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Hossein Movasati

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Annales mathématiques Blaise Pascal **19**, 307-377 (2012)

# Quasi-modular forms attached to elliptic curves, I

Hossein Movasati

**Abstract**

In the present text we give a geometric interpretation of quasi-modular forms using moduli of elliptic curves with marked elements in their de Rham cohomologies. In this way differential equations of modular and quasi-modular forms are interpreted as vector fields on such moduli spaces and they can be calculated from the Gauss-Manin connection of the corresponding universal family of elliptic curves. For the full modular group such a differential equation is calculated and it turns out to be the Ramanujan differential equation between Eisenstein series. We also explain the notion of period map constructed from elliptic integrals. This turns out to be the bridge between the algebraic notion of a quasi-modular form and the one as a holomorphic function on the upper half plane. In this way we also get another interpretation, essentially due to Halphen, of the Ramanujan differential equation in terms of hypergeometric functions. The interpretation of quasi-modular forms as sections of jet bundles and some related enumerative problems are also presented.

*Formes quasimodulaires attachées aux courbes elliptiques, I*

**Résumé**

Dans ce texte, on donne une interprétation géométrique des formes quasimodulaires en utilisant les modules des courbes elliptiques avec un point marqué dans leurs cohomologies de de Rham. De cette façon, les équations différentielles des formes modulaires et quasimodulaires sont interprétées comme des champs de vecteurs de ces espaces de modules. Elles peuvent être établies grâce à la connection de Gauss-Manin de la famille universelle de courbes elliptiques correspondante. Pour le groupe modulaire, on calcule une telle équation différentielle qui apparaît être celle de Ramanujan qui relie entre elles les séries d'Eisenstein. On explique aussi la notion de périodes construites à partir des intégrales elliptiques. Elles apparaissent comme le pont entre la notion algébrique de forme quasimodulaire et la définition en terme de fonction holomorphe sur le demi-plan de Poincaré. De cette façon, nous obtenons aussi une autre interprétation, essentiellement due à Halphen, de l'équation différentielle de Ramanujan en termes de fonctions hypergéométriques. L'interprétation des formes quasimodulaires comme sections de fibrés des jets et des problèmes de combinatoire énumérative sont aussi présentés.

## Contents

## 1. Introduction

The objective of this note is to develop the theory of quasi-modular forms in the framework of Algebraic Geometry. The request for an algebra which contains the classical modular forms and which is closed under derivation leads in a natural way to the theory of quasi-modular forms and this can be the main reason why the name *differential modular form* is more natural (see [22, 23]). The literature on modular forms and their applications is huge and a naive mind may look for similar applications of quasi-modular forms. At the beginning of our journey, we may think that we are dealing with a new theory. However, the main examples of quasi-modular forms and their differential equations go back to 19th century, due to G. Darboux (1878) and G. Halphen (1881).

The development of the theory of modular forms as holomorphic functions in the Poincaré upper half plane has shown that many of problems related to modular forms can be proved if we take a modular form into Algebraic Geometry and interpret it in the following way: a modular form of weight $m$ is a section of $m$-times tensor power of the line bundle $F$ on compactified moduli spaces of elliptic curves, where the fiber of $F$ at the elliptic curve $E$ is defined to be $\mathrm{Lie}(E)^\vee$. This is equivalent to say that a modular form is a function from the pairs $(E,\omega)$ to $\mathsf{k}$, where $E$ is an elliptic curve defined over a field $\mathsf{k}$ of arbitrary characteristic and $\omega$ is a regular differential form on $E$, such that $f(E,a\omega)=a^{-m}f(E,\omega)$ for all $a\in\mathsf{k}^*$. Some additional properties regarding the degeneration of the pair $(E,\omega)$ is also required. The first interpretation can be generalized to the context of quasi-modular forms using the notion of jet bundles (see Appendix A). We found it much more convenient for calculations to generalize the later interpretation to the context of quasi-modular forms. The differential form $\omega$ is replaced with an element in the first algebraic de Rham cohomology of $E$ such that it is not represented by a regular differential form. Algebraic de Rham cohomology is introduced for an arbitrary smooth variety by A. Grothendieck (1966) in [11] after a work of Atiyah and Hodge (1955). Apart from the multiplicative group $\mathsf{k}^*$, we have also the additive group of $\mathsf{k}$ acting on such pairs and the corresponding functional equation of a quasi-modular form. In order to use the algebraic de Rham cohomology we have to assume that the field $\mathsf{k}$ is of characteristic zero. It turns out that the Ramanujan relations between Eisenstein series can be derived from the Gauss-Manin connection of families of elliptic curves and such series in the $q$-expansion form are uniquely and recursively determined by the Ramanujan relations. Looking in this way, we observe that the theory is not so much new. We find the Darboux-Halphen differential equation which gives rise to the theory of quasi-modular forms for $\Gamma(2)$.

We have given some applications of our effort; in order to find differential and polynomial equations for modular forms attached to congruence groups it is sufficient to construct explicit affine coordinates on the moduli of elliptic curves enhanced with certain torsion point structure, and then to calculate its Gauss-Manin connection. As the geometrization of modular forms was an important tool in understanding many difficult problems in number theory, such as the Taniyama-Shimura conjecture and its solution which is known under modularity theorem, I hope that the geometrization presented in this text helps to understand many enumerative problems related to quasi-modular, and even modular, forms (see Appendix B). However, the main justification in our mind for writing a text which apparently deals with the mathematics of a century ago, is to prepare the ground for a dreaming program: to develop a theory attached to an arbitrary family of varieties similar to the modular form theory attached to elliptic curves. Apart from Siegel and Hilbert modular forms attached to Abelian varieties, such a theory is under construction for a certain family of Calabi-Yau varieties which appears in mathematical physics, see for instance [24]. The text is written in such a way that the way for generalizations becomes smoother.

We assume that the reader has a basic knowledge in algebraic geometry, complex analysis in one variable and Riemann surfaces. At the end of each sub section the reader finds many exercises with different degrees of difficulty. The reader who is interested on the content of this note and who is not worry with the details, may skip them. We have collected such exercises in order to avoid statements like *it is left to the reader*, *it is easy to check* and so on. The students are highly recommended to do some good piece of the exercises.

The reader who wants to have a fast overview of the text is invited to read the small introductions at the beginning of each section. There, we have tried to write the content of each section in a down-to-earth way. In this way it turns out that the mathematics presented in this text go back to a century ago, to mathematicians like Gauss, Halphen, Ramanujan, Abel, Picard, Poincaré and many others. The reader gets only a flavor of the history behind the mathematics of the present text. A full account of the works of all these respected mathematicians would require a deep reading of many treatises that they have left to us. Another relatively fast reading of the present text would be in the following order: §2.1, §3.1, §4.1, §4.3, §4.4, §B, §5.1, §6.1, §7.1 and the entire §8.

The text is organized in the following way. For the definition of a geometric quasi-modular form we need to introduce the algebraic de Rham cohomology of an elliptic curve and its intersection form. This is done in §2. We have also included the residue calculus in this chapter. We need this for the definition of the intersection form and also for the definition of Eisenstein series in the algebraic context. §3 is dedicated to the definition and calculation of the Gauss-Manin connection of families of elliptic curves. Using this, in §4 we describe how we get some of our main protagonists, namely the differential equations satisfied by quasi-modular forms. Therefore, we can work with such differential equations and formal $q$-expansions uniquely determined by them without knowing a word about quasi-modular forms. For this reason the definition of a quasi-modular form comes later in §6. This is done after §5 in which we describe some classical facts such as Weierstrass form, group structure, torsion points and so on. The algebraic definition of a quasi-modular form is done in §6. In §7 we give the classical definition of a quasi-modular form as a holomorphic function in the upper half plane. The relation between the algebraic and holomorphic notions of quasi-modular forms is explained at the end of §8 using the inverse of the period map. In this section we explain how $\mathrm{SL}(2, \mathbb{Z})$ can be interpreted as the monodromy group of families of elliptic curves, and how elliptic integrals are hypergeometric functions with special parameters. Appendix A is dedicated to the description of quasi-modular forms in terms of jet bundles over the moduli of elliptic curves and Appendix B is dedicated to examples of quasi-modular forms with enumerative aspects.

**Acknowledgments.** The present text arose from my lecture notes at Besse summer school on quasi-modular forms, 2010. Here, I would like to thank the organizers, and in particular, Emmanuel Royer and François Martin for the wonderful job they did. The text is also used in a mini course given at IMPA during the summer school 2011. Prof. P. Deligne sent me some comments on the first and final draft of the present text which is essentially included in Appendix A. Here, I would like to thank him for his clarifying comments. My sincere thanks go to Prof. E. Ghys who drew my attention to the works of Halphen. Finally, I would like to apologize from all the people whose works is related to the topic of the present text and I do not mention them. There is a huge literature on elliptic curves and modular forms and I am sure that I have missed many related works.

## 2. De Rham cohomology of smooth varieties

### 2.1. Introduction

In many calculus books we find tables of integrals and there we never find a formula for elliptic integrals

$$\int_a^b \frac{Q(x)dx}{\sqrt{P(x)}}, \tag{2.1}$$

where $P(x)$ is a degree three polynomial in one variable $x$ and with real coefficients, for simplicity we assume that it has real roots, and $a, b$ are two consecutive roots of $P$ or $\pm\infty$. Since Abel and Gauss it was known that if we choose $P$ randomly (in other words for generic $P$) such integrals cannot be calculated in terms of until then well-known functions (for a polynomial $P$ with a double root we can calculate the elliptic integrals, see Exercise 2.1. For other particular examples of $P$ we have some formulas calculating elliptic integrals in terms of the values of the Gamma function on rational numbers. The Chowla-Selberg theorem, see for instance Gross's article [10], describes this phenomenon in a complete way). It was also well known that any such elliptic integral, say it $\int_\delta Q(x)\omega$ with $\omega := dx/\sqrt{P(x)}$, with the integration domain $\delta = [a, b]$ and polynomial $P$ fixed, is a linear combination of two integrals $\int_\delta \omega$ and $\int_\delta x\omega$, that is, it is possible to calculate effectively two numbers $r_1, r_2 \in \mathbb{R}$ such that

$$\int_\delta Q(x)\omega = r_1 \int_\delta \omega + r_2 \int_\delta x\omega. \tag{2.2}$$

For instance, take

$$P(x) = 4(x - t_1)^3 - t_2(x - t_1) - t_3, \ t_1, t_2, t_3 \in \mathbb{R} \tag{2.3}$$

(in §5 we explain why we write $P$ in this form). For examples of the equality (2.2) see Exercise 2.4. The equalities there, are written just by neglecting $\int_\delta$. The next historical step was to consider the integration on the complex domain instead of integration on the real interval. Now, the integration is on any path which connects two roots or $\infty$ to each other. Soon after it was invented the new variable $y$ representing the quantity $\sqrt{P(x)}$ and the integration took place on a closed path in the topological space $E := \{(x, y) \in \mathbb{C}^2 | y^2 = P(x)\}$. The integrand $Q(x)\frac{dx}{y}$ is now called a differential 1-form and the topological space $E$ is called an elliptic curve (for a brief description of how the study of integrals contributed to the development of Algebraic Topology, see the introduction of §8). The study of elliptic, and later Abelian and multiple, integrals by Abel, Picard and Poincaré prepared the ground for the definition of the de Rham cohomology of varieties and its algebraic version due to Atiyah, Hodge and Grothendieck. Nowadays, algebraic geometers prefer to avoid the usage of integrals for expressing an equality like (2.2). Instead, they say that the algebraic de Rham cohomology of the elliptic curve $y^2 = P(x)$ is a two dimensional vector space generated by two differential forms $\frac{x^i dx}{y}$, $i = 0, 1$.

In this section we want to explain equalities like (2.2) without mentioning integrals and domains of integration. Moreover, we work with a ring $\mathsf{R}$ instead of the field of real numbers. We assume a basic knowledge in Algebraic Geometry. The reader may consult the book of Hartshorne [14], for any lacking definition or proof. We need to work with families of elliptic curves and so we use projective and affine schemes over a commutative ring $\mathsf{R}$ with multiplicative identity element 1. We assume that $\mathsf{R}$ is an integral domain, that is, it is without zero divisors. For simplicity, we take a field of characteristic zero and we consider $\mathsf{R}$ as a finitely generated $\mathsf{k}$-algebra. We denote by $\mathsf{k}_1$ the fractional field of $\mathsf{R}$.

Let $E$ be an elliptic curve over $\mathsf{k}$. The algebraic de Rham cohomologies

$$H^i_{\mathrm{dR}}(E), i = 0, 1, 2$$

are $\mathsf{k}$-vector spaces of dimensions respectively $1, 2$ and 1, see Proposition 2.2, Proposition 2.4 and Proposition 2.5. We have $H^0_{\mathrm{dR}}(E) = \mathsf{k}$, an isomorphism

$$\mathrm{Tr} : H^2_{\mathrm{dR}}(E) \cong \mathsf{k}$$

and a bilinear map

$$H^1_{\mathrm{dR}}(E) \times H^1_{\mathrm{dR}}(E) \to H^2_{\mathrm{dR}}(E)$$

The map Tr composed with the bilinear map gives us:

$$\langle \cdot, \cdot \rangle : H^1_{\mathrm{dR}}(E) \times H^1_{\mathrm{dR}}(E) \to \mathsf{k}$$

which is non-degenerate and anti symmetric, see §2.10. We call it the intersection bilinear form. We have also a natural filtration of $H^1_{\mathrm{dR}}(E)$ which is called the Hodge filtration:

$$\{0\} = F^2 \subset F^1 \subset F^0 = H^1_{\mathrm{dR}}(E)$$

Its non-trivial piece $F^1$ is generated by a regular differential form (a differential form of the first kind). The objective of this section is to define all these in a down-to-earth manner. Grothendieck's original article [11] is still a main source for the definition of algebraic de Rham cohomology.

**Exercise.** (1) Calculate the integral (2.1) for $P$ with a double root.

(2) For particular examples of $P$ there are some formulas for elliptic integrals in terms of the values of the Gamma function on rational numbers. Collect some of these formulas and explain why the formula holds.

(3) Show that $\int_\delta 1/\sqrt{P(x)}\,dx$, where $\delta$ is an interval between two consecutive real roots of $P$ or $\pm\infty$, and $\deg(P) = 4$, can be calculated in terms of elliptic integrals.

## 2.2. Differential forms

Let $A$ be a commutative $\mathsf{k}$-algebra and $\mathsf{R} \to A$ be a morphism of $\mathsf{k}$-algebras. Using this morphism, $A$ can be seen as an $\mathsf{R}$-algebra. We assume that $A$ as an $\mathsf{R}$-algebra is finitely generated.

Let $\Omega_{A/\mathsf{R}}$ denote the module of relative (Kähler) differentials, that is, $\Omega_{A/\mathsf{R}}$ is the quotient of the $A$-module freely generated by symbols $dr$, $r \in A$, modulo its submodule generated by

$$dr,\ r \in \mathsf{R},\ d(ab) - adb - bda,\ d(a+b) - da - db,\ a, b \in A.$$

The $A$-module $\Omega_{A/\mathsf{R}}$ is finitely generated and it is equipped with the derivation

$$d : A \to \Omega_{A/\mathsf{R}},\ r \mapsto dr.$$

It has the universal property that for any $\mathsf{R}$-linear derivation $D : A \to M$ with the $A$-module $M$, there is a unique $A$-linear map $\psi : \Omega_{A/\mathsf{R}} \to M$ such that $D = \psi \circ d$.

Let $X = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(\mathsf{R})$ be the corresponding affine varieties over $\mathsf{k}$ and $X \to T$ be the map obtained by $\mathsf{R} \to A$. We will mainly use the Algebraic Geometry notation $\Omega^1_{X/T} := \Omega_{A/\mathsf{R}}$. Let

$$\Omega^i_{X/T} = \bigwedge_{k=1}^{i} \Omega_{A/\mathsf{R}},$$

be the $i$-th wedge product of $\Omega_{X/T}$ over $A$, that is, $\Omega^i_{X/T}$ is the quotient of the $A$-module freely generated by the symbols $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_i$ modulo its submodule generated by elements which make $\wedge$ $A$-linear in each $\omega_i$ and

$$\omega_1 \wedge \cdots \wedge \omega_j \wedge \omega_{j+1} \wedge \cdots \wedge \omega_i = 0, \quad \text{for } \omega_j = \omega_{j+1}.$$

It is convenient to define

$$\Omega^0_{X/T} := A.$$

The differential operator

$$d_i : \Omega^i_{X/T} \to \Omega^{i+1}_{X/T}$$

is defined by assuming that it is $\mathsf{R}$-linear and

$$d_i(a da_1 \wedge \cdots \wedge da_i) = da \wedge da_1 \wedge \cdots \wedge da_i, \quad a, a_1, \ldots, a_i \in A.$$

Sometimes it is convenient to remember that $d_i$'s are defined relative to $\mathsf{R}$. One can verify easily that $d_i$ is in fact well-defined and satisfy all the properties of the classical differential operator on differential forms on manifolds. From now on we simply write $d$ instead of $d_i$. If $\mathsf{R} = \mathsf{k}$ is a field then we write $X$ instead of $X/T$.

**Exercise.** (1) Prove the universal property of the differential map $d : A \to \Omega_{A/\mathsf{R}}$.

(2) Prove the following properties of the wedge product: For $\alpha \in \Omega^i_{X/T},\ \beta \in \Omega^j_{X/T},\ \gamma \in \Omega^r_{X/T}$

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma),$$

$$\alpha \wedge \beta \wedge \gamma = (-1)^{ij+jr+ir} \gamma \wedge \beta \wedge \alpha,$$

(3) Prove that $d \circ d = 0$.

(4) For $\alpha \in \Omega^i_{X/T},\ \beta \in \Omega^j_{X/T}$ we have:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^i \alpha \wedge (d\beta).$$

## 2.3. **De Rham cohomology**

After the definition of differential forms, we get the de Rham complex of $X/T$, namely:

$$\Omega^0_{X/T} \to \Omega^1_{X/T} \to \cdots \Omega^i_{X/T} \to \Omega^{i+1}_{X/T} \to \cdots$$

Since $d \circ d = 0$, we can define the de Rham cohomologies

$$H^i_{\mathrm{dR}}(X/T) := \frac{\ker(\Omega^i_{X/T} \overset{d}{\to} \Omega^{i+1}_{X/T})}{\mathrm{Im}(\Omega^{i-1}_{X/T} \overset{d}{\to} \Omega^i_{X/T})}.$$

**Exercise.** (1) Let $m$ be the number of generators of the $\mathsf{R}$-algebra $A$. Show that for $i \geq m+1$ we have $\Omega^i_{X/T} = 0$ and hence $H^i_{\mathrm{dR}}(X/T) = 0$.

(2) Let $A = \mathsf{R}[x_1, x_2, \ldots, x_n]$. In this case, we use the notation $\mathbb{A}^n_{\mathsf{R}} := \mathrm{Spec}(A)$. The $A$-module $\Omega^1_{\mathbb{A}^n_{\mathsf{R}}}$ is freely generated by the elements $dx_1, dx_2, \ldots, dx_n$. Prove that

$$H^i(\mathbb{A}^n_{\mathsf{R}}) = 0, \; i = 1, 2, \ldots$$

This is Exercise 16.15 c, p. 414. of [7].

(3) Let us come back to the case of an arbitrary $A$. Let $a_1, a_2, \ldots, a_m \in A$ generate the $\mathsf{R}$-algebra $A$. Define

$$I = \{P \in \mathsf{R}[x_1, x_2, \ldots, x_m] \mid P(a_1, a_2, \ldots, a_m) = 0\}.$$

The set $I$ is an ideal of $\mathsf{R}[x_1, x_2, \ldots, x_m]$ and we have

$$A \cong \mathsf{R}[x_1, x_2, \ldots, x_m]/I,$$

$$\Omega^i_{X/T} \cong \Omega^i_{\mathbb{A}^n_{\mathsf{R}}}/(dI \wedge \Omega^{i-1}_{\mathbb{A}^n_{\mathsf{R}}} + I\Omega^i_{\mathbb{A}^n_{\mathsf{R}}}),$$

where by $dI \wedge \Omega^{i-1}_{\mathbb{A}^n_{\mathsf{R}}} + I\Omega^i_{\mathbb{A}^n_{\mathsf{R}}}$ we mean the $A$-module generated by

$$dr_1 \wedge \omega_1 + r_2\omega_2, \; r_1, r_2 \in I, \; \omega_1 \in \Omega^{i-1}_{\mathbb{A}^n_{\mathsf{R}}}, \; \omega_2 \in \Omega^i_{\mathbb{A}^n_{\mathsf{R}}}.$$

(4) Discuss conditions on $A$ such that $H^0(X/T) = \mathsf{R}$. For instance, show that if $\mathsf{R} = \mathsf{k}$ is an algebraically closed field of characteristic zero and $X$ is an irreducible reduced variety over $\mathsf{k}$ then $H^0(X) = \mathsf{k}$.

## 2.4. An incomplete elliptic curve, I

In this section we find an explicit basis for the de Rham cohomology of the main examples of this text, that is, affine elliptic curves in Weierstrass form. The general theory uses the notion of a Brieskorn module which is essentially the same as de Rham cohomology. Our main source for this section is [25, 19].

Let $t_1, t_2, t_3 \in \mathsf{R}$, $P(x) = 4(x-t_1)^3 - t_2(x-t_1) - t_3 \in \mathsf{R}[x]$ and $f = y^2 - P(x)$. Define

$$A = \mathsf{R}[x, y]/\langle f \rangle.$$

where $\langle f \rangle$ is the ideal generated by $f$. We have

$$\Omega^1_{X/T} = \Omega^1_{\mathbb{A}^2_{\mathsf{R}}}/\langle f\Omega^1_{\mathbb{A}^2_{\mathsf{R}}} + \Omega^0_{\mathbb{A}^2_{\mathsf{R}}} df\rangle,\ \Omega^2_{X/T} = \Omega^2_{\mathbb{A}^2_{\mathsf{R}}}/\langle f\Omega^2_{\mathbb{A}^2_{\mathsf{R}}} + df \wedge \Omega^1_{\mathbb{A}^2_{\mathsf{R}}}\rangle.$$

We have to say some words about $\Omega^2_{X/T}$. We define the auxiliary $\mathsf{R}$-module:

$$V := \Omega^2_{\mathbb{A}^2_{\mathsf{R}}}/df \wedge \Omega^1_{\mathbb{A}^2_{\mathsf{R}}},$$

and

$$\Delta := 27t_3^2 - t_2^3.$$

The quotient $\mathsf{R}[x, y]/\langle f_x, f_y\rangle$ is isomorphic to $V$ by sending $P \in V$ to $Pdx \wedge dy$.

**Proposition 2.1.** *We have*

$$\Delta\Omega^2_{X/T} = 0$$

*Proof.* Using the explicit form of $f$, we can easily verify that the $\mathsf{R}$-module $V$ is freely generated by $dx \wedge dy, xdx \wedge dy$ (here we use the fact that 2 and 3 are invertible in $\mathsf{R}$). Let $M : V \to V,\ M(\omega) = f\omega$. We write $M$ in the basis $dx \wedge dy, xdx \wedge dy$:

$$M = \begin{pmatrix} \frac{2}{3}t_1t_2 + t_3 & -\frac{2}{3}t_1^2t_2 + \frac{1}{18}t_2^2 \\ \frac{2}{3}t_2 & -\frac{2}{3}t_1t_2 + t_3 \end{pmatrix}.$$

Let $p(z) := z^2 - \mathrm{tr}(M)z + \det(M) = \det(M - zI_{2\times 2})$ be the characteristic polynomial of $M$. We have $P(f)V = 0$ and since $\Omega^2_{X/T} = V/\langle f\rangle$, we conclude that $\det(M)\Omega^2_{X/T} = 0$. From another side $\det(M) = \frac{-1}{27}\Delta$. □

From now on we assume that $\Delta$ is irreducible in $\mathsf{R}$ and we replace $\mathsf{R}$ with its localization on its multiplicative group generated by $\Delta$. Therefore,

$\Delta$ is invertible in $\mathsf{R}$ and we can talk about the pole or zero order along $\Delta$ of an element in any $\mathsf{R}$-module. In this way $\Omega^2_{X/T}=0$ and

$$H^1_{\mathrm{dR}}(X/T)\cong \Omega^1_{\mathbb{A}^2_{\mathsf{R}}}/\langle f\Omega^1_{\mathbb{A}^2_{\mathsf{R}}}+df\Omega^0_{\mathbb{A}^2_{\mathsf{R}}}+d\Omega^0_{\mathbb{A}^2_{\mathsf{R}}}\rangle.$$

There are two polynomials $A,B\in\mathsf{R}[x]$ such that $AP+BP'=\Delta$. We define

$$\omega=\frac{1}{\Delta}(Aydx+2Bdy)$$

which satisfies:

$$dx=y\omega, dy=\frac{1}{2}P'\omega \tag{2.4}$$

We denote by $\frac{dx}{y}$ and $\frac{xdx}{y}$ the elements $\omega$, respectively $x\omega$. Note that these two elements have poles of order at most one along $\Delta$.

**Proposition 2.2.** *The* $\mathsf{R}$*-module* $H^1_{\mathrm{dR}}(X/T)$ *is freely generated by the elements* $\frac{dx}{y}$ *and* $\frac{xdx}{y}$*.*

*Proof.* Using Equation (2.4) and $y^2=P(x)$, every element of $H^1_{\mathrm{dR}}(X/T)$ can be written in the form $(C+yD)\omega,\ C,D\in\mathsf{R}[x]$. Since $Dy\omega=Ddx$ is exact, this reduces to $C\omega$. From another side the elements

$$d(x^ay)=(\frac{1}{2}P'x^a+ax^{a-1}P)\omega$$

are cohomologous to zero. If $\deg(C)\geq 2$, we can choose a monomial $F=x^a$ in such a way that the leading coefficient of $(\frac{1}{2}P'F+F'P)$ is equal to the leading coefficient of $C$. We subtract $d(Fy)$ from $C\omega$ and we get smaller degree for $C$. We repeat this until getting a degree one $C$. □

**Exercise.** Verify the following equalities in $H^1_{\mathrm{dR}}(X/T)$:

$$\frac{x^2dx}{y}=(2t_1)\frac{xdx}{y}+(-t_1^2+\frac{1}{12}t_2)\frac{dx}{y},$$

$$\frac{x^3dx}{y}=(3t_1^2+\frac{3}{20}t_2)\frac{xdx}{y}+(-2t_1^3+\frac{1}{10}t_1t_2+\frac{1}{10}t_3)\frac{dx}{y},$$

$$\frac{x^4dx}{y}=(4t_1^3+\frac{3}{5}t_1t_2+\frac{1}{7}t_3)\frac{xdx}{y}+(-3t_1^4-\frac{1}{10}t_1^2t_2+\frac{9}{35}t_1t_3+\frac{5}{336}t_2^2)\frac{dx}{y},$$

$$\frac{x^5dx}{y} = (5t_1^4 + \frac{3}{2}t_1^2t_2 + \frac{5}{7}t_1t_3 + \frac{7}{240}t_2^2)\frac{xdx}{y}$$
$$+ (-4t_1^5 - \frac{2}{3}t_1^3t_2 + \frac{2}{7}t_1^2t_3 + \frac{19}{420}t_1t_2^2 + \frac{1}{30}t_2t_3)\frac{dx}{y}.$$

## 2.5. An incomplete elliptic curve, II

Let $P(x) \in \mathsf{R}[x]$ be as in the previous section and

$$A = \mathsf{R}[x, y, z]/\langle y^2 - P(x), yz - 1\rangle$$

We will simply write $\frac{1}{y}$ instead of $z$.

**Proposition 2.3.** *The* $\mathsf{R}$ *module* $H^1_{\mathrm{dR}}(X/T)$ *is freely generated by*

$$\frac{dx}{y}, \frac{xdx}{y}, \frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{x^2dx}{y^2},$$

*Proof.* In this example $dy = \frac{1}{2y}P'dx$ and so every element $\omega$ of $H^1_{\mathrm{dR}}(X/T)$ can be written in the form $Cy^{-k}dx + Cy^{-k+1}$, $C, D \in \mathsf{R}[x]$, $k \geq 1$. We use the equality

$$d(x^ay^{-b}) = ax^{a-1}y^{-b}dx + \frac{-b}{2}x^ay^{-b-2}P'dx$$

for $b = -1, -2, \dots$ and see that $\omega$ is reduced to a form with $k = 1$ (each time we multiply $\omega$ with $\Delta = BP' + Ay^2$). Now, for terms $Cy^{-2}dx$ we make the division of $C$ by $P$ and we are thus left with the generators $\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{x^2dx}{y^2}$. For terms $Dy^{-1}dx$ we proceed as in Proposition 2.2 and we are left with the generators $\frac{dx}{y}, \frac{xdx}{y}$. □

## 2.6. De Rham cohomology of projective varieties

Let $X$ be a projective reduced variety over $\mathsf{R}$. We have the complex of sheaves of differential forms $(\Omega^\bullet_{X/T}, d)$ and we define the $i$-th de Rham cohomology of $X$ as the $i$-th hypercohomology of the complex $(\Omega^\bullet_{X/T}, d)$, that is

$$H^i_{\mathrm{dR}}(X) = \mathbb{H}^i(\Omega^\bullet_{X/T}, d).$$

The reader is referred to [34] for the full categorical definition of the hypercohomology. In this section we take an open covering of $X$ and define the hypercohomology using such a covering. In this way, we explain how its elements look like and how to calculate it.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be any open covering of $X$ by affine subsets, where $I$ is a totally ordered finite set. We have the following double complex

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \Omega_2^0 & \to & \Omega_2^1 & \to & \Omega_2^2 & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ \Omega_1^0 & \to & \Omega_1^1 & \to & \Omega_1^2 & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ \Omega_0^0 & \to & \Omega_0^1 & \to & \Omega_0^2 & \to & \cdots \end{array} \tag{2.5}$$

Here $\Omega_j^i$ is the product over $I_1 \subset I$, $\#I_1 = j+1$ of the set of global sections $\omega_\sigma$ of $\Omega^i_{X/T}$ in the open set $\sigma = \cap_{i \in I_1} U_i$. The horizontal arrows are usual differential operator $d$ of $\Omega^i_{X/T}$'s and vertical arrows are differential operators $\delta$ in the sense of Cech cohomology, that is,

$$\delta : \Omega_j^i \to \Omega_{j+1}^i, \quad \{\omega_\sigma\}_\sigma \mapsto \{\sum_{k=0}^{j+1} (-1)^k \omega_{\tilde{\sigma}_k} \mid_{\tilde{\sigma}}\}_{\tilde{\sigma}}. \tag{2.6}$$

Here $\tilde{\sigma}_k$ is obtained from $\tilde{\sigma}$, neglecting the $k$-th open set in the definition of $\tilde{\sigma}$. The $k$-th piece of the total chain of (2.5) is

$$\mathcal{L}^k := \oplus_{i=0}^k \Omega_{k-i}^i$$

with the differential operator

$$d' = d + (-1)^k \delta : \mathcal{L}^k \to \mathcal{L}^{k+1}. \tag{2.7}$$

The hypercohomology $\mathbb{H}^k(M, \Omega^\bullet)$ is the total cohomology of the double complex (2.5), that is

$$\mathbb{H}^k(M, \Omega^\bullet) = \frac{\ker(\mathcal{L}^k \xrightarrow{d} \mathcal{L}^{k+1})}{\operatorname{Im}(\mathcal{L}^{k-1} \xrightarrow{d} \mathcal{L}^k)}.$$

**Exercise.** (1) Show that the above definition does not depend on the choice of covering, that is, if $\mathcal{U}_1$ and $\mathcal{U}_2$ are two open covering of $X$ then the corresponding hypercohomologies are isomorphic in a canonical way.

(2) For which varieties $X$, we have $H^0_{\mathrm{dR}}(X) = \mathsf{R}$.

## 2.7. Complete elliptic curve, I

Let us consider the projective variety

$$E = \mathrm{Proj}(\mathsf{R}[x,y,z]/\langle zy^2 - 4(x-t_1z)^3 + t_2(x-t_1z)z^2 + t_3z^3\rangle)$$

which is covered by two open sets

$$U_0 = \mathrm{Spec}(\mathsf{R}[x,y]/\langle y^2 - 4(x-t_1)^3 + t_2(x-t_1) + t_3\rangle),$$
$$U_1 = \mathrm{Spec}(\mathsf{R}[x,z]/\langle z - 4(x-t_1z)^3 + t_2(x-t_1z)z^2 + t_3z^3\rangle).$$

Note that $U_0$ and $U_0 \cap U_1$ are the affine varieties in §2.4, respectively §2.5. For simplicity, we will cut $/\mathsf{R}$ and $/T$ from our notations. The variety $E$ has a closed point $O := [0;1;0]$ which is in the affine chart $U_1$. It is sometimes called the point at infinity. By definition, we have

$$H^1_{\mathrm{dR}}(E) = \{(\omega_0,\omega_1) \in \Omega^1_{U_0} \times \Omega^1_{U_1} \mid \omega_1 - \omega_0 \in d(\Omega^0_{U_0\cap U_1})\}/d\Omega^0_{U_0} \times d\Omega^0_{U_1},$$
$$H^2_{\mathrm{dR}}(E) = \Omega^1_{U_0\cap U_1}/(\Omega^1_{U_0} + \Omega^1_{U_1} + d\Omega^0_{U_0\cap U_1}).$$

In the definition of $H^1_{\mathrm{dR}}(E)$ as above, we sometimes take $U_1$ smaller but always containing $O$.

## 2.8. Residue calculus

We need to carry out some residue calculus near the closed smooth point $O$, see for instance [33]. Such a machinery is usually developed for curves over a field and so it seems to be necessary to consider the elliptic curve $E$ over the fractional field $\mathsf{k}_1$ of $\mathsf{R}$, that is, we use $E \otimes_{\mathsf{R}} \mathsf{k}_1$ instead of $E$. However, most of our calculations lead to elements in $\mathsf{R}$ which will be used later in the theory of quasi-modular forms.

A regular function $t$ in a neighborhood of $O = [0;1;0]$ is called a coordinate system at $O$ if $t(O) = 0$ and $t$ generates the one dimensional $\mathsf{k}_1$-space $m_O/m_O^2$, where $m_O$ is the ring of regular functions in a neighborhood of $O$ such that they vanish at $O$, and $m_O^2$ is the $\mathcal{O}_{X,O}$-module generated by $ab,\ a,b \in m_O$. Recall that $O$ is a smooth point of $E$. Any meromorphic function $f$ (meromorphic 1-form $\omega$) near $O$ has an expansion in $t$:

$$f = \sum_{i=-a}^{\infty} f_i t^i, \text{ resp. } \omega = (\sum_{i=-a}^{\infty} f_i t^i)dt,\ f_i \in \mathsf{k}_1, \tag{2.8}$$

where $a$ is some integer. The stalk of the ring of meromorphic differential 1-forms at $O$ is a $\mathcal{O}_{X,O}$-module generated by $dt$ and so $\omega = fdt$ for some meromorphic function near $O$. Therefore, it is enough to explain the first equality. Let $a$ be the pole order of $f$ at $O$. We work with $t^a f$ and so without loosing the generality we can assume that $f$ is regular at $O$. Let $f_0 = f(O)$. For some $f_1 \in \mathsf{k}_1$ we have $f - f_0 - f_1 t \in m_O^2$. We repeat this process and get a sequence $f_0, f_1, f_2, \ldots, f_m, f_{m+1}, \ldots \in \mathsf{k}_1$ such that

$$f - \sum_{i=0}^{m} f_i t^i \in m_O^{m+1}.$$

Another way of reformulating the above statement is:

$$f = \sum_{i=0}^{m} f_i t^i + O(t^{m+1}),$$

where $O(t^i)$ means a sum $\sum_{j\geq i} a_j t^j$. This is what we have written in (2.8).

The residue of $\omega$ at $O$ is defined to be $f_{-1}$. It is independent of the choice of the coordinate $t$. In our example, we take the coordinate $t = \frac{x}{y}$ with the notation of chart $U_0$ (in the chart $U_1$ we have $t = x$). The expansions of $x$ and $y$ in $t$ are of the form:

$$x = \frac{1}{4}t^{-2} + O(t^0),\ y = \frac{1}{4}t^{-3} + O(t^{-1}). \tag{2.9}$$

**Exercise.** (1) Show that $O$ is a smooth point of $E$, that is, the $\mathsf{k}_1$-vector space $m_O/m_O^2$ is one dimensional.

(2) Verify the equalities (2.9) and prove that the notion of residue does not depend on the coordinate system $t$.

(3) Calculate the residue of $\frac{x^n dx}{y^2}$, $n = 0, 1, 2, 3, 4, 5$ at $O$.

(4) Calculate the first 4 coefficients of the expansion of $\frac{dx}{y}$ in the coordinate $t = \frac{x}{y}$.

(5) Let us take the coordinates $(x, z)$ in which the elliptic curve $E$ is given by $z - 4(x - t_1 z)^3 - t_2(x - t_1 z)z^2 - t_3 z^3$ and we have $O = (0, 0)$, $t = x$. Consider $E$ over the ring $\mathsf{R}$. A regular function $f$ at $O$ can be written as $\frac{P(x,z)}{Q(x,z)}$ with $Q(0, 0) \neq 0$. Show that if $Q(0, 0)$ is invertible in $\mathsf{R}$ and $P, Q \in \mathsf{R}[x, z]$ then all the coefficients in the expansion of $f$ belong to $\mathsf{R}$ (Hint: Verify this for $f = z$.)

## 2.9. Complete elliptic curve, II

Recall the open covering $\{U_0, U_1\}$ of $E$ introduced in §2.7. In this section we prove the following proposition:

**Proposition 2.4.** *The canonical restriction map*

$$H^1_{dR}(E) \to H^1_{dR}(U_0),\ (\omega_0, \omega_1) \to \omega_0$$

*is an isomorphism of* $\mathsf{R}$*-modules.*

*Proof.* First we check that it is injective. Let us take an element $(\omega_0, \omega_1) \in H^1_{\mathrm{dR}}(X)$ with $\omega_0 = 0$. By definition $\omega_1 = \omega_1 - \omega_0 = df,\ f \in \Omega^0_{U_0 \cap U_1}$. Since $\omega_1$ has not poles at the closed point $O \in X$, $f$ has not too, which implies that $(\omega_0, \omega_1)$ is cohomologous to zero.

Now, we prove the surjectivity. The restriction map is $\mathsf{R}$-linear and so by Proposition 2.2, it is enough to prove that $\frac{dx}{y}$, $\frac{xdx}{y}$ are in the image of the restriction map. In fact, the corresponding elements in $H^1_{\mathrm{dR}}(E)$ are respectively

$$(\frac{dx}{y}, \frac{dx}{y}),\ \ (\frac{xdx}{y}, \frac{xdx}{y} - \frac{1}{2}d(\frac{y}{x})).$$

We prove this affirmation for $\frac{xdx}{y}$. We define $\tilde{U}_1 = U_1 \backslash \{x = 0\}$ and use the definition of hypercohomology with the covering $\{U_0, \tilde{U}_1\}$. We compute $x$ and $y$ in terms of the local coordinate $t = \frac{y}{x}$ around the point at infinity $O$ and we have (2.9). Substituting this in $\frac{xdx}{y}$, we get the desired result. $\square$

Let $U_0, U_1$ be an arbitrary covering of $E$. We have a well-defined map

$$\mathrm{Tr}: H^2_{\mathrm{dR}}(X) \to \mathsf{R},$$
$$\mathrm{Tr}(\omega) = \text{ sum of the residues of } \omega_{01} \text{ around the points } X \backslash U_0,$$

where $\omega$ is represented by $\omega_{01} \in \Omega^1_{U_0 \cap U_1}$. As usual, we take the canonical charts of $X$ described in §2.7. The map Tr turns out to be an isomorphism of $\mathsf{R}$-modules.

**Proposition 2.5.** *The* $\mathsf{R}$*-module* $H^2_{\mathrm{dR}}(X)$ *is of rank one.*

*Proof.* According to Proposition 2.3 any element in $\Omega^1_{U_0 \cap U_1}$ modulo exact forms can be reduced to an $\mathsf{R}$-linear combination of 5 elements. The classes of all these elements in $H^2_{\mathrm{dR}}(X)$ is zero, except the last one $\frac{x^2dx}{y^2}$. The first two elements are regular forms in $U_0$ and the next two forms are regular in $U_1$. We have proved that any element $\omega \in H^2_{\mathrm{dR}}(X)$ is reduced to

$r\frac{x^2dx}{y^2}$, $r \in \mathsf{R}$. Since $\frac{x^2dx}{y^2}$ at $O$ has the residue $\frac{-1}{2}$ (use the local coordinate $t = \frac{x}{y}$ and the equalities (2.9)), we get the desired result. $\square$

**Exercise.** (1) Let us take two open sets $U_1, \tilde{U}_1 \subset E$ which contain $O$. Show that the definition of de Rham cohomologies of $E$ attached to the coverings $\{U_0, U_1\}$ and $\{U_0, \tilde{U}_1\}$ are canonically isomorphic.

(2) By our definition of residue, it takes values in $\mathsf{k}_1$, the fractional field of $E$. Show that the map Tr has values in $\mathsf{R}$.

## 2.10. The intersection form

Let $X$ be a smooth irreducible reduced projective variety over $\mathsf{R}$. One can define the cup product

$$H^i_{\mathrm{dR}}(X) \times H^j_{\mathrm{dR}}(X) \to H^{i+j}_{\mathrm{dR}}(X) \tag{2.10}$$

which is the translation of the usual wedge product for the de Rham cohomologies of real manifolds. Further, we can define an isomorphism

$$\mathrm{Tr} : H^{2n}_{\mathrm{dR}}(X) \cong \mathsf{R} \tag{2.11}$$

which imply that $H^{2n}_{\mathrm{dR}}(X)$ is a rank one $\mathsf{R}$-module. For $i = j = \dim(X)$, the map (2.10) composed with (2.11) gives us a bilinear maps

$$\langle \cdot, \cdot \rangle : H^n_{\mathrm{dR}}(X) \times H^n_{\mathrm{dR}}(X) \to \mathsf{R}.$$

(for all these see Deligne's lectures in [4]). We have already defined Tr in the case of elliptic curves over the ring $\mathsf{R}$. In this section we are going to define the cup product in the case of a curve defined over $\mathsf{R}$.

Let us take two elements $\omega, \alpha \in H^1_{\mathrm{dR}}(X)$. We take an arbitrary covering $X = \cup_i U_i$ of $X$ and we assume that $\omega$ and $\alpha$ are given by $\{\omega_{ij}\}_{i,j\in I}$and $\{\alpha_{ij}\}_{i,j\in I}$with

$$\omega_j - \omega_i = df_{ij},\ \alpha_j - \alpha_i = dg_{ij}.$$

We define

$$\gamma := \omega \cup \alpha \in H^2_{\mathrm{dR}}(X)$$

which is given by:

$$\gamma_{ij} = g_{ij}\omega_j - f_{ij}\alpha_j + f_{ij}dg_{ij}. \tag{2.12}$$

Let us consider the situation of §2.7. In this case

$$\frac{dx}{y} \cup \frac{xdx}{y} = \{\omega_{01}\},\ \omega_{01} = \frac{-1}{2}\frac{dx}{x},$$

and

$$\langle \frac{dx}{y}, \frac{xdx}{y} \rangle = 1. \tag{2.13}$$

**Exercise.** (1) Show that the definition of $\omega \cup \alpha$ does not depend on the covering of the curve $X$ and that $\cup$ is non-degenerate.

(2) For a curve over complex numbers show its algebraic de Rham cohomology, cup product and intersection form are essentially the same objects defined by $C^\infty$-functions.

## 3. **Gauss-Manin connection**

### 3.1. **Introduction**

In 1958 Yu. Manin solved the Mordell conjecture over function fields and A. Grothendieck after reading his article invented the name Gauss-Manin connection. I did not find any simple exposition of this subject, the one which could be understandable by Gauss's mathematics. I hope that the following explains the presence of the name of Gauss on this notion. Our story again goes back to integrals. Many times an integral depends on some parameter and so the resulting integration is a function in that parameter. For instance take the elliptic integral (2.2) and assume that $P$ and $Q$ depends on the parameter $t$ and the interval $\delta$ does not depend on $t$. In any course in calculus we learn that the integration and derivation with respect to $t$ commute:

$$\frac{\partial}{\partial t}\int_\delta \frac{Q(x)}{\sqrt{P(x)}}dx = \int_\delta \frac{\partial}{\partial t}(\frac{Q(x)}{\sqrt{P(x)}})dx.$$

As before we know that the right hand side of the above equality can be written as a linear combination of two integrals $\int_\delta \omega$ and $\int_\delta x\omega$. This is the historical origin of the notion of Gauss-Manin connection, that is, derivation of integrals with respect to parameters and simplifying the result in terms of integrals which cannot be simplified more. For instance, take $P(x)$ as in (2.3) which depends on three parameters $t_1, t_2, t_3$. We have

$$\frac{\partial}{\partial t_i} I = IA_i, \quad i = 1, 2, 3, \quad \text{where } I := [\int_\delta \frac{dx}{y} \int_\delta \frac{xdx}{y}], \tag{3.1}$$

and $A_i$ is a $2 \times 2$ matrix whose coefficients can be calculated effectively. When the integrand depends on many parameters the best way to put the information of derivations with respect to all parameters in one object is by using differential forms (recall that differential forms are also used to represent the integrand). We define

$$A = A_1 dt_1 + A_2 dt_2 + A_3 dt_3,$$

that is, $A_{ij} := (A_1)_{ij}dt_1 + (A_2)_{ij}dt_2 + (A_3)_{ij}dt_3,\ i, j = 1, 2$. Now, we write (3.1) in the form $dI = I \cdot A$. In this section we calculate the Gauss-Manin connection, that is, we calculate the matrix $A$, see Proposition 3.1.

## 3.2. Gauss-Manin connection

What we do in this section in the framework of Algebraic Geometry is as follows: Let $X$ be a smooth reduced variety over $\mathsf{R}$. We construct a connection

$$\nabla : H^i_{\rm dR}(X) \to \Omega^1_T \otimes_{\mathsf{R}} H^i_{\rm dR}(X),$$

where $\Omega^1_T$ is by definition $\Omega^1_{\mathsf{R}/\mathsf{k}}$, that is, the $\mathsf{R}$-module of differential attached to $\mathsf{R}$. By definition of a connection, $\nabla$ is $\mathsf{k}$-linear and satisfies the Leibniz rule

$$\nabla(r\omega) = dr \otimes \omega + r\nabla\omega.$$

A vector field $v$ in $T$ is an $\mathsf{R}$-linear map $\Omega^1_T \to \mathsf{R}$. We define

$$\nabla_v : H^i_{\rm dR}(X) \to H^i_{\rm dR}(X)$$

to be $\nabla$ composed with

$$v \otimes \mathrm{Id} : \Omega^1_T \otimes_{\mathsf{R}} H^i_{\rm dR}(X) \to \mathsf{R} \otimes_{\mathsf{R}} H^i_{\rm dR}(X) = H^i_{\rm dR}(X).$$

If $\mathsf{R}$ is a polynomial ring $\mathbb{Q}[t_1, t_2, \ldots]$ then we have vector fields $\frac{\partial}{\partial t_i}$ which are defined by the rule

$$\frac{\partial}{\partial t_i}(dt_j) = 1 \text{ if } i = j \text{ and } = 0 \text{ if } i \neq j.$$

In this case we simply write $\frac{\partial}{\partial t_i}$ instead of $\nabla_{\frac{\partial}{\partial t_i}}$.

Sometimes it is useful to choose a basis $\omega_1, \omega_2, \ldots, \omega_h$ of the $\mathsf{R}$-modular $H^i(X/T)$ and write the Gauss-Manin connection in this basis:

$$\nabla \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_h \end{pmatrix} = A \otimes \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_h \end{pmatrix}, \tag{3.2}$$

where $A$ is a $h \times h$ matrix with entries in $\Omega_T^1$.

## 3.3. Construction

Recall the notation of §2.6. Let us take a covering $\mathcal{U} = \{U_i\}_{i\in I}$ of $X$ by affine open sets and $\omega \in H_{\rm dR}^k(X)$. By our definition $\omega$ is represented by $\oplus_{i=0}^k \omega_i,\ \omega_i \in \Omega_{k-i}^i$ and $\omega_i$ is a collection of $i$-forms $\{\omega_{i,\sigma}\}_\sigma$. By definition we have $d'\omega = 0$, where $d'$ is given by (2.7). Recall that the differential map $d$ used in the definition of $d'$ is relative to $\mathsf{R}$, that is, by definition $dr = 0, r \in \mathsf{R}$. Now, let us consider $d$ in the double complex (2.5) relative to $\mathsf{k}$ and not $\mathsf{R}$. The condition $d'\omega = 0$ turns to be

$$d'\omega = \eta,\ \eta = \oplus_{i=0}^{k+1}\eta_i \in \mathcal{L}^{k+1},\ \eta_i \in \Omega_{k+1-i}^i$$

and each $\eta_i,\ i \neq 0$ is a collection of $i$-forms that is a finite sum of the form $\sum_j dr_j \otimes \omega_j$. Since $d$ has no contribution in $\Omega_{k+1}^0$, we know that $\eta_0 = 0$. We make the tensor product $\otimes_{\mathsf{R}}$ of the double complex (2.5) with $\Omega_T^1$ and finally get an element in $\Omega_T^1 \otimes_{\mathsf{R}} H_{\rm dR}^k(X)$. Of course we have to verify that everything is well-defined.

Let us now assume that $\mathsf{k} = \mathbb{C}$. The main motivation, which is also the historical one, for defining the Gauss-Manin connection is the following: For any $\omega \in H_{\rm dR}^i(X)$ and a continuous family of cycles $\delta_t \in H_i(X_t, \mathbb{Z})$ we have

$$d\left(\int_{\delta_t} \omega\right) = \int_{\delta_t} \nabla\omega. \tag{3.3}$$

Here, by definition

$$\int_{\delta_t} \alpha \otimes \beta = \alpha \int_{\delta_t} \beta,$$

where $\beta \in H_{\rm dR}^i(X)$ and $\alpha \in \Omega_T^1$. Integrating both side of the equality (3.2) over a a basis $\delta_1, \delta_2, \dots, \delta_h \in H_i(X_t, \mathbb{Q})$ we conclude that

$$d([\int_{\delta_j} \omega_i]) = [\int_{\delta_j} \omega_i] \cdot A. \tag{3.4}$$

## 3.4. Gauss-Manin connection of families of elliptic curves

Let us consider the ring $\mathsf{R} = \mathbb{Q}[t_1, t_2, t_3, \frac{1}{\Delta}]$ and the family of elliptic curves in §2.4. By Proposition 2.4 we know that the first de Rham cohomology of $E$ is isomorphic to the first de Rham cohomology of the affine variety $U_0$. Therefore, we calculate the Gauss-Manin connection attached to $U_0$.

**Proposition 3.1.** *The Gauss-Manin connection of the family of elliptic curves* $y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3$ *written in the basis* $\frac{dx}{y}$, $\frac{xdx}{y}$ *is given as bellow:*

$$\nabla \begin{pmatrix} \frac{dx}{y} \\ \frac{xdx}{y} \end{pmatrix} = A \begin{pmatrix} \frac{dx}{y} \\ \frac{xdx}{y} \end{pmatrix} \tag{3.5}$$

*where*

$$A = \frac{1}{\Delta} \begin{pmatrix} -\frac{3}{2}t_1\alpha - \frac{1}{12}d\Delta & \frac{3}{2}\alpha \\ \Delta dt_1 - \frac{1}{6}t_1 d\Delta - (\frac{3}{2}t_1^2 + \frac{1}{8}t_2)\alpha & \frac{3}{2}t_1\alpha + \frac{1}{12}d\Delta \end{pmatrix},$$

$$\Delta = 27t_3^2 - t_2^3, \ \alpha = 3t_3dt_2 - 2t_2dt_3.$$

*Proof.* The proof is a mere calculation which is classical and can be found in ([30] p. 304, [29] ). We explain only the calculation of $\frac{\partial}{\partial t_3}(\frac{dx}{y})$ . For $p(x) = 4t_0(x-t_1)^3 - t_2(x-t_1) - t_3$ we have:

$$\Delta = -p' \cdot a_1 + p \cdot a_2,$$

where

$$a_1 = -36t_0^3x^4 + 144t_0^3t_1x^3 + (-216t_0^3t_1^2 + 15t_0^2t_2)x^2 + (144t_0^3t_1^3 - 30t_0^2t_1t_2)x - 36t_0^3t_1^4 + 15t_0^2t_1^2t_2 - t_0t_2^2$$

$$a_2 = (-108t_0^3)x^3 + (324t_0^3t_1)x^2 + (-324t_0^3t_1^2 + 27t_0^2t_2)x + (108t_0^3t_1^3 - 27t_0^2t_1t_2 - 27t_0^2t_3)$$

We have

$$\begin{aligned} \frac{\partial}{\partial t_3}(\frac{dx}{y}) &= \frac{-dy \wedge dx}{y^2} = \frac{1}{2}\frac{dx}{py} \\ &= \frac{1}{\Delta}\frac{(-p'a_1 + pa_2)dx}{2py} = \frac{1}{\Delta}(\frac{1}{2}a_2 - a_1')\frac{dx}{y} \\ &= (3t_0^2t_1t_2 - \frac{9}{2}t_0^2t_3)\frac{dx}{y} - 3t_0^2t_2\frac{xdx}{y}. \end{aligned}$$

Note that in the fourth equality above we use $y^2 = p(x)$ and the fact that modulo exact forms we have

$$\frac{p'a_1dx}{2py} = \frac{a_1dp}{2py} = \frac{a_1dy}{p} = -a_1d(\frac{1}{y}) = \frac{a_1'dx}{y}.$$

□

**Exercise.** In a similar way as in the proof of Proposition 3.1 calculate $\frac{\partial}{\partial t_i}(\frac{dx}{y}),\ i=1,2$.

## 3.5. Another family of elliptic curves

We modify a little bit the parameter space of our family of elliptic curves. Let

$$E_{\mathrm{R}} : y^2 - 4(x-t_1)^3 + t_2(x-t_1) + t_3 = 0,$$

$$(t_1,t_2,t_3) \in T_{\mathrm{R}} := \{(t_1,t_2,t_3) \in \mathsf{k}^3 \mid 27t_3^2 - t_2^3 \neq 0\}$$

be our previous family of elliptic curves and

$$E_{\mathrm{H}} : y^2 - 4(x-t_1)(x-t_2)(x-t_3) = 0, \tag{3.6}$$

$$(t_1,t_2,t_3) \in T_{\mathrm{H}} := (t_1,t_2,t_3) \in \mathsf{k}^3 \mid t_1 \neq t_2 \neq t_3\}$$

Note that we have used the same notation $t_1,t_2,t_3$ for the coordinates system both in $T_{\mathrm{H}}$ and $T_{\mathrm{R}}$. I hope that this will not make any confusion. The algebraic morphism $\alpha : T_{\mathrm{H}} \to T_{\mathrm{R}}$ defined by

$$\alpha : (t_1,t_2,t_3) \mapsto (T, 4\sum_{1\leq i<j\leq 3}(T-t_i)(T-t_j), 4(T-t_1)(T-t_2)(T-t_3)),$$

where

$$T := \frac{1}{3}(t_1+t_2+t_3),$$

connects two families, that is, if in $E_{\mathrm{R}}$ we replace $t$ with $\alpha(t)$ we obtain the family $E_{\mathrm{H}}$. The Gauss-Manin connection matrix associated to $E_{\mathrm{H}}$ is just the pull-back of the Gauss-Manin connection associate to $E$. In this way we obtain

$$\alpha^* A_{\mathrm{R}} = A_{\mathrm{H}} = \frac{dt_1}{2(t_1-t_2)(t_1-t_3)}\begin{pmatrix} -t_1 & 1 \\ t_2t_3 - t_1(t_2+t_3) & t_1 \end{pmatrix} + \tag{3.7}$$

$$\frac{dt_2}{2(t_2-t_1)(t_2-t_3)}\begin{pmatrix} -t_2 & 1 \\ t_1t_3 - t_2(t_1+t_3) & t_2 \end{pmatrix}$$
$$+ \frac{dt_3}{2(t_3-t_1)(t_3-t_2)}\begin{pmatrix} -t_3 & 1 \\ t_1t_2 - t_3(t_1+t_2) & t_3 \end{pmatrix}.$$

**Exercise.** Verify the equality (3.7).

## 4. Modular differential equations

### 4.1. Introduction

After calculation of the Gauss-Manin connection of families of elliptic curves, we immediately calculate the Ramanujan and Darboux-Halphen differential equation (Proposition 4.1). The history of these differential equations is full of rediscoveries. The first example of differential equations which has a particular solution given by theta constants was studied by Jacobi in 1848. Later, in 1978 G. Darboux studied the system of differential equations

$$\begin{cases} \dot t_1 + \dot t_2 = 2t_1t_2 \\ \dot t_2 + \dot t_3 = 2t_2t_3 \\ \dot t_1 + \dot t_3 = 2t_1t_3 \end{cases}, \tag{4.1}$$

in connection with triply orthogonal surfaces in $\mathbb{R}^3$(see [3]). G. Halphen (1881), M. Brioschi (1881), and J. Chazy (1909) contributed to the study of the differential equation (4.1). In particular, Halphen expressed a solution of the system (4.1) in terms of the logarithmic derivatives of the null theta functions, see §4.4. Halphen generalized also (4.1) to a differential equation with three parameters corresponding to the three parameters of the Gauss hypergeometric function. His method of calculating such differential equations is essentially described in §8.7 and it is near in spirit to the methods used in the present text. The history from number theory point of view is different. S. Ramanujan, who was a master of convergent and formal series and who did not know about the geometry of differential equations, in 1916 observed that the derivation of three Eisenstein series $E_2, E_4$ and $E_6$ are polynomials in $E_i$'s. Therefore, he had the solution and he found the corresponding differential equation. This is opposite to the work of Halphen who had the differential equation and he calculated a solution (the later in general is more difficult than the former). Halphen even calculated the Ramanujan differential equation years before Ramanujan and apparently without knowing about Eisenstein series (see [13] page 331). It is remarkable to say that Darboux-Halphen differential equations was rediscovered in mathematical physics by M. Atiyah and N. Hitchin in 1985. Even the author of the present text calculated Ramanujan and Darboux-Halphen differential equations independently without knowing about Ramanujan, Darboux and Halphen's work. All these rediscoveries were useful because they have helped us to understand the importance and applications of such differential equations and also to find the general context of such differential equations in relation with Gauss-Manin connections and in general multi dimensional linear differential systems.

### 4.2. **Ramanujan vector field**

Our main observation in this section is the following:

**Proposition 4.1.** *In the parameter space of the family of elliptic curves* $y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3$ *there is a unique vector field* R*, such that*

$$\nabla_{\mathrm{R}}(\frac{dx}{y}) = -\frac{xdx}{y}, \ \nabla_{\mathrm{R}}(\frac{xdx}{y}) = 0. \tag{4.2}$$

*The vector field* R *is given by*

$$\mathrm{R} = (t_1^2 - \frac{1}{12}t_2)\frac{\partial}{\partial t_1} + (4t_1t_2 - 6t_3)\frac{\partial}{\partial t_2} + (6t_1t_3 - \frac{1}{3}t_2^2)\frac{\partial}{\partial t_3}. \tag{4.3}$$

*Proof.* The proof is based on explicit calculations. □

**Exercise.** Perform the calculations leading to a proof of Proposition 4.1.

### 4.3. **Vector field or ordinary differential equation?**

Any vector field in $\mathbb{C}^n$ represent an ordinary differential equation, for which we can study the dynamics of its solutions. For instance, the vector field R of the previous section can be seen as the following ordinary differential equation:

$$\mathrm{R}: \begin{cases} \dot{t}_1 = t_1^2 - \frac{1}{12}t_2 \\ \dot{t}_2 = 4t_1t_2 - 6t_3 \\ \dot{t}_3 = 6t_1t_3 - \frac{1}{3}t_2^2 \end{cases} \tag{4.4}$$

where dot means derivation with respect to a variable. S. Ramanujan verfied that the Eisenstein series:

$$E_{2i}(q) := \left(1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1}\right) q^n\right), \quad i = 1, 2, 3, \tag{4.5}$$

$$(b_1, b_2, b_3) = (-24, 240, -504)$$

with the derivation $\dot{E}_{2i} = q\frac{\partial E_{2i}}{\partial q}$ satisfy an ordinary differential equation which is obtained from (4.4) after the affine transformation

$$(t_1, t_2, t_3) \mapsto (\frac{1}{12}t_1, \frac{1}{12}t_2, \frac{2}{3(12)^2}t_3),$$

see [26], p. 4 or [22].

For a moment forget all what we have done to obtain R. We are going to explain how the differential equation (4.4) determines uniquely the Eisenstein series (4.5) up to a constant. We write each $t_i$ as a formal power series in $q$, $t_i = \sum_{n=0}^{\infty} t_{i,n}q^n,\ i = 1, 2, 3$ and substitute in the above differential equation. We define the derivation to be:

$$\dot{t} = aq\frac{\partial t}{\partial q}$$

for a fixed non zero number $a$. Comparing the coefficients of $q^0$ we have

$$(t_{1,0}, t_{2,0}, t_{3,0}) = (b, 12b^2, 8b^3),\ \text{for some } b \in \mathbb{C}.$$

Comparing the coefficients of $q^1$ we have

$$MV = aV, \quad \text{where },\quad M := \begin{pmatrix} 2b & -\frac{1}{12} & 0 \\ 48b^2 & 4b & -6 \\ 48b^3 & -8b^2 & 6b \end{pmatrix},\ V := \begin{pmatrix} t_{1,1} \\ t_{2,1} \\ t_{3,1} \end{pmatrix}.$$

Assume that $V$ is not zero and so we have $\det(M - aI_{3\times 3}) = (12b - a)a^2 = 0$ which implies that $a = 12b$. We also calculate $V$ up to multiplication by a constant:

$$V^{\mathsf{tr}} = c[-24b, 240(12b^2), -504(8b^3)], \quad \text{for some } c \in \mathbb{C}.$$

We realize that all the coefficients $t_{i,n},\ n > 1$ are determined uniquely and recursively:

$$(12nbI_{3\times 3} - M)\begin{pmatrix} t_{1,n} \\ t_{2,n} \\ t_{3,n} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n-1} t_{1,i}t_{1,n-i} \\ 4\sum_{i=1}^{n-1} t_{1,i}t_{2,n-i} \\ \sum_{i=1}^{n-1} 6t_{1,i}t_{3,n-i} - \frac{1}{3}t_{2,i}t_{2,n-i} \end{pmatrix}$$

**Proposition 4.2.** *We have $t_i = a_iE_{2i}(cq)$, where $E_{2i}$'s are the Eisenstein series (4.5) and $(a_1, a_2, a_3) = (b, 12b^2, 8b^3)$.*

Note that the solutions $t_i,\ i = 1, 2, 3$ of (4.4) depend on a parameter $c$ which cannot be calculated from the differential equation (the parameter $b$ can be fixed by fixing the derivation). Later, we will see that $t_i$'s are convergent in the unit disc. If we set $q = e^{2\pi iz}$ and look at $t_i$'s as holomorphic functions in $z$, which varies in the upper half plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$, then they are basic examples of quasi-modular forms. Note that for $b = \frac{2\pi i}{12}$ and $a = 2\pi i$ we have $2\pi i q\frac{\partial}{\partial q} = \frac{\partial}{\partial z}$ and so if we define

$$(g_1(z), g_2(z), g_3(z)) = (\frac{2\pi i}{12}E_2, 12(\frac{2\pi i}{12})^2 E_4, 8(\frac{2\pi i}{12})^3 E_6) \qquad (4.6)$$

then $g_i$'s satisfy the Ramanujan differential equation (4.4), where dot means derivation with respect to $z$.

**Exercise.** (1) For a moment forget Proposition 4.2. Let

$$t_i = \sum_{n=0}^{\infty} t_{i,n} q^n, \qquad i = 1, 2, 3$$

be formal power series in $q$ with unknown coefficients as before and assume that $t_i$ satisfy the differential equation 4.4, where dot is $12q\frac{\partial}{\partial q}$ and we know the initial value $t_{1,2} = -24$. Calculate the first five coefficients $t_{k,n},\ k = 1, 2, 3,\ n = 1, 2, \dots 5$ and compare them with the coefficients in the Eisenstein series.

(2) Find a proof for Proposition 4.2 in the literature (see [26, 22]).

## 4.4. Halphen vector field

Let us consider the family of elliptic curves considered in §3.5. A vector field with the properties (4.2) is given by

$$\mathrm{H} = (t_1(t_2+t_3) - t_2t_3)\frac{\partial}{\partial t_1} + (t_2(t_1+t_3) - t_1t_3)\frac{\partial}{\partial t_2} + (t_3(t_1+t_2) - t_1t_2)\frac{\partial}{\partial t_3}.$$

The corresponding ordinary differential equation is

$$\mathrm{H}: \begin{cases} \dot{t}_1 = t_1(t_2+t_3) - t_2t_3 \\ \dot{t}_2 = t_2(t_1+t_3) - t_1t_3 \\ \dot{t}_3 = t_3(t_1+t_2) - t_1t_2 \end{cases} \qquad (4.7)$$

which is the same as (4.1). Halphen expressed a solution of the system (4.7) in terms of the logarithmic derivatives of the null theta functions:

**Proposition 4.3.** *The holomorphic functions*

$$u_1 = 2(\ln\theta_4(0|z))', u_2 = 2(\ln\theta_2(0|z))', u_3 = 2(\ln\theta_3(0|z))'$$

*where*

$$\begin{cases} \theta_2(0|z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ \theta_3(0|z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \\ \theta_4(0|z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \end{cases}, \quad q = e^{2\pi i z}, \ z \in \mathbb{H}$$

*satisfy the ordinary differential equation (4.7), where the derivation is with respect to $z$.*

**Exercise.** (1) Give a proof of Proposition 4.3, see [12, 27].

## 4.5. Relations between theta and Eisenstein series

As we mentioned in §3.5, the map $\alpha : T_{\mathrm{H}} \to T_{\mathrm{R}}$ maps the Gauss-Manin connection matrix of $E_{\mathrm{H}}$ to the Gauss-Manin connection matrix of $E_{\mathrm{R}}$, both written in the basis $\frac{dx}{y}, \frac{xdx}{y}$. We know that $X = \mathrm{R}$ or H are both determined uniquely by $\nabla_X \frac{dx}{y} = -\frac{xdx}{y}, \ \nabla_X \frac{xdx}{y} = 0$. All these imply that the vector field H is mapped to the vector field R through the map $\alpha$. Let $u : \mathbb{H} \to \mathbb{C}^3, \ u(z) = (u_1(z), u_2(z), u_3(z))$ be the solution of H by the logarithmic derivative of theta functions. It turns out that $\alpha(u(z))$ is a solution of R, and we claim that it is $(g_1, g_2, g_3)$ defined in (4.6). Two solutions of (4.4) with the same $t_1$ coordinates are equal (this follows by the explicit expression of (4.4)). From this and the discussion in §4.3 it follows that it is enough to prove that both $\frac{1}{3}(u_1+u_2+u_3)$ and $g_1(z)$ have the form $\frac{2\pi i}{12}(1 - 24q + \cdots)$. Finally, we get the equalities:

$$\frac{2}{3}\ln(\theta_2(0|z)\theta_3(0|z)\theta_4(0|z))' = \frac{2\pi i}{12}E_2(z),$$

$$4 \cdot 4 \sum_{2\le i<j\le 4} \ln\Big(\frac{(\theta_2(0|z)\theta_3(0|z)\theta_4(0|z))^{\frac{1}{3}}}{\theta_i(0|z)}\Big)' \ln\Big(\frac{(\theta_2(0|z)\theta_3(0|z)\theta_4(0|z))^{\frac{1}{3}}}{\theta_j(0|z)}\Big)' = 12\Big(\frac{2\pi i}{12}\Big)^2 E_4(z),$$

$$4 \cdot 8 \prod_{i=1}^{3} \ln\Big(\frac{(\theta_2(0|z)\theta_3(0|z)\theta_4(0|z))^{\frac{1}{3}}}{\theta_i(0|z)}\Big)' = 8\Big(\frac{2\pi i}{12}\Big)^3 E_6(z).$$

## 4.6. **Automorphic properties of the special solutions**

Let us define

$$\mathrm{SL}(2,\mathbb{Z}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=1\}$$

and

$$\Gamma(d) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{mod}\ d\},\ d \in \mathbb{N}.$$

For a holomorphic function $f(z)$ let also define:

$$(f\mid_m^0 A)(z) := (cz+d)^{-m} f(Az),$$

$$(f\mid_m^1 A)(z) := (cz+d)^{-m} f(Az) - c(cz+d)^{-1},$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}),\ m \in \mathbb{N}.$$

**Exercise.** If $\phi_i(z),\ i=1,2,3$ with $\dot{\phi}_i = \frac{\partial \phi_i}{\partial z}$ are the coordinates of a solution of R(resp. H) then

$$\phi_1 \mid_2^1 A,\ \phi_2 \mid_4^0 A,\ \phi \mid_6^0 A$$

(resp.

$$\phi_i \mid_2^1 A,\ i=1,2,3,$$

) are also coordinates of a solution of R (resp. H) for all $A \in \mathrm{SL}(2,\mathbb{C})$. The subgroup of $\mathrm{SL}(2,\mathbb{C})$ which fixes the solution given by Eisenstein series (resp. theta series) is $\mathrm{SL}(2,\mathbb{Z})$ (resp. $\Gamma(2)$). The second part of the exercise will be verified in §8.

## 4.7. **Another example**

Let

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),\ q = e^{2\pi i z}$$

be the Dedekind's $\eta$-function. In [28] Y. Ohyama has found that

$$W = (3\log\eta(\frac{z}{3}) - \log\eta(z))' \tag{4.8}$$

$$X = (3\log\eta(3z) - \log\eta(z))' \tag{4.9}$$

$$Y = (3\log\eta(\frac{z+2}{3}) - \log\eta(z))' \tag{4.10}$$

$$Z = (3\log\eta(\frac{z+1}{3}) - \log\eta(z))' \tag{4.11}$$

satisfy the equations:

$$\begin{cases} \dot t_1 + \dot t_2 + \dot t_3 = t_1t_2 + t_2t_3 + t_3t_1 \\ \dot t_1 + \dot t_3 + \dot t_4 = t_1t_3 + t_3t_4 + t_4t_1 \\ \dot t_1 + \dot t_2 + \dot t_4 = t_1t_2 + t_2t_4 + t_4t_1 \\ \dot t_2 + \dot t_3 + \dot t_4 = t_2t_3 + t_3t_4 + t_4t_2 \\ \zeta_3^2(t_2t_4 + t_3t_1) + \zeta_3(t_2t_1 + t_3t_4) + (t_2t_3 + t_4t_1) = 0 \end{cases},$$

where $\zeta_3 = e^{\frac{2\pi i}{3}}$ and $\dot t_i = \frac{\partial t_i}{\partial z}$. We write the first four lines of the above equation as a solution to a vector field $V$ in $\mathbb{C}^4$ and let $F(t_1, t_2, t_3, t_4)$ be the polynomial in the fifth line. Using a computer, or by hand if we have a good patience for calculations, we can verify the equality $dF(V) = 0$ and so $V$ is tangent to $T := \{t \in \mathbb{C}^4 \mid F(t) = 0\}$. Our discussion leads to a problem which is presented in Exercise 4.7. For more examples, see the article [35].

**Exercise.** Show that there is a family of elliptic curves $E \to T$ and a basis $\omega_1, \omega_2 \in H^1_{\mathrm{dR}}(E/T)$ such that $\omega_1$ is a regular differential formal and

$$\nabla_{\mathrm{R}}(\omega_1) = -\omega_2, \quad \nabla_{\mathrm{R}}(\omega_2) = 0,$$

where R is the restriction of $V$ to $T$.

## 5. **Weierstrass form of elliptic curves**

### 5.1. **Introduction**

We can think of an elliptic curve over rational numbers as the Diophantine equation

$$y^2 = 4P(x),$$

where $P$ is a monic degree three polynomial in $x$ with rational coefficients and without double roots in $\mathbb{C}$. In fact, this is the Weierstrass form of any elliptic curve in the framework of algebraic geometry. The moduli of elliptic curves is one dimensional and we have apparently three independent parameters in the polynomial $P$. It turns out that there is an algebraic group of dimension two which acts on the coefficients space of $P$ and the resulting quotient is the moduli of elliptic curves, see §6.3. In this section, we remark that the coefficients space of $P$ has also a moduli interpretation. We consider elliptic curves with elements in their algebraic de Rham cohomologies and we ask for normal forms of such objects. It turns out that the three independent coefficients of the polynomial $P$ are the coordinates system of the moduli of such enhanced elliptic curves.

## 5.2. Elliptic curves

Let $\mathsf{k}$ be a field of characteristic zero. For a reduced smooth curve $C$ over $\mathsf{k}$ we define its genus to be the dimension of the space of regular differential forms on $C$.

**Definition 5.1.** An elliptic curve over $\mathsf{k}$ is a pair $(E, O)$, where $E$ is a genus one complete smooth curve and $O$ is a $\mathsf{k}$-rational point of $E$.

Therefore, by definition an elliptic curve over $\mathsf{k}$ has at least a $\mathsf{k}$-rational point. A smooth projective curve of degree 3 is therefore an elliptic curve if it has a $\mathsf{k}$-rational point. For instance, the Fermat curve

$$F_3 : x^3 + y^3 = z^3$$

is an elliptic curve over $\mathbb{Q}$. It has $\mathbb{Q}$-rational points $[0; 1; 1]$ and $[1; 0; 1]$. However

$$E : 3x^3 + 4y^3 + 5z^3 = 0$$

has not $\mathbb{Q}$-rational points and so it is not an elliptic curve defined over $\mathbb{Q}$. It is an interesting fact to mention that $E(\mathbb{Q}_p)$ for all prime $p$ and $E(\mathbb{R})$ are not empty. This example is due to Selmer (see [2, 31]).

## 5.3. Weierstrass form

In this section we prove the following proposition:

**Proposition 5.2.** *Let $E$ be an elliptic curve over a field $\mathsf{k}$ of characteristic $\neq 2, 3$ and let $\omega$ be a regular differential form on $E$. There exist unique functions $x, y \in \mathsf{k}(E)$ such that the map*

$$E \to \mathbb{P}^2, \ a \mapsto [x(a); y(a); 1]$$

*gives an isomorphism between the curve $E$ and the curve in $\mathbb{P}^2$ given by*

$$y^2 = 4x^3 - t_2 x - t_3,\ \ t_2, t_3 \in \mathsf{k}$$

*Under this isomorphism $O$ is identified with $[0;1;0]$ and $\omega = \frac{dx}{y}$.*

We call $x$ and $y$ the Weierstrass coordinates of $E$. Since $x, y \in \mathsf{k}(E)$ the above isomorphism is defined over $\mathsf{k}$. Note that $\frac{x}{y}$ has a zero of order one at $O$ and hence the map $E \to \mathbb{P}^2$ is well-defined at $O$ and it takes the value $[\frac{x}{y}(O); 1; \frac{1}{y}(O)] = [0;1;0]$. For a curve $E$ of genus one defined over $\mathsf{k}$ and with $O \in E(\mathsf{k})$, there is a unique abelian algebraic group structure on $E$ with the neutral element $O$, and so, the above isomorphism is automatically an isomorphism of algebraic groups.

*Proof.* For a divisor $D$ on a curve $C$ over $\bar{\mathsf{k}}$ define the linear system

$$\mathcal{L}(D) = \{f \in \bar{\mathsf{k}}(C),\ f \neq 0 \mid div(f) + D \geq 0\} \cup \{0\}$$

and

$$l(D) = \dim_{\bar{\mathsf{k}}}(\mathcal{L}(D)).$$

We know by Riemann-Roch theorem that

$$l(D) - l(K - D) = \deg(D) - g + 1,$$

where $K$ is the canonical divisor of $C$ and $g$ is the genus of $C$. We have $\deg(K) = 2g-2$ and so for $\deg(D) > 2g-2$, equivalently $\deg(K-D) < 0$, we have

$$l(D) = \deg(D) - g + 1.$$

For $g = 1$ and $D = nO$ we get $l(D) = n$. Using this for $n = 2, 3$, we can choose $x, y \in \mathsf{k}(E)$ such that $1, x$ form a basis of $\mathcal{L}(2O)$ and $1, x, y$ form a basis of $\mathcal{L}(3O)$. The function $x$ (resp. $y$) has a pole of order 2 (resp. 3) at $O$. In fact, we need the following choice of $y$, $y := \frac{dx}{\omega}$. Note that $\omega$ is regular and vanishes nowhere. The map $\sigma : E \to E,\ P \mapsto -P$ acts trivially on any $x$ in $\mathcal{L}(2O)$ because $\sigma x - x$ has a simple pole, and hence by residue formula for $(\sigma x - x)\omega$, has no pole and so $\sigma x = x$. We have $l(6O) = 6$ and so there is a linear relation between $1, x, x^2, x^3, y^2, y, xy$. The last two terms $y, xy$ does not appear on such a linear relation: a point $p$ is a non-zero 2-torsion point of $E$ if and only if $p$ is a double root of $x - x(p)$ (equivalently $y(p) = 0$). We can further assume that the coefficient of $y^2$ is one and of $x^3$ is 4. After a substitution of $x$ with $x + a$ for some $a \in \mathsf{k}$, we get the desired polynomial relation between $x$ and $y$.

□

For an elliptic curve $E$ defined over $\mathsf{k}$, let $F^1 \subset H^1_{\rm dR}(E)$ be the one dimensional $\mathsf{k}$-vector space of regular differential 1-forms on $E$.

**Proposition 5.3.** *Let $E$ be an elliptic curve over a field $\mathsf{k}$ of characteristic zero and $\omega \in H^1_{\rm dR}(E)\backslash F^1$. There exist unique functions $x, y \in \mathsf{k}(E)$ such that the map*

$$E \to \mathbb{P}^2, \ a \mapsto [x(a); y(a); 1]$$

*gives an isomorphism between the curve $E$ and the curve in $\mathbb{P}^2$ given by*

$$y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3, \ t_1, t_2, t_3 \in \mathsf{k}$$

*Under this isomorphism $O$ is identified with $[0;1;0]$ and $\omega = \frac{xdx}{y}$.*

*Proof.* We have a regular differential form $\omega_1$ on $E$ which is determined uniquely by $\langle \omega_1, \omega \rangle = 1$. We apply Proposition 5.2 for the pair $(E, \omega_1)$. In the corresponding Weierstrass coordinates by Proposition 2.2 we can write $\omega = t_1 \frac{dx}{y} + t_0 \frac{xdx}{y}, \ t_1, t_0 \in \mathsf{k}$. Using (2.13) and $\langle \omega_1, \omega \rangle = 1$, we have $t_0 = 1$. We now substitute $x$ by $x - t_1$. ☐

**Exercise.** Prove that the map $E \to \mathbb{P}^2$ in Proposition 5.2 and Proposition 5.3 gives an isomorphism between the curve $E$ and its image.

## 5.4. Group structure

From now on we will simply write $E$ instead of $(E, O)$. An elliptic curve carries a structure of an abelian group. We explain this for a smooth cubic in $\mathbb{P}^2$.

Let $E$ be a smooth cubic curve in $\mathbb{P}^2$ and $O \in E(\mathsf{k})$. Let also $P, Q \in E(\mathsf{k})$ and $L$ be the line in $\mathbb{P}^2$ connecting two points $P$ and $Q$. If $P = Q$ then $L$ is the tangent line to $E$ at $P$. The line $L$ is defined over $\mathsf{k}$ and it is easy to verify that the third intersection $R := PQ$ of $E(\bar{\mathsf{k}})$ with $L(\bar{\mathsf{k}})$ is also in $E(\mathsf{k})$. Define

$$P + Q = O(PQ)$$

For instance, for an elliptic curve in the Weierstrass form take $O = [0;1;0]$ the point at infinity. By definition $O + O = O$. The above construction turns $E(\mathsf{k})$ into a commutative group with the zero element $O$.

**Exercise.** (1) Let $E$ be an elliptic curve over $\mathsf{k}$, $a \in E(\mathsf{k})$ and $f : E \to E, \ f(x) = x + a$. Prove that the induced map in the de Rham cohomology is identity.

(2) For $g : E \to E,\ g(x) = nx,\ n \in \mathbb{N}$, prove that the induced map in the de Rham cohomology is multiplication by $n$. In fact in the Weierstrass coordinates, and using Weierstrass $p$ function, we have the equality:

$$f^*(\frac{xdx}{y}) = (\frac{1}{4}(\frac{y-y_0}{x-x_0})^2 - x - x_0)\frac{dx}{y}, \quad a = (x_0, y_0).$$

This is a real equality between differential forms and not modulo exact forms.

## 5.5. Moduli spaces of elliptic curves

Let $\mathsf{R} = \mathsf{k}[t_1, t_2, t_3, \frac{1}{\Delta}],\ \Delta := 27t_3^2 - t_2^3$ and $T = \mathrm{Spec}(\mathsf{R})$. Let also $E$ be the subvariety of $\mathbb{P}^2 \times T$ given by:

$$E : zy^2 - 4(x-t_1)^3 + t_2z^2(x-t_1) + t_3z^3 = 0, [x;y;z] \times (t_1,t_2,t_3) \in \mathbb{P}^2 \times T \tag{5.1}$$

and

$$E \to T$$

be the projection on $T$. The differential forms $\frac{dx}{y}, \frac{xdx}{y}$ form a free basis of the $\mathsf{R}$-module $H^1_{\mathrm{dR}}(E/T)$, $\frac{dx}{y}$ is a regular differential form on $E$, and $\langle \frac{dx}{y}, \frac{xdx}{y} \rangle = 1$. The discussion in §5 leads to:

**Proposition 5.4.** *The affine variety $T$ is the moduli of the pairs $(\tilde{E}, \omega)$, where $\tilde{E}$ is an elliptic curve over $\mathsf{k}$ and $\omega \in H^1_{\mathrm{dR}}(\tilde{E}) \backslash F^1$, where $F^1$ is the one dimensional subspace of $H^1_{\mathrm{dR}}(\tilde{E})$ containing regular differential 1-forms.*

Note that from the beginning we could work with the elliptic curve $E$ in the Weierstrass form with $t_1 = 0$. We have the isomorphism

$$(\{y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3\}, \frac{xdx}{y}) \cong (\{y^2 = 4x^3 - t_2x - t_3\}, \frac{xdx}{y} + t_1\frac{dx}{y}),$$

$$(x,y) \mapsto (x - t_1, y).$$

## 5.6. Torsion points

For an elliptic curve over $\mathsf{k}$ we define $E[N]$ to be the set of $N$-torsion points of $E$:

$$E[N](\mathsf{k}) := \{p \in E(\mathsf{k}) \mid Np = 0\}.$$

When the base field is clear from the text, we simply write $E[N] = E[N](\mathsf{k})$.

**Proposition 5.5.** *Let $E$ be an elliptic curve over $\mathsf{k}$. We have an isomorphism of groups*

$$E[N](\bar{\mathsf{k}}) \cong (\mathbb{Z}/N\mathbb{Z})^2$$

*and so the cardinality of $E[N](\mathsf{k})$ is less than $N^2$.*

For a proof see [32], Theorem 6.1 page 165.

## 6. Quasi-modular forms

### 6.1. Introduction

In the present section we introduce quasi-modular forms in the framework of Algebraic Geometry. For an elliptic curve with an element in its de Rham cohomology, which is not represented by a regular differential form, we can associate three quantities $t_1, t_2, t_3$ which appear in Weierstrass form of $E$, see Proposition 5.3. These quantities satisfy a simple functional property with respect to the action of an algebraic group which turn them our first examples of quasi-modular forms. These are algebraic version of the Eisenstein series. In fact, we can describe the algebraic version of all Eisenstein series by using residue calculus, see §6.5. For future applications, we introduce quasi-modular forms for elliptic curves enhanced with certain torsion elements structure.

### 6.2. Enhanced elliptic curves

Let $N$ be a positive integer. In this section we use the notation of groups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ without using them, see §7 for their definitions. Their appearance in this section is for the sake of following the terminology in the literature. The reason for all these, which have apparently nothing to do with our algebraic context, will be clear in §8.

An enhanced elliptic curve for $\Gamma_0(N)$ is a 3-tuple $(E, C, \omega)$, where $E$ is an elliptic curve over $\mathsf{k}$, $C$ is a cyclic subgroup of $E(\mathsf{k})$ of order $N$ and $\omega$ is an element in $H^1_{\mathrm{dR}}(E)\backslash F^1$. An enhanced elliptic curve for $\Gamma_1(N)$ is a 3-tuple $(E, Q, \omega)$, where $E, \omega$ are as before and $Q$ is a point of $E(\mathsf{k})$ of order $N$. Let us fix a primitive root of unity of order $N$ in $\mathsf{k}$, say $\zeta$. An enhanced elliptic curve for $\Gamma(N)$ is a 3-tuple $(E, (P, Q), \omega)$, where $E, \omega$

are as before and $P$ and $Q$ are a pair of points of $E(\mathsf{k})$ that generates the $N$-torsion subgroup $E[N]$ with Weil pairing $e_N(P,Q)=\zeta$. For the definition of Weil pairing see Chapter 3, Section 8 of Silverman [32]. We call $C,Q$ or $(P,Q)$ a torsion structure on $E$. The number of enhanced elliptic curves for $\Gamma$ with fixed $E$ and $\omega$ is finite and it can be shown that it is the cardinality of the quotient $\Gamma\backslash \mathrm{SL}(2,\mathbb{Z})$. For $N=1$ an enhanced elliptic curve for all $\Gamma_0(N),\Gamma_1(N)$ and $\Gamma(N)$ is the same and so we write $\Gamma=\mathrm{SL}(2,\mathbb{Z})=\Gamma_0(N)=\Gamma_1(N)=\Gamma(N)$.

The choice of $\omega\in H^1_{\mathrm{dR}}(E)\backslash F^1$ determines in a unique way a regular differential 1-form $\omega_1\in F^1$ with $\langle\omega_1,\omega\rangle=1$. This is because $F^1$ is a one dimensional subspace of $H^1_{\mathrm{dR}}(E)$ and any non-zero element in $F^1$ together with $\omega$ form a basis of $H^1_{\mathrm{dR}}(E)$, and hence, it has non-zero intersection with $\omega$ (otherwise the intersection form would be identically zero). In this way, $\omega_1,\omega$ form a basis of the $\mathsf{k}$-vector space $H^1_{\mathrm{dR}}(E)$.

In a similar way we can define a family of enhanced elliptic curves (see [14], Chapter III, Section 10).

## 6.3. Action of algebraic groups

Let $\Gamma$ be one of the $\Gamma_0(N),\Gamma_1(N)$ and $\Gamma(N)$ and $T_\Gamma$ be the set of enhanced elliptic curves for $\Gamma$ modulo canonical isomorphisms. The additive group $\mathbb{G}_a=(\mathsf{k},+)$ and the multiplicative group $\mathbb{G}_m=(\mathsf{k}^*,\cdot)$ acts in a canonical way on $T_\Gamma$:

$$(*,*,\omega)\bullet k=(*,*,k^{-1}\omega),\ k\in\mathbb{G}_m,\ (*,*,\omega)\in T_\Gamma,$$

$$(*,*,\omega)\bullet k=(*,*,k'\omega_1+\omega),\ k'\in\mathbb{G}_a,\ (*,*,\omega)\in T_\Gamma.$$

Both these actions can be summarized in the action of the algebraic group

$$G=\left\{\begin{pmatrix}k & k'\\ 0 & k^{-1}\end{pmatrix}\mid\ k'\in\mathsf{k},k\in\mathsf{k}-\{0\}\right\}\cong\mathbb{G}_a\times\mathbb{G}_m \tag{6.1}$$

i.e.

$$(*,*,\omega)\bullet g=(*,*,k'\omega_1+k^{-1}\omega),\ g\in G,\ (*,*,\omega)\in T_\Gamma.$$

It is expected that $T_\Gamma$ is the set of $\mathsf{k}$-rational points of an affine variety equipped with an action of the algebraic group $G$, all defined over $\mathsf{k}$, and compatible with the action of $G$ on $T_\Gamma$. In the present text we have verified this in the case $N=1$, where an enhanced elliptic curve becomes a pair $(E,\omega)$, $\omega\in H^1_{\mathrm{dR}}(E)\backslash F^1$ (see §5.5), and $\Gamma=\Gamma(2)$. Since the main example of the present text is the case $N = 1$, the study of such affine varieties and maps between them is postponed to forthcoming articles.

**Proposition 6.1.** *Let* $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. *The canonical map*

$$T(\mathsf{k}) \to T_\Gamma,\ t \mapsto (E_{\pi^{-1}t}, \frac{xdx}{y})$$

*is an isomorphism. Under this isomorphism the action of the algebraic group $G$ is given by*

$$t \bullet g := (t_1k^{-2} + k'k^{-1}, t_2k^{-4}, t_3k^{-6}), t = (t_1,t_2,t_3), g = \begin{pmatrix} k & k' \\ 0 & k^{-1}\end{pmatrix} \in G.$$

*Proof.* The first part follow from Proposition 5.4. The proof of the second part is as follows: Let

$$\alpha : \mathbb{A}^2_{\mathsf{k}} \to \mathbb{A}^2_{\mathsf{k}},\ (x,y) \mapsto (k^2x - k'k, k^3y)$$

and $f = y^2 - 4(x-t_1)^3 + t_2(x-t_1) + t_3$. We have

$$k^{-6}\alpha^*(f) = y^2 - 4k^{-6}(k^2x - k'k - t_1)^3 + t_2k^{-6}(k^2x - k'k - t_1) + t_3k^{-6} =$$

$$y^2 - 4(x - k'k^{-1} - t_1k^{-2})^3 + t_2k^{-4}(x - k'k^{-1} - t_1k^{-2}) + t_3k^{-6}.$$

This implies that $\alpha$ induces an isomorphism of elliptic curves

$$\alpha : (E_{t\bullet g}, \alpha^*(\frac{xdx}{y})) \to (E_t, \frac{xdx}{y}).$$

Since

$$\alpha^*\frac{xdx}{y} = k\frac{xdx}{y} - k'\frac{dx}{y}$$

we get the result. □

## 6.4. Quasi-modular forms

The algebraic group $G$ acts from the right on $T_\Gamma$ and so it acts from the left on the space of functions on $T_\Gamma$. A quasi-modular form $f$ of weight $m$ and differential order or depth $n$ for $\Gamma$ is a function $T_\Gamma \to \mathsf{k}$ with the following properties:

(1) With respect to the action of $\mathbb{G}_m$, $f$ satisfies

$$k \bullet f = k^m f,\ \mathsf{k} \in \mathbb{G}_m. \tag{6.2}$$

(2) With respect to the action of $\mathbb{G}_a$, $f$ satisfies the following condition: there are functions $f_i : T_\Gamma \to \mathsf{k},\ i = 0, 1, 2, \ldots, n$ such that

$$k' \bullet f = \sum_{i=0}^{n} \binom{n}{i} k'^i f_i,\ k' \in \mathbb{G}_a. \tag{6.3}$$

(3) (Growth condition)?

We were not able to formulate a growth condition for quasi-modular forms in a purely algebraic and intrinsic way using degeneration of curves. Such a condition would correspond to the classical growth condition for holomorphic quasi-modular forms. In [18], this condition is formulated in terms of Tate curves and Eisenstein series. This does not seem to be a natural one because it assumes a priori that we know Eisenstein series. The formulation in [15] allows modular forms to have poles on cusps. We are going to introduce this condition using one of its main consequences, namely, the $\mathsf{k}$-algebra of quasi-modular forms for $\mathrm{SL}(2,\mathbb{Z})$ is generated by three Weierstrass coordinates.

Note that combining both actions (6.2) and (6.3) we have:

$$f \bullet g = k^{-m} \sum_{i=0}^{n} \binom{n}{i} k'^i k^i f_i,\ \forall g = \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \in G. \tag{6.4}$$

Let us consider the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. We are going to describe the growth condition in this case. Using Proposition 6.1, the Weierstrass coordinate $t_i,\ i = 1, 2, 3$ of an enhanced elliptic curve $(E, \omega)$ satisfies the functional equations (6.2) and (6.3) with weight $m = 2i$ and differential order $n = 1$ for $t_1$ and $n = 0$ for $t_2$ and $t_4$. The growth condition for $f$ in this case is that $f$ is an element in the $\mathsf{k}$-algebra

$$\mathsf{k}[t_1, t_2, t_3],\ \mathrm{weight}(t_i) = 2i,\ i = 1, 2, 3.$$

It follows that it is homogeneous with $\deg(f) = m$ and $\deg_{t_1} f \leq n$. A quasi-modular form for $\mathrm{SL}(2,\mathbb{Z})$ is also called a full quasi-modular form.

Let us now describe the growth condition for arbitrary $\Gamma$. Assume that $\mathsf{k}$ is algebraically closed and consider $f : T_\Gamma \to \mathsf{k}$ with the properties (6.2) and (6.3). For simplicity we write $T = T_{\mathrm{SL}(2,\mathbb{Z})}$. We define $g_i : T \to \mathsf{k},\ i = 1, 2, \ldots, a = \Gamma\backslash\mathrm{SL}(2,\mathbb{Z})$ satisfying (6.2) and (6.3) with weight $mi$ and differential order $ni$ in the following way:

$$g_i : T \to \mathsf{k},$$

$$g_i(E,\omega) := \sum_{*_1,*_2,*_2,\dots,*_i} f(E,*_1,\omega)f(E,*_2,\omega)\cdots f(E,*_i,\omega),$$

where $*_1, *_2, *_2, \dots, *_i$ runs through $i$-tuples of torsion structures on $E$ and attached to $\Gamma$. It follows that $f$ is a root of the polynomial

$$X^a - g_1X^{a-1} + g_2X^{a-2} - \cdots + (-1)^{a-1}g_{a-1}X + (-1)^a g_a. \tag{6.5}$$

The growth condition for the form $f$ is that the corresponding $g_i$'s are quasi-modular forms for $\mathrm{SL}(2,\mathbb{Z})$. It follows that

$$g_i \in \mathsf{k}[t_1,t_2,t_3],\ g_i \text{ homogeneous},\ \deg(g_i) = mi,\ \deg_{t_1}(g_i) \le ni.$$

For $n = 0$ we recover the definition of modular forms of weight $m$, see [18]. A modular form of weight $m$ is a function from the set of enhanced elliptic curves as before but with this difference that $\omega \in F^1$ is a regular differential form and not an element in $H^1_{\mathrm{dR}}(E)\backslash F^1$. The action of $\mathbb{G}_m$ is given by $(*,*,\omega)\bullet k = (*,*,k\omega)$ and $f$ satisfies $k\bullet f = k^{-m}f,\ \mathsf{k}\in\mathbb{G}_m$. The growth condition in this case can be also expressed using Tate curves.

We denote by $M^n_m(\Gamma)$ the set of quasi-modular forms of weight $m$ and differential order $n$ and we set

$$M(\Gamma) = \sum_{m\in\mathbb{Z},n\in\mathbb{N}_0} M^n_m(\Gamma).$$

When there is no confusion we simply write $M = M(\Gamma)$ and so on. If $n \le n'$ then $M^n_m \subset M^{n'}_m$ and

$$M^n_m M^{n'}_{m'} \subset M^{n+n'}_{m+m'},\ M^n_m + M^{n'}_m = M^{n'}_m.$$

We see that $M$ has a structure of a graded $\mathsf{k}$-algebra. The $\mathsf{k}$-algebra of full quasi-modular forms has also a differential structure which is given by:

$$M^n_m \to M^{n+1}_{m+2},\ t \mapsto dt(\mathrm{R}) = \sum_{i=1}^3 \frac{\partial t}{\partial t_i}\mathrm{R}_i$$

where $\mathrm{R} = \sum_{i=1}^3 \mathrm{R}_i\frac{\partial}{\partial t_i}$ is the Ramanujan vector field. We sometimes use $\mathrm{R}: M \to M$ to denote this differential operator.

The family $y^2 - 4(x-s_1)(x-s_2)(x-s_3) = 0$ is the universal family for the moduli of 3-tuple $(E,(P,Q),\omega)$, where $E,\omega$ are as before and $P$ and $Q$ are a pair of points of $E(\mathsf{k})$ that generates the 2-torsion subgroup $E[N]$ with Weil pairing $e(P,Q) = -1$. The points $P$ and $Q$ are given by $(s_1,0)$ and $(s_2,0)$. In this case each $s_i,\ i = 1,2,3$ is a quasi-modular form of weight 2 and differential order 1. They generate the algebra $M(\Gamma(2))$

freely. The corresponding differential structure is defined by the Halphen vector field.

**Exercise.** (1) There is a canonical bijection between modular forms of weight $m$ and quasi-modular forms of weight $m$ and differential order 0.

(2) Verify that $f_i$ is a quasi-modular form of weight $m-2i$ and differential order $n-i$. In particular, $f_n$ is a modular form of weight $m-2n$.

(3) Show that the family mentioned in Exercise 4.7 is the universal family of enhanced elliptic curves for $\Gamma(3)$.

(4) The algebra $M(\Gamma(2))$ is freely generated by three quasi-modular form $s_1, s_2, s_3$ of weight 2 and differential order 1. Show that the polynomial in (6.5) for each $s_i$ is

$$((X-t_1)^3 - \frac{1}{4}t_2(X-t_1) - \frac{1}{4}t_3)^2.$$

## 6.5. **Eisenstein modular forms**

Let $E$ be an elliptic curve over $\mathsf{k}$ and $\omega \in H^1_{\mathrm{dR}}(E)\backslash F^1$. We take the Weierstrass coordinates of the pair $(E, \omega)$ as we have described it in Proposition 5.3. Let also $t$ be a coordinate system around the point $O$, for instance take $t = \frac{x}{y}$. We have $\frac{dx}{y} = Pdt$ for some regular function $P$ in a neighborhood of $O$. Let us write the formal series of $P$ at $O$ and then write it as a derivation of some other formal power series $z = z(t) = \sum_{i=1}^{\infty} z_i t^i$. We have

$$\frac{dx}{y} = dz, \ z_1 = -2.$$

We call $z$ the analytic coordinate system on $E$. Note that the first coefficients in the formal power series of $t^3 y, t^2 x$ in $t$ are invertible and so $z(t)$ has coefficients in $\mathsf{R}$.

**Proposition 6.2.** *We have*

$$x = \frac{1}{z^2} + \sum_{k=1}^{\infty} g_{2k+2} z^{2k},$$

*and*

$$y = \frac{\partial x}{\partial z} = \frac{-2}{z^3} + \sum_{k=1}^{\infty} 2k \cdot g_{2k+2} z^{2k}, \ \ g_{2k+2} \in \mathsf{R}.$$

*Proof.* We have $z(t) = -2t + O(t^2)$ and write $t$ in terms of $z$, that is, $t = t(z) = \frac{-1}{2} z + O(z^2)$. Since the coefficients of $z(t)$ are in $\mathsf{R}$ and $z(t)$ starts with $-2t$, the coefficients of $t(z)$ are also in $\mathsf{R}$. We write $x$ in terms of $z$ and we have: $x = \sum_{k=-2}^{\infty} g_k z^k$ for some $g_k \in \mathsf{R}$. The elliptic curve $E$ is invariant under the involution $(x, y) \mapsto (x, -y)$. The coordinates $t$ and $z$ are mapped to $-t$ and $-z$, respectively, and $x$ is invariant. This implies that $g_k = 0$ for all odd integers $k$. Calculating $g_0, g_2$ we see that $g_0 = g_2 = 0$. The expansion of $y$ follows from the equality $dx = ydz$. $\square$

**Proposition 6.3.** *The mapping* $(E, \omega) \to g_{2k+2}$ *is a full modular form of weight* $2k + 2$*.*

We denote this modular form with $G_{2k+2}$ and we call it the Eisenstein modular form of weight $2k + 2$.

*Proof.* The growth condition in the definition of a quasi-modular form follows from the fact that in the process of defining $G_{2k+2}$, all the coefficients are in $\mathsf{R}$. For $k \in \mathbb{G}_a$, the Weierstrass coordinates system of $(E, \omega) \bullet k$ is $(x+k, y)$ and so $\frac{dx}{y}$ does not change. This implies that $g_{2k+2}$'s are invariant under the action of $\mathbb{G}_a$. For $k \in \mathbb{G}_m$, the Weierstrass coordinates system of $(E, \omega) \bullet k$ is $(\tilde{x}, \tilde{y}) = (k^{-2}x, k^{-3}y)$. In this coordinates system $\tilde{t} = kt$ and $\tilde{z} = kz$ which give us the desired functional property of $g_{2k+2}$'s with respect to the action of $\mathbb{G}_m$. $\square$

**Exercise.** Show the last piece of the proof of Proposition 6.2, that is, $g_0 = g_2 = 0$. Calculate $G_4$ and $G_6$.

## 7. Quasi-modular forms over $\mathbb{C}$

### 7.1. Introduction

The name quasi-modular form seems to appear for the first time in the work [17] of M. Kaneko and D. Zagier. In this article they give a direct proof for a formula stated by R. Dijkgraaf in [6] which deals with counting ramified covering of elliptic curves, see §B.1. In fact if we want to extend the algebra of full modular forms to an algebra which is closed under the canonical derivation then we naturally suspect the existence of the Eisenstein series $E_2$. The full functional equation can be derived by consecutive derivations of the functional equation of a modular form. This is the way in which the author of the present text rediscovered all these, see [23], and for this reason called them differential modular forms.

The classical modular or quasi-modular forms are holomorphic functions on the Poincaré upper half plane which satisfy a functional property with respect to the action of a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ and have some growth condition at infinity. The aim of this section is to show that what we we have developed so far in the context of algebraic geometry is essentially the same as its complex counterpart. The bridge between two notions is the period map which is constructed by elliptic integrals.

## 7.2. Quasi-modular forms

In this section we recall the definition of a quasi-modular form. For more details see [22, 23]. We use the notations $A = \begin{pmatrix} a_A & b_A \\ c_A & d_A \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

When there is no confusion we will simply write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We denote by $\mathbb{H}$ the Poincaré upper half plane and

$$\mathrm{j}(A,z) := c_A z + d_A.$$

For $A \in \mathrm{SL}(2,\mathbb{R})$ and $m \in \mathbb{Z}$ we use the slash operator

$$f|_m A = \mathrm{j}(A,z)^{-m} f(Az).$$

Let $\Gamma$ be a subgroup of $\mathrm{SL}(2,\mathbb{Z})$. For instance, take a congruence group of level $N$. This is by definition any subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ which contains:

$$\Gamma(N) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (\bmod N)\}.$$

It follows that $\Gamma$ has finite index in $\mathrm{SL}(2,\mathbb{Z})$. Our main examples are $\Gamma(N)$ itself and

$$\Gamma_0(N) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \ (\bmod N)\},$$

$$\Gamma_1(N) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \ (\bmod N)\}.$$

We define the notion of an $M_m^n(\Gamma)$-function, a quasi-modular form of weight $m$ and differential order $n$ for $\Gamma$. For simplicity we write $M_m^n(\Gamma) = M_m^n$. For $n = 0$ an $M_m^0$-function is a classical modular form of weight $m$ on $\mathbb{H}$ (see below). A holomorphic function $f$ on $\mathbb{H}$ is called $M_m^n$ if the following two conditions are satisfied:

(1) There are holomorphic functions $f_i,\ i = 0, 1, \dots, n$ on $\mathbb{H}$ such that

$$f|_m A = \sum_{i=0}^{n} \binom{n}{i} c_A^i \mathrm{j}(A,z)^{-i} f_i,\ \forall A \in \Gamma. \tag{7.1}$$

(2) $f_i \mid_m A, i = 0, 1, 2, \dots, n$ have finite growths when $\Im(z)$ tends to $+\infty$ for all $A \in \mathrm{SL}(2,\mathbb{Z})$, that is

$$\lim_{\Im(z)\to+\infty} (f_i \mid_m A)(z) = a_{i,A} < \infty,\ a_{i,A} \in \mathbb{C}.$$

We will also denote by $M_m^n$ the set of $M_m^n$-functions and we set

$$M := \sum_{m\in\mathbb{Z}, n\in\mathbb{N}_0} M_m^n.$$

For an $f \in M_m^n$ we have $f|_m I = f_0$ and so $f_0 = f$. Note that for an $M_m^n$-function $f$ the associated functions $f_i$ are unique. If $f$ is $M_m^n$-function with the associated functions $f_i$ then $f_i$ is an $M_{m-2i}^{n-i}$-function with the associated functions $f_{ij} := f_{i+j}$. The set $M$ is a differential $\mathbb{C}$-algebra:

$$\frac{d}{dz} : M_m^n \to M_{m+2}^{n+1}$$

If $n \le n'$ then $M_m^n \subset M_m^{n'}$ and $M_m^n M_{m'}^{n'} \subset M_{m+m'}^{n+n'}$. It is useful to define

$$f||_m A := (\det A)^{m-n-1} \sum_{i=0}^{n} \binom{n}{i} c_{A^{-1}}^i \mathrm{j}(A,z)^{i-m} f_i(Az), \tag{7.2}$$

for $A \in \mathrm{GL}(2,\mathbb{R})$, $f \in M_m^n$. The equality (7.1) is written in the form

$$f = f||_m A, \forall A \in \Gamma \tag{7.3}$$

One can prove that

$$f||_m A = f||_m (BA),\ \forall A \in \mathrm{GL}(2,\mathbb{R}),\ B \in \Gamma,\ f \in M_m^n.$$

Using this we can prove that the growth condition on $f$ is required only for a finite number of cases $f_i||_m\alpha,\ \alpha \in \Gamma\backslash \mathrm{SL}(2,\mathbb{Z}),\ i = 0, 1, 2 \dots, n$. The relation of $||_m$ with $\frac{d}{dz}$ is given by:

$$\frac{d(f||_m A)}{dz} = \frac{df}{dz}||_{m+2}A,\ \forall A \in \mathrm{GL}(2,\mathbb{R}). \tag{7.4}$$

Let $A \in \mathrm{SL}(2,\mathbb{Z})$. If $f \in M_m^n(\Gamma)$ with associated functions $f_i$ then $f||_m A \in M_m^n(A^{-1}\Gamma A)$ with associated functions $f_i||_m A \in M_{m-2i}^{n-i}(A^{-1}\Gamma A)$.

## 7.3. $q$-expansion

Let us assume that there is $h \in \mathbb{N}$ such that $T_h := \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. Take $h > 0$ the smallest one. Recall that $\Gamma$ is a normal subgroup of $\mathrm{SL}(2,\mathbb{Z})$. For an $f \in M_m^n(\Gamma)$ and $A \in \mathrm{SL}(2,\mathbb{Z})$ with $[A] = \alpha \in \Gamma\backslash \mathrm{SL}(2,\mathbb{Z})$ we have $(f||_m A)|_m T_h = f$ and so we can write the Fourier expansion of $f||_m A$ at $\alpha$

$$f||_m A = \sum_{n=0}^{+\infty} a_n q_h^n,\ a_n \in \mathbb{C},\quad q_h := e^{2\pi i h z}.$$

We have used the growth condition on $f$ to see that the above function in $q_h$ is holomorphic at 0.

## 7.4. Period domain

Quasi-modular forms are best viewed as holomorphic functions on the period domain

$$\mathcal{P} := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mid x_i \in \mathbb{C},\ x_1x_4 - x_2x_3 = 1,\ \Im(x_1\overline{x_3}) > 0 \right\}. \tag{7.5}$$

We let the group $\mathrm{SL}(2,\mathbb{Z})$ (resp. $G$ in (6.1) with $\mathsf{k} = \mathbb{C}$) act from the left (resp. right) on $\mathcal{P}$ by usual multiplication of matrices. The Poincaré upper half plane $\mathbb{H}$ is embedded in $\mathcal{P}$ in the following way:

$$z \to \tilde{z} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}.$$

We denote by $\tilde{\mathbb{H}}$ the image of $\mathbb{H}$ under this map. Note that any element of $\mathcal{P}$ is equivalent to an element of $\tilde{\mathbb{H}}$ under the action of $G$ because:

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{x_3} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_3 & x_4 \\ 0 & \frac{\det(x)}{x_3} \end{pmatrix}. \tag{7.6}$$

The map

$$J : \mathrm{GL}(2,\mathbb{R}) \times \mathbb{H} \to G,\ J(A,z) = \begin{pmatrix} \mathrm{j}(A,z) & -c \\ 0 & \mathrm{j}(A,z)^{-1}\det(A) \end{pmatrix}$$

is an automorphy factor, that is, it satisfies the functional equation:

$$J(AB,z) = J(A,Bz)J(B,z),\ A,B \in \mathrm{GL}(2,\mathbb{R}),\ z \in \mathbb{H}.$$

This follows from the equality

$$A\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} Az & -1 \\ 1 & 0 \end{pmatrix} J(A,z),\ A \in \mathrm{GL}(2,\mathbb{R}),\ z \in \mathbb{H}.$$

**Proposition 7.1.** *Quasi-modular forms $f \in M_m^n$ are in a one to one correspondence with holomorphic functions $F = \phi(f) : \mathcal{P} \to \mathbb{C}$ with the following properties:*

(1) *The function $F$ is $\Gamma$-invariant.*

(2) *There are holomorphic functions $F_i : \mathcal{P} \to \mathbb{C},\ i = 0,1,\ldots,n$ such that*

$$F(x \cdot g) = k^{-m}\sum_{i=0}^{n}\binom{n}{i}k'^i k^i F_i(x),\ \forall x \in \mathcal{P},\ g \in G, \tag{7.7}$$

(3) *For all $\alpha \in \mathrm{SL}(2,\mathbb{Z})$ the restriction of $F_i$ to $\tilde{\mathbb{H}}_\alpha$ has finite growth at infinity, where $\tilde{\mathbb{H}}_\alpha$ is be the image of $\tilde{\mathbb{H}}$ under the action of $\alpha$ from the left on $\mathcal{P}$.*

In fact we have $F_i = \phi(f_i)$. The proof is a mere calculation and can be found in [23], Proposition 6.

**Exercise.** (1) *Verify that the vector field*

$$X := -x_2\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_3} \tag{7.8}$$

*is invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$ and hence it induces a vector field $\tilde{X}$ in the quotient $\Gamma\backslash\mathcal{P}$.*

(2) *Show that under the the correspondence in Proposition 7.1, the differential operator on quasi-modular forms as functions on $\Gamma\backslash\mathcal{P}$ is given by the vector field $\tilde{X}$. Note that $X$ restricted to the loci $\tilde{\mathbb{H}}$ is $\frac{\partial}{\partial z}$.*

# 8. **Elliptic integrals**

## 8.1. **Introduction**

The study of multiple integrals requires an independent study of their domain of integration. For the elliptic integral (2.1) we have apparently five different domain of integration: $[-\infty, a_1], [a_1, a_2], [a_2, a_3],\ [a_3, +\infty]$, where $a_1, a_2, a_3$ are three consecutive roots of $P$. However, the complexification and then the geometrization process of such integrals, showed that if we fix the integrand $\omega$ then all such integrals can be written in terms of just two of them. All such integrals up to some constants, which can be calculated effectively, can be written in the form $\int_\delta \omega$, where $\delta$ is an element in the homotopy group $G := \pi_1(E, b)$ of the elliptic curve $E : y^2 = P(x)$ with a base point $b \in E$. The integral is zero over the group $[G, G]$ generated by the commutators of $G$, that is, elements of the form $\delta_1\delta_2\delta_1^{-1}\delta_2^{-1}$, and so we can choose $\delta$ from the first homology group:

$$\delta \in H_1(E, \mathbb{Z}) := G/[G, G].$$

The homology group $H_1(E, \mathbb{Z})$ is a $\mathbb{Z}$-module of rank two and in this way we get our affirmation. This simple observation can be considered as the beginning of singular homology. H. Poincaré in his book named *Analysis Situs* founded the Algebraic Topology and if we look more precisely for his motivation, we find his articles on multiple integrals. E. Picard was a first who wrote the treatise *Théorie des fonctions algébriques de deux variables indépendantes* on double integrals. For him the study of the integration domain was justified by the study of integrals, but after him and in particular for S. Lefschetz, it was done as an independent subject. The classical theorems of Lefschetz on the topology of algebraic varieties are an evidence to this fact. One of the beautiful topological theories which arose from the study of elliptic and multiple integrals is the so called Picard-Lefschetz theory. A brief description of this is done in §8.2. It can be considered as the complex counterpart of Morse theory, however, it is older than it. In families of elliptic curves that we consider the topology is fixed, all elliptic curves are tori. However, some interesting phenomena occur when we turn around degenerated elliptic curves. This is described by Picard-Lefschetz theory. It says that when we are dealing with families of elliptic curves, $\mathrm{SL}(2, \mathbb{Z})$ and its subgroups are there, even we do not mention them explicitly.

I have avoided to use Weierstrass uniformization theorem for elliptic curves. Instead, I have tried to use Hodge theoretic arguments in order to derive many statements which follow from it. The main reason for this is the possibility of the generalizations for other type of varieties. For an introduction to Hodge theory (Hodge structures, period maps, Torelli problem and etc.) the reader is referred to the books of C. Voisin (2003). In §8.5 we describe how Eisenstein series appear in the inverse of the period map. The case of $E_4$ and $E_6$ follow from the Weierstrass uniformization theorem, however, the case of $E_2$ is not covered by this theorem. In order to calculate the $q$-expansion of a quasi-modular form we need first its differential equation and second, its first coefficients. In order to calculate such coefficients we need to write down the Taylor series of elliptic integrals in a degenerated elliptic curve. Doing this, we get the formulas of elliptic integrals in terms of hypergeometric functions (see 8.6) and immediately after, we give another characterization of the Ramanujan differential equation in terms of hypergeometric functions (see §8.7). This is in fact the method which G. Halphen used to derive a differential equation depending on three parameters of the Gauss hypergeometric function. We also get the Schwarz map whose image gives us the well-known triangulation of the upper half plane such that each triangle is a fundamental domain for $\mathrm{SL}(2,\mathbb{Z})$. The bridge between the analytic and algebraic versions of quasi-modular forms is the notion of period map constructed from elliptic integrals (see §8.4). In this section we also explain this.

**Exercise.** Calculate the integrals (2.2), where $\delta$ is one of

$$[-\infty, a_1], [a_1, a_2], [a_2, a_3],\ [a_3, +\infty]$$

and $a_1, a_2, a_3$ are three consecutive roots of $P$ in terms of two of them.

## 8.2. The monodromy group

In the framework of algebraic geometry, the arithmetic group $\mathrm{SL}(2,\mathbb{Z})$ is hidden and it appears as the monodromy group of the family of elliptic curves $E_t$ over $\mathbb{C}$. In order to calculate such a monodromy group we need a machinery called Picard-Lefschetz theory. See for instance [1] for the local version of such a theory and see [20] for a global version.

The elliptic curve $E_t,\ t \in T$ given by (5.1) as a topological space is a torus and hence $H_1(E_t,\mathbb{Z})$ is a free rank two $\mathbb{Z}$-module. Smooth variations of $t$, gives us the monodromy representation

$$\pi_1(T, b) \to \mathrm{Iso}(H_1(E_b, \mathbb{Z})),$$

where $b$ is a fixed point in $T$. We would like to calculate the image of the monodromy representation in a fixed basis of $\delta_1, \delta_2$ of $H_1(E_b, \mathbb{Z})$. A classical way for choosing such a basis is given by the Picard-Lefschetz theory. Fix the parameters $t_1$ and $t_2 \neq 0$ and let $t_3$ varies. Exactly for two values $\tilde{t}_3,\ \check{t}_3 = \pm\sqrt{\frac{t_2^3}{27}}$ of $t_3$, the curve $E_t$ is singular. In $E_b - \{\infty\}$ we can take two cycles $\delta_1$ and $\delta_2$ such that $\langle \delta_1, \delta_2 \rangle = -1$ and $\delta_1$ (resp. $\delta_2$) vanishes along a straight line connecting $b_3$ to $\tilde{t}_3$ (resp. $\check{t}_3$). The corresponding anti-clockwise monodromy around the critical value $\tilde{t}_3$ (resp $\check{t}_3$) can be computed using the Picard-Lefschetz formula:

$$\delta_1 \mapsto \delta_1,\ \delta_2 \mapsto \delta_2 + \delta_1 \ (\text{ resp. } \delta_1 \mapsto \delta_1 - \delta_2,\ \delta_2 \mapsto \delta_2).$$

The canonical map $\pi_1(\mathbb{C}\backslash\{\tilde{t}_3, \check{t}_3\}, t) \to \pi_1(T, t)$ induced by inclusion is a surjection and so the image of $\pi_1(T, t)$ under the monodromy representation is

$$\mathrm{SL}(2, \mathbb{Z}) = \langle A_1, A_2 \rangle, \quad \text{where } A_1 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ A_2 := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let us explain the above topological picture by the following one parameter family of elliptic curves:

$$E_\psi : y^2 - 4x^3 + 12x - 4\psi = 0.$$

For $b$ a real number between 2 and $-2$ the elliptic curve $E_b$ intersects the real plane $\mathbb{R}^2$ in two connected pieces which one of them is an oval and we can take it as $\delta_2$ with the anti clockwise orientation. In this example as $\psi$ moves from $-2$ to 2, $\delta_2$ is born from the point $(-1, 0)$ and ends up in the $\alpha$-shaped piece which is the intersection of $E_2$ with $\mathbb{R}^2$. The cycle $\delta_1$ lies in the complex domain and it vanishes on the critical point $(1, 0)$ as $\psi$ moves to 2. It intersects each connected component of $E_b \cap \mathbb{R}^2$ once and it is oriented in such away that $\langle \delta_1, \delta_2 \rangle = -1$.

**Exercise.** If we fix $t_1$ and $t_3$ and let $t_2$ vary then we get three critical curves. Describe the intersection number between the corresponding vanishing cycles $\delta_i,\ i = 1, 2, 3$, linear relations between $\delta_i$'s and the monodromy around each critical fiber.

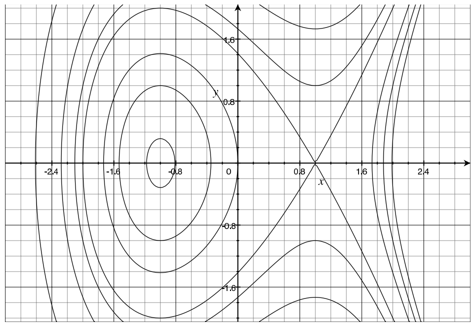


Figure 8.1. Elliptic curves: $y^2 - x^3 + 12x - 4\psi,\ \psi = -1.9, -1, 0, 2, 3, 5, 10$

## 8.3. **Hodge structure of elliptic curves**

Let $E$ be an elliptic curves over $\mathbb{C}$. We can regard $E$ as a complex manifold and so we have de Rham cohomologies of $H^i_{\rm dR}(E),\ i = 0, 1, 2$ defined using $C^\infty$ and $\mathbb{C}$-valued differential forms. In the $C^\infty$ context we have also the wedge product $H^1_{\rm dR}(E) \times H^1_{\rm dR}(E) \to H^2_{\rm dR}(E)$ bilinear map and the isomorphism

$$H^2_{\rm dR}(E) \cong \mathbb{C},\ \omega \mapsto \frac{1}{2\pi i}\int_E \omega.$$

The $2\pi i$ factor is there, because in this way the above isomorphism is the complexification of an isomorphism of $\mathbb{Z}$-modules $H^2(E, \mathbb{Z}) \cong \mathbb{Z}$. The translation of all these in the algebraic context is done in §2 and we leave it to the reader the details of comparison of algebraic and $C^\infty$ contexts. For this and the details of what we are going to describe the reader is referred to the classical Book of Griffiths and Harris (1978) or to the two volume book of C. Voisin (2002) on Hodge theory.

There is a one dimensional subspace $F^1 \subset H^1_{\rm dR}(E)$ which is spanned by regular differential forms on $E$. We have the complex conjugation in $H^1_{\rm dR}(E)$ and it turns out that $F^1 \cap \overline{F^1} = \{0\}$ and so

$$H^1_{\rm dR}(E) = F^1 \oplus \overline{F^1},$$

which is called the Hodge decomposition. The bilinear map $\langle\cdot,\cdot\rangle$ constructed in §2.10 turns out to be

$$\langle\cdot,\cdot\rangle : H^1_{\mathrm{dR}}(E)\times H^1_{\mathrm{dR}}(E)\to\mathbb{C},\ (\omega,\alpha)\mapsto \frac{1}{2\pi i}\int_E \omega\wedge\alpha.$$

It satisfies the following inequality

$$-\langle\omega,\overline{\omega}\rangle>0,\ \omega\in F^1. \tag{8.1}$$

From all these we want to conclude two well-known facts about elliptic integrals.

**Proposition 8.1.** *Let $E$ be an elliptic curve in the Weierstrass form and let $\delta_1,\delta_2\in H_1(E,\mathbb{Z})$ with $\langle\delta_1,\delta_2\rangle=-1$. We have*

(1) *The integral $\int_\delta \frac{dx}{y},\ \delta\in H_1(E,\mathbb{Z})$ never vanishes and*

$$\Im\left(\frac{\int_{\delta_1}\frac{dx}{y}}{\int_{\delta_2}\frac{dx}{y}}\right)>0. \tag{8.2}$$

(2) *We have*

$$\int_{\delta_2}\frac{dx}{y}\int_{\delta_1}\frac{xdx}{y}-\int_{\delta_1}\frac{dx}{y}\int_{\delta_2}\frac{xdx}{y}=2\pi i. \tag{8.3}$$

*Proof.* The first part follows, for instance, from [32], Proposition 5.2. The second part is known as Legendre relations between elliptic integrals, see for instance [32], Exercise 6.4 d. We give another proof of all these based on de Rham cohomology arguments presented in this section.

Let $\check{\delta}_i\in H^1(E_t,\mathbb{Z}),\ i=1,2$ be the Poincaré dual of $\delta_i$, that is, $\int_\delta\check{\delta}_i=\langle\delta_i,\delta\rangle$ for all $\delta\in H_1(E,\mathbb{Z})$. The bilinear map $\langle\cdot,\cdot\rangle$ in cohomology is dual to the intersection linear map in cohomology and so $\langle\check{\delta}_1,\check{\delta}_2\rangle=-1$. In the de Rham cohomology $H^1_{\mathrm{dR}}(E)$ we have

$$\omega=-(\int_{\delta_2}\omega)\check{\delta}_1+(\int_{\delta_1}\omega)\check{\delta}_2,\ \omega=\frac{dx}{y},\ \frac{xdx}{y}.$$

We use this for $\omega = \frac{dx}{y}$ and we see that the inequality (8.1) is equivalent to the first part of the proposition. The second part follows from

$$\begin{aligned} 2\pi i &= 2\pi i \langle \frac{dx}{y}, \frac{xdx}{y} \rangle \\ &= \langle -(\int_{\delta_2} \frac{dx}{y})\check{\delta}_1 + (\int_{\delta_1} \frac{dx}{y})\check{\delta}_2, -(\int_{\delta_2} \frac{xdx}{y})\check{\delta}_1 + (\int_{\delta_1} \frac{xdx}{y})\check{\delta}_2 \rangle \\ &= (\int_{\delta_2} \frac{dx}{y} \int_{\delta_1} \frac{xdx}{y} - \int_{\delta_1} \frac{dx}{y} \int_{\delta_2} \frac{xdx}{y}) \end{aligned}$$

□

## 8.4. **Period map**

Recall the notations of §6 for the base field $\mathsf{k} = \mathbb{C}$. Recall also that that for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ we have

$$T := T_\Gamma = \{(t_1, t_2, t_3) \in \mathbb{C}^3 \mid 27t_3^2 - t_2^3 \neq 0\}.$$

If $\Gamma$ is one of $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$ then we know that the projection map $\beta : T_\Gamma \to T$ (neglecting the torsion point structure) is a covering of degree $\#(\Gamma\backslash\mathrm{SL}(2, \mathbb{Z}))$ (see Exercise 8.4) and so $T_\Gamma$ has a natural structure of a complex manifold. We define $\mathrm{R}_\Gamma$ to be the pull-back of the Ramanujan vector field in $T_\Gamma$.

Let us fix $b \in T_\Gamma$ and a basis $\delta_1^0, \delta_2^0$ of the $\mathbb{Z}$-module $H_1(E_{\beta(b)}, \mathbb{Z})$ with $\langle \delta_1^0, \delta_2^0 \rangle = -1$. For any path $\gamma$ which connects $b$ to an arbitrary point $t \in T_\Gamma$ we define $\delta_1, \delta_2 \in H_1(E_t, \mathbb{Z})$ to be the monodromy of $\delta_1^0$ and $\delta_2^0$ along the path $\gamma$. The period map is defined by

$$\mathsf{pm} : T_\Gamma \to \Gamma\backslash\mathcal{P},\ t \mapsto \left[ \frac{1}{\sqrt{-2\pi i}} \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix} \right].$$

Brackets $[\cdot]$ means the equivalence class in the quotient $\Gamma\backslash\mathcal{P}$. It is well-defined because of Proposition 8.1 and the following fact: different choices of the path $\gamma$ lead to the action of $\Gamma$ from the left on $\mathcal{P}$ which is already absorbed in the quotient $\Gamma\backslash\mathcal{P}$. Different choices of $b$ and $\delta_1^0, \delta_2^0$ lead to the composition of the period map with canonical automorphisms of $\Gamma\backslash\mathcal{P}$ (see Exercise 8.4, 2). The factor $\frac{1}{\sqrt{-2\pi i}}$ is inserted so that the determinant of the matrix is one (Legendre relation between elliptic integrals).

**Proposition 8.2.** *We have*

(1) *The period map is a local biholomorphism;*

(2) *It satisfies*

$$\mathsf{pm}(t \bullet g) = \mathsf{pm}(t) \cdot g, \ t \in T_\Gamma, \ g \in G; \tag{8.4}$$

(3) *The push forward of the vector field* $\mathrm{R}_\Gamma$ *by the period map* $\mathsf{pm}$ *is the vector field* $X$ *in (7.8).*

*Proof.* It is enough to prove the Proposition for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ (Exercise 8.4, 4). The equality (8.4) follows from Proposition 6.1. The last statement follows from Proposition 3.1 and (3.3) as follows:

$$d\mathsf{pm}(\mathrm{R}) = \mathsf{pm}(t) \cdot A^{\mathsf{tr}}(\mathrm{R}) = \mathsf{pm} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -x_2 & 0 \\ -x_4 & 0 \end{pmatrix}.$$

We have used the notation $\mathsf{pm} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. Using the equality (3.4), (8.3) we have:

$$\begin{aligned} dx_1 \wedge dx_3 \wedge dx_2 &= A_{11} \wedge A_{12} \wedge (x_1 A_3 + x_2 A_{22}) \\ &= \frac{1}{\Delta^3}(-\frac{1}{12} d\Delta) \wedge (\frac{3}{2}\alpha) \wedge (x_1 \Delta dt_1) \\ &= \frac{3x_1}{4\Delta} dt_1 \wedge dt_2 \wedge dt_3, \end{aligned}$$

where $A = [A_{ij}]$ is the Gauss-Manin connection in the basis in Proposition 3.1. Using (8.2) we conclude that $\mathsf{pm}$ is a local biholomorphism. □

**Exercise.** (1) For $\Gamma = \Gamma_0(N), \ \Gamma_1(N), \ \Gamma(N)$ show that the cardinality of $\Gamma \backslash \mathrm{SL}(2, \mathbb{Z})$ is the number of enhanced elliptic curves for $\Gamma$ with $(E, \omega)$ fixed.

(2) Show that the period map $\mathsf{pm}$ is well-defined.

(3) For $A \in \Gamma \backslash \mathrm{SL}(2, \mathbb{Z})$ we have the well-defined map $F_A : \Gamma \backslash \mathcal{P} \to \Gamma \backslash \mathcal{P}, \ x \mapsto Ax$. A different choice of $\delta_1^0, \delta_2^0$ in the definition of the period map leads to the composition $\mathsf{pm} \circ F_A$.

(4) Proposition 8.2 for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ implies the same proposition for arbitrary $\Gamma$.

## 8.5. **Inverse of the period map**

In this section we consider the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Let

$$g = (g_1, g_2, g_3) : \mathbb{H} \to T$$

be the composition $\mathbb{H} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P} \stackrel{\mathsf{pm}^{-1}}{\to} T$. Here, $\mathsf{pm}^{-1}$ is the local inverse of the period map, however, since $\mathbb{H}$ is simply connected, $g$ is a well-defined one valued holomorphic function on $\mathbb{H}$. For a moment, we assume that the period map is a global biholomorphism. From Proposition 8.2 part 2 it follows that $g_i$'s satisfy

$$(cz+d)^{-2i} g_i\left(\frac{az+b}{cz+d}\right) = g_i(z), \ i = 2, 3, \tag{8.5}$$

$$(cz+d)^{-2} g_1\left(\frac{az+b}{cz+d}\right) = g_1(z) + c(cz+d)^{-1}, \ z \in \mathbb{H}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

From Proposition 8.2 part 3 it follows also that $g$ is a solution of the vector field R, that is,

$$\frac{dg_1}{dz} = g_1^2 - \frac{1}{12} g_2, \ \frac{dg_2}{dz} = 4g_1g_2 - 6g_3, \ \frac{dg_3}{dz} = 6g_1g_3 - \frac{1}{3} g_2^2 \tag{8.6}$$

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, the functions $g_i$ are invariant under $z \mapsto z + 1$, and so, they can be written in terms of the new variable $q = e^{2\pi i z}$. Later, we will prove that $g_i$'s have a finite growth at infinity and hence as functions in $q$ are holomorphic at $q = 0$.

## 8.6. **Hypergeometric functions**

Let us consider the following one parameter family of elliptic curves

$$E_\psi : y^2 - 4x^3 + 12x - 4\psi = 0$$

and the cycles $\delta_1, \delta_2 \in H_1(E_\psi, \mathbb{Z})$ described in §8.2: for $\psi$ a real number between $-2$ and 2, $\delta_2$ is the closed curve inside $E_\psi \cap \mathbb{R}^2$ which encircles $(-1, 0)$ and $\delta_1 \in H_1(E_\psi, \mathbb{Z})$ vanishes on the nodal point $(1, 0)$. Whenever we need to emphasize that $\delta_i, i = 1, 2$ depends on $\psi$ we write $\delta_i = \delta_i(\psi)$. The cycles $\delta_i, \ i = 1, 2$ form a basis of $H_1(E_\psi, \mathbb{Z})$ and it follows from Proposition 3.1 and the equality (3.4) that the matrix

$$Y = \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_2} \frac{dx}{y} \\ \int_{\delta_1} \frac{xdx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}$$

forms a fundamental system of the linear differential equation:

$$Y' = \frac{1}{\psi^2-4}\begin{pmatrix} \frac{-1}{6}\psi & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{6}\psi \end{pmatrix} Y, \tag{8.7}$$

that is, any solution of (8.7) is a linear combination of the columns of $Y$. This example shows a little bit the historical aspects of the Gauss-Manin connection. From (8.7) it follows that the elliptic integral $\int_{\delta_2}\frac{dx}{y}$ (resp. $\int_{\delta_2}\frac{xdx}{y}$) satisfies the differential equation

$$\frac{5}{36}I+2\psi I'+(\psi^2-4)I''=0 \qquad (\text{ resp. } \frac{-7}{36}I+2\psi I'+(\psi^2-4)I''=0) \tag{8.8}$$

which is called a Picard-Fuchs equation. We make a linear transformation

$$\tau = \frac{\psi+2}{4}$$

which sends the singularities $\psi=-2,2$ of (8.7) to $\tau=0,1$. We write (8.7) in the variable $\tau$. The integrals $\int_{\delta_2}\frac{dx}{y}$ and $\int_{\delta_2}\frac{xdx}{y}$ are holomorphic around $\tau=0$. We write $X := [\int_{\delta_2}\frac{dx}{y}, \int_{\delta_2}\frac{xdx}{y}]^{\mathsf{tr}}$ as a formal power series in $\tau$ $X=\sum_{i=0}^{\infty} Y_i\tau^i$, substitute it in (8.7) and obtain a recursive formula for $Y_i$'s. We also obtain $Y_0=[a_0,-a_0]^{\mathsf{tr}}$, where $a_0$ is the value of $\int_{\delta_2}\frac{dx}{y}$ at $\psi=-2$. This must be calculated separately. The intersection of the elliptic curve $E_\psi, -2<\psi<2$ with the real plane $\mathbb{R}^2$ has two connected component, one of them is $\delta_2$ and the other $\tilde{\delta}_2$ is a closed path in $E_\psi$ which crosses the point at infinity $[0;1;0]$. It turns out that if we give the clockwise orientation to $\tilde{\delta}_2$ then it is homotopic to $\delta_2$ in $E_\psi$ and

$$a_0 = \int_{\tilde{\delta}_2}\frac{dx}{y}\Big|_{\psi=-2} = 2\int_2^\infty \frac{dx}{2(x+1)\sqrt{x-2}} = \frac{2\mathrm{tang}^{-1}(\frac{\sqrt{x-2}}{\sqrt{3}})}{\sqrt{3}}\Bigg|_2^\infty = \frac{\pi}{\sqrt{3}}.$$

Note that for $\psi$ a real number near $-2$, by Stokes formula we have $\int_{\delta_2}\frac{dx}{y} = \int_{\Delta_2}\frac{dx\wedge dy}{y^2}>0$, where $\Delta_2$ is the region in $\mathbb{R}^2$ bounded by $\delta_2$, and so we already knew that $a_0\geq 0$. This explain the fact that why $\delta_2$ is homotopic to clockwise oriented $\tilde{\delta}_2$. The result of all these calculations is:

$$\int_{\delta_2}\frac{dx}{y} = \frac{\pi}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|\frac{\psi+2}{4}), \tag{8.9}$$

$$\int_{\delta_2}\frac{xdx}{y} = -\frac{\pi}{\sqrt{3}}F(\frac{-1}{6},\frac{7}{6},1|\frac{\psi+2}{4}),$$

where

$$F(a,b,c|z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,\ \ c\notin\{0,-1,-2,-3,\ldots\},\qquad (8.10)$$

is the Gauss hypergeometric function and $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$.

Let us now calculate the integrals $\int_{\delta_1}\frac{x^i dx}{y},\ i=0,1$. We have the isomorphism $E_{-\psi}\to E_{\psi},\ (x,y)\mapsto(-x,iy)$ which sends the cycle $\delta_2(-\psi)$ to $\delta_1(\psi)$ and $\delta_1(-\psi)$ to $-\delta_2(\psi)$. This gives us the equalities:

$$\int_{\delta_1(\psi)}\frac{x^j dx}{y}=(-1)^j i\int_{\delta_2(-\psi)}\frac{x^j dx}{y}$$

Finally, we have calculated all the entries of $Y$:

$$Y=\begin{pmatrix}\frac{\pi i}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|\frac{-\psi+2}{4}) & \frac{\pi}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|\frac{\psi+2}{4})\\ \frac{\pi i}{\sqrt{3}}F(\frac{-1}{6},\frac{7}{6},1|\frac{-\psi+2}{4}) & -\frac{\pi}{\sqrt{3}}F(\frac{-1}{6},\frac{7}{6},1|\frac{\psi+2}{4})\end{pmatrix}$$

The monodromy around $\tau=0$ leaves $\delta_2$ invariant and takes $\delta_1$ to $\delta_1+\delta_2$. From this it follows that for a fixed complex number $a$:

$$\int_{\delta_1}\frac{dx}{y}=\frac{\ln(a\tau)}{2\pi i}(\int_{\delta_2}\frac{dx}{y})+\frac{1}{2i\sqrt{3}}f(\tau)=\frac{1}{2i\sqrt{3}}(F(\frac{1}{6},\frac{5}{6},1|\tau)\ln(a\tau)+f(\tau)),\qquad (8.11)$$

where $f$ is a one valued function in a neighborhood of $\tau=0$. From Exercises 8.6, 4 it follows that $f$ is holomorphic at $\tau=0$. We choose $a$ in such a way that the value of $f$ at $\tau=0$ is 0. This is equivalent to the following formula for $a$:

$$a=\exp(2\pi i(\lim_{\tau\to 0}\int_{\delta_1}\frac{dx}{y}-\frac{\ln\tau}{2\pi i}\int_{\delta_2}\frac{dx}{y})).$$

According to Exercise 8.6, 4 we have

$$a=\frac{1}{432}.$$

We write $f=\sum_{i=1}^{\infty}f_n\tau^n$ and substitute (8.11) in the Picard-Fuchs equation (8.9) and we obtain the following recursion for $f_n$'s: $f_0=0$,

$$f_{n+1}=\frac{(n-\frac{1}{6})(n-\frac{5}{6})}{(n+1)^2}f_n+\frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(n!)^2}\frac{2n+1}{(n+1)^2}-\frac{2}{n+1}\frac{(\frac{1}{6})_{n+1}(\frac{5}{6})_{n+1}}{((n+1)!)^2}.$$

We will need the value $f_1=\frac{13}{18}$.

**Exercise.** (1) Deduce (8.8) from (8.7).

(2) The integrals $\int_{\delta_2} \frac{dx}{y}$ and $\int_{\delta_2} \frac{xdx}{y}$ are holomorphic at $\tau = 0$.

(3) Do the details of the calculations which lead to the equalities (8.9).

(4) Prove

$$\lim_{\tau \to 0} \quad \frac{\pi i}{\sqrt{3}} F(\frac{1}{6}, \frac{5}{6}, 1|1-\tau) - \frac{\pi}{\sqrt{3}} F(\frac{1}{6}, \frac{5}{6}, 1|\tau) \frac{\ln \tau}{2\pi i} = \frac{\ln(432)}{2\pi i}.$$

## 8.7. **Periods and Ramanujan**

In this section consider the full modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and the corresponding period map. We are interested in the image $L$ of the map $g$ constructed in §8.5. This is the locus $L$ of parameters $t \in T$ such that:

$$\int_{\delta_1} \frac{xdx}{y} = -\sqrt{-2\pi i}, \ \int_{\delta_2} \frac{xdx}{y} = 0, \quad \text{for some } \delta_1, \delta_2 \in H_1(E_t, \mathbb{Z}) \tag{8.12}$$

with $\langle \delta_1, \delta_2 \rangle = -1$. Using Proposition 8.2, part 2 and and the equality (7.6), we know that the locus of such parameters is given by:

$$I = (I_1, I_2, I_3) := (t_1, t_2, t_3) \bullet \begin{pmatrix} (\frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \frac{dx}{y})^{-1} & -\frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \frac{xdx}{y} \\ 0 & \frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \frac{dx}{y} \end{pmatrix} =$$

$$\Big( -t_1 (2\pi i)^{-1} (\int_{\delta_2} \frac{dx}{y})^2 + (2\pi i)^{-1} \int_{\delta_2} \frac{xdx}{y} \int_{\delta_2} \frac{dx}{y}, \ t_2 \cdot (2\pi i)^{-2} (\int_{\delta_2} \frac{dx}{y})^4, \\ - t_3 (2\pi i)^{-3} (\int_{\delta_2} \frac{dx}{y})^6 \Big)$$

The mentioned locus is one dimensional and the above parametrization is by using three parameters $t_1, t_2, t_3$. We may restrict it to a one dimensional subspace $t = (0, 12, -4\psi)$ as in §8.6, use the formulas of elliptic integrals in terms of hypergeometric functions (8.9) and obtain the following parametrization of $L$:

$$I = \left(a_1F(-\frac{1}{6},\frac{7}{6},1 \mid \tau)F(\frac{1}{6},\frac{5}{6},1 \mid \tau), a_2F(\frac{1}{6},\frac{5}{6},1 \mid \tau)^4,\right.$$
$$\left. a_3(1-2\tau)F(\frac{1}{6},\frac{5}{6},1 \mid \tau)^6\right)$$

where

$$(a_1,a_2,a_3)=(\frac{2\pi i}{12},12(\frac{2\pi i}{12})^2,8(\frac{2\pi i}{12})^3).$$

From $\nabla_{\mathrm{R}}\frac{xdx}{y}=0$ and (8.12) it follows that $L$ is tangent to the vector field R. In other words, it is a leaf of the foliation induced by R. Since the period map sends R to $X$, and the canonical map $\mathbb{H}\to\mathcal{P}$ sends $\frac{\partial}{\partial z}$ to $X$, we conclude that $I_i$'s can be written in terms of the new variable

$$z=\frac{\int_{\delta_1}\frac{dx}{y}}{\int_{\delta_2}\frac{dx}{y}}=i\frac{F(\frac{1}{6},\frac{5}{6},1|1-\tau)}{F(\frac{1}{6},\frac{5}{6},1|\tau)}$$

that is

$$(I_1,I_2,I_3)=(g_1(z),g_2(z),g_3(z)),$$

where $(g_1,g_2,g_3):\mathbb{H}\to\mathbb{C}^3$ is given in §8.5. Let us define

$$E_{2i}(z)=a_i^{-1}g_i(z),\ \ i=1,2,3.$$

We get the equalities:

$$F(-\frac{1}{6},\frac{7}{6},1 \mid \tau)F(\frac{1}{6},\frac{5}{6},1 \mid \tau)=E_2(i\frac{F(\frac{1}{6},\frac{5}{6},1|1-\tau)}{F(\frac{1}{6},\frac{5}{6},1|\tau)}), \tag{8.13}$$

$$F(\frac{1}{6},\frac{5}{6},1 \mid \tau)^4=E_4(i\frac{F(\frac{1}{6},\frac{5}{6},1|1-\tau)}{F(\frac{1}{6},\frac{5}{6},1|\tau)}),$$

$$(1-2\tau)F(\frac{1}{6},\frac{5}{6},1 \mid \tau)^6=E_6(i\frac{F(\frac{1}{6},\frac{5}{6},1|1-\tau)}{F(\frac{1}{6},\frac{5}{6},1|\tau)}).$$

## 8.8. **Torelli problem**

In Hodge theory the global injectivity of the period map is known as global Torelli problem. As the reader may have noticed we need only the local injectivity of the period map in order to extract quasi-modular forms from the inverse of the period map.

The period map $\mathsf{pm}$ in §8.4 is a global biholomorphism if and only if the induced map $p : T/G \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P}/G \cong \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is a biholomorphism. The last statement follows from Weierstrass uniformization theorem. We give another proof based on a $q$-expansion argument. The quotient $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ has a canonical structure of a Riemann surface such that the map $p$ is a local biholomorphism. Let $U$ be a subset of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ containing all $z$ with $\Im(z) > 1$. The map

$$U \to D(0, e^{-2\pi}),\ z \mapsto q = e^{2\pi i z},$$

where $D(0,r)$ is a disk in $\mathbb{C}$ with center 0 and radius $r$, is a coordinate system around each point of $U$. Using this map $\bar{S} := \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \cup \{\infty\}$ becomes a compact Riemann surface, where the value of the above coordinate at $\infty$ is $q = 0$. From another side $T/G$ admits also the canonical compactification $\overline{T/G} := \mathbb{C}^3/G$ which is obtained by adding the single point $p = \{\Delta = 0\}/G$ to $T/G$. A coordinate system around $p$ for $\overline{T/G}$ is given by $(\mathbb{C},0) \to \overline{T/G},\ \tau \mapsto (0, 12, -4(4\tau - 2))$ (recall the one parameter family of elliptic curves in §8.6). The map $p$ written in these coordinates is

$$\tau \mapsto q = e^{2\pi i \frac{\int_{\delta_1} \frac{dx}{y}}{\int_{\delta_2} \frac{dx}{y}}} = \frac{1}{432}\tau e^{\frac{f(\tau)}{F(\frac{1}{6},\frac{5}{6},1|\tau)}} = \frac{1}{432}\tau e^{\frac{\frac{13}{18}\tau + \cdots}{1+\frac{5}{36}\tau+\cdots}} \tag{8.14}$$

This is an invertible map at $\tau = 0$. This implies that $p$ extends to a local biholomorphism $\overline{T/G} \to \bar{S}$ without critical points. Since both the image and domain of this map are compact Riemann surfaces of genus zero, we conclude that $p$ is a global biholomorphism.

## 8.9. $q$-**expansion**

The subgroup of $\mathrm{SL}(2,\mathbb{Z})$ which leaves the set $\{\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \mid z \in \mathbb{H}\}$ invariant is generated by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, therefore, the variable $q$ gives us a biholomorphism between the image $\tilde{\mathbb{H}}$ of $\mathbb{H}$ in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P}$ and the punctured disc of radius one. In other words, $q$ is a global coordinate system on $\tilde{\mathbb{H}}$. Therefore, we can write $g_i$'s of §8.7 in terms of $q$:

$$g_i := I_i(p^{-1}(q)).$$

where $p : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is the map given by (8.14). It follows that $g_i$ as a function in $q$ is one valued and holomorphic in the disc of radius one and center 0. If we write $g_i$ as a formal Laurent series in $q$, and substitute in (8.13), then we get a recursion for the coefficients of $g_i$'s. There is a better way to calculate such formal power series. We calculate the first two coefficients of $g_1$ as above:

$$E_2 = 1 - 24.q + \cdots$$

The functions $g_i = a_i E_i$'s as formal power series satisfy the Ramanujan differential equation (4.4) and so according to discussion in §4.3, we can calculate all the coefficients of $E_i$ knowing the initial values 1 and $-24$ as above and the recursion given by (4.4).

## 8.10. **Schwarz map**

The multivalued function

$$p : \mathbb{C} - \{0, 1\} \to \mathbb{H}, \ \tau \mapsto \frac{\int_{\delta_1} \frac{dx}{y}}{\int_{\delta_2} \frac{dx}{y}} = i \frac{F(\frac{1}{6}, \frac{5}{6}, 1|1-\tau)}{F(\frac{1}{6}, \frac{5}{6}, 1|\tau)}$$

is called the Schwarz map. We summarize its global behavior in the following proposition:

**Proposition 8.3.** *Let*

$$U := \{z \in \mathbb{C} \mid \Re(z) < \frac{1}{2}\}\backslash\{z \in \mathbb{R} \mid z \leq 0\}.$$

*and consider the branch of the Schwarz map in $U$ which has pure imaginary values in $0 < \tau < \frac{1}{2}$. Its image is the interior of the classical fundamental domain of the action of $\mathrm{SL}(2,\mathbb{Z})$ in $\mathbb{H}$ depicted in Picture (8.2). Its analytic continuation result in the triangulation of $\mathbb{H}$ as in Picture (8.2).*

Basic ingredients of the proof are the global injectivity of the period map discussed in §8.8 and the following exercise:

**Exercise.** *Let $p$ be the branch of the Schwarz map described in Proposition 8.3. Prove the following:*

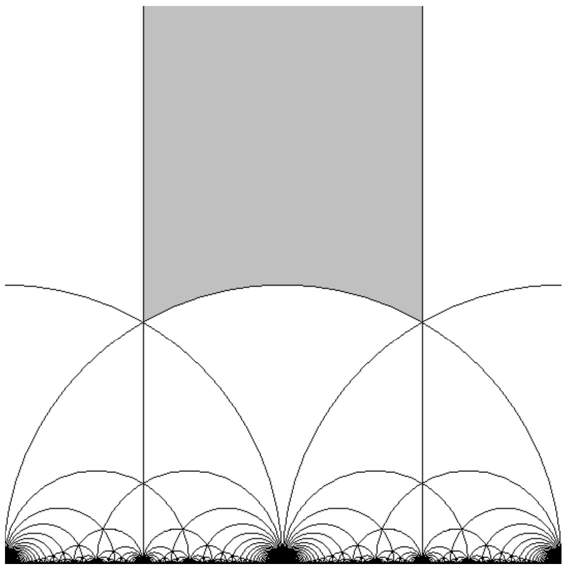

Figure 8.2. Fundamental domain

(1)
$$\lim_{\tau\in\mathbb{R},\ \tau\to 0^+} p(\tau) = +\infty.$$

(2)
$$|p(\frac{1}{2} + ix)| = 1,\ \ x \in \mathbb{R}.$$

(3)
$$\lim_{x\in\mathbb{R},\ x\to\pm\infty} p(\frac{1}{2} + ix) = \pm\frac{1}{2} + \frac{\sqrt{3}}{2}.$$

(4) *The analytic continuation of $p$ from the upper half (resp. lower half) of $\mathbb{C}$ to $\mathbb{R}^-$ has the constant real part $\frac{1}{2}$ (resp. $-\frac{1}{2}$).*

## 8.11. **Comparison theorem**

Now, we are in a position to prove that the algebraic and analytic notions of quasi-modular forms are equivalent.

**Theorem 8.4.** *The differential graded algebra of quasi-modular forms in the Poincaré upper half plane together with the differential operator* $\frac{d}{dz}$ *is isomorphic to the graded differential algebra of quasi-modular forms defined in §6 together with the differential operator* $\mathrm{R}_\Gamma$.

*Proof.* According to Proposition 7.1, quasi-modular forms can be viewed as functions on $\Gamma\backslash\mathcal{P}$. Now, the period map which is a biholomorphism gives us the desired isomorphism of algebras.

□

## Appendix A. Quasi-modular forms as sections of jet bundles

In this appendix we explain the geometric interpretation of quasi-modular forms in terms of sections of jet bundles of tensor powers of line bundles on the moduli of elliptic curves. The main ingredients of this appendix are taken from a private communication with Prof. P. Deligne. We leave to the reader the comparison of the material of this appendix with Lee's article [21] and in particular the notion of quasi-modular polynomial. For simplicity, we work with elliptic curves over $\mathbb{C}$.

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ be one of the modular groups $\Gamma_0(N), \Gamma(N), \Gamma_0(N)$. As we noticed $M = \Gamma\backslash\mathbb{H}$ is the moduli space of elliptic curves over $\mathbb{C}$ with a certain torsion structure. Let $G$ be any algebraic group, for instance take $G$ the multiplicative group $(\mathbb{C}^*, \cdot)$ or the group (6.1) for $\mathsf{k} = \mathbb{C}$. We usually identify $G$ with a linear subgroup of $\mathrm{GL}(n,\mathbb{C})$, for some $n \in \mathbb{N}$, and hence assume that $G$ acts on $\mathbb{C}^n$. A $G$-automorphy factor on $\mathbb{H}$ is a map:

$$j : \Gamma \times \mathbb{H} \to G$$

which satisfies

$$j(AB, z) = j(A, Bz)j(B, z),\ \ A, B \in \Gamma,\ \ z \in \mathbb{H}.$$

Any $G$-automorphy factor gives us a $G$-vector bundle on $M$ and vice verse: the quotient $\mathbb{H} \times \mathbb{C}^n / \sim$, where

$$(z, v) \sim (Az, j(A, z)v),\ \ \forall z \in \mathbb{H},\ \ A \in \Gamma,\ \ v \in \mathbb{C}^n,$$

gives us a $G$-bundle in $M$. For $G = (\mathbb{C}^*, \cdot)$ we get line bundles on $M$. Holomorphic functions $f : \mathbb{H} \to \mathbb{C}^n$ with the functional equation

$$f(Az) = j(A, z)f(z),\ \ A \in \Gamma,\ \ z \in \mathbb{H}$$

are in one to one correspondence with holomorphic sections of the $G$-bundle associated to the automorphy factor $j$.

For $z \in \mathbb{H}$ we have the elliptic curve $\mathbb{C}/\langle 1, z\rangle$ and the one dimensional vector space $\mathbb{C}d\tau$, where $\tau$ is the canonical coordinate on $\mathbb{C}$. This can be also identified with $\mathrm{Lie}(E)^\vee$, the linear dual of the Lie algebra of $E$, that is, the linear dual of the tangent space of $E$ at $0 \in E$. This gives us a line bundle, say it $\omega$. The corresponding automorphy factor is given by

$$j(A, z) = (cz + d),\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ z \in \mathbb{H}. \tag{A.1}$$

(Exercise A,1). We have also the canonical line bundle $\Omega$ of $M$ (dual of the tangent bundle of $M$) which is given by the automorphy factor $j(A, z)^2$. It follows that

$$\Omega = \omega \otimes \omega. \tag{A.2}$$

For a vector bundle $F \to M$ over a complex manifold $M$, the $n$-th jet bundle $J_nF$ of $F$ is defined as follows: the fiber $J_nF_x$ of $J_nF$ at $x \in M$ is defined to be the set of sections of $F$ in a neighborhood of $x$ modulo those which vanish at $x$ of order $n$. In other words, $J_nF_x$ is the set of Taylor series at $x$ of the sections of $F$ up to order $n$. A section of $J_nF$ in a small open set $U \subset M$ with a coordinate system $(z_1, z_2, \ldots, z_m)$ is a sum

$$\sum_{0\le k_1,k_2,\ldots,k_m\le n} f_{k_1,k_2,\ldots,k_m}(z)(w_1 - z_1)^{k_1}(w_2 - z_2)^{k_2} \cdots (w_m - z_m)^{k_m},$$

where $f_{k_1,k_2,\ldots,k_m}(z)$ are holomorphic sections of $F$ in $U$ and $w$ is an extra multi variable.

Let $M := \Gamma\backslash\mathbb{H}$ and $\omega^m := \omega \otimes \omega \otimes \cdots \otimes$, $m$ times, be as before. A section of $J_n\omega^m$ corresponds to a sum

$$F(z, w) = \sum_{i=0}^{n} \frac{f_i(z)}{i!}(w - z)^i,\ z \in \mathbb{H},\ w \in \mathbb{C},$$

where $f_i$'s are holomorphic functions on $\mathbb{H}$ such that

$$F(Az, Aw) = F(z, w)j(A, w)^m + O((Aw - Az)^{n+1}).$$

We perform $i$-times the derivation $\frac{\partial}{\partial(Aw)} = (cw + d)^2\frac{\partial}{\partial w}$ in both sides of the above equality and then put $w = z$. We get functional equations for $f_i$'s:

$$f_i(Az) = ((cz + d)^2\frac{\partial}{\partial z})^{(i)}(f(z)j(A, z)^m).$$

Here $f = f_0$ and we redefine the derivation of $f_i$'s: $\frac{\partial f_i}{\partial z} := f_{i+1}$ (the derivation in other terms is the usual one). Neglecting the growth condition for $f_i$'s we conclude that $f_i$ is a quasi-modular form of weight $m + 2i$ and differential order $i$.

We discuss briefly the growth condition. From $j(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z) = 1$ it follows that the line bundles $\omega$ and $\Omega$ have canonical extensions $\bar{\omega}$ and $\bar{\Omega}$ to $\bar{M} = \Gamma\backslash(\mathbb{H} \cup \mathbb{Q})$. Sections of $\bar{\omega}^m$ are in one to one correspondence with modular forms of weight $m$. We end this section with the following affirmation: the following map is a bijection

$$\begin{aligned} \text{global sections of } J_n\bar{\omega}^{m-2n} &\to M_m^n, \\ \sum_{i=0}^{n} \frac{f_i(z)}{i!}(w-z)^i &\mapsto f_n. \end{aligned} \tag{A.3}$$

**Exercise.** (1) *The automorphy factor associated to $\omega$ and $\Omega$ are respectively $j$ and $j^2$, where $j$ is given by (A.1).*

(2) *The cohomology bundle $H$ on $\Gamma\backslash\mathbb{H}$ associates to each point $z \in \Gamma\backslash\mathbb{H}$ the two dimensional vector space $H^1(\mathbb{C}/\langle 1, z\rangle, \mathbb{C})$. We have a canonical inclusion $\omega \subset H$ and an isomorphism of bundles $H/\omega \cong \omega^{-1}$.*

(3) *Calculate the automorphy factor of $J_1\omega^{-1}$ and conclude that*

$$J_1\omega^{-1} = H.$$

(4) *Prove that the Gauss-Manin connection induces an isomorphism $\omega \tilde{\to} \Omega \otimes H/\omega$.*

(5) *Prove the bijection (A).*

## Appendix B. Examples of quasi-modular forms as generating functions

The field $\mathbb{Q}(E_2, E_4, E_6)$ generated by three Eisenstein series

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \left( \sum_{d|n} d^{2k-1} \right) q^n, \quad k = 1, 2, 3, \tag{B.1}$$

where $B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ \ldots$ are Bernoulli numbers, and its algebraic closure contain many interesting generating functions. We list some of them without proofs. It is convenient to use the weights

$$\text{weight}(E_{2k}) = 2k,\ k = 1, 2, 3$$

for the ring $\mathbb{Q}[E_2, E_4, E_6]$.

## B.1. **Ramified elliptic curves**

Let $E$ be a complex elliptic curve and let $p_1, \ldots, p_{2g-2}$ be distinct points of $E$, where $g > 1$. We will discuss the case $g = 1$ separately. The set $X_g(d)$ of equivalence classes of holomorphic maps $\phi : C \to E$ of degree $d$ from compact connected smooth complex curves $C$ to $E$, which have only one double ramification point over each point $p_i \in E$ and no other ramification points, is finite. By the Hurwitz formula the genus of $C$ is equal to $g$. Define

$$N_{g,d} := \sum_{[\phi] \in X_g(d)} \frac{1}{|\text{ Aut } (\phi)\ |} \tag{B.2}$$

and

$$F_g := \sum_{d=1}^{\infty} N_{g,d} q^d.$$

After R. Dijkgraaf, M. Douglas, D. Zagier, M. Kaneko, see [6, 17], we know that

$$F_g \in \mathbb{Q}[E_2, E_4, E_6],$$

For instance,

$$F_2(q) = \frac{1}{103680}(10E_2^3 - 6E_2E_4 - 4E_6),$$

$$F_3(q) = \frac{1}{35831808}(-6E_2^6 + 15E_2^4E_4 - 12E_2^2E_4^3 + 7E_4^3 + 4E_2^3E_6 - 12E_2E_4E_6 + 4E_6^2).$$

For $g = 1$ we do note have ramification points and for $\phi : C \to E$ as before, $\mathrm{A}ut(\phi)$ consists of translations by elements of $\phi^{-1}(0)$ and so $\#\mathrm{A}ut(\phi)) = d$. Therefore, $d \cdot N_{d,1} = \sum_{i|d} i$ is the number of group plus Riemann surface morphisms $C \to E$ of degree $d$. In this case we have the contribution of constant maps which is given by $N_{1,0}\log q = -\frac{1}{24}\log q$. Therefore,

$$q\frac{\partial F_1}{\partial q} = -\frac{1}{24}E_2.$$

**Exercise.** Calculate $N_{2,2}$ and $N_{3,3}$ from the formulas for $F_2(q)$ and $F_3(q)$ respectively and prove that in fact they satisfy (B.2).

## B.2. Elliptic curves over finite field

For an elliptic curve $E$ over a finite field $\mathbb{F}_q$, $q = p^n$ and $p$ a prime number, a theorem of Hasse tells us that there is an algebraic integer $\alpha \in \bar{\mathbb{Q}}$ with $|\alpha| = \sqrt{q}$ and such that

$$\#E(F_q) = q + 1 - (\alpha + \bar{\alpha}).$$

The expression

$$\sigma_k(q) = -\sum_{E/\mathbb{F}_q} \frac{(\alpha^{k+1} - \bar{\alpha}^{k+1})/(\alpha - \bar{\alpha})}{\#Aut_{\mathbb{F}_q}(E)}$$

can be considered as average of the quantities $(\alpha^{k+1} - \bar{\alpha}^{k+1})/(\alpha - \bar{\alpha})$ for all elliptic curves over $\mathbb{F}_q$. Here, $Aut_{\mathbb{F}_q}(E)$ is the group of $\mathbb{F}_q$-automorphisms of E. Let

$$\begin{aligned}\frac{1}{1728}(E_4^3 - E_6^2) &= q\prod_{n=1}^{\infty}(1 - q^n)^{24}\\ &= (q - 24q^2 + 253q^3 - 3520q^4 + 4830q^5 + \cdots + \tau(n)q^n + \cdots).\end{aligned}$$

It turns out that

$$\sigma_{10}(p) = \tau(p), \ \forall p \text{ prime}.$$

See the article of van der Geer [9] for more history behind this phenomenon.

## B.3. Monstrous moonshine conjecture

We write the $q$ expansion of the $j$-function

$$j = 1728\frac{E_4^3}{E_4^3 - E_6^2} =$$

$$q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots.$$

In 1978 MacKay noticed that $196884 = 196883 + 1$ and 196883 is the number of dimensions in which the Monster group can be most simply represented. Based on this observation J.H. Conway and S.P. Norton in 1979 formulated the Monstrous moonshine conjecture which relates all the coefficients in the $j$-function to the representation dimensions of the Monster group. In 1992 R. Borcherds solved this conjecture and got fields medal. See [8] for more information on this conjecture.

## B.4. Modularity theorem

Let $E$ be an elliptic curve defined by the equation $g(x, y) = 0$, where $g$ is a polynomial with integer coefficients and with the discriminant $\Delta \neq 0$. For instance, take $a_2$ and $a_3$ integers with $\Delta := a_2^3 - 27a_3^2 \neq 0$ and let $E : y^2 = 4x^3 - a_2x - a_3$. Let also $p$ be a prime, $N_p$ be the number of solutions of $E$ working modulo $p$ and $a_p(E) := p - N_p$ (we have not counted the point at infinity $[0; 1; 0]$). A version of modularity theorem says that there is an element $f = \sum_{n=0}^{\infty} a_n q^n$ in the algebraic closure of $\mathbb{Q}(E_4, E_6)$ such that $a_p = a_p(E)$ for all primes $p \not| \Delta$. In fact $f$ is a cusp form of weight 2 associated to some $\Gamma_0(N)$. Here, $N$ is the conductor of $E$. This was originally known as Taniyama-Shimura conjecture and it is solved by A. Wiles, R. Taylor, C. Breuil, B. Conrad, F.Diamond. For further information see [5]. As an example consider $E : y^2 + y = x^3 - x^2$. This has conductor $N = 11$. The corresponding modular form is

$$\eta(q)^2\eta(q^{11})^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 \\ - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + \cdots ,$$

where $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$ is the Dedekind eta function. For further examples see [16].

## B.5. Rational curves on $K3$ surfaces

A $K3$ surface by definition is a simply connected complex surface with trivial canonical bundle. Projective $K3$ surfaces fall into countable many families $\mathcal{F}_k,\ k \in \mathbb{N}$. A surface in $\mathcal{F}_k$ admits a $k$-dimensional linear system $|L|$ of curves of genus $k$. A curve $C$ in $|L|$ depends on $k$-parameters and so if we put $k$ conditions on that curve, we would get an isolated curve and so we can count the number of such curves. For instance take $k = n+g,\ n, g \in \mathbb{N}$

and assume that $C$ passes through $g$ generic fixed points and it is singular with $n$ nodal singularities (and hence the geometric genus of $C$ is $g$). In fact for a generic $K3$ surface the number $N_n(g)$ of such curves turns out to be finite. The generating function for the numbers $N_n(0)$, that is the number of rational curves in the linear system $|L|$, was first discovered by Yau-Zaslow (1996), Beauville (1999) and Göttsche (1994):

$$\sum_{n=0}^{\infty} N_n(0)q^n = \frac{1728q}{E_4^3 - E_6^2} = 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + 176256q^5 + 1073720q^6 + \cdots$$

(by definition $N_0(0) = 1$). For instance, a smooth quadric $X$ in $\mathbb{P}^3$ is $K3$ and for such a generic $X$ the number of planes tangent to $X$ in three points is 3200.

For arbitrary genus $g$ we have the following generalization of Bryan-Leung (1999):

$$\sum_{n=0}^{\infty} N_n(g)q^n = (\frac{-1}{24}\frac{\partial E_2}{\partial q})^g \frac{1728q}{E_4^3 - E_6^2}.$$

Some first coefficients for $g = 1$ and $g = 2$ are given respectively by

$$1 + 30q + 480q^2 + 5460q^3 + \cdots,\ 1 + 36q + 672q^2 + 8728q^3 + \cdots .$$

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Hossein Movasati
Instituto de Matemática Pura e
Aplicada, IMPA
Estrada Dona Castorina, 110
22460-320, Rio de Janeiro, RJ
Brazil
hossein@impa.br