \mathbf{y}_t=1+\sum_{I;|I|\geq 1} e_I\mathbf{x}_{0,t}^{\overleftarrow{I}} \text{ with } \mathbf{x}_{s,t}^{Ij}=\int_s^t \mathbf{x}_{s,r}^I\,\mathrm{d}x_r^j.$$

Besides, $\log(\mathbf{x}_{s,t})$ *and* $\log(\mathbf{y}_{s,t})$ *are Lie elements in* $\mathfrak{T}(\mathbb{R}^d)$, *and for* $e_{[I]}=[e_{i_1},[e_{i_2},\dots,[e_{i_{k-1},e_{i_k}}]\cdots]]$,

$$\log(\mathbf{x}_{s,t})=\sum_{k\geq 1}\sum_{I;|I|=k}\gamma_I(s,t)e_{[I]} \text{ with } \gamma_I(\bullet):=\sum_{\sigma\in\mathrm{Perm}_k}\frac{(-1)^{e(\sigma)}}{k^2\binom{k-1}{e(\sigma)}}\mathbf{x}_\bullet^{\sigma(I)} \tag{2.7}$$

where Perm_k *is the set of permutations of* $\{1,\dots,k\}$,

$$\sigma(I):=(\sigma(i_1),\dots,\sigma(i_k)) \text{ for } I=(i_1,\dots,i_k)$$
$$\text{and } e(\sigma):=\#\{j\in\{1,\dots,k-1\};\sigma(j)>\sigma(j+1)\}.$$

Let us consider a family $\mathsf{A}^1,\dots,\mathsf{A}^d$ of elements in $\mathfrak{B}$ as well as a path $x:\mathbb{R}_+\to\mathbb{R}^d$ of bounded variation. Set $\mathsf{A}_t=\sum_{i=1}^d\mathsf{A}^i x_t^i$ for $t\in[0,T]$. The linear equation

$$\mathsf{Y}_t=\mathsf{Id}+\int_0^t\mathsf{Y}_r\,\mathrm{d}\mathsf{A}_r=\mathsf{Id}+\int_0^t\mathsf{Y}_r\sum_{i=1}^d\mathsf{A}^i\,\mathrm{d}x_r^i \tag{2.8}$$

is easily solved by considering the algebra homomorphism $\psi:\mathfrak{T}(\mathbb{R}^d)\to\mathfrak{B}$ by $\psi(e_I)=\mathsf{A}^I$ with the conventions of Section 2.2.6. Indeed, for $\mathbf{x}$ solution to (2.6) with $x_t=\sum_{i=1}^d e_i x_t^i$, $\mathsf{Y}_t=\psi(\mathbf{x}_t)$, $t\in[0,T]$.

The proof of eq. 2.7 relies on the Baker-Campbell-Hausdorff-Dynkin formula, and contains it. For this, simply set $x_t^1=t\mathbf{1}_{t\in[0,1]}+1\mathbf{1}_{t\in(1,2]}$ and $x_t^2=(t-1)\mathbf{1}_{t\in[1,2]}$ for $t\in[0,2]$. Then $\log(\mathsf{Y}_2)=\log(\exp(A^1)\exp(A^2))$ with $\mathsf{Y}_\bullet$ solution to (2.8) by applying the homomorphism ϕ to (2.7).

For an operators-valued path $(\mathsf{A}_t)_{t\in[0,T]}$ of bounded variation and a partition $\{t_i^n\}_{i=0}^n$ of $[0,T]$, we may consider solving

$$\mathsf{Y}_t^n=\mathsf{Id}+\int_0^t\mathsf{Y}_r^n\sum_{i=0}^{n-1}\mathbf{1}_{r\in[t_i^n,t_{i+1}^n)}\mathsf{A}_{t_i^n,t_{i+1}^n}. \tag{2.9}$$

This means that (2.9) may be seen as a special case of (2.8) with

$$d=n,\ e_i=\mathsf{A}_{t_i^n,t_{i+1}^n} \text{ and } x_t^i=(t-t_i^n)\mathbf{1}_{t\in[t_i^n,t_{i+1}^n)}+(t_{i+1}^n-t_i^n)\mathbf{1}_{t\in[t_i^n,t_{i+1}^n]}.$$

With Theorem 2.12, Y_t may be expressed as the exponential of operators defined as series defined in the Lie algebra containing the A_t, $t \in [0,T]$. This remains true as $n \to \infty$ [53].

The Magnus formula, and its variant, gives an explicit expression for this formula, and could be thought as a "continuous analogue of the Baker-Campbell-Hausdorff-Dynkin formula". We refer to [47, 7, 48] on this topic.

2.4. **Functions of finite p-variation**

For $p \geq 1$, let $\mathcal{R}^p(\mathrm{V})$ be the set of continuous paths with values in V such that $|x_0| < +\infty$ and $x_\bullet \prec C\omega^{1/p}$, which means with our conventions of Section 2.1 and Remark 2.7, that

$$\|x\|_p := \sup_{(s,t)\in\Delta_2^+(T)} \frac{|x_{s,t}|}{\omega(s,t)^{1/p}} < +\infty \text{ with } x_{s,t} := x_t - x_s.$$

Note that $\|\cdot\|_p$ is just a semi-norm, but $x \mapsto \|x\|_p + |x_0|$ defines a norm. An element in $\mathcal{R}^p(\mathrm{V})$ is called a function of *finite p-variation controlled by ω*. Under the condition that $\omega(s,t) = t - s$, then paths in $\mathcal{R}^p(\mathrm{V})$ are α-Hölder continuous with $\alpha = 1/p$.

For $a \in \mathrm{V}$, we set $\mathcal{R}^p_a(\mathrm{V})$ the subset of paths $x \in \mathcal{R}^p(\mathrm{V})$ with $x_0 = a$.

For $\theta > 1$, if $|x_{s,t}| \leq C\omega(s,t)^\theta$ for $(s,t) \in \Delta_2^+(T)$, then x is constant since for any partition $\{t_i\}_{i=0,\dots,n}$ of $[s,t]$,

$$|x_{s,t}| \leq \sum_{i=0}^{n-1} |x_{t_i,t_{i+1}}| \leq \omega(0,T) \sup_{i=0,\dots,n-1} \omega(t_i,t_{i+1})^{\theta-1}.$$

2.5. **Rough paths**

Let us consider a control $\omega : \Delta_2(T) \to \mathbb{R}_+$.

Definition 2.14 (p-rough path)**.** A p-rough path $\mathbf{x}$ of order ℓ is a path taking its values in $\mathrm{T}_\ell(\mathrm{V})$ with

(i) The order ℓ satisfies $\ell \geq \lfloor p \rfloor$, where $\lfloor a \rfloor$ is the integer part of a.
(ii) For any $t \in [0,T]$ $\pi_0(\mathbf{x}_t) = 1$ and $\mathbf{x}_t$ is invertible in $\mathrm{T}_\ell(\mathrm{V})$.
(iii) With (2.4), $\|\mathbf{x}_\bullet\|_{\mathrm{hom}} \prec C\omega^{1/p}$.

We write

$$\mathbf{x}_{s,t} = \sum_{k\geq 0} \mathbf{x}^{(k)}_{s,t} \text{ with } \mathbf{x}^{(k)}_{s,t} = \sum_{\text{words } I,\ |I|=k} e_I \mathbf{x}^I_{s,t}$$

with $e_\emptyset = 1$ and $\mathbf{x}^\emptyset_{s,t} = 1$ for the null word $\emptyset$.

We denote by $\mathcal{RP}^p_\ell(\mathbb{R}^d)$ the set of p-rough paths. This space is equipped with the semi-norm

$$\|\mathbf{x}\|_p := \sup_{(s,t)\in\Delta_2^+(T)} \sup_{k=1,\dots,\ell} \frac{|\mathbf{x}^{(k)}_{s,t}|}{\omega(s,t)^{k/p}}$$

and the norm $\|\mathbf{x}\|_{\infty,p} := \sup_{t\in[0,T]} |\mathbf{x}_t| + \|\mathbf{x}\|_p$.

As we can see, a rough path is an extension of a function of finite p-variation and $\mathcal{RP}^p(\mathbb{R}^d) = \mathcal{R}^p(\mathbb{R}^d)$ for $p \in [1,2)$.

Let us present briefly some particular classes of rough paths (See [28, 42] for a detailed account on these notions).

- A *smooth rough path* is a rough path $\mathbf{x} \in \mathcal{RP}^p_\ell(\mathbb{R}^d)$ whose projection on $\mathbb{R}^d$ is a smooth path from $[0,T]$ to $\mathbb{R}^d$ and such that

$$\mathbf{x}^I_{s,t} = \iint_{0\leq s_1<\cdots<s_k\leq t} \mathrm{d}x^{i_1}_{s,t_1} \cdots \mathrm{d}x^{i_k}_{s,s_k}$$

  for any word $I = i_1 \cdots i_k$, $k \leq \ell$. Such a path takes its values in $\pi_{\leq\ell}\mathfrak{gr}(\mathbb{R}^d)$. It is the projection onto $\mathrm{T}_\ell(\mathbb{R}^d)$ of the Chen series of Theorem 2.13 above a given path.
- A *geometric rough path* is the closure in $\mathcal{RP}^p_\ell(\mathbb{R}^d)$ with respect $\|\cdot\|_{\infty,p}$ of the set of smooth rough paths.
- A *weak geometric rough path* is a rough path taking its values in $\mathfrak{G}_\ell(\mathbb{R}^d)$. The set of such paths is denoted by $\mathcal{WGRP}^p_\ell(\mathbb{R}^d)$.

Indeed, any path in $\mathcal{WGRP}^p_\ell(\mathbb{R}^d)$ is the limit of a sequence of smooth rough paths in $\mathcal{RP}^q_\ell(\mathbb{R}^d)$ for $q > p$. Besides, for $p \in [2,3)$, non-geometric rough paths may be interpreted as geometric ones lying above a path with values in $\mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$ [40]. With the appropriate algebraic structure involving trees, a similar result also holds for any value of p [32].

We refer to [43, 42, 28, 35, 37] among others for the properties of rough paths.

2.6. Young integrals

The theory of rough paths is an extension of the theory of Young integrals [42, 54, 27].

Let X, Y and Z be Banach spaces with a product $(x,y) \in \mathrm{X}\times\mathrm{Y} \mapsto xy \in \mathrm{Z}$, such that $|xy| \leq |x|\cdot|y|$.

Let x and y be two paths of finite p-variation controlled by ω with $p < 2$ respectively with values in X and Y.

In the sequel, we make use of the following inequalities:

$$\|xy\|_p \leq \|x\|_p y^\sharp_\bullet + \|y\|_p x^\sharp_\bullet \quad \text{and} \quad x^\sharp_\bullet \leq |x_0| + \omega(0,T)^{1/p}\|x\|_p. \tag{2.10}$$

For any $(s,t) \in \Delta_2^+(T)$, $\int_s^t y_r \,\mathrm{d}x_r$ may be defined as a *Young integral*, that is as limit of Riemann sums $\sum_{i=0}^{n-1} y_{t_i^n}(x_{t_{i+1}^n} - x_{t_i^n})$ for partitions $\{t_i^n\}_{i=0}^n$ whose meshes decrease to 0.

Let us recall some standard results about Young integrals.

Proposition 2.15. *With the above setting,*

(i) *If $u \in \mathcal{R}^p(\mathrm{Z})$ satisfies for $C \geq 0$ and $\theta > 1$,*

$$u_0 = 0 \quad \textit{and} \quad (u_t - u_s - y_s x_{s,t})_{(s,t)\in\Delta_2^+(T)} \prec C\omega^\theta,$$

then $u_t = \int_0^t y_s \,\mathrm{d}x_s$ for $t \in [0,T]$.

(ii) *For any $(s,t) \in \Delta_2^+(T)$,*

$$\left|\int_s^t y_r \,\mathrm{d}x_r - y_s(x_t - x_s)\right| \leq \zeta(2p)\|x\|_p\|y\|_p\omega(s,t)^{2/p} \tag{2.11}$$

with $\zeta(q) := \sum_{n\geq 1} 1/n^q$, $q > 1$.

(iii) *The following control holds:*

$$\left\|\int_0^\cdot y_r \,\mathrm{d}x_r\right\|_p \leq (|y_0| + \|y\|_p\omega(0,T)^{1/p})\|x\|_p + \zeta(2p)\|x\|_p\|y\|_p\omega(0,T)^{1/p}. \tag{2.12}$$

This construction is very general. We will either use $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = \mathfrak{L}$ for an algebra $\mathfrak{L}$ defined as in Section 3.3, or $\mathrm{X} = \mathrm{Z} = \mathrm{V}$ and $\mathrm{Y} = L(\mathrm{V},\mathrm{V})$ for a Banach space V.

We present only Young integrals. From the results in Proposition 2.15, differential equations driven by finite p-variation paths with $p < 2$ are easy to consider [38].

2.7. The Gamma function and the neo-classical inequality

Let $\Gamma(z) := \int_0^{+\infty} e^{-t}t^{z-1}\,\mathrm{d}t$ be the Gamma function [52, Chap. 6, p .76]. We use a majoration as in [1].

Lemma 2.16. *Let x be a positive real and $p \geq 1$. Then for $\ell \geq \lfloor p \rfloor$,*

$$\sum_{k \geq \ell} \frac{x^{k/p}}{\Gamma(k/p)} \leq (1 + \lfloor p \rfloor) \exp(x) \frac{x^{\lfloor \ell/p \rfloor + 1}}{\Gamma(\lfloor \ell/p \rfloor + 1)} (1 \vee x).$$

Proof. For an integer i, the set of integers k such that $\lfloor k/p \rfloor = i$ is included in $\{\lceil ip \rceil, \dots, \lfloor p(i+1) \rfloor\}$. This set contains at most $(\lfloor p \rfloor + 1)$ elements. Hence,

$$\sum_{k \geq \ell} \frac{x^{k/p}}{\Gamma(k/p)} \leq \sum_{i \geq \lfloor \ell/p \rfloor + 1} \sum_{k \in \mathbb{N} \text{ s.t. } \lfloor k/p \rfloor = i} \frac{x^i (x \vee 1)}{\Gamma(i + (k/p - i))}.$$

The Γ function is increasing, so that

$$\sum_{k \geq \ell} \frac{x^{k/p}}{\Gamma(k/p)} \leq \sum_{i \geq \lfloor \ell/p \rfloor} (\lfloor p \rfloor + 1) \frac{x^i (x \vee 1)}{\Gamma(i)} \leq (\lfloor p \rfloor + 1)(x \vee 1) \sum_{i \geq \lfloor \ell/p \rfloor} \frac{x^i}{\Gamma(i)}.$$

Again with the properties of the Gamma function,

$$\sum_{i \geq k} \frac{x^i}{\Gamma(i)} \leq \frac{x^k}{\Gamma(k)} \exp(x).$$

since $\Gamma(i+k) \geq \Gamma(i)\Gamma(k)$ for $i, k \geq 1$ [52]. Hence the result. □

We give give now the so-called neo-classical inequality, proved first by T. Lyons [43] and then improved by K. Hara and H. Masanori [34].

Proposition 2.17. *[Neo-classical inequality,* [42, Theorem 3.1.1, p.35], [34]*] For any $p \geq 1$, $n \in \mathbb{N}$ and $a, b \geq 0$,*

$$\sum_{i=0}^{n} \frac{a^{i/p} b^{(n-i)/p}}{\Gamma\left(\frac{i}{p}\right) \Gamma\left(\frac{n-i}{p}\right)} \leq p \frac{(a+b)^{n/p}}{\Gamma\left(\frac{n}{p}\right)}. \tag{2.13}$$

3. Rough resolvent

Before introducing our main result, we present the features of linear RDE in the case $p < 2$ which motivates our definition of a rough resolvent.

3.1. **Linear RDE in the Young case**

Let $\mathcal{A} : [0,T] \to \mathfrak{L}$ be a path of finite p-variation with $1 \leq p < 2$, that is $\mathcal{A} \in \mathcal{R}^p(\mathfrak{L})$.

Let us consider the equations

$$\mathsf{Y}_t = \mathsf{Id} + \int_0^t \mathsf{Y}_r \,\mathrm{d}\mathcal{A}_r \tag{3.1}$$

$$\text{and } \mathsf{Z}_t = \mathsf{Id} - \int^t \mathrm{d}\mathcal{A}_r \mathsf{Z}_r \text{ for } t \in [0,T]. \tag{3.2}$$

Proposition 3.1. *The following properties hold:*

(i) *There exist unique solutions* Y *and* Z *to* (3.1) *and* (3.2).

(ii) *For some constant* C *and* $(s,t) \in \Delta_2^+(T)$,

$$\|\mathsf{Y}_t - \mathsf{Y}_s - \mathsf{Y}_s\mathcal{A}_{s,t}\| \leq C\omega(s,t)^{2/p} \text{ and } \|\mathsf{Z}_t - \mathsf{Z}_s + \mathcal{A}_{s,t}\mathsf{Z}_s\| \leq C\omega(s,t)^{2/p} \tag{3.3}$$

and $\mathsf{Y}, \mathsf{Z} \in \mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$.

(iii) *Any paths* Y *and* Z *in* $\mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$ *satisfying* (3.3) *are solutions to equations* (3.1) *and* (3.2).

(iv) *For any* $t \in [0,T]$, $\mathsf{Y}_t\mathsf{Z}_t = \mathsf{Z}_t\mathsf{Y}_t = \mathsf{Id}$.

(v) *For* $\mathsf{A}_{s,t} := \mathsf{Y}_s^{-1}\mathsf{Y}_t$ *with* $(s,t) \in [0,T]^2$, $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ *satisfies the right/left multiplicative property.*

(vi) *For* $(s,t) \in \Delta_2^\pm(T)$,

$$\|\mathsf{A}_{s,t} - \mathsf{Id} - \mathcal{A}_{s,t}\| \leq C\omega(s,t)^{2/p}. \tag{3.4}$$

Proof. Fix $T_1 \leq T$ and consider only $t \in [0,T_1]$. Set $\mathsf{Y}^{(0)}_t = \mathsf{Id}$ and $\mathsf{Y}^{(k+1)}_t = \mathsf{Id} + \int_0^t \mathsf{Y}^{(k)}_s \,\mathrm{d}\mathcal{A}_s$. From the properties of the Young integral, since $p \leq 2$ and $\mathsf{Y}^{(0)} \in \mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$, an induction proves that $\mathsf{Y}^{(k)} \in \mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$. Besides, the Young integral is linear, so that

$$\|\mathsf{Y}^{(k+1)} - \mathsf{Y}^{(k)}\|_p \leq \zeta(2p)\|\mathsf{Y}^{(k)} - \mathsf{Y}^{(k-1)}\|_p\|\mathcal{A}\|_p\omega(0,T_1)^{1/p}. \tag{3.5}$$

For T_1 such that $\zeta(2p)\|\mathcal{A}\|_p\omega(0,T_1)^{1/p} < 1$, $(\mathsf{Y}^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{R}^p(\mathfrak{L})$ which converges in q-variation for any $q > p$ to $\mathsf{Y} \in \mathcal{R}^p(\mathfrak{L})$ which is solution to (3.1).

With an inequality similar to (3.5), the solution Y to (3.1) is unique, and by linearity, the solution to $\mathsf{y}_t = \mathsf{a} + \int_0^t \mathsf{y}_s \,\mathrm{d}\mathcal{A}_s$ is given by $\mathsf{y}_t = \mathsf{a}\mathsf{Y}_t$, $t \in [0,T_1]$.

As the choice of the maximal time T_1 depends only on ω, $\|\mathcal{A}\|_p$ and p, it is possible to extend the solution to $[T_1, T_2]$ with $\zeta(2p)\|\mathcal{A}\|_p \omega(T_1, T_2)^{1/p}$ by solving $\mathsf{y}_t = \mathsf{Y}_{T_1} + \int_0^t \mathsf{y}_s \, \mathrm{d}\mathcal{A}_{T_1+s}$ for $t \in [0, T_2 - T_1]$, then $\mathsf{Y}_t = \mathsf{y}_{t-T_1}$ for $t \in [T_1, T_2]$, and so on... Since ω is continuous close to its diagonal, it is possible to find a finite family $(T_i)_{i \geq 0}$ such that $\omega(T_i, T_{i+1}) < \delta$ for any $\delta > 0$ and $T_j = T$ for some $j \geq 1$.

A similar reasoning applies to $\mathsf{Z}_\bullet$.

Inequalities (3.3) in (ii) follow immediately from (2.11). It is then immediate that Y and Z belong to $\mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$.

Conversely, that (3.3) characterizes uniquely (3.1) and (3.2) follows from Proposition 2.15(i).

For $(s,t) \in \Delta_2^+(T)$,

$$\mathsf{Y}_t\mathsf{Z}_t - \mathsf{Y}_s\mathsf{Z}_s = (\mathsf{Y}_t - \mathsf{Y}_s - \mathsf{Y}_s\mathcal{A}_{s,t})\mathsf{Z}_t - \mathsf{Y}_s(\mathsf{Z}_t - \mathsf{Z}_s + \mathcal{A}_{s,t}\mathsf{Z}_s) + \mathsf{Y}_s\mathcal{A}_{s,t}(\mathsf{Z}_t - \mathsf{Z}_s).$$

Thus $t \mapsto \mathsf{Y}_t\mathsf{Z}_t$ is a continuous path of $p/2$-finite variation and then constant and equal to Id since $\mathsf{Y}_0 = \mathsf{Z}_0 = \mathsf{Id}$.

The multiplicative properties in (iv) are immediate from the construction of $\mathsf{A}_{s,t}$.

Finally, (3.4) in (vi) follows by multiplying $\mathsf{Y}_t - \mathsf{Y}_s - \mathsf{Y}_s\mathcal{A}_{s,t}$ by $\mathsf{Y}_t\mathsf{Y}_s^{-1}$ and using the fact that $\mathsf{Y}_\bullet$ is bounded. □

3.2. **Rough resolvent**

Based on the previous computations, we define, using the vocabulary of differential equations, the notion of *rough resolvent*, which is a *multiplicative functional* [42, 43] taking its values in the Banach space $\mathfrak{L}$.

Definition 3.2 (Rough resolvent)**.** A family $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ of elements in $\mathfrak{L}$ satisfying the right/left multiplicative property (2.1) and

$$\mathsf{A}_\bullet - \mathsf{Id} \prec C\omega^{1/p} \tag{3.6}$$

for some constant C is called a *right/left p-rough resolvent.*

Let us start with some simple properties.

Lemma 3.3 (Extension property)**.** *Assuming that on a partition* $0 = T_0 < T_1 < \cdots < T_k = T$ *of* $[0,T]$, *we have a family of p-rough resolvents* $(\mathsf{A}^i_{s,t})_{(s,t)\in\Delta_2^\pm(T_i,T_{i+1})}$. *Then there is a right resolvent* $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ *such that* $\mathsf{A}_{s,t} = \mathsf{A}^i_{s,t}$ *for* $(s,t) \in \Delta_2^\pm(T_i, T_{i+1})$, $i = 0, \ldots, k-1$.

Proof. For $s,t \in \Delta_2^{\pm}(T_i, T_{i+1})$, set $\mathsf{A}_{s,t} = \mathsf{A}^i_{s,t}$. For $s \in [T_i, T_{i+1})$ and $t \in [T_j, T_{j+1})$ with $i < j$, set $\mathsf{A}_{s,t} = \mathsf{A}^i_{s,T_{i+1}} \cdots \mathsf{A}^j_{T_j,t}$. Then $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^+(T)}$ satisfies the multiplicative property (2.1) and

$$\mathsf{A}_{s,t} - \mathsf{Id} = (\mathsf{A}^i_{s,T_{i+1}} - \mathsf{Id})\mathsf{A}^{i+1}_{T_{i+1},T_{i+2}} \cdots \mathsf{A}^j_{T_j,t} + \mathsf{A}^{i+1}_{T_{i+1},T_{i+2}} \cdots \mathsf{A}^j_{T_j,t}.$$

Iterating this procedure leads to (3.6). A similar construction holds for left p-rough resolvent. □

Lemma 3.4 (Existence of an Inverse)**.** *For any $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^{\pm}(T)}$ that is a right/left p-rough resolvent, there exists a left/right p-rough resolvent $(\mathsf{A}_{t,s})_{(s,t)\in\Delta_2^{\pm}(T)}$ such that $\mathsf{A}_{s,t}\mathsf{A}_{t,s} = \mathsf{A}_{t,s}\mathsf{A}_{s,t} = \mathsf{Id}$, that is $\mathsf{A}_{t,s} = \mathsf{A}^{-1}_{s,t}$ for $(s,t) \in \Delta_2^{\pm}(T)$.*

Proof. Using the extension property of Lemma 3.3, it is sufficient to prove the existence of an inverse of $\mathsf{A}_{s,t}$ for $\omega(s,t)$ small enough, whose existence follows from (2.2) and (3.6). □

The next proposition is the converse of Lemma 2.4.

Proposition 3.5. *A path $(\mathsf{A}_t)_{t\in[0,T]} \in \mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$ may be associated to a right/left p-rough resolvent $\mathsf{A}_\bullet$.*

Proof. For a right p-rough resolvent $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^+(T)}$, $\mathsf{A}_t := \mathsf{A}_{0,t}$ is such that $\mathsf{A}_{s,t} = \mathsf{A}_s^{-1}\mathsf{A}_t$, since $\mathsf{A}_{0,s}\mathsf{A}_{s,t} = \mathsf{A}_{0,t}$ for $(s,t) \in \Delta_2^+(T)$. Besides,

$$\|\mathsf{A}_t - \mathsf{A}_s\| = \|\mathsf{A}_s\mathsf{A}_s^{-1}(\mathsf{A}_t - \mathsf{A}_s)\| \leq \mathsf{A}^\sharp_\bullet\|\mathsf{A}_{s,t} - \mathsf{Id}\| \leq \mathsf{A}^\sharp_\bullet C\omega(s,t)^{1/p}.$$

Thus, $t \mapsto \mathsf{A}_t \in \mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$. Similar results hold for left p-rough multiplicative resolvent with $\mathsf{A}_t = \mathsf{A}_{t,0}$, $t \in [0,T]$. □

Lemma 3.6 (Uniqueness of a p-rough resolvent)**.** *Let $\mathsf{A}_\bullet$ and $\mathsf{B}_\bullet$ be two left/right p-rough resolvents such that $\mathsf{A}_\bullet \simeq \mathsf{B}_\bullet$. Then $\mathsf{A}_\bullet = \mathsf{B}_\bullet$.*

Proof. Let us consider that $\mathsf{A}_\bullet$ and $\mathsf{B}_\bullet$ are right p-rough resolvents. Then for $f(t) := \mathsf{A}_{0,t}\mathsf{B}^{-1}_{0,t}$,

$$f(t) - f(s) = \mathsf{A}_{0,s}(\mathsf{A}_{s,t}\mathsf{B}^{-1}_{s,t} - \mathsf{Id})\mathsf{C}^{-1}_{0,s}$$

and then

$$|f(t) - f(s)| \leq \mathsf{A}^\sharp_\bullet(\mathsf{B}^{-1}_\bullet)^\sharp\|\mathsf{A}_{s,t}\mathsf{B}^{-1}_{s,t} - \mathsf{Id}\|.$$

Since

$$\mathsf{A}_{s,t}\mathsf{B}^{-1}_{s,t} - \mathsf{Id} = (\mathsf{A}_{s,t}\mathsf{B}^{-1}_{s,t} - \mathsf{Id})\mathsf{B}_{s,t}\mathsf{B}^{-1}_{s,t} = (\mathsf{A}_{s,t} - \mathsf{B}_{s,t})\mathsf{B}^{-1}_{s,t},$$

it follows easily that f is a finite θ-variation with $\theta > 1$ and then that $\mathsf{A}_{0,t} = \mathsf{B}_{0,t}$ for any $t \in [0,T]$. This leads to $\mathsf{A}_\bullet = \mathsf{B}_\bullet$. □

3.3. From almost rough resolvent to rough resolvent: the sewing lemma

As in this article, we focus on linear RDE, we introduce some vocabulary which refers to the theory of linear differential equations. Thus, $\mathfrak{L}$ is thought as a space of operators. However, the proofs may be used for tensor algebras.

Following the construction proposed by T. Lyons, we construct a rough resolvent from an almost rough resolvent. However, our proof borrows some ideas from the elegant proof of Theorem 10 in [27] where $\omega(s,t) = V(t-s)$ provided that for some $\theta > 2$, $\sum_{n\geq 0} \theta^n V(t2^{-n}) < +\infty$ for any $t > 0$, as well as the ones from [2] regarding the composition of flows.

Definition 3.7 (Almost rough resolvent)**.** A family $(\mathsf{B}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ of elements of $\mathfrak{L}$ satisfying for some constants $p \geq 1$, $\theta > 1$, $C \geq 0$, $B \geq 0$ and for any $(s,r,t) \in \Delta_3^\pm(T)$,

$$\mathsf{B}_\bullet - \mathsf{Id} \prec B\omega^{1/p}, \tag{3.7}$$

$$\|\mathsf{B}_{s,r,t}\| \leq C\omega(s,t)^\theta \text{ for } (s,r,t) \in \Delta_3^\pm(T) \text{ with } \mathsf{B}_{s,r,t} := \mathsf{B}_{s,r}\mathsf{B}_{r,t} - \mathsf{B}_{s,t} \tag{3.8}$$

is called an *almost right/left p-rough resolvent.*

The next lemma is similar to the extension Lemma 3.3.

Lemma 3.8. *Let us consider $\eta > 0$ and a family $(T_i)_{i=0,\dots,N}$ of times such that $T_0 = 0$, $T_N = T$ and $\omega(T_i, T_{i+1}) \leq \eta$. For each i, consider a family $(\mathsf{B}^{(i)}_{s,t})_{(s,t)\in\Delta_2^+(T_i,T_{i+1})}$ of almost right p-rough resolvents, each with the same constant C and B in* (3.7)–(3.8)*. Now*

$$\mathsf{A}_{s,t} := \begin{cases} \mathsf{B}^{(i-1)}_{s,T_i}\mathsf{B}^{(i)}_{T_i,T_{i+1}}\cdots\mathsf{B}^{(j)}_{T_{j-1},T_j}\mathsf{B}_{T_j,t} \text{ if } T_{i-1} \leq s < T_i \leq T_j < t \leq T_{j+1}, \\ \mathsf{B}_{s,t} \text{ if } T_i \leq s \leq t \leq T_{i+1}. \end{cases}$$

Then $\mathsf{A}_\bullet$ is an almost right p-rough resolvent with

$$\mathsf{A}_\bullet - \mathsf{Id} \prec BN(1+B\eta^{1/p})^{N-1}\omega^{1/p} \text{ and } \|\mathsf{A}_{r,s,t}\| \leq C(1+B\eta^{1/p})^{N-1}\omega(r,t)^\theta$$

for $(s,r,t) \in \Delta_3^+(T)$.

Proof. Fix $(r,s,t) \in \Delta_3^+(T)$ and consider the indexes i,j,k such that $r \in [T_i, T_{i+1}]$, $s \in [T_j, T_{j+1}]$, $t \in [T_k, T_{k+1}]$, Then

$$\mathsf{A}_{r,s}\mathsf{A}_{s,t} - \mathsf{A}_{r,t} = \mathsf{A}_{r,T_i}(\mathsf{A}_{T_j,r}\mathsf{A}_{r,T_{j+1}} - \mathsf{A}_{T_j,T_{j+1}})\mathsf{A}_{T_{j+1},t},$$

so that

$$\|\mathsf{A}_{r,s,t}\| \leq (1+B\eta^{1/p})^{N-1}\omega(T_j,T_{j+1})^\theta \leq (1+B\eta^{1/p})^{N-1}C\omega(r,t)^\theta.$$

In addition

$$\mathsf{A}_{r,t} - \mathsf{Id} = (\mathsf{A}_{r,T_i} - \mathsf{Id})\mathsf{A}_{T_i,T_{i+1}}\cdots\mathsf{A}_{T_k,t} + \mathsf{A}_{T_i,T_{i+1}}\cdots\mathsf{A}_{T_k,t} - \mathsf{Id}$$

so that

$$\begin{aligned}\|\mathsf{A}_{r,t} - \mathsf{Id}\| &\leq B\omega(r,T_i)^{1/p}(1+B\eta^{1/p})^{N-1} + \|\mathsf{A}_{T_i,T_{i+1}}\cdots\mathsf{A}_{T_k,t} - \mathsf{Id}\| \\ &\leq B(1+B\eta^{1/p})^{N-1}\left(\omega(r,T_i)^{1/p} + \sum_{\ell=i}^{j}\omega(T_\ell,T_{\ell+1})^{1/p} + \omega(T_j,t)^{1/p}\right) \\ &\leq BN(1+B\eta^{1/p})^{N-1}\omega(r,t)^{1/p}.\end{aligned}$$

Hence the result. □

For an almost right/left p-rough resolvent $(\mathsf{B}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ and a family $(\mathsf{C}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ we write $\mathsf{B}_\bullet \simeq \mathsf{C}_\bullet$ when there exist constants C and $\theta > 1$ such that

$$\mathsf{B}_\bullet - \mathsf{C}_\bullet \prec C\omega^\theta.$$

Lemma 3.9. *If $\mathsf{B}_\bullet$ is an almost p-rough resolvent and $\mathsf{C}_\bullet \simeq \mathsf{B}_\bullet$ for a family $\mathsf{C}_\bullet$, then $\mathsf{C}_\bullet$ is an almost p-rough resolvent.*

Proposition 3.10. *Let $\mathsf{B}_\bullet$ be an almost right p-rough resolvent. Then there exists a constant L such that for any partition $\pi = \{t_i\}_{i=0,\dots,n}$ of $[0,T]$,*

$$\mathsf{B}^\pi_{s,t} := \begin{cases}\mathsf{B}_{s,t_i}\mathsf{B}_{t_i,t_{i+1}}\cdots\mathsf{B}_{t_{j-1},t_j}\mathsf{B}_{t_j,t} & \text{with } s\in(t_i,t_{i+1}],\ t\in(t_j,t_{j+1}],\ i\leq j,\\ \mathsf{B}_{s,t} & \text{if } t_i\leq s<t\leq t_{i+1}\end{cases} \tag{3.9}$$

satisfies $\mathsf{B}^\pi_\bullet - \mathsf{B}_\bullet \prec L\omega^\theta$, so that B^π is an almost right p-rough resolvent. A similar result holds for almost left p-rough resolvents.

Proof. Set $f(\eta,L) := 1 + B\eta^{1/p} + L\eta^\theta$ for $\eta, L > 0$ and consider L solution to

$$C(2^\theta\zeta(\theta)f(\eta,L)^2 + 1) = L,$$

where ζ is the zeta function defined in Proposition 2.15. This is a second order polynomial equation in L which has some solutions provided that

$$(1 - 2(1 + B\eta^{1/p})\eta^\theta C 2^\theta \zeta(\theta))^2 \geq 4(1 + 2^\theta \zeta(\theta)(1 + B\eta^{1/p})^2) C^2 2^\theta \zeta(\theta)\eta^{2\theta}.$$

For η small enough, depending only on θ, B and C, this is always possible.

Now, let us construct recursively a sequence $(T_k)_{k\geq 0}$ such that $T_0 = 0$ and $\omega(T_k, T_{k+1}) < \eta$. As ω is continuous close to its diagonal, there exists a choice of such a sequence which forms a finite partition of $[0, T]$.

Let us fix $(s,t) \in \Delta_2^+(T_k, T_{k+1})$. Let us proceed by induction over the number of points in the partition π that belongs to $[s,t]$.

Our induction hypothesis is that

$$\|\mathsf{B}^\pi_{s,t} - \mathsf{B}_{s,t}\| \leq L_n \omega(s,t)^\theta \text{ with } L_n := C\left(1 + \sum_{k=1}^{n-1} \frac{f(\eta, L)^2 2^\theta}{k^\theta}\right) \leq L,$$

as soon as $\#(\pi \cap [s,t]) = n$ for some $n \geq 1$.

For $i = j$, that is $n = 1$, then $|\mathsf{B}^\pi_{s,t} - \mathsf{B}_{s,t}| \leq C\omega(s,t)^\theta$ and $L_1 = C$.

If the induction hypothesis is satisfied at step n, then

$$\|\mathsf{B}^\pi_{s,t}\| \leq 1 + B\omega(T_k, T_{k+1})^{1/p} + L_n \omega(T_k, T_{k+1})^\theta \leq f(\eta, L)$$

for any partition π with $\#(\pi \cap [s,t]) = n$.

Let us consider π be a partition of size $n+1$. As soon as $n \geq 1$, it is possible to choose $t_\ell \in (t_i, t_j)$ (where t_i and t_j are defined by (3.9)) such that

$$\omega(t_{\ell-1}, t_{\ell+1}) \leq \frac{2\omega(\sigma, \tau)}{n} \tag{3.10}$$

(see equation (1.4) page 11 of [44]). Set $\widehat{\pi} = \pi \setminus \{t_\ell\}$. Then from the induction hypothesis,

$$\begin{aligned}
\|\mathsf{B}^\pi_{s,t} - \mathsf{B}_{s,t}\| &\leq \|\mathsf{B}^\pi_{s,t} - \mathsf{B}^{\widehat{\pi}}_{s,t}\| + L_n \omega(s,t)^\theta \\
&\leq \|\mathsf{B}^\pi_{s,t_\ell - 1}\| \times \|\mathsf{B}_{t_{\ell-1}, t_\ell, t_{\ell+1}}\| \times \|\mathsf{B}^\pi_{s,t_\ell - 1}\| + L\omega(s,t)^\theta \\
&\leq \left(\frac{2^\theta \cdot f(T,\eta)^2 \cdot C}{n^\theta} + L_n\right)\omega(s,t)^\theta \leq L_{n+1}\omega(s,t)^\theta.
\end{aligned}$$

The induction hypothesis is then true at step $n+1$. Combined to Lemma 3.8, this leads to the conclusion. □

Theorem 3.11 (The sewing lemma). *Let $(\mathsf{B}_{t,s})_{(s,t)\in\Delta_2^+(T)}$ be an almost right p-rough resolvent as in Definition 3.7. Then there exists a unique* *right p-rough resolvent* $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^+(T)}$ *such that*

$$\mathsf{A}_\bullet - \mathsf{B}_\bullet \prec L\omega^\theta \tag{3.11}$$

for some constant L that depends only on B and C in (3.7)–(3.8) *and T. Of course, a similar result holds for almost left p-rough resolvents.*

Proof. The uniqueness follows from Lemma 3.6.

We consider only the proof for right p-rough resolvents. We use the trick introduced in [26, Remark 2, p. 862] to reduce the analysis to the case of $\omega(s,t) = t - s$. Let us consider $\phi(t) = \omega(0,t)$ for $t \in [0,T]$ and

$$\phi^{-1}(u) := \inf\{t \in [0,\omega(0,T)] \mid \phi(t) \geq u\},$$

the generalized left-continuous inverse which is such that $\phi(\phi^{-1}(t)) = t$.

As ω is super-additive, it holds that $\omega(s,t) \leq \omega(0,t) - \omega(0,s)$ and then

$$\omega(\phi^{-1}(s), \phi^{-1}(t)) \leq t - s.$$

Set $\mathsf{C}_{s,t} := \mathsf{B}_{\phi^{-1}(s),\phi^{-1}(t)}$ for $(s,t) \in \Delta_2^+(\phi(T))$ so that

$$\|\mathsf{C}_{s,r,t}\| \leq C|t-s|^\theta \text{ and } \|\mathsf{C}_{s,t} - \mathsf{Id}\| \leq B|t-s|^{1/p}.$$

Define recursively

$$\mathsf{C}^1_{s,t} = \mathsf{C}_{s,t} \text{ and } \mathsf{C}^{n+1}_{r,t} = \mathsf{C}^n_{r,s}\mathsf{C}^n_{s,t} \text{ with } s = \frac{r+t}{2}.$$

We prove by induction that for any n,

$$\|\mathsf{C}^n_{s,t} - \mathsf{C}^{n-1}_{s,t}\| \leq C\kappa^{n-2}|t-s|^\theta \tag{3.12}$$

with

$$\kappa := \frac{D_T}{2^{\theta-1}} \text{ and } D_T := \frac{C}{1-\kappa}T^\theta + BT^{1/p} + 1,$$

provided that T is small enough so that $\kappa < 1$. Here, B and C are the constants in (3.7)–(3.8). The existence of such a choice for $\kappa < 1$ is proved by considering solving

$$\kappa = \frac{\alpha_T}{(1-\kappa)} + \beta_T \text{ with } \alpha_T =: \frac{CT^\theta}{2^{\theta-1}} \text{ and } \beta_T := \frac{1+BT^{1/p}}{2^{\theta-1}}.$$

This equation has a solution $\kappa < 1$ as soon as

$$1 + \beta_T - \sqrt{(1+\beta_T)^2 - 4\alpha_T} < 2,$$

and then if $1 + BT^{1/p} < 2^{\theta-1}$, which is possible since $\theta > 1$ and then $2^{\theta-1} > 1$.

This choice implies that if (3.12) is true for up to order n, then

$$\|\mathsf{C}^n_{s,t}\| \leq \|\mathsf{C}^n_{s,t}-\mathsf{C}_{s,t}\|+\|\mathsf{C}_{s,t}-\mathsf{Id}\|+1 \leq 1+B|t-s|^{1/p}+\frac{C}{1-\kappa}|t-s|^{\theta} \leq D_T.$$

Clearly, $\mathsf{C}^2_{r,t} - \mathsf{C}^1_{r,t} = \mathsf{C}_{r,s,t}$ so that the induction hypothesis is true at step 1. If it is true at step n,

$$\mathsf{C}^{n+1}_{r,t} - \mathsf{C}^n_{r,t} = \mathsf{C}^n_{r,s}(\mathsf{C}^n_{s,t} - \mathsf{C}^{n-1}_{s,t}) - (\mathsf{C}^{n-1}_{r,s} - \mathsf{C}^n_{r,s})\mathsf{C}^{n-1}_{s,t}$$

and then, because $s - r = t - r = (t - s)/2$,

$$\|\mathsf{C}^{n+1}_{r,t} - \mathsf{C}^n_{r,t}\| \leq C\frac{D_T}{2^{\theta-1}}\kappa^{n-1}|t-r|^{\theta} \leq C\kappa^n|t-r|^{\theta}.$$

This means that our induction step is true at step $n + 1$. Thus, $(\mathsf{C}^n_{r,t})_{n\in\mathbb{N}}$ is a Cauchy sequence for any $(r, t) \in \Delta_2(T)$. Let $\mathsf{D}_{r,t}$ be its limit. To proof that $\mathsf{D}_{r,t}$ is a multiplicative functional is the same as in [27] so that we skip it.

Finally, we set $\mathsf{A}_t := \mathsf{D}_{0,\phi(t)}$ and $\mathsf{A}_{s,t} = \mathsf{D}^{-1}_{0,\phi(s)}\mathsf{D}_{0,\phi(t)}$ for $(s, t) \in \Delta^+_2(T)$. As such, it holds that for some universal constant C' and $\theta > 0$,

$$\|\mathsf{A}_{s,t} - \mathsf{B}_{r,t}\| \leq C'|\omega(0, t) - \omega(0, s)|^{\theta},\ (s, t) \in \Delta^+_2(0, T).$$

Fixing $r \in [0, T)$ and applying the same argument to $(\mathsf{B}_{s,t})_{(s,t)\in\Delta^+_2(r,T)}$ leads to the existence of right p-rough resolvent $(\mathsf{A}_{s,t}(s))_{(s,t)\in\Delta^+_2(\alpha,T)}$ such that

$$\|\mathsf{A}_{s,t}(r) - \mathsf{B}_{s,t}\| \leq C'|\omega(r, t) - \omega(r, s)|^{\theta},\ (s, t) \in \Delta^+_2(r, T).$$

From Lemma 3.6 on the uniqueness, $\mathsf{A}_{r,t}(\alpha) = \mathsf{A}_{s,t}$ for any $(s, t) \in \Delta^+_2(r, T)$. In particular, for $s = r$,

$$\|\mathsf{A}_{r,t}(r) - \mathsf{B}_{r,t}\| = \|\mathsf{A}_{r,t}(r) - \mathsf{B}_{r,t}\| \leq C'|\omega(r, t) - \omega(r, r)|^{\theta}$$

for $0 \leq r \leq t \leq T$. Then $\mathsf{A}_\bullet \simeq \mathsf{B}_\bullet$.

As the choice of T depends only on ω, B and C, the extension property (Lemma 3.3) proves the existence of $\mathsf{A}_\bullet$ on $[0, T]$ for any time horizon T. □

From the very construction of a p-rough resolvent from a resolvent, we deduce the following results.

Corollary 3.12. *Let $\mathfrak{G}$ be a close subgroup of $\mathfrak{L}$ for the multiplication and an almost p-rough path $\mathsf{B}_\bullet$ which takes its values in $\mathfrak{G}$.*

(i) *The p-rough resolvent $\mathsf{A}_\bullet$ generated by $\mathsf{B}_\bullet$ also takes its values in $\mathfrak{G}$.*

(ii) *Let* $\rho : \mathfrak{G} \to \mathbb{R}_+$ *be a continuous function such that* $\rho(\mathsf{ab}) \leq \rho(\mathsf{a}) + \rho(\mathsf{b})$, $\mathsf{a}, \mathsf{b} \in \mathfrak{G}$. *If* $\rho(\mathsf{B}_\bullet) \prec \omega$, *then* $\rho(\mathsf{A}_\bullet) \prec \omega$.

Remark 3.13. Using the group isomorphism between a vector space $(\mathrm{V}, +)$ and the group $(\exp(\mathrm{V}), \otimes)$ of the Banach algebra $(\mathrm{T}_1(\mathrm{V}), +, \otimes)$ given in Remark 2.7, Theorem 3.11 contains the additive sewing lemma which implies 2.15(i) and from which rough integrals are constructed.

Corollary 3.14. *Let* $\mathsf{B}_\bullet$ *be an almost p-rough resolvent and let* $\mathsf{C}_\bullet \simeq \mathsf{B}_\bullet$. *Then* $\mathsf{C}_\bullet$ *generates the same p-rough resolvent as* $\mathsf{B}_\bullet$.

Proof. With Lemma 3.9, $\mathsf{C}_\bullet$ is an almost p-rough resolvent. The proof follows then from the uniqueness in Theorem 3.11, which follows from Lemma 3.6. □

The proof of the next continuity result is similar to the one of Theorem 3.11 so we skip it. It is an extension of Theorem 3.2.2 in [42].

Corollary 3.15 (Continuity property). *Let* $\mathsf{B}_\bullet$ *and* $\mathsf{B}'_\bullet$ *be two almost p-rough resolvents such that for some* $\epsilon \geq 0$,

$\mathsf{B}_\bullet - \mathsf{B}'_\bullet \prec \epsilon\omega^{1/p}$ *and* $\|\mathsf{B}_{s,r,t} - \mathsf{B}'_{s,r,t}\| \leq \epsilon\omega(s,t)^\theta$ *for any* $(s,r,t) \in \Delta_3^\pm(T)$.

Then there exists a constant L *such that*

$$\|\mathsf{A}_{s,t} - \mathsf{B}_{s,t} - (\mathsf{A}'_{s,t} - \mathsf{B}'_{s,t})\| \leq \epsilon L\omega(s,t)^\theta \text{ for any } (s,t) \in \Delta_2^\pm(T),$$

where A *(resp* A'*) is the p-rough resolvent associated to* B *(resp.* B'*).*

Corollary 3.16. *Let* $(\mathsf{A}_t)_{t\in[0,T]}$ *and* $(\mathsf{B}_t)_{t\in[0,T]}$ *in* $\mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$ *such that* $(\mathsf{AB})_\bullet \simeq \mathsf{Id}$ *with* $(\mathsf{AB})_{s,t} = \mathsf{A}_s^{-1}\mathsf{A}_t\mathsf{B}_t\mathsf{B}_s^{-1}$ *for* $(s,t) \in \Delta_2^\pm(T)$. *Then* $\mathsf{B}_t = \mathsf{A}_t^{-1}$ *for* $t \in [0,T]$.

Proof. For $(s,t) \in \Delta_2^\pm(T)$,

$$\mathsf{A}_t\mathsf{B}_t - \mathsf{A}_s\mathsf{B}_s = \mathsf{A}_s\mathsf{A}_s^{-1}(\mathsf{A}_t\mathsf{B}_t - \mathsf{A}_s\mathsf{B}_s)\mathsf{B}_s^{-1}\mathsf{B}_s = \mathsf{A}_s(\mathsf{A}_{s,t}\mathsf{B}_{t,s} - \mathsf{Id})\mathsf{B}_s.$$

Thus,

$$\|\mathsf{A}_t\mathsf{B}_t - \mathsf{A}_s\mathsf{B}_s\| \leq \mathsf{A}^\sharp_\bullet\mathsf{B}^\sharp_\bullet C\omega(s,t)^\theta.$$

As $\theta > 1$, this implies that $\mathsf{A}_t\mathsf{B}_t$ is constant and equal to Id since $\mathsf{A}_0 = \mathsf{B}_0 = \mathsf{Id}$. Thus, $\mathsf{A}_t = \mathsf{B}_t^{-1}$ for $t \in [0,T]$. □

Lemma 3.17. *Let* $(\mathsf{B}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ *be an almost p-rough resolvent. Let* δ *be such that* $\mathsf{B}_{s,t}$ *is invertible when* $\omega(s,t) < \delta$ *and* $(\mathsf{B}_{s,t}^{-1})_{(s,t)\in\Delta_2^\pm(T),\omega(s,t)\leq\delta}$

is uniformly bounded. Let $(\mathsf{C}_{t,s})_{(s,t)\in\Delta^p m_2(T)}$ *be a family in* $\mathfrak{L}$ *such that* $(\mathsf{C}_{t,s}\mathsf{B}_{s,t})_{(s,t)\in\Delta_2^\pm(T)} \simeq \mathsf{Id}$. *Then* $\mathsf{C}_\bullet \simeq \mathsf{B}_\bullet^{-1}$.

Proof. Set $\epsilon_{s,t} = \mathsf{C}_{t,s}\mathsf{B}_{s,t} - \mathsf{Id}$, so that $\epsilon_\bullet \simeq 0$ and $\epsilon_{s,t}\mathsf{B}_{s,t}^{-1} = \mathsf{C}_{t,s} - \mathsf{B}_{s,t}^{-1}$ when $\omega(s,t) \leq \delta$. Hence the result. □

Proposition 3.18. *Let* $(\mathsf{B}_{s,t})_{(s,t)\in\Delta_2^+(T)}$ *be an almost right* p*-rough resolvent generating the right* p*-rough resolvent* $(\mathsf{A}_{s,t})_{(s,t)\in\Delta_2^+(T)}$. *Then there exists* $\delta>0$ *such that for* $\omega(s,t) \leq \delta$, $\mathsf{B}_{s,t}$ *is invertible and* $(\mathsf{B}_{s,t}^{-1})_{(s,t)\in\Delta_2^+(T);\omega(s,t)\leq\delta}$ *is an almost left* p*-rough resolvent generating the left* p*-rough resolvent* $(\mathsf{A}_{t,s}^{-1})_{(s,t)\in\Delta_2^+(T)}$.

A similar result holds for almost left p*-rough resolvent.*

Proof. For $(s,r,t) \in \Delta_3^\pm(T)$,

$$\|\mathsf{B}_{s,r}\mathsf{B}_{r,t} - \mathsf{Id}\| \leq C\omega(s,t)^{1/p}(1 + \mathsf{B}_\bullet^\sharp).$$

Choose δ such that

$$\|\mathsf{B}_{s,r}\mathsf{B}_{r,t} - \mathsf{Id}\| + \|\mathsf{B}_{s,t} - \mathsf{Id}\| < 1 \text{ for } (s,r,t) \in \Delta_3^+(T),\ \omega(s,t) \leq \delta.$$

For this choice of δ, $\mathsf{B}_{s,t}$ is invertible when $\omega(s,t) \leq \delta$ and $\mathsf{B}_\bullet^{-1} - \mathsf{Id} \prec C(1 - C\delta)^{-1}\omega^{1/p}$. Besides,

$$\|\mathsf{B}_{s,t}^{-1} - \mathsf{B}_{r,t}^{-1}\mathsf{B}_{s,r}^{-1}\| = \|\mathsf{B}_{s,t}^{-1} - (\mathsf{B}_{s,r}\mathsf{B}_{r,t})^{-1}\| \leq \|\mathsf{B}_{s,t}^{-1} - (\mathsf{B}_{s,r,t} + \mathsf{B}_{s,t})^{-1}\|.$$

With Lemma 2.5 and the definition of $\mathsf{B}_{s,r,t}$ in (3.8),

$$\|\mathsf{B}_{s,t}^{-1} - \mathsf{B}_{r,t}^{-1}\mathsf{B}_{s,r}^{-1}\| \leq \frac{C\omega(s,t)^\theta}{1 - \|\mathsf{B}_{s,t} - \mathsf{Id}\| - \|\mathsf{B}_{s,r,t} + \mathsf{B}_{s,t} - \mathsf{Id}\|} \leq \frac{C}{1-\delta}\omega(s,t)^\theta.$$

Let $(\mathsf{C}_{t,s})_{(s,t)\in\Delta_2(T)}$ be the p-rough resolvent associated to $(\mathsf{B}_{s,t}^{-1})_{(s,t)\in\Delta_2(T)}$. For $(s,t) \in \Delta_2^\pm(T)$,

$$\mathsf{A}_{s,t}\mathsf{C}_{t,s} = \mathsf{A}_{s,t}(\mathsf{C}_{t,s} - \mathsf{B}_{s,t}^{-1}) + (\mathsf{A}_{s,t} - \mathsf{B}_{s,t})\mathsf{B}_{s,t}^{-1} + \mathsf{Id}$$

from which we deduce that $(\mathsf{AC})_\bullet \simeq \mathsf{Id}$ and then that $\mathsf{C}_{t,s} = \mathsf{A}_{s,t}^{-1}$ from Corollary 3.16. □

Remark 3.19. If $\mathcal{A} \in \mathcal{R}^p(\mathfrak{L})$ with $1 \leq p < 2$ is a path of finite p-variation and Y and Z are the solutions to $\mathsf{Y}_t = \mathsf{Id} + \int_0^t \mathsf{Y}_r\,\mathrm{d}\mathcal{A}_r$ and $\mathsf{Z}_t = \mathsf{Id} - \int_0^t \mathrm{d}\mathcal{A}_r\mathsf{Z}_r$ understood in the Young sense, then $\mathsf{Y}_\bullet$ and $\mathsf{Z}_\bullet$ are paths of finite p-variation in $\mathcal{R}_{\mathsf{Id}}^p(\mathfrak{L})$ associated to the almost p-rough resolvents $(\mathsf{Id} \pm \mathcal{A}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ or $\exp(\pm\mathcal{A}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$, respectively.

3.4. Using the Baker-Campbell-Hausdorff-Dynkin formula

Unless $\mathfrak{L}$ is nilpotent where we set $\mathfrak{L}(\delta) := \mathfrak{L}$, we then consider a subset

$$\mathfrak{L}(\delta) := \{\mathsf{b} \in \mathfrak{L} \mid \|\mathsf{b}\| < \delta \text{ with } \delta < \log(2)/4\}.$$

This way, $\log(\mathsf{b})$ exists and (2.5) holds for any $\mathsf{a}, \mathsf{b} \in \mathfrak{L}(\delta)$.

The next theorem is a direct consequence of Lemma 2.5.

Theorem 3.20. *Let $\rho \in (0, +\infty]$. Let us consider a family $(\Omega_{s,t})_{(s,t)\in\Delta_2^+(T)}$ such that for $\omega(s,t) \leq \rho$, $\Omega_{s,t} \in \mathfrak{L}(\delta)$.*

We assume that for some constants C_1, C_2 and $\theta > 1$,

$$\Omega_\bullet \prec C_1 \omega^{1/p},$$

$$\|\Omega_{s,r} \star \Omega_{r,t} - \Omega_{s,t}\| \leq C_2 \omega(s,t)^\theta \text{ for } (s,r,t) \in \Delta_3^+(T).$$

Then there exists a family $(\Theta_{s,t})_{(s,t)\in\Delta_2(T),\omega(s,t)\leq\rho}$ such that $\Theta_{s,r} \star \Theta_{r,t} = \Theta_{s,t}$ and for some constants K_1 and K_2, $\Theta_\bullet \prec K_1 \omega^{1/p}$ and $\Theta_\bullet - \Omega_\bullet \prec K_2 \omega^\theta$.

Remark 3.21. Of course, one may also deduce the construction of the family $(\Theta_{t,s})_{(t,s)\in\Delta_2^+(T),\omega(s,t)\leq\rho}$ satisfying $\Theta_{t,r} \star \Theta_{r,s} = \Theta_{t,s}$ for $(s,r,t) \in \Delta_3^+(T)$ from a family $\Omega_\bullet$ satisfying $\|\Omega_{t,r} \star \Omega_{r,s} - \Omega_{t,s}\| \leq C_2 \omega(s,t)^\theta$.

Remark 3.22. Using Theorem 2.8, if $\Omega_{s,t}$ belongs to a free Lie algebra $\mathfrak{l}$, then $\Theta_{s,t}$ also belongs to $\mathfrak{l}$ for any $(s,t) \in \Delta_2^\pm(T)$.

Remark 3.23. Despite $\Theta_{s,t}$ may be defined for sufficiently small (s,t), $\mathsf{A}_\bullet = \exp(\Theta_\bullet)$ may be extended to a p-rough resolvent for any $(s,t) \in \Delta_2^\pm(T)$ thanks to Lemma 3.3.

3.5. Linear differential equations in the Heisenberg group

We consider the simplest non-trivial example. The results of this section may be extended to Carnot groups which are described for example in [8].

We now consider that $\mathfrak{L} = \mathfrak{H}$, the Heisenberg group introduced in Section 2. As $\mathfrak{H}$ is a nilpotent group of step 2,

$$\log(\mathcal{A} - \mathsf{Id}) = \mathcal{A} - \frac{1}{2}\mathcal{A}^2 \text{ for } \mathcal{A} \in \mathfrak{H} \text{ and } \exp(\mathcal{B}) = \mathsf{Id} + \mathcal{B} + \frac{1}{2}\mathcal{B}^2 \text{ for } \mathcal{B} \in \mathfrak{h}.$$

It is then easily checked that for $\mathcal{A}, \mathcal{B} \in \mathfrak{h}$,

$$\mathcal{A} \star \mathcal{B} = \mathcal{A} + \mathcal{B} + \frac{1}{2}[\mathcal{A}, \mathcal{B}]. \tag{3.13}$$

Let $x, y : [0,T] \to \mathbb{R}$ be two paths of finite p-variation, $1 \leq p < 2$, and consider the controlled linear differential equation

$$U_t = \mathsf{Id} + \int_0^t \mathcal{A} U_s \, \mathrm{d}x_s + \int_0^t \mathcal{B} U_s \, \mathrm{d}y_s, \ U_t \in \mathcal{M}_{3\times 3}(\mathbb{R}) \tag{3.14}$$

for $\mathcal{A}$ and $\mathcal{B}$ in $\mathfrak{h}$.

Proposition 3.24. *The solution* $(U_t)_{t\in[0,T]} \in \mathcal{R}^p_{\mathsf{Id}}(\mathfrak{H})$ *to* (3.14) *is defined by* $U_s^{-1} U_t = \exp(\Theta_{s,t})$ *with*

$$\Theta_{s,t} := \mathcal{A} x_{s,t} + \mathcal{B} y_{s,t} - \frac{1}{2}[\mathcal{A}, \mathcal{B}] \left(\int_s^t x_{s,r} \, \mathrm{d}y_r - \int_s^t y_{s,r} \, \mathrm{d}x_r \right). \tag{3.15}$$

Proof. For a constant matrix $\mathcal{F} \in \mathcal{M}_{3\times 3}(\mathbb{R})$, the linear equation $U_t = \mathsf{Id} + \int_0^t \mathcal{F} U_s \, \mathrm{d}s$ is solved by $U_t = \exp(t\mathcal{F})$ [3]. When $\mathcal{F} \in \mathfrak{h}$, then $U_t \in \mathfrak{H}$ for any $t \geq 0$.

This suggests to use as an approximation of the flow $\exp(\Omega_{s,t})$ with $\Omega_{s,t} := \mathcal{A} x_{s,t} + \mathcal{B} y_{s,t}$. Then

$$\|\Omega_{s,t}\| \leq (\|\mathcal{A}\| \|x\|_p + \|\mathcal{B}\| \|y\|_p) \omega(s,t)^{1/p}$$

and

$$\Omega_{r,t} \star \Omega_{s,r} = \Omega_{s,t} - \frac{1}{2}[\mathcal{A}, \mathcal{B}](x_{r,t} y_{s,r} - x_{s,r} y_{r,t})$$

so that

$$\|\Omega_{r,t} \star \Omega_{s,r} - \Omega_{s,t}\| \leq \|\mathcal{A}\| \|\mathcal{B}\| \|x\|_p \|y\|_p \omega(s,t)^{2/p}$$

and $2/p > 1$. This gives the existence of $\Xi_\bullet$ for any $(s,t) \in \Delta_2^+(T)$ such that $\Xi_{r,t} \star \Xi_{s,r} = \Xi_{r,t}$ and $\|\Xi_{s,t} - \Omega_{s,t}\| \leq K\omega(s,t)^{2/p}$. On the other hand, it is easily checked that $\Theta_\bullet$ defined by (3.15) satisfies these conditions so that $\Xi_\bullet = \Theta_\bullet$. □

Remark 3.25. A slightly different construction of the flow may be given by the Trotter-Kato formula, which is slightly different: For this, we would have considered as an approximation of the flow

$$\mathsf{B}_{s,t} = \exp(\mathcal{A} x_{s,t}) \exp(\mathcal{B} y_{s,t}) = \exp\left(\mathcal{A} x_{s,t} + \mathcal{B} y_{s,t} + \frac{1}{2}[\mathcal{A}, \mathcal{B}] x_{s,t} y_{s,t} \right).$$

This would however have led to similar computations.

For a short proof of the Trotter-Kato formula in the context of p-rough paths, $p < 2$, see [27].

Using approximations of weak geometric p-rough paths by smooth rough paths and the results of [40] for non-weak geometric rough paths, we get the following corollary.

Corollary 3.26. *For* $\mathbf{x}_\bullet \in \mathcal{RP}_2^p(\mathbb{R}^2)$ *with* $p \in [2,3)$, *write*

$$\mathbf{x}_{s,t} := 1 + \sum_{i=1,2} \mathbf{x}_{s,t}^i e_i + \sum_{i,j=1,2} \mathbf{x}_{s,t}^{ij} e_i \otimes e_j$$

(i) *If* $\mathbf{x} \in \mathcal{WGRP}_2^p(\mathbb{R}^2)$ *and*

$$\Theta_{s,t} := \mathcal{A}\mathbf{x}_{s,t}^1 + \mathcal{B}\mathbf{x}_{s,t}^2 - \frac{1}{2}[\mathcal{A},\mathcal{B}](\mathbf{x}_{s,t}^{12} - \mathbf{x}_{s,t}^{21}),$$

then $\exp(\Theta_\bullet)$ *defines a left p-rough resolvent in* $\mathfrak{H}$.

(ii) *If* $\mathbf{x} \in \mathcal{RP}_2^2(\mathbb{R}^2) \setminus \mathcal{WGRP}_2^2(\mathbb{R}^2)$, *then* $\mathbf{x}_{s,t}$ *may be decomposed as* $\mathbf{x}_{s,t} = \mathbf{y}_{s,t} + \phi_{s,t}$ *for* $\phi \in \mathcal{R}^{2p}(\mathbb{R}^2 \otimes \mathbb{R}^2)$ *which is symmetric, that is* $\phi_{s,t} = \sum_{i,j=1,2} \phi_{s,t}^{ij} e_i \otimes e_j$ *with* $\phi^{ij} = \phi^{ji}$. *With*

$$\Theta_{s,t} := \mathcal{A}\mathbf{y}_{s,t}^1 + \mathcal{B}\mathbf{y}_{s,t}^2 - \frac{1}{2}[\mathcal{A},\mathcal{B}](\mathbf{y}_{s,t}^{12} - \mathbf{y}_{s,t}^{21}) + (\mathcal{AB} + \mathcal{BA})\phi_{s,t}^{12} + \mathcal{A}^2\phi_{s,t}^{11} + \mathcal{B}^2\phi_{s,t}^{22},$$

$\exp(\Theta_{s,t})$ *defines a left p-rough resolvent in* $\mathfrak{H}$.

4. Series expansions

4.1. Extension theorem

Let us fix $p \geq 1$ and $\beta \geq p$. For $\mathsf{a} \in \mathfrak{L}$, set

$$\gamma\mathsf{a} = (\mathsf{a}_0, \beta\Gamma(1/p)\mathsf{a}_1, \beta\Gamma(2/p)\mathsf{a}_2, \dots).$$

Recall that $\overline{\mathfrak{L}}$ and $\mathfrak{S}$ have been defined in Section 2.2.3.

Let us consider the following subspaces of $\overline{\mathfrak{L}}$:

$$\mathfrak{B} := \left\{ \mathsf{a} \in \overline{\mathfrak{L}};\ \mathsf{a}_0 = 1,\ \|\gamma\mathsf{a}\|_{\mathrm{hom}} := \sup_{i\geq 1} (\beta\Gamma(i/p)\|\mathsf{a}_i\|)^{1/i} < +\infty \right\},$$

$$\mathfrak{D} := \left\{ \mathsf{a} \in \overline{\mathfrak{L}};\ \mathsf{a}_0 = 0,\ \|\mathsf{a}\|_{\mathrm{hom}} := \sup_{i\geq 1} \|\mathsf{a}_i\|^{1/i} < +\infty \right\}.$$

The next lemma is immediate from the properties given in Section 2.7.

Lemma 4.1. *The space* $\mathfrak{B}$ *is a close subset of* $\mathfrak{S}$ *for* $\|\cdot\|_{\mathrm{sum}}$ *and stable under* $\boxtimes$. *Besides,*

$$\|\gamma(\mathsf{a}\boxtimes\mathsf{b})\|_{\mathrm{hom}} \leq (\|\gamma\mathsf{a}\|_{\mathrm{hom}}^p + \|\gamma\mathsf{b}\|_{\mathrm{hom}}^p)^{1/p} \leq \|\gamma\mathsf{a}\|_{\mathrm{hom}} + \|\gamma\mathsf{b}\|_{\mathrm{hom}} \tag{4.1}$$

for $\mathsf{a},\mathsf{b}\in\mathfrak{B}$.

With Lemma 2.16, $\|\mathsf{a}\|_{\mathrm{sum}} < +\infty$ when $\|\gamma\mathsf{a}\|_{\mathrm{hom}} < +\infty$. However, the finiteness of $\|\mathsf{a}\|_{\mathrm{hom}}$ is not sufficient to ensure that $\|\mathsf{a}\|_{\mathrm{sum}}$ is finite.

Lemma 4.2. *If* $\mathsf{a}\in\mathfrak{D}$, $\|\mathsf{a}\|_{\mathrm{hom}} < 1$, *then* $\mathsf{a}\in\mathfrak{S}$ *and* $\|\mathsf{a}\|_{\mathrm{sum}} \leq \|\mathsf{a}\|_{\mathrm{hom}}/(1-\|\mathsf{a}\|_{\mathrm{hom}})$. *In addition,*

$$\|\mathsf{a}-\pi_{\leq\ell}(\mathsf{a})\|_{\mathrm{sum}} \leq \frac{\|\mathsf{a}\|_{\mathrm{hom}}^{\ell+1}}{1-\|\mathsf{a}\|_{\mathrm{hom}}}. \tag{4.2}$$

Lemma 4.3. *Let* $f(z)=\sum_{m\geq 0} c_m z^m$ *be an analytic power series with* $c_0=0$. *Assume that for some constant* L,

$$\frac{1}{\beta\Gamma\left(\frac{k}{p}\right)}\sum_{m=0}^{k}|c_m|m^{k/p} \leq L^k \text{ for all } k\in\mathbb{Z}^+. \tag{4.3}$$

Then for $\mathsf{a}\in\overline{\mathfrak{L}}$ *with* $\mathsf{a}_0=0$ *and* $\|\gamma\mathsf{a}\|_{\mathrm{hom}} < 1/L$, $f(\mathsf{a})$ *belongs to* $\mathfrak{D}\cap\mathfrak{S}$ *with*

$$\|f(\mathsf{a})\|_{\mathrm{hom}} \leq L\|\gamma\mathsf{a}\|_{\mathrm{hom}} \text{ and } \|f(\mathsf{a})\|_{\mathrm{sum}} \leq L\|\gamma\mathsf{a}\|_{\mathrm{hom}}/(1-L\|\gamma\mathsf{a}\|_{\mathrm{hom}}).$$

Proof. Let $\mathsf{a}\in\overline{\mathfrak{L}}$ with $\mathsf{a}_0=0$ and $\|\gamma\mathsf{a}\|_{\mathrm{hom}} < +\infty$. Then

$$\pi_k((\mathsf{a})^{\boxtimes m}) = \sum_{\substack{i_1+\dots+i_m=k\\ i_1,\dots,i_m\in\{0,\dots,k\}}} \mathsf{a}_{i_1}\cdots\mathsf{a}_{i_m} \text{ and } \pi_k(f(\mathsf{a})) = \sum_{m\geq 1} c_m\pi_k((\mathsf{a})^{\boxtimes m}).$$

Since $\mathsf{a}_0=0$, any product $\mathsf{a}_{i_1}\cdots\mathsf{a}_{i_m}$ for which one of the i_j's is equal to 0 vanishes. This is necessarily the case as soon as $i_1+\dots+i_m=k$ and $m>k$. Using the neo-classical inequality (2.13),

$$\left\|\sum_{\substack{i_1+\dots+i_m=k\\ i_1,\dots,i_m\in\{0,\dots,k\}}} \mathsf{a}_{i_1}\cdots\mathsf{a}_{i_m}\right\| \leq \begin{cases} \|\gamma\mathsf{a}\|_{\mathrm{hom}}^k \dfrac{m^{k/p}}{\beta\Gamma\left(\frac{k}{p}\right)} & \text{if } m\leq k,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that

$$\|\pi_k(f(\mathsf{a}))\| \leq \frac{\|\gamma\mathsf{a}\|_{\mathrm{hom}}^k}{\beta\Gamma\left(\frac{k}{p}\right)}\sum_{m=1}^{k}|c_m|m^{k/p} \leq \|\gamma\mathsf{a}\|_{\mathrm{hom}}^k L^k.$$

The result follows then from Lemma 4.2. □

Corollary 4.4. *There exists a constant ρ, depending only on p, such that $z \mapsto z^{-1} - 1$ and $z \mapsto \log(z)$ map $B := \{\mathsf{a} \in \mathfrak{B}; \|\gamma\mathsf{a}\|_{\mathrm{hom}} \leq \rho\}$ into $\mathfrak{D} \cap \mathfrak{S}$ with*

$$\|\mathsf{a}^{-1} - \mathsf{1}\|_{\mathrm{hom}} \leq C_1\|\gamma\mathsf{a}\|_{\mathrm{hom}} \text{ and } \|\log(\mathsf{a})\|_{\mathrm{hom}} \leq C_2\|\gamma\mathsf{a}\|_{\mathrm{hom}}.$$

In addition, if for $\mathsf{a}, \mathsf{b} \in B$ and some $\epsilon > 0$,

$$\|\gamma(\mathsf{a} - \mathsf{b})\|_{\mathrm{hom}} \leq \epsilon \text{ then } \|f(\mathsf{a}) - f(\mathsf{b})\|_{\mathrm{hom}} \leq \epsilon C_3$$

for $f(z) = \log(z)$ and $f(z) = z^{-1} - 1$.

Proof. Note first that $\|\mathsf{a}\|_{\mathrm{hom}} = \|\mathsf{a} - \mathsf{1}\|_{\mathrm{hom}}$ and $\|\gamma\mathsf{a}\|_{\mathrm{hom}} = \|\gamma(\mathsf{a} - \mathsf{1})\|_{\mathrm{hom}}$ since $\|\cdot\|_{\mathrm{hom}}$ does not depend on the first term.

For $f(z) = \log(1 + z) = \sum_{k\geq 1}(-1)^k z^k/k$, it holds that

$$\sum_{m=1}^{k} |c_m| m^{k/p} \leq k^{k/p}.$$

On the other hand (see *e.g.* [52, 6.1.39, p. 78]),

$$\Gamma\left(\frac{k}{p}\right) \sim \sqrt{2\pi} e^{-k/p} \left(\frac{k}{p}\right)^{\frac{k}{p} - \frac{1}{2}}.$$

Thus, for $\eta > 1$,

$$\begin{aligned}\frac{1}{k}\log(k^{k/p}) &- \frac{1}{k}\log\left(\eta\beta^{-1}\sqrt{2\pi}e^{-k/p}\left(\frac{k}{p}\right)^{\frac{k}{p}-\frac{1}{2}}\right)\\ &= \frac{1}{p} + \frac{1}{p}\log(p) + \frac{\log(k)}{2k} + \frac{1}{k}\left(-\frac{1}{2}\log(p) - \log(\eta\beta^{-1}\sqrt{2\pi})\right)\\ &\xrightarrow[k\to\infty]{} \gamma_p := \frac{1}{p} + \frac{1}{p}\log(p).\end{aligned}$$

Hence, for some $\kappa > 0$, the condition (4.3) is satisfied with $L = e^{\kappa\gamma_p}$.

For $f(z) = (1 + z)^{-1} - 1$, $c_0 = 0$ and $c_k = (-1)^k$ for $k \geq 0$, the conclusions are the same.

For the continuity, note first that when $i_1 + \cdots + i_m = k$,

$$\mathsf{a}_{i_1} \cdots \mathsf{a}_{i_m} - \mathsf{b}_{i_1} \cdots \mathsf{b}_{i_m} = \prod_{j=1}^{m} \mathsf{a}_{i_1} \cdots \mathsf{a}_{i_{j-1}}(\mathsf{a}_{i_j} - \mathsf{b}_{i_j})\mathsf{b}_{i_{j+1}} \cdots \mathsf{b}_{i_m}.$$

Iterating the neo-classical inequality,

$$\begin{aligned}\|\mathsf{a}_{i_1}\cdots\mathsf{a}_{i_m}-\mathsf{b}_{i_1}\cdots\mathsf{b}_{i_m}\| &\leq \epsilon^k\frac{km^{k/p}}{\beta\Gamma\left(\frac{k}{p}\right)}\prod_{j=1}^m \|\gamma\mathsf{a}\|_{\mathrm{hom}}^{i_1+\dots+i_{j-1}}\|\gamma\mathsf{b}\|_{\mathrm{hom}}^{i_{j+1}+\dots+i_m}\\ &\leq \epsilon^k\frac{km^{k/p}}{\beta\Gamma\left(\frac{k}{p}\right)}(1\vee\|\gamma\mathsf{a}\|_{\mathrm{hom}}\vee\|\gamma\mathsf{b}\|_{\mathrm{hom}})^k.\end{aligned}$$

Hence the result since $\rho<1$. □

For $\ell\in\mathbb{N}$, set

$$\mathsf{a}\boxtimes_\ell\mathsf{b}:=\pi_{\leq\ell}(\mathsf{a}\boxtimes\mathsf{b}),\ \ \mathsf{a},\mathsf{b}\in\overline{\mathfrak{L}}.$$

Lemma 4.5. *For $\ell\in\mathbb{N}$, $\pi_{\leq\ell}\mathfrak{B}$ is a close subset of the Banach algebra $(\pi_{\leq\ell}\mathfrak{S},+,\boxtimes_\ell)$ stable under $\boxtimes_\ell$ and $\|\gamma(\mathsf{a}\boxtimes_\ell\mathsf{b})\|_{\mathrm{hom}}\leq\|\gamma\mathsf{a}\|_{\mathrm{hom}}+\|\gamma\mathsf{b}\|_{\mathrm{hom}}$ for $\mathsf{a},\mathsf{b}\in\pi_{\leq\ell}\mathfrak{B}$. Besides, there exists $\rho>0$ such that for any $\mathsf{a}\in\pi_{\leq\ell}\mathfrak{B}$ with $\|\gamma\mathsf{a}\|_{\mathrm{hom}}<\rho$, then $\pi_{\leq\ell}(\mathsf{a}^{-1})\in\pi_{\leq\ell}\mathfrak{B}$.*

Proof. Of course, the properties of Lemma 4.1 are still true when $\mathfrak{B}$ is replaced by $\pi_{\leq\ell}\mathfrak{B}$. The existence of an inverse in a small ball follows from Corollary 4.4. □

Proposition 4.6. *For $\lfloor p\rfloor\leq\ell<+\infty$, let $(\mathsf{b}_{s,t})_{(s,t)\in\Delta_2^\pm(T)}$ be a p-rough resolvent in $(\pi_{\leq\ell}\mathfrak{S},+,\boxtimes_\ell)$ taking its values in $\pi_{\leq\ell}\mathfrak{B}$ with $\|\gamma\mathsf{b}_\bullet\|_{\mathrm{hom}}\leq\omega^{1/p}$.*

(i) *The family $\mathsf{b}_\bullet$ is an almost p-rough resolvent in $(\mathfrak{S},+,\boxtimes)$ generating a p-rough resolvent $\mathsf{a}_\bullet$ that satisfies $\|\gamma\mathsf{a}_\bullet\|_{\mathrm{hom}}\prec\omega^{1/p}$ and $\pi_{\leq\ell}(\mathsf{a}_\bullet)=\mathsf{b}_\bullet$.*

(ii) *For $p\geq1$, $k\geq\ell$ and $(s,t)\in\Delta_2^\pm(T)$,*

$$\|\mathsf{a}_{s,t}-\pi_{\leq k}(\mathsf{a}_{s,t})\|_{\mathrm{sum}}\leq(1+\lfloor p\rfloor)(1\vee\omega(s,t))\frac{\omega(s,t)^{(\lfloor(k+1)/p\rfloor+1}}{\Gamma(\lfloor k/p\rfloor+1)}\exp(\omega(s,t)).$$

(iii) *The p-rough resolvent $\mathsf{a}_\bullet^{-1}$ satisfies $\|\gamma(\mathsf{a}_\bullet^{-1}-1)\|_{\mathrm{hom}}\prec C\omega^{1/p}$ for some constant C.*

Remark 4.7. This theorem is a variation of the extension theorem of T. Lyons [42, Theorem 3.2.1] which allows one to pass from a p-rough path of finite order to a p-rough path of infinite order. This theorem itself may be seen as a generalization of the notion of Chen series (see Theorem 2.13).

Remark 4.8. Of course, the extension is continuous and the proof relies on similar arguments so that we skip it. However, we could consider estimating the distance between two extensions in $(\mathfrak{S},+,\boxtimes)$ with $\|\cdot\|_{\mathrm{sum}}$ or with $\|\gamma\cdot\|_{\mathrm{hom}}$. In the later case, stronger constraints on β could lead to different bounds [46].

Proof. For $(s,r,t)\in\Delta_3^{\pm}(T)$,

$$\mathsf{b}_{s,r}\boxtimes\mathsf{b}_{r,t}=\mathsf{b}_{s,t}+\mathsf{c}_{s,r,t}$$
$$\text{where } \pi_k\mathsf{c}_{s,r,t}=\begin{cases}\sum_{j=0}^{k}\pi_j(\mathsf{b}_{s,r})\pi_{k-j}(\mathsf{b}_{r,t}) & \text{if } k=\ell+1,\dots,2\ell,\\ 0 & \text{otherwise.}\end{cases}$$

As $\ell\geq\lfloor p\rfloor$, it follows that $\mathsf{b}_\bullet$ is an almost p-rough resolvent in $\mathfrak{S}$. Hence, (i) follows from Lemma 4.1 together with Corollary 3.12 used with $\rho(\mathsf{a})=\|\gamma\mathsf{a}\|_{\mathrm{hom}}$.

The control (ii) follows from Lemma 2.16.

When $\omega(s,t)$ is small enough, $\mathsf{b}_{s,t}$ is invertible in $\mathfrak{S}$. With Lemma 4.5 and (4.2),

$$\pi_{\leq\ell}(\mathsf{b}_{s,t}^{-1})\in\pi_{\leq\ell}\mathfrak{B}\text{ and }\mathsf{b}_\bullet^{-1}\simeq\pi_{\leq\ell}(\mathsf{b}_\bullet^{-1}).$$

Then $\pi_{\leq\ell}(\mathsf{b}_\bullet^{-1})$ is an almost p-rough resolvent in $(\pi_{\leq\ell}\mathfrak{S},+,\boxtimes_\ell)$ which gives rise to a p-rough resolvent in $(\pi_{\leq\ell}\mathfrak{S},+,\boxtimes_\ell)$ still taking its values in $\pi_{\leq\ell}\mathfrak{B}$ by Corollary 3.12. Then (i) may be applied to give rise to a p-rough resolvent taking its values in $\mathfrak{S}\cap\mathfrak{B}$. Necessarily, this p-rough resolvent is $\mathsf{b}_\bullet^{-1}$ by the uniqueness in the sewing Lemma, so (iii) is proved. □

4.2. Linear differential equations, $1\leq p<2$

We may now consider linear differential equations in the Young sense in $\mathfrak{S}$.

Assume that $1\leq p<2$ and let $\mathcal{A}\in\mathcal{R}_0^p(\mathfrak{L})$.

Set $\mathsf{a}(t)=(\mathsf{Id},\mathcal{A}_t,0,\dots)$, which defines a path in $\mathcal{R}_1^p(\mathfrak{S})$ that takes its values in $\pi_{\leq 1}\mathcal{B}_{\mathsf{Id}}$.

Let us consider the linear differential equation in the Young sense

$$\mathsf{y}(t)=1+\int_0^t\mathsf{y}(s)\boxtimes\,\mathrm{d}\mathsf{a}(s),\ \ \mathsf{y}(\bullet)\in\mathcal{R}_1^p(\mathfrak{S}).\tag{4.4}$$

Proposition 4.9. *Let $\mathcal{A}\in\mathcal{R}^p(\mathfrak{L})$ for $p\in[1,2)$. There exists a unique solution $\mathsf{y}(\bullet)$ of (4.4) in $\mathcal{R}_1^p(\mathfrak{S})$. This solution satisfies*

$$\|\gamma\mathsf{y}_\bullet\|_{\mathrm{hom}}\prec\beta\Gamma(1/p)\|\mathcal{A}\|_p\omega^{1/p}$$

and $\mathsf{y}_k(\bullet) = \pi_k(\mathsf{y}(\bullet))$ *is given by the recursive relation*

$$\mathsf{y}_0(t) = \mathsf{Id} \text{ and } \mathsf{y}_{k+1}(t) = \int_0^t \mathsf{y}_k(s)\,\mathrm{d}\mathcal{A}_s. \tag{4.5}$$

Proof. Let us set $\mathsf{a}_{s,t} := (\mathsf{Id}, \mathcal{A}_{s,t}, 0, \dots)$ for $(s,t) \in \Delta_2^+(T)$.

With Proposition 3.1, there exists a unique solution $\mathsf{y} \in \mathcal{R}_1^p(\mathfrak{S})$ to (4.4) and $\|\mathsf{y}_\bullet - \mathsf{a}_\bullet\| \prec C\omega_\bullet^{2/p}$. This equation is easily solved recursively by (4.5).

On the other hand, $\mathsf{a}_\bullet$ defines a p-rough resolvent in $(\pi_{\leq 1}\mathfrak{S}, +, \boxtimes_1)$ that takes its values in $\pi_{\leq 1}\mathfrak{B}$. Thanks to Proposition 4.6, it generates an almost p-rough resolvent $\mathsf{c}_\bullet \in \mathfrak{S}$ with $\pi_{\leq 1}\mathsf{c}_\bullet = \mathsf{a}_\bullet$ and $\|\mathsf{c}_\bullet - \mathsf{a}_\bullet\|_{\mathrm{sum}} \prec C\omega^{2/p}$. This proves that $\mathsf{c}_\bullet = \mathsf{y}_\bullet$. ☐

The next result is an immediate consequence of Proposition 4.9 and Corollary 4.4.

Corollary 4.10. *There exists $C > 0$ and $\delta > 0$ such that for $(s,t) \in \Delta_2^+(T)$, $\omega(s,t) < \delta$, $\log(\mathsf{y}_{s,t}) \in \mathfrak{D} \cap \mathfrak{S}$ and $\|\log(\mathsf{y}_{s,t})\|_{\mathrm{hom}} \leq C\omega(s,t)^{1/p}$.*

When applying the algebra homomorphism $\phi : \mathfrak{S} \to \mathfrak{L}$ defined by $\phi(\mathsf{a}) = \sum_{k\geq 0} \pi_k(\mathsf{a})$, Proposition 4.9 yields a series representations of Dyson type [25] for the solution to $\mathsf{Y}_t = \mathsf{Id} + \int_0^t \mathsf{Y}_s\,\mathrm{d}\mathcal{A}_s$, and the series is normally convergent in $[0,T]$.

4.3. Magnus and Chen-Strichartz formula

Combining the previous results with Theorem 2.13 and using a continuity argument, we may extend the results on Chen series to weak geometric rough paths, as well as the Chen-Strichartz formula (2.7).

Proposition 4.11. *Let $\mathbf{x} \in \mathcal{WGRP}_\ell^p(\mathbb{R}^d)$ with $\ell \geq \lfloor p \rfloor$, and $\|\mathbf{x}\|_p < C$, then $\mathbf{x}$ may be extended to a rough path $\mathbf{z}$ in $\mathcal{RP}_\infty^p(\mathbb{R}^d)$ such that for some $\delta > 0$ and any $(s,t) \in \Delta_2^+(T)$ with $\omega(s,t) < \delta$, $\mathbf{z}_{s,t}$ is a Lie element, $\log(\mathbf{z}_{s,t})$ belongs to $\mathfrak{D} \cap \mathrm{T}_\infty(\mathbb{R}^d)$ and (2.7) holds.*

For $\mathcal{B}^1, \dots, \mathcal{B}^d \in \mathfrak{L}$, let us consider the algebra homomorphism ψ from $(\mathfrak{L}, +, \otimes)$ to $(\mathfrak{S}, +, \cdot)$ defined by

$$\pi_\ell(\psi(e_I)) = \begin{cases} \mathcal{B}^I & \text{if } \ell = |I|, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathbf{x} \in \mathcal{WGRP}^p_\ell(\mathbb{R}^d)$, we then get that for some $\delta > 0$ and that for $(s,t) \in \Delta_2^+(T)$ with $\omega(s,t) < \delta$, there exists $\Omega_{s,t} \in \mathfrak{L}$ such that

$$\mathsf{X}_{s,t} := \sum_{I;|I|\geq 0} \mathcal{B}^I \mathbf{x}^I_{s,t} = \exp(\Omega_{s,t}) \tag{4.6}$$

and $\Omega_{s,t} = \phi(\log(\mathbf{z}_{s,t}))$. When $\mathbf{x}_\bullet$ lies above a smooth path $x : [0,T] \to \mathbb{R}^d$, then $\mathsf{X}_\bullet$ is the solution to

$$\mathsf{X}_t = \mathsf{Id} + \sum_{i=1}^d \int_0^t \mathsf{X}_s \mathcal{B}^i \,\mathrm{d}x^i_s.$$

Remark 4.12. When $\mathcal{B}^i$ are the entries of a matrix $\mathcal{A}$, then one recovers in (2.7) the Magnus formula [47], which is extensively presented in [7].

Other formulas, with various names, may be used to represent this logarithm [7]. In any case, the Baker-Campbell-Hausdorff-Dynkin formula is the main tool [9].

Remark 4.13. A large amount of literature is devoted to the construction of numerical procedures relying on this kind of computation for smooth paths [7, 31], or for Brownian paths (See [13, 41] for example).

Remark 4.14. Clearly, our estimates for the existence of a logarithm are not optimal. However, unless one consider nilpotent algebras, one cannot expect in general that (4.6) converges for any time (s,t). The radius of convergence of the logarithms of Chen series of paths of bounded variations is finite in general [45].

In the case of Magnus series, see for example [7, 49]. For stochastic processes, see [5] regarding the fractional Brownian motion and [6] for the Brownian motion.

5. Perturbed linear RDE when $1 \leq p < 2$

Throughout all this section, we consider two paths $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{R}^p(\mathfrak{L})$ with $p \in [1,2)$.

Proposition 5.1 (Duhamel principle/Variation of constant formula)**.** *Fix $p \in [1,2)$. Let Y and Z be the solutions to* (3.1) *and* (3.2). *Let us consider that either*

(a) *$\mathcal{B}^*, \mathcal{B} \in \mathcal{R}^p(\mathfrak{L})$, or*
(b) *$\mathcal{B}^* \in \mathcal{R}^p(\mathrm{V}^*)$ and $\mathcal{B} \in \mathcal{R}^p(\mathrm{V})$.*

Then

$$\mathsf{S}_t = \left(\int_0^t \mathrm{d}\mathcal{B}^*_s \mathsf{Y}_s^{-1}\right)\mathsf{Y}_t \text{ and } \mathsf{T}_t = \mathsf{Z}_t\left(\int_0^t \mathsf{Z}_s^{-1}\,\mathrm{d}\mathcal{B}_s\right),$$

are the unique solutions to

$$\mathsf{S}_t = \int_0^t \mathsf{S}_s\,\mathrm{d}\mathcal{A}_s + \mathcal{B}^*_{0,t} \text{ and } \mathsf{T}_t = \int_0^t \mathrm{d}\mathcal{A}_s \mathsf{T}_s + \mathcal{B}_{0,t}, \tag{5.1}$$

where $\mathsf{S}, \mathsf{T} \in \mathcal{R}^p(\mathfrak{L})$ *in case* (a) *and* $\mathsf{S} \in \mathcal{R}^p(\mathrm{V}^*)$, $\mathsf{T} \in \mathcal{R}^p(\mathrm{V})$ *in case* (b).

Besides, there exists a constant C *depending only on* $\omega(0,T)$ *and* p *such that*

$$\|\mathsf{S}\|_p \leq C(1 + \|\mathcal{A}\|_p + \|\mathcal{A}\|_p^2)\|\mathcal{B}^*\|_p \text{ and } \|\mathsf{T}\|_p \leq C(1 + \|\mathcal{A}\|_p + \|\mathcal{A}\|_p^2)\|\mathcal{B}\|_p.$$

Proof. The existence and uniqueness of solutions to (5.1) follows from the same arguments as in Proposition 3.1.

We only consider S, the computations being similar for T. Using the characterization of the Young integral of Proposition 2.15(i), it is sufficient to show that

$$(\mathsf{S}_t - \mathsf{S}_s - \mathsf{S}_s\mathcal{A}_{s,t} - \mathcal{B}^*_{s,t})_{(s,t)\in\Delta_2^+(T)} \simeq 0.$$

We have

$$\begin{aligned}
\mathsf{S}_t - \mathsf{S}_s &= \left(\int_s^t \mathrm{d}\mathcal{B}^*_r \mathsf{Y}_r^{-1}\right)\mathsf{Y}_t + \left(\int_0^s \mathrm{d}\mathcal{B}^*_r \mathsf{Y}_r^{-1}\right)(\mathsf{Y}_t - \mathsf{Y}_s)\\
&= \mathcal{B}^*_{s,t}\mathsf{Y}_s^{-1}\mathsf{Y}_t + \left(\int_0^s \mathrm{d}\mathcal{B}^*_r \mathsf{Y}_r^{-1}\right)\mathsf{Y}_s\mathcal{A}_{s,t} + \epsilon^{(1)}_{s,t}\\
&= \mathcal{B}^*_{s,t} + \mathcal{B}^*_{s,t}\mathsf{Y}_s^{-1}(\mathsf{Y}_t - \mathsf{Y}_s) + \mathsf{Y}_s\mathcal{A}_{s,t} + \epsilon^{(1)}_{s,t}\\
&= \mathcal{B}^*_{s,t} + \mathsf{Y}_s\mathcal{A}_{s,t} + \epsilon^{(2)}_{s,t}
\end{aligned}$$

with $\epsilon^{(i)}_\bullet \prec C\omega^{2/p}$, $i = 1, 2$. Hence the result with Proposition 3.1(vi). The control on the p-variation follows from (2.10). □

Corollary 5.2. *For* $\mathcal{B}^* \in \mathcal{R}^p(\mathrm{V}^*)$ *and* $\mathcal{B} \in \mathcal{R}^p(\mathrm{V}))$, $\mathcal{A} \in \mathcal{R}^p(\mathfrak{L})$. *Then the solutions* $p \in 1,2$ *and* $\mathsf{S}^a_\bullet$, $\mathsf{T}^a_\bullet$ *(defined by linearity) to*

$$\mathsf{S}^a_t = a + \int_0^t \mathsf{S}^a_s\,\mathrm{d}\mathcal{A}_s + \mathcal{B}^*_{0,t} \text{ and } \mathsf{T}^a_t = a + \int_0^t \mathrm{d}\mathcal{A}_s\mathsf{T}^a_s + \mathcal{B}_{0,t}$$

defined flows of homeomorphims $a \in \mathrm{V}^* \mapsto \mathsf{S}^a_t$ *and* $a \in \mathrm{V} \mapsto \mathsf{T}^a_t$.

Proof. By linearity for $a, b \in \mathrm{V}^*$, $\mathsf{C}_\bullet = \mathsf{S}^a_\bullet - \mathsf{S}^b_\bullet$ is solution to $\mathsf{C}_t = (b - a) + \int_0^t \mathsf{C}_s \,\mathrm{d}\mathcal{A}_s$. This proves that $\mathsf{C}_t = (b-a)\mathsf{D}_t$ with $\mathsf{D}_t = \mathsf{Id} + \int_0^t \mathsf{D}_s \,\mathrm{d}\mathcal{A}_s$ and D is invertible by Lemma 3.4. This proves that C_t is one-to-one for any $t \in [0, T]$. The proof is similar for $\mathsf{T}^\cdot_t$. □

6. Perturbed linear RDE when $2 \leq p < 3$

6.1. A Duhamel principle

Let us consider now $\mathcal{A} \in \mathcal{R}^p(\mathfrak{L})$ with $2 \leq p < 3$.

Let $(\mathcal{A}^{(2)}_{s,t})_{(s,t)\in\Delta_2^+(T)}$ be a family in $\mathfrak{L}$ such that $\mathcal{A}^{(2)}_\bullet \prec C\omega^{2/p}$ and for $\underline{\mathcal{A}}_{s,t} := \mathsf{Id} + \mathcal{A}_{s,t} + \mathcal{A}^{(2)}_{s,t}$ satisfies

$$\|\underline{\mathcal{A}}_{s,t} - \underline{\mathcal{A}}_{s,r}\underline{\mathcal{A}}_{r,t}\| \leq C\omega^{3/p}(s,t) \text{ for } (s,r,t) \in \Delta_3^+(T).$$

This way, $\underline{\mathcal{A}}_\bullet$ is an almost right p-rough resolvent, to which is associated a right p-rough resolvent $\mathsf{R}_\bullet$ which satisfies

$$\mathsf{R}_\bullet - \underline{\mathcal{A}}_\bullet \prec C\omega^{3/p},$$

and so that $\mathsf{R}_t := \mathsf{R}_{0,t} \in \mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$ is formally the solution to

$$\mathsf{R}_t = \mathsf{Id} + \int_0^t \mathsf{R}_s \,\mathrm{d}\underline{\mathcal{A}}_s.$$

Let us also consider

$$\underline{\mathcal{D}}_{t,s} := \mathsf{Id} - \mathcal{A}_{s,t} - \mathcal{A}^{(2)}_{s,t} + \mathcal{A}_{s,t} \cdot \mathcal{A}_{s,t} \text{ for } (s,t) \in \Delta_2^+(T).$$

It is easily checked that $\underline{\mathcal{D}}_\bullet$ is an almost left p-rough resolvent and that

$$\|\underline{\mathcal{D}}_{t,s}\underline{\mathcal{A}}_{s,t}\| \leq C\omega(s,t)^{3/p}.$$

Thus, $\underline{\mathcal{D}}_\bullet \simeq \underline{\mathcal{A}}_\bullet^{-1}$ when $\omega(s,t)$ is small enough and $\underline{\mathcal{D}}_\bullet$ generates a left p-rough resolvent $\mathsf{L}_\bullet$ which satisfies

$$\mathsf{L}_\bullet - \underline{\mathcal{D}}_\bullet \prec C\omega^{3/p}.$$

The path defined by $\mathsf{L}_t := \mathsf{L}_{t,0} \in \mathcal{R}^p_{\mathsf{Id}}(\mathfrak{L})$ is formally the solution to

$$\mathsf{L}_t = \mathsf{Id} + \int_0^t \mathrm{d}\underline{\mathcal{D}}_s \mathsf{L}_s$$

and $\mathsf{L}_t = \mathsf{R}_t^{-1}$ for $t \in [0, T]$.

Definition 6.1 (p-rough lift)**.** For $\mathfrak{P} = \mathfrak{L}$ or $\mathfrak{P} = \mathrm{V}^*$, a *right p-rough lift $(\underrightarrow{\mathcal{B}}_{s,t})_{(s,t)\in\Delta_2^+(T)}$ of $\mathcal{B} \in \mathcal{R}^p(\mathfrak{P})$ with respect to $\mathcal{A}_\bullet$* is defined as $\underrightarrow{\mathcal{B}}_\bullet := \mathcal{B}_\bullet + \mathcal{B}_\bullet^{(2)}$ where $\mathcal{B}_{s,t}^{(2)} \in \mathfrak{P}$ and for some $\theta > 1$,

$$\mathcal{B}_\bullet \prec C\omega^{1/p},\ \mathcal{B}_\bullet^{(2)} \prec C\omega^{2/p}, \tag{6.1}$$

$$\|\underrightarrow{\mathcal{B}}_{s,t} - \underrightarrow{\mathcal{B}}_{s,r} - \underrightarrow{\mathcal{B}}_{r,t} - \mathcal{B}_{s,r}\mathcal{A}_{r,t}\| \leq C\omega(s,t)^\theta \text{ for } (s,r,t) \in \Delta_3^+(T). \tag{6.2}$$

The smallest constant C for which (6.1) holds is denoted by $\|\underrightarrow{\mathcal{B}}\|_p$.

For $\mathfrak{P} = \mathfrak{L}$ or $\mathfrak{P} = \mathrm{V}$, a *left p-rough lift* $(\underleftarrow{\mathcal{B}}_{t,s})_{(s,t)\in\Delta_2^+(T)}$ is defined similarly with (6.2) replaced by

$$|\underleftarrow{\mathcal{B}}_{t,s} - \underleftarrow{\mathcal{B}}_{r,s} - \underleftarrow{\mathcal{B}}_{r,s} - \mathcal{A}_{r,t}\mathcal{B}_{s,r}\| \leq C\omega(s,t)^\theta \text{ for } (s,r,t) \in \Delta_3^+(T).$$

Proposition 6.2. *Set $\mathfrak{P} = \mathfrak{L}$ or $\mathfrak{P} = \mathrm{V}$. Assume that $\mathcal{B} \in \mathcal{R}^p(\mathfrak{P})$ admits a right p-rough lift $\underrightarrow{\mathcal{B}}_\bullet$ with respect to $\mathcal{A}_\bullet$. Then there exists a path S in $\mathcal{R}^p(\mathfrak{P})$ with which satisfies $\mathsf{S}_0 = 0$ and*

$$\|\mathsf{S}_t - \mathsf{S}_s\mathcal{A}_{s,t} - \underrightarrow{\mathcal{B}}_{s,t}\| \leq C\|\underrightarrow{\mathcal{B}}\|_p\omega(s,t)^{3/p} \text{ and } \|\mathsf{S}\|_p \leq C\|\underrightarrow{\mathcal{B}}\|_p. \tag{6.3}$$

This path is denoted by $\mathsf{S}_t = \int_0^t \mathrm{d}^{\mathrm{b}}\underrightarrow{\mathcal{B}}_r\mathsf{R}_{r,t}$, where the b *stands for "backward integration".*

Proof. Set for $0 \leq s \leq t \leq u \leq T$,

$$\mu_{s,t}(u) := \underrightarrow{\mathcal{B}}_{s,t}\mathsf{R}_{t,u} \text{ so that } \mu_\bullet(u) \prec \mathsf{R}_\bullet^\#\|\underrightarrow{\mathcal{B}}\|_p\omega^{1/p}. \tag{6.4}$$

With (6.4),

$$\begin{aligned}
\mu_{s,r,t}(u) &:= \mu_{s,t}(u) - \mu_{s,r}(u) - \mu_{r,t}(u)\\
&= -(\underrightarrow{\mathcal{B}}_{s,r}\mathsf{R}_{r,t} + \underrightarrow{\mathcal{B}}_{r,t} - \underrightarrow{\mathcal{B}}_{s,t})\mathsf{R}_{t,u}\\
&= -(\underrightarrow{\mathcal{B}}_{s,r}(\mathsf{R}_{r,t} - \mathsf{Id}) - \mathcal{B}_{s,r}\mathcal{A}_{r,t})\mathsf{R}_{t,u}\\
&= -(\mathcal{B}_{s,r}(\mathsf{R}_{r,t} - \mathsf{Id} - \mathcal{A}_{r,t}) - \mathcal{B}_{s,r}^{(2)}(\mathsf{R}_{r,t} - \mathsf{Id}))\mathsf{R}_{t,u}.
\end{aligned}$$

Hence $\|\mu_{s,r,t}(u)\| \leq C\|\underrightarrow{\mathcal{B}}\|_p\omega(s,t)^{3/p}$ for $(s,t) \in \Delta_2^+(T)$.
With the additive Sewing Lemma, there exists a path $\nu_t(u)$ such that $(\mu_{s,t}(u) - \nu_t(u) - \nu_s(u))_{(s,t)\in\Delta_2^+(T)} \prec C\|\underrightarrow{\mathcal{B}}\|_p\omega^{3/p}$.

The integral $\int_0^t \mathrm{d}^{\mathrm{b}}\underrightarrow{\mathcal{B}}_r\mathsf{R}_{r,t}$ is defined as $\nu_t(t)$.

For $s \leq t \leq v \leq u$, $\mu_{s,t}(u) = \mu_{s,t}(v)\mathsf{R}_{v,u}$ so that again by the uniqueness in the sewing lemma, $\mu_t(u) = \mu_t(v)\mathsf{R}_{v,u}$ for any $t \leq v \leq u$.

It remains to check the regularity of $t \mapsto \mu_t(t)$ and (6.3). For $(s,t) \in \Delta_2^+(T)$,

$$\mu_t(t) - \mu_s(s) = \mu_t(t) - \mu_s(t) + \mu_s(s)(\mathsf{R}_{s,t} - \mathsf{Id}).$$

With (6.4), this proves that $(\nu_t(t) - \nu_s(s))_{(s,t)\in\Delta_2^+(T)} \prec C\omega^{1/p}$. In addition, $\mathsf{R}_\bullet - \underleftarrow{\mathcal{A}}_\bullet \simeq 0$, hence (6.3). □

The proofs of the next propositions follow the same lines so that we skip them.

Proposition 6.3. *Set $\mathfrak{P} = \mathfrak{L}$ or $\mathfrak{P} = \mathrm{V}$. Assume that $\mathcal{B} \in \mathcal{R}^p(\mathfrak{P})$ admits a left p-rough lift $\underleftarrow{\mathcal{B}}_\bullet$ with respect to $\mathcal{A}_\bullet$. Then there exists a path $\mathsf{T} \in \mathcal{R}^p(\mathfrak{P})$ which satisfies $\mathsf{T}_0 = 0$ and*

$$\|\mathsf{T}_t - \underline{\mathcal{D}}_{t,s}\mathsf{T}_s - \underleftarrow{\mathcal{B}}_{t,s}\| \leq C\|\underleftarrow{\mathcal{B}}\|_p\omega(s,t)^{3/p} \text{ and } \|\mathsf{T}\|_p \leq C\|\underleftarrow{\mathcal{B}}\|_p. \tag{6.5}$$

This path is denoted by $\mathsf{T}_t = \int_0^t \mathsf{L}_{t,r}\,\mathrm{d}^{\mathrm{b}}\underleftarrow{\mathcal{B}}_r$.

Proposition 6.4. *Assume that in Propositions 6.2 and 6.3, $\mathcal{B} \in \mathcal{R}^q(\mathfrak{P})$ with $q^{-1} + p^{-1} > 1$. Then one may take $\mathcal{B}_\bullet^{(2)} = 0$ and*

$$\left\|\int_0^t \mathrm{d}^{\mathrm{b}}\mathcal{B}_r\mathsf{R}_{r,t}\right\|_p \leq C\|\mathcal{B}\|_q \text{ and } \left\|\int_0^t \mathsf{L}_{t,r}\,\mathrm{d}^{\mathrm{b}}\mathcal{B}_r\right\|_p \leq C\|\mathcal{B}\|_q.$$

In addition, the "backward integrals" are indeed Young integrals.

Of course, an equivalent of Corollary 5.2 is also true for $2 \leq p < 3$.

6.2. Application to rough differential equations

Let U, V and W be Banach spaces. The space W is assumed to be finite-dimensional.

Let $\mathbf{x} \in \mathcal{RP}_2^p(\mathrm{U})$ be a p-rough path controlled by ω with $p \in [2,3)$.

Let us consider a function $g : \mathrm{V} \to L(\mathrm{U},\mathrm{V})$ such that g is bounded whose first and second order derivatives ∇g and $\nabla^2 g$ are bounded and $\nabla^2 g$ is γ-Hölder continuous, $2 + \gamma > p$.

These conditions are sufficient [28, 39] to ensure the existence of a unique solution $y^a \in \mathcal{R}^p(\mathrm{V})$ to

$$y_t^a = a + \int_0^t g(y_s^a)\,\mathrm{d}\mathbf{x}_s \tag{6.6}$$

which satisfies $y_0^a = a$ and

$$|y_t^a - y_s^a - g(y_s^a)\mathbf{x}_{s,t}^{(1)} - g\nabla g(y_s^a)\mathbf{x}_{s,t}^{(2)}| \leq C\omega(s,t)^{(2+\gamma)/p} \text{ for } (s,t) \in \Delta_2^+(T).$$

In addition, there exists a family $(y \ltimes x_{s,t})_{(s,t)\in\Delta_2^+(T)}$ with values in $\mathrm{V}\otimes\mathrm{U}$ for which

$$y^a \ltimes x_{s,t} = y^a \ltimes x_{s,r} + y^a \ltimes x_{r,t} + y^a_{s,r} \otimes x_{r,t} \text{ and } |y^a \ltimes x_\bullet| \leq C\omega^{2/p} \quad (6.7)$$

for all $(s,r,t) \in \Delta_3^+(T)$.

The map $\mathscr{I} : a \in \mathrm{V} \mapsto y \in \mathcal{R}^p(\mathrm{V})$, called the *Itô map*, is Lipschitz continuous [28, 39]. Set $\mathscr{I}_t(a) := y_t$. When $\mathbf{x}$ is a smooth rough path, then y is solution to an ODE and $\mathscr{I}_t : \mathrm{V} \to \mathrm{V}$ defines a flow of diffeomorphisms for any $t \in [0,T]$. Using an approximation argument, P. Friz and N. Victoir proved in [28] that $\mathscr{I}_t$ is a flow of diffeomorphisms when $\mathbf{x} \in \mathcal{WGRP}_2^p(\mathbb{R}^d)$. The gradient of $\mathscr{I}_t$ is solution to the Jacobi flow

$$\nabla\mathscr{I}_t(a) = \mathsf{Id} + \int_0^t \nabla g(y^a_s)\nabla\mathscr{I}_s(a)\,\mathrm{d}\mathbf{x}_s, \quad (6.8)$$

which is a linear RDE.

In a forthcoming article [19], we provide an alternative proof of these facts without relying on a regularization argument, which allows to consider $\mathbf{x} \in \mathcal{RP}_2^p(\mathrm{U})$. Moreover, we show that $\mathscr{I}_t$ is Hölder continuous. The core idea is the following. For a, h in U, $\epsilon > 0$, set $\Delta_t^\epsilon(a) = \mathscr{I}_t(a+\epsilon h) - \mathscr{I}_t(a)$, write a first order Taylor expansion of $g(\mathscr{I}_t(a))$ as

$$g(\mathscr{I}_t(a+\epsilon h)) - g(\mathscr{I}_t(a)) = \nabla g(\mathscr{I}_t(a))\Delta_t^\epsilon(a) + G(\mathscr{I}_t(a+\epsilon h), \mathscr{I}_t(a))$$

and show that

$$\Delta_t^\epsilon(a) = \epsilon h + \int_0^t \nabla g(\mathscr{I}_t(a))\Delta_t^\epsilon(a)\,\mathrm{d}\mathbf{x}_s + \int_0^t G(\mathscr{I}_t(a+\epsilon h), \mathscr{I}_t(a))\,\mathrm{d}\mathbf{x}_s.$$

Thus, $\Delta_t^\epsilon(a)$ is solution to a perturbed linear RDE and the convergence of $\epsilon^{-1}\Delta_t^\epsilon(a)$ towards $\nabla\mathscr{I}_t(a)h$ with $\nabla\mathscr{I}_t(a)$ given by (6.8) is studied by giving bounds on $\int_0^t G(\mathscr{I}_t(a+\epsilon h), \mathscr{I}_t(a))\,\mathrm{d}\mathbf{x}_s$.

In [19], we also study differentiability properties with respect to perturbation of a parametrized family of vector fields $f = V(\cdot,\lambda)$ or of the driving path $\mathbf{x}$ by a path in $\mathcal{R}^q(\mathrm{U})$, $q^{-1} + p^{-1} > 1$. This leads to study

$$y^a_t = a + \int_0^t V(y^a_s,\lambda)\,\mathrm{d}\mathbf{x}_s + \int_0^t V(y^a_s,\lambda)\,\mathrm{d}h_s,\ h \in \mathcal{R}^q(\mathrm{V}) \text{ with } 1/p + 1/q > 1$$

with respect to a, λ and h. In this case, $f(\cdot) = \nabla V(\cdot,\lambda)$. Again, Propositions 6.3 and 6.4 lead to the conclusion.

Let us now consider a function $f : \mathrm{V} \to L(\mathrm{W}\otimes\mathrm{U},\mathrm{W})$ such that f is bounded with a bounded derivative ∇f, which is γ-Hölder continuous, $1+\gamma>p$. We also consider a path $y\in\mathcal{R}^p(\mathrm{V})$ for which a family $(y\ltimes x_{s,t})_{(s,t)\in\Delta_2^+(T)}$ satisfying (6.7) exists. Typically, $f=\nabla g$ and y is the solution to (6.6) as above.

Our first aim is to consider the solution to the differential equation

$$z_t = a + \int_0^t f(y_s) z_s \,\mathrm{d}\mathbf{x}_s. \tag{6.9}$$

If $\mathrm{U}=\mathbb{R}^d$, $\mathrm{V}=\mathbb{R}^n$ and $\mathrm{W}=\mathbb{R}^m$ and $\mathbf{x}$ is a smooth rough path living above x, by an equation of type (6.9), we mean

$$z_t^i = a^i + \sum_{\substack{k=1,\dots,d\\ j=1,\dots,n}} \int_0^t f_k^{i,j}(y_s) z_s^j \,\mathrm{d}x_s^k,\ i=1,\dots,m.$$

In the sense of Davie, a solution of (6.9) is sought as satisfying for constants $C\geq 0$ and $\theta>1$, with $\mathbf{x}_{s,t}^{(1)} := x_t - x_s$ and $\mathbf{x}_{s,t}^{(2)} := \int_s^t [x_r - x_s]\otimes dx_r$,

$$\sup_{i=1,\dots,m}\Bigg| z_t^i - z_s^i - \sum_{\substack{k=1,\dots,d\\ j=1,\dots,n}} f_k^{i,j}(y_s) z_s^j \mathbf{x}_{s,t}^{(1),k} - \sum_{\substack{k=1,\dots,d\\ j=1,\dots,n\\ r=1,\dots,n}} \frac{\partial f_k^{i,j}}{\partial y^r}(y_s) z_s^j (y\ltimes x)_{s,t}^{r,k}$$
$$- \sum_{\substack{k=1,\dots,d\\ j=1,\dots,n\\ q=1,\dots,d\\ p=1,\dots,n}} f_k^{i,j}(y_s) f_q^{j,p}(y_s) z_s^p \mathbf{x}_{s,t}^{(2),q,k}\Bigg| \leq C\omega(s,t)^\theta,\ \forall (s,t)\in\Delta_2^+(T),$$

which we write under the more compact form

$$|z_t - z_s - f(y_s) z_s \mathbf{x}_{s,t}^{(1)} - \nabla f(y_s) z_s y\ltimes x_{s,t} - F(y_s) z_s \mathbf{x}_{s,t}^{(2)}| \leq C\omega(s,t)^\theta, \tag{6.10}$$

with $F_{k,\ell}(y_s) = f_k(y_s) f_\ell(y_s)$ is a matrix in $\mathbb{R}^m$ for $k,\ell = 1,\dots,d$, and then F may be identified with a map from V to $L(\mathrm{W}\otimes\mathrm{U}\otimes\mathrm{U},\mathrm{W})$.

Proposition 6.5. *Let $\mathcal{A}_t = \int_0^t f(y_s)\,\mathrm{d}\mathbf{x}_s$ be the path in $\mathcal{R}^p(L(\mathrm{W},\mathrm{W}))$ defined by the rough integral $\mathcal{A}_t z := \pi_{\mathrm{W}}\left(\int_0^t f(y_s) z \,\mathrm{d}\mathbf{x}_s\right) \in \mathcal{R}^p(\mathrm{W})$ for any $z\in\mathrm{W}$. Then there exists a left p-rough lift $\underline{\mathcal{A}}$ of $\mathcal{A}$.*

Proof. For $(s,t) \in \Delta_2^+(T)$, let $\mathsf{A}_{s,t}$ be the linear operator in $L(\mathrm{W},\mathrm{W})$ defined by

$$\mathsf{A}_{t,s}z = z + f(y_s)[z]\mathbf{x}_{s,t}^{(1)} + \nabla f(y_s)[z]y\ltimes x_{s,t} + F(y_s)[z]\mathbf{x}_{s,t}^{(2)},\ z \in \mathrm{W}. \quad (6.11)$$

Then

$$\begin{aligned}\mathsf{A}_{t,r}\mathsf{A}_{r,s}z = z &+ f(y_s)[z]\mathbf{x}_{s,r}^{(1)} + f(y_r)[z]\mathbf{x}_{r,t}^{(1)} \\ &+ \nabla f(y_s)[z]y\ltimes x_{s,r} + \nabla f(y_r)[z]y\ltimes x_{r,t} \\ &+ F(y_s)[z]\mathbf{x}_{s,r}^{(2)} + F(y_r)[z]\mathbf{x}_{r,t}^{(2)} \\ &+ f(y_r)f(y_s)[z]\mathbf{x}_{s,r}^{(1)} \otimes \mathbf{x}_{r,t}^{(1)} + \sigma(s,r,t)[z],\end{aligned}$$

where $\sigma(s,r,t)[z]$ contains all the terms which are smaller than $C|z|\omega(s,t)^{3p}$ for some constant C. Since $\mathbf{x}_{s,t} = \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t}$ and then $\mathbf{x}_{s,t}^{(2)} = \mathbf{x}_{s,r}^{(2)} + \mathbf{x}_{r,t}^{(2)} + \mathbf{x}_{s,r}^{(1)} \otimes \mathbf{x}_{r,t}^{(1)}$ and using standard computations, this proves that $|\mathsf{A}_{t,r,s}| \leq C\omega(s,t)^{\theta}$ and then that A is an almost left p-rough resolvent. Then $\underline{\mathcal{A}}_{\bullet}$ is the p-rough resolvent associated to $\mathsf{A}_{\bullet}$ through Theorem 3.11. ☐

Proposition 6.6. *Under the above hypotheses on f, $\mathbf{x}$ and y, the three notions (linear RDE, in the sense of A.M. Davie* [20] *and in the sense of Lyons) of solution to* (6.9) *are equivalent.*

Proof. The equivalence between solutions in the sense of Davie and in the sense of Lyons have been proved in [39]. The equivalence with solutions of linear RDE follows from the sewing lemma and the definition of $\mathsf{A}_{\bullet}$ in (6.11). ☐

Let us consider now bounded function $g : \mathrm{V} \to L(\mathrm{U},\mathrm{W})$ with a bounded derivative which is γ-Hölder continuous and set

$$b_t := \int_0^t g(y_s)\,\mathrm{d}\mathbf{x}_s.$$

which means that $b_{\bullet}$ is the unique path in $\mathcal{R}^p(\mathrm{W})$ satisfying

$$|b_{s,t} - g(y_s)\mathbf{x}_{s,t}^{(1)} - \nabla g(y_s)y\ltimes x_{s,t}| \leq C\omega(s,t)^{\theta}$$

for some $\theta > 1$.

Our second aim is to consider the solution to the differential equation

$$z_t = a + \int_0^t f(y_s)z_s\,\mathrm{d}\mathbf{x}_s + \int_0^t g(y_s)\,\mathrm{d}\mathbf{x}_s. \quad (6.12)$$

Proposition 6.7. *Under the above hypotheses on f, $\mathbf{x}$ and y, there exists a unique solution to (6.12), and for any $t \in [0,T]$, $a \mapsto z_t$ defines an homeomorphism from* W *to* W*. Besides,*

$$\|z\|_p \le C(|a| + \|g\|_\infty + \|\nabla g\|_\infty + H_\gamma(\nabla g)), \tag{6.13}$$

where C depends only on $\|f\|_\infty$ $\|\nabla f\|_\infty$, $H_\gamma(\nabla f)$, $\mathbf{x}$, $\omega(0,T)$, p and γ, and $H_\gamma(\nabla g)$ is the γ-Hölder norm of ∇g.

Proof. Let us set for $(s,t) \in \Delta_2^+(T)$,

$$\mu_{s,t} := 1 + \int_s^t f(y_r)\,\mathrm{d}\mathbf{x}_r + \int_s^t g(y_r)\,\mathrm{d}\mathbf{x}_r + g(y_s) \otimes f(y_s)\mathbf{x}_{s,t}^{(2)}.$$

Let us consider $\mathrm{X} := 1 \oplus L(\mathrm{W},\mathrm{W}) \oplus \mathrm{W} \oplus (\mathrm{W} \otimes L(\mathrm{W},\mathrm{W})))$ which is a Banach sub-algebra of $(\mathrm{T}_2(L(\mathrm{W},\mathrm{W}) \oplus \mathrm{W}), +, \otimes)$ when equipped with $a \otimes b = \pi_{\mathrm{X}}(a \otimes b)$. Thus, working only in X,

$$\begin{aligned}\mu_{s,t} - \mu_{s,r} \otimes \mu_{r,t} = {} & (g(y_s) \otimes f(y_s) - g(y_r) \otimes f(y_r))\mathbf{x}_{r,t}^{(2)} \\ & + \left(g(y_s)\mathbf{x}_{s,r}^{(1)} - \int_s^r g(y_u) d\mathbf{x}_u\right) \otimes f(y_s)\mathbf{x}_{r,t}^{(1)} \\ & + \int_s^r g(y_u) d\mathbf{x}_u \otimes \left(f(y_s)\mathbf{x}_{r,t}^{(1)} - \int_r^t f(y_u) d\mathbf{x}_u\right).\end{aligned}$$

Using the regularity of f and g and the properties of the rough integrals, it follows that $(\mu_{s,t})_{(s,t)\in\Delta_2^+(T)}$ is an almost right p-rough resolvent with values in X. Let $(\nu_{s,t})_{(s,t)\in\Delta_2^+(T)}$ be the corresponding p-rough resolvent with values in X given by the sewing lemma. The multiplicative property $\nu_{s,t} = \nu_{s,r} \otimes \nu_{r,t}$ implies that

$$\begin{aligned}\pi_{L(\mathrm{W},\mathrm{W})\oplus\mathrm{W}}(\nu_{s,t}) &= \int_s^t f(y_s)\,\mathrm{d}\mathbf{x}_s + \int_s^t g(y_s)\,\mathrm{d}\mathbf{x}_s \\ \text{and } \nu_{s,t}^{(2)} &= \nu_{s,r}^{(2)} + \nu_{r,t}^{(2)} + b_{s,r} \otimes \mathcal{A}_{t,r}^{(1)} \text{ with } \nu_{s,t}^{(2)} := \pi_{\mathrm{W}\otimes L(\mathrm{W},\mathrm{W}))}(\nu_{s,t}).\end{aligned}$$

Let us now introduce the linear map from $(\mathrm{X},+)$ to $(\mathrm{W},+)$ defined by $\Phi(a) = a$, $\Phi(b) = \Phi(c) = 0$ and $\Phi(a \otimes b) := b[a]$ for $a \in \mathrm{W}$, $b \in L(\mathrm{W},\mathrm{W})$ and $c \in \mathbb{R}$.

Then $\overleftarrow{\mathcal{B}}_{t,s} := \Phi(\nu_{s,t})$ for $(s,t) \in \Delta_2^+(T)$ defines a left p-rough lift $\overleftarrow{\mathcal{B}}_\bullet$ of b. The result follows then from Proposition 6.3 □

References

[1] S. Aida – "Notes on proofs of continuity theorem in rough path analysis", 2006, Unpublished note, Osaka University.

[2] I. Bailleul – "Flows driven by rough paths", 2012, Preprint `arxiv:1203.0888`.

[3] A. Baker – *Matrix groups. An introduction to Lie group theory*, Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London, 2002.

[4] F. Baudoin – *An introduction to the geometry of stochastic flows*, Imperial College Press, London, 2004.

[5] F. Baudoin and X. Zhang – "Taylor expansion for the solution of a stochastic differential equation driven by fractional Brownian motions", *Electron. J. Probab.* **17** (2012), p. no. 51, 21.

[6] G. Ben Arous – "Flots et séries de Taylor stochastiques", *Probab. Theory Related Fields* **81** (1989), no. 1, p. 29–77.

[7] S. Blanes, F. Casas, J. A. Oteo and J. Ros – "The Magnus expansion and some of its applications", *Phys. Rep.* **470** (2009), no. 5-6, p. 151–238.

[8] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni – *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[9] A. Bonfiglioli and R. Fulci – *Topics in noncommutative algebra. The theorem of Campbell, Baker, Hausdorff and Dynkin*, Lecture Notes in Mathematics, vol. 2034, Springer, Heidelberg, 2012.

[10] M. Caruana, P. K. Friz and H. Oberhauser – "A (rough) pathwise approach to a class of non-linear stochastic partial differential equations", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** (2011), no. 1, p. 27–46.

[11] M. Caruana and P. Friz – "Partial differential equations driven by rough paths", *J. Differential Equations* **247** (2009), no. 1, p. 140–173.

[12] F. Castell – "Asymptotic expansion of stochastic flows", *Probab. Theory Related Fields* **96** (1993), no. 2, p. 225–239.

[13] F. Castell and J. Gaines – "The ordinary differential equation approach to asymptotically efficient schemes for solution of stochastic differential equations", *Ann. Inst. H. Poincaré Probab. Statist.* **32** (1996), no. 2, p. 231–250.

[14] K.-T. Chen – "Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula", *Ann. of Math. (2)* **65** (1957), p. 163–178.

[15] ______, "Integration of paths—a faithful representation of paths by non-commutative formal power series", *Trans. Amer. Math. Soc.* **89** (1958), p. 395–407.

[16] ______, "Formal differential equations", *Ann. of Math. (2)* **73** (1961), p. 110–133.

[17] ______, "Expansion of solutions of differential systems", *Arch. Rational Mech. Anal.* **13** (1963), p. 348–363.

[18] L. Coutin – "Rough paths *via* sewing lemma", *ESAIM Probab. Stat.* **16** (2012), p. 479–526.

[19] L. Coutin and A. Lejay – "Sensitivity of rough differential equations", 2013, Preprint.

[20] A. Davie – "Differential equations driven by rough signals: an approach via discrete approximation", *Appl. Math. Res. Express. AMRX* **2** (2007), p. Art. ID abm009.

[21] A. Deya, M. Gubinelli and S. Tindel – "Non-linear rough heat equations", *Probab. Theory Related Fields* **153** (2012), no. 1-2, p. 97–147.

[22] A. Deya and S. Tindel – "Rough Volterra equations. I. The algebraic integration setting", *Stoch. Dyn.* **9** (2009), no. 3, p. 437–477.

[23] ______, "Rough Volterra equations 2: Convolutional generalized integrals", *Stochastic Process. Appl.* **121** (2011), no. 8, p. 1864–1899.

[24] R. G. Douglas – *Banach algebra techniques in operator theory*, second éd., Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998.

[25] F. J. Dyson – "The radiation theories of Tomonaga, Schwinger, and Feynman", *Physical Rev. (2)* **75** (1949), p. 486–502.

[26] D. Feyel and A. de La Pradelle – "Curvilinear integrals along enriched paths", *Electron. J. Probab.* **11** (2006), p. no. 34, 860–892 (electronic).

[27] D. Feyel, A. de La Pradelle and G. Mokobodzki – "A non-commutative sewing lemma", *Electron. Commun. Probab.* **13** (2008), p. 24–34.

[28] P. K. Friz and N. B. Victoir – *Multidimensional stochastic processes as rough paths. Theory and applications*, Cambridge Studies in Advanced Mathematics, vol. 120, Cambridge University Press, Cambridge, 2010.

[29] M. Gubinelli – "Abstract integration, combinatorics of trees and differential equations", Combinatorics and physics, Contemp. Math., vol. 539, Amer. Math. Soc., Providence, RI, 2011, p. 135–151.

[30] M. Gubinelli, A. Lejay and S. Tindel – "Young integrals and SPDEs", *Potential Anal.* **25** (2006), no. 4, p. 307–326.

[31] E. Hairer, C. Lubich and G. Wanner – *Geometric numerical integration. structure-preserving algorithms for ordinary differential equations*, Springer Series in Computational Mathematics, vol. 31, Springer, Heidelberg, 2010, Reprint of the second (2006) edition.

[32] M. Hairer and D. Kelly – "Geometric versus non-geometric rough paths", 2012, Preprint `arxiv:1210.9294`.

[33] B. C. Hall – *Lie groups, Lie algebras, and representations. an elementary introduction*, Graduate Texts in Mathematics, vol. 222, Springer-Verlag, New York, 2003.

[34] K. Hara and M. Hino – "Fractional order Taylor's series and the neo-classical inequality", *Bull. Lond. Math. Soc.* **42** (2010), no. 3, p. 467–477.

[35] A. Lejay – "An introduction to rough paths", Séminaire de Probabilités XXXVII, Lecture Notes in Math., vol. 1832, Springer, Berlin, 2003, p. 1–59.

[36] ______, "On rough differential equations", *Electron. J. Probab.* **14** (2009), p. no. 12, 341–364.

[37] ______, "Yet another introduction to rough paths", Séminaire de Probabilités XLII, Lecture Notes in Math., vol. 1979, Springer, Berlin, 2009, p. 1–101.

[38] ______, "Controlled differential equations as Young integrals: a simple approach", *J. Differential Equations* **249** (2010), no. 8, p. 1777–1798.

[39] ______, "Global solutions to rough differential equations with unbounded vector fields", Séminaire de Probabilités XLIV, Lecture Notes in Math., vol. 2046, Springer, Heidelberg, 2012, p. 215–246.

[40] A. Lejay and N. Victoir – "On (p, q)-rough paths", *J. Differential Equations* **225** (2006), no. 1, p. 103–133.

[41] G. Lord, S. J. A. Malham and A. Wiese – "Efficient strong integrators for linear stochastic systems", *SIAM J. Numer. Anal.* **46** (2008), no. 6, p. 2892–2919.

[42] T. Lyons and Z. Qian – *System control and rough paths*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2002.

[43] T. J. Lyons – "Differential equations driven by rough signals", *Rev. Mat. Iberoamericana* **14** (1998), no. 2, p. 215–310.

[44] T. J. Lyons, M. Caruana and T. Lévy – *Differential equations driven by rough paths (Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004)*, Lecture Notes in Mathematics, vol. 1908, Springer, Berlin, 2007.

[45] T. J. Lyons and N. Sidorova – "On the radius of convergence of the logarithmic signature", *Illinois J. Math.* **50** (2006), no. 1-4, p. 763–790 (electronic).

[46] T. J. Lyons and W. Xu – "A uniform estimate for rough paths", *Bull. Sci. Math.* **137** (2013), no. 7, p. 867–879.

[47] W. Magnus – "On the exponential solution of differential equations for a linear operator", *Comm. Pure Appl. Math.* **7** (1954), p. 649–673.

[48] B. Mielnik and J. Plebański – "Combinatorial approach to Baker-Campbell-Hausdorff exponents", *Ann. Inst. H. Poincaré Sect. A (N.S.)* **12** (1970), p. 215–254.

[49] P. C. Moan and J. Niesen – "Convergence of the Magnus series", *Found. Comput. Math.* **8** (2008), no. 3, p. 291–301.

[50] R. Ree – "Lie elements and an algebra associated with shuffles", *Ann. of Math. (2)* **68** (1958), p. 210–220.

[51] C. Reutenauer – *Free Lie algebras*, London Mathematical Society Monographs. New Series, vol. 7, The Clarendon Press Oxford University Press, New York, 1993.

[52] I. A. Stegun (éd.) – *Pocketbook of mathematical functions (abridged edition of Handbook of mathematical functions edited by Milton Abramowitz and Irene A. Stegun)*, Verlag Harri Deutsch, Thun, 1984.

[53] R. S. Strichartz – "The Campbell-Baker-Hausdorff-Dynkin formula and solutions of differential equations", *J. Funct. Anal.* **72** (1987), no. 2, p. 320–345.

[54] L. C. Young – "An inequality of the Hölder type, connected with Stieltjes integration", *Acta Math.* **67** (1936), no. 1, p. 251–282.

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