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Mutating seeds: types \mathbb{A} and $\widetilde{\mathbb{A}}$.

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Abstract

In the cases \mathbb{A} and $\widetilde{\mathbb{A}}$, we describe the seeds obtained by sequences of mutations from an initial seed. In the $\widetilde{\mathbb{A}}$ case, we deduce a linear representation of the group of mutations which contains as matrix entries all cluster variables obtained after an arbitrary sequence of mutations (this sequence is an element of the group). Nontransjective variables correspond to certain subgroups of finite index. A non-commutative rational series is constructed, which contains all this information.

1. Introduction

Our basic motivation in this paper arises from the theory of cluster algebras of Fomin and Zelevinsky [25]. We begin by recalling some definitions and results. Let Q_0 be a quiver with set of vertices $\{1, \dots, n\}$, and a variable x_i associated to each vertex i . This data is called the *initial seed*, denoted by S_0 . We consider the pairs $S = (Q, \{y_1, \dots, y_n\})$ where Q is a quiver with set of vertices $\{1, \dots, n\}$ and y_1, \dots, y_n generate the field $\mathbb{Q}(x_1, \dots, x_n)$ (in particular, they are algebraically independent over \mathbb{Q} , since the transcendence degree of $\mathbb{Q}(x_1, \dots, x_n)$ is n). A *seed* is such a pair which is obtained from the initial seed by a sequence of operations called *mutations*. If $S = (Q, \{y_1, \dots, y_n\})$ is a seed, then y_i is called the *i -th cluster variable of S* , or the *cluster variable at the vertex i* and is denoted by $y_i(S)$.

The mutation at the vertex i on a seed has the following property: it replaces the quiver Q by another quiver, with the same set of vertices, and with a new cluster variable y'_i at vertex i instead of y_i , the other variables being unchanged. Mutation at vertex i is involutive, which means that if one performs it twice, then one recovers the original seed. Denote by μ_i the mutation at vertex i .

Keywords: Cluster algebras, mutations, seeds, quivers.

Math. classification: 13F60, 16G20, 16G99.

Our first objective in this paper is to describe precisely the seeds in euclidean type $\tilde{\mathbb{A}}$. Note that the quivers of mutation type $\tilde{\mathbb{A}}$ are known, see [7] (for instance, Figure 2 in that article) or [3]. What we add to these descriptions is the explicit computation of the cluster variable associated to each vertex in one of these mutated quivers. We do this, on one hand, by embedding the *cyclic part* of the quiver into an SL_2 -tiling of the plane, which contains all transjective cluster variables (that is, those which correspond to indecomposable transjective objects of the associated cluster category, see [11]); and, on the other hand, by describing the remaining parts of the quiver in terms of continuant trees, which are tree-like quivers whose vertices are indexed by *signed continuant polynomials*, the latter give directly the nontransjective cluster variables. The signed continuant polynomials are a variant of the ordinary continuant polynomials, which go back to Euler (see [34] p. 133, [16] p. 116, [28] p. 302, [10] p. 186). They have been considered implicitly by Coxeter [19] Eq. (7.5), in [9], and explicitly by Grégoire Dupont in [21], where he uses them to study the nontransjective cluster variables (see also [23, 22]).

Returning to the cyclic part of a mutated quiver of type $\tilde{\mathbb{A}}$, the main tool allowing to compute the transjective cluster variables is the notion of *SL_2 -tiling* (see [6]), which is related to that of *friezes*. Friezes were introduced by Coxeter in [16] for type \mathbb{A} and behave like mutations on sinks and sources of the given quiver, hence their use in computing the corresponding cluster variables. Friezes gave rise to *SL_2 -tilings*, which yield the transjective variables in euclidean type (see [6]), and were further generalized to SL_k -tilings in [9]. Friezes were also used in [4] to find an algorithm for computing the nontransjective cluster variables in terms of the transjective ones, and in [5], to give an explicit expression for the cluster variables associated to the string modules over a cluster-tilted algebra (or, more generally, over a 2-Calabi-Yau tilted algebra), see also [21, 8, 27].

Our second objective is to explore the connection between mutated seeds of type $\tilde{\mathbb{A}}$ and representative functions. Recall that a *representative function* f on a group G is a function from G into some field K which is the composition of a linear representation $G \rightarrow K^{n \times n}$ followed by a linear form on $K^{n \times n}$. Equivalently, the set of translates $f.g$, $g \in G$ (with the natural right action of $G : (f.g)(g_1) = f(gg_1)$ for any g, g_1 in G), spans a finite dimensional subspace of the vector space of functions on G . See [30] I.1, [1, 20]. As an example, take the additive group $G = \mathbb{Z}$. A

representative function on this group is a sequence indexed by \mathbb{Z} which satisfies a linear recursion which works in both directions (for instance, the Fibonacci sequence extended to negative integers).

The interest of representative functions is illustrated for example by: the theorem of Peter-Weyl, which asserts that the representative functions on a compact group are dense in the space of continuous functions on this group (see [33], where representative functions are called *matrix elements*); their role in the theory of affine algebraic groups, see [1, 20]; the theorem of Kleene-Schützenberger which asserts that the representative functions on a free monoid coincide with noncommutative rational series, see [10]. Moreover, representative functions on an algebra (a generalization of those on groups) are the elements of the dual coalgebra of this algebra, or Sweedler dual, see [32, 1, 20].

Since we need a slightly more general notion of representative function (with values in a ring instead of a field), we have developed their theory in Section 3 below.

In the present paper, we consider the group \mathcal{M} generated by the set $\{\mu_1, \dots, \mu_n\}$ subject to the relation that the generators are involutive. We call this group the *group of mutations*. It acts naturally on seeds. If m is an element of \mathcal{M} , and S a seed, we denote by S^m the seed obtained by applying the sequence of mutations determined by m to the seed S .

Suppose that the initial quiver Q is of type $\tilde{\mathbb{A}}_{n-1}$ with an acyclic orientation (note that $\tilde{\mathbb{A}}_{n-1}$ has n vertices). Fix i . We shall show that the function from \mathcal{M} into $\mathbb{Q}(x_1, \dots, x_n)$ (actually its subring of Laurent polynomials) which associates to m the i -th cluster variable of S_0^m is a representative function of the mutation group. Moreover, we show that if y is a fixed nontransjective cluster variable, then the set of $m \in \mathcal{M}$ such that $y_1(S_0^m) = y$ is a finite union of cosets of a normal subgroup of finite index of the mutation group.

As a byproduct of the concept of continuant trees, which are shown to correspond to triangulations of a regular polygon, we obtain a description of the mutated seeds in type \mathbb{A} . This may be of some interest, since it presents some novelty and is completely elementary. The mutation formula turns out to be a consequence of a formula on continuant polynomials. Recall that the quivers of mutation type \mathbb{A} (that is mutation equivalent to an orientation of the Dynkin diagram of type \mathbb{A}) are known, see [13].

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2. Preliminaries

2.1. Signed continuant polynomials

The ordinary continuant polynomials are defined, for any elements a_1, \dots, a_n of a ring R by the recursion

$$p_n(a_1, \dots, a_n) = p_{n-1}(a_1, \dots, a_{n-1})a_n + p_{n-2}(a_1, \dots, a_{n-2}),$$

with initial conditions $p_{-1} := 0$ and $p_0 := 1$. The terminology comes from their link with continued fractions as shows the following identity, valid if R is commutative and if the inversions are defined in R :

$$\frac{p(a_1, \dots, a_n)}{p(a_2, \dots, a_n)} = a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}. \quad (2.1)$$

The *signed continuant polynomials* are a variant of the continuant polynomials. They are defined as follows. Let a_1, \dots, a_n be as above. Define for $n \geq 1$,

$$q_n(a_1, \dots, a_n) = q_{n-1}(a_1, \dots, a_{n-1})a_n - q_{n-2}(a_1, \dots, a_{n-2}), \quad (2.2)$$

setting $q_{-1} := 0$ and $q_0 := 1$. We omit indices when possible, writing simply $q(x_1, \dots, x_n)$ for $q_n(x_1, \dots, x_n)$. Let us now consider the particular SL_2 matrices

$$Q(a) := \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}.$$

One has the following result, see [9] 8.1.

Lemma 2.1.

$$Q(a_1)Q(a_2)\cdots Q(a_n) = \begin{pmatrix} -q(a_2, \dots, a_{n-1}), & -q(a_2, \dots, a_n) \\ q(a_1, \dots, a_{n-1}), & q(a_1, \dots, a_n) \end{pmatrix}. \quad (2.3)$$

It follows from this matrix equation that one has also

$$q(a_1, \dots, a_n) = a_1q(a_2, \dots, a_n) - q(a_3, \dots, a_n). \quad (2.4)$$

The q 's satisfy the following identity (a consequence of [19] Eq.(7.4)), which is a variant of Eq.(2.1) and which holds under the same assumptions:

$$\frac{q(a_1, \dots, a_n)}{q(a_2, \dots, a_n)} = a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_n}}}.$$

Indeed, this holds for $n = 1$. Assume that it holds for n . Then the continued fraction for $n + 1$ is equal by induction to

$$\begin{aligned} a_1 - \frac{1}{\frac{q(a_2, \dots, a_{n+1})}{q(a_3, \dots, a_{n+1})}} &= a_1 - \frac{q(a_3, \dots, a_{n+1})}{q(a_2, \dots, a_{n+1})} \\ &= \frac{a_1 q(a_2, \dots, a_{n+1}) - q(a_3, \dots, a_{n+1})}{q(a_2, \dots, a_{n+1})}. \end{aligned}$$

Thus the statement follows from Eq.(2.4).

It will be useful to adopt the following notation, which avoids the use of indices: let w be a finite sequence (a *word* on R) of elements of the ring R . For example, $w = a_1 \cdots a_n$ (not the product in the ring); then we write $q(w)$ for $q(a_1, \dots, a_n)$. If u, v are two such sequences, we denote by uv their concatenation. Then we have the following result, valid when R is *commutative*, which is assumed from now on.

Lemma 2.2. *Let u, v, w be words on R and a, b be in R . Then*

$$q(uav)q(vbw) = q(u)q(w) + q(uavbw)q(v).$$

In the case of ordinary continuant polynomials, there is an analogue of this identity, due to Euler and given in [28] Eq.(6.134) p.303.

Proof. We prove first this equation when $w = 1$ (*empty word*), that is $q(uav)q(vb) = q(u) + q(uavb)q(v)$. We adopt the notation $Q(u)$ for the matrix product corresponding to u . Then we have by Eq.(2.3):

$$Q(uavb) = \begin{pmatrix} * & * \\ q(uav) & q(uavb) \end{pmatrix}, \quad Q(vb) = \begin{pmatrix} * & * \\ q(v) & q(vb) \end{pmatrix}, \quad Q(ua) = \begin{pmatrix} * & * \\ q(u) & * \end{pmatrix}.$$

Thus

$$Q(ua) = Q(uavb)Q(vb)^{-1} = \begin{pmatrix} * & * \\ q(uav) & q(uavb) \end{pmatrix} \begin{pmatrix} q(vb) & * \\ -q(v) & * \end{pmatrix}.$$

Thus $q(u) = q(uav)q(vb) - q(uavb)q(v)$ which proves the lemma when $w = 1$.

Suppose now that w is of length 1, that is $w = c$, $c \in R$. Then the left-hand side of the equation in the lemma is by Eq.(2.2) equal to

$$q(uav)q(vbc) = q(uav)((q(vb)c - q(v)) = q(uav)q(vb)c - q(uav)q(v).$$

By the $w = 1$ case and by Eq.(2.2), this is equal to

$$\begin{aligned} q(u)c + q(uavb)q(v)c - q(uav)q(v) &= q(u)c + (q(uavb)c - q(uav))q(v) \\ &= q(u)c + q(uavbc)q(v), \end{aligned}$$

which proves the $w = c$ case.

Otherwise, we may write $w = w'cd$ for c, d in R . Then by Eq.(2.2)

$$\begin{aligned} q(uav)q(vbw) &= q(uav)q(vbw'cd) = q(uav)(q(vbw'c)d - q(vbw')) \\ &= q(uav)q(vbw'c)d - q(uav)q(vbw') \end{aligned}$$

By induction (cases $w = w'$ and $w = w'c$), this is equal to

$$\begin{aligned} q(u)q(w'c)d + q(uavbw'c)q(v)d - q(u)q(w') - q(uavbw')q(v) \\ &= q(u)(q(w'c)d - q(w')) + (q(uavbw'c)d - q(uavbw'))q(v) \\ &= q(u)q(w'cd) + q(uavbw'cd)q(v) = q(u)q(w) + q(uavbw)q(v). \end{aligned}$$

Note that commutativity is used in the second and third case. □

Lemma 2.3. *Suppose that*

$$Q(a_1)Q(a_2) \cdots Q(a_{n+3}) = -1.$$

Then for any i with $1 \leq i \leq n + 3$, we have

$$q(a_1, \dots, a_{i-1}) = q(a_{i+1}, \dots, a_{n+2}).$$

Proof. By hypothesis, we have

$$(Q(a_1) \cdots Q(a_{i-1}))^{-1} = -Q(a_i) \cdots Q(a_{n+3}).$$

Using Eq. (2.3) and the fact that the matrices have determinant 1, we obtain

$$\begin{pmatrix} q(a_1, \dots, a_{i-1}) & * \\ * & * \end{pmatrix} = - \begin{pmatrix} -q(a_{i+1}, \dots, a_{n+2}) & * \\ * & * \end{pmatrix}.$$

□

2.2. Continuant trees

We call *pre-continuant tree* a planar quiver T which is constructed from a planar binary, not necessarily complete, tree τ as follows:

- An edge between a node and its left child is oriented towards this node;
- an edge between a node and its right child is oriented towards this child;
- If a node has two children, then an arrow from the right child towards its left child is created.

See Figure 2.1 for an example, disregarding the labels. These quivers are already known: they are described in [13] and [3] (Th.2.7). Their tree-like shape comes from the classification of the tilted algebras of the linearly oriented quivers of type \mathbb{A}_n , see [29]. The additional arrows from right to left children arise because of [2, 12].

Note that since the original tree τ is planar, we may use for T the terminology of planar binary trees: subtree at a vertex (which is a pre-continuant tree), left and right child, parent. . . .

Now we call *continuant tree* a pre-continuant tree T together with a labelling of its vertices by words on R , as follows:

- the length of the label of a vertex is the number of vertices of the subtree having this vertex as root;
- if a vertex is a left (resp. right) child, then its label is a prefix (resp. suffix) of its parent;
- finally, to each vertex labelled u is associated the signed continuant polynomial $q(u)$.

See Figure 2.1 for an example. Note that a continuant tree is completely determined by the underlying pre-continuant tree together with the label w (of length the number of vertices) of the root. For later use, we call the continuant tree a w -continuant tree, or an (a_1, \dots, a_n) -continuant tree, if $w = a_1 \dots a_n$.

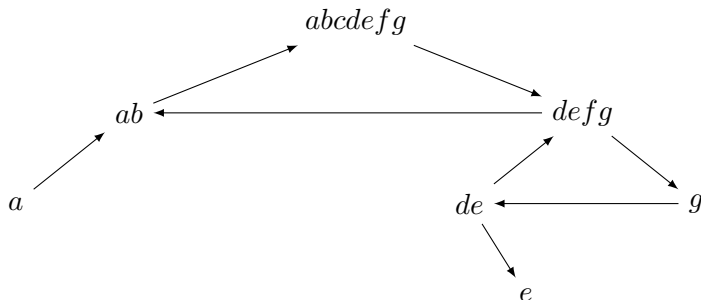


FIGURE 2.1. A continuant tree

2.3. Mutation of a continuant tree outside the root

Recall the definition of the mutation of a quiver. We remind the reader that to each vertex k of this quiver is associated an element $y_k \in R$, called *variable at k* . We assume that the quiver has no cyclic path of length 1 or 2. The mutation at a vertex k is performed as follows (we follow the definition of [31] 3.2.):

- For each pair of arrows $i \rightarrow k \rightarrow j$, add an arrow $i \rightarrow j$.
- Reverse all arrows incident with k .
- Remove pairs of opposite arrows, until no such pair exists.

Now, the variables of the new quiver are the same at each vertex, except at vertex k where the new variable y'_k (the *mutated variable*) must satisfy the *exchange relation*

$$y_k y'_k = \prod_{i \rightarrow k} y_i + \prod_{k \rightarrow j} y_j,$$

where the products are taken over all arrows ending or starting at k , respectively. Note that uniqueness of y'_k is ensured if R has no zero divisors, an hypothesis which will be made in the sequel.

Assume now that T is a continuant tree. The variables at the vertices of T are the corresponding continuant polynomials, that is, if k is a vertex with associated word u , then the variable at k is $q(u)$.

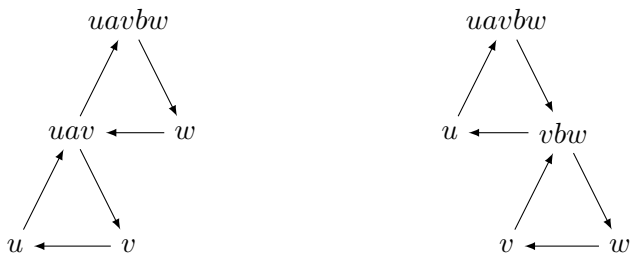


FIGURE 2.2. mutation of a continuant tree at a vertex different from the root

Lemma 2.4. *Mutation of a continuant tree at a vertex k which is not the root gives another continuant tree.*

Proof. The lemma is illustrated in Figure 2.2, where only the vertices involved in the mutation are represented. The vertex k is the one with uav on the left and the one with vbw on the right. By inspection and by definition of mutation, it is seen that the quiver on the right is mutated from the quiver at vertex k . The mutation formula for the labels is a consequence of Lemma 2.2, which ensures the existence of the mutated variable. Note the limiting cases where some vertices among u, v, w are missing; in other words, the corresponding word is empty: they are still covered by this proof since the continuant polynomial $q(x)$ equals 1 if x is the empty word. Moreover, the mutation from right to left in Figure 2.2 follows from the fact that mutation is an involution. \square

2.4. SL_2 -tilings of the plane

Following [6], we call SL_2 -tiling of the plane a mapping $t : \mathbb{Z}^2 \mapsto K$, for some field K , such that, for any x, y in \mathbb{Z} ,

$$\begin{vmatrix} t(x, y) & t(x+1, y) \\ t(x, y+1) & t(x+1, y+1) \end{vmatrix} = 1.$$

Here we represent the discrete plane \mathbb{Z}^2 with coordinates as illustrated in Figure 2.4, so that the y -axis points downwards, and the x -axis points to the right. Note that the x -coordinates therefore represent the column indices, and the y -coordinates represent the row indices.

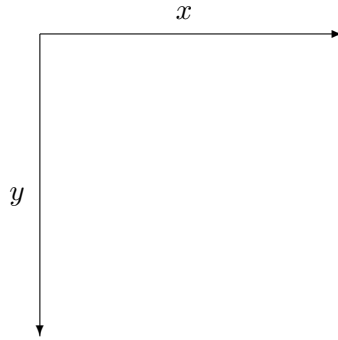


FIGURE 2.3. Coordinate convention

An example is given below, with $K = \mathbb{Q}$.

									1	1	1	1		
								1	1	2	3	4		
								1	2	5	8	11		
								1	3	8	13	18		
					1	1	4	11	18	25	...	
							1	2	9	25	41	57		
							1	3	14	39	64	89		
			1	1	1	1	1	4	19	53	87	121		
1	1	1	1	2	3	4	5	6	25	119	332	545	758	
1	2	3	4	9	14	19	24	29	121	576	1607	2368	3669	...
						

Here is another example, with K the field of fractions over \mathbb{Q} in the variables a, b, c, d, e, f, \dots

				...										
			$\frac{d+b+bce}{cd}$	$\frac{1+ce}{d}$		e				f				
...			b	c		d				$\frac{1+df}{e}$...
			a	$\frac{1+ac}{b}$	$\frac{b+d+acd}{bc}$	$\frac{bce+b+ddf+dac+d^2acf+d+d^2f}{bcde}$								
					...									

In these two examples, the dots indicate that the tiling may be extended to the whole plane; in the first case, by positive integers (as follows from [6] Theorem 3) and in the second case, by Laurent polynomials with numerators having coefficients in \mathbb{N} (as follows from [6] Theorem 4).

2.5. Tameness

Note that an SL_2 -tiling of the plane, viewed as an infinite matrix, has necessarily rank at least 2. Following [9], we say that the tiling is *tame* if its rank is 2.

Given three *successive* columns C_0, C_1, C_2 of a tame SL_2 -tiling t , there is a unique coefficient α such that

$$C_0 + C_2 = \alpha C_1. \quad (2.5)$$

This is proved as follows: consider two consecutive rows, with elements a, b, c in the first row and in columns C_0, C_1, C_2 respectively, and elements d, e, f in the second row; now let x, y, z be elements of these 3 columns, respectively, located on some arbitrary row; then the 3 by 3 matrix $\begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}$ is a submatrix of the tiling; hence its determinant is 0, so that, by expanding the determinant with respect to the first row, and noting that the 2 by 2 adjacent minors are equal to 1, we obtain that Eq.(2.5) holds, with $\alpha = \det \begin{pmatrix} a & c \\ d & f \end{pmatrix}$.

We call α the *linearization coefficient* of column C_1 . Similarly for rows. The following result extends Eq.(2.5) (the $n = 1$ case).

Lemma 2.5. *Let t be a tame SL_2 -tiling of the plane and C_0, \dots, C_{n+1} successive columns of t , with linearization coefficients $\alpha_0, \dots, \alpha_{n+1}$. Then for any i in $\{1, \dots, n\}$*

$$q(\alpha_{i+1}, \dots, \alpha_n)C_0 + q(\alpha_1, \dots, \alpha_{i-1})C_{n+1} = q(\alpha_1, \dots, \alpha_n)C_i.$$

Proof. We use the identity

$$C_j = -q(\alpha_2, \dots, \alpha_{j-1})C_0 + q(\alpha_1, \dots, \alpha_{j-1})C_1, \quad (2.6)$$

which is proved as follows. First, we have by definition of the linearization coefficients, $C_{j+2} = -C_j + \alpha_{j+1}C_{j+1}$. This implies, with the matrix notation $Q(\alpha)$ of Subsection 2.1, that $(C_j, C_{j+1})Q(\alpha_{j+1}) = (C_{j+1}, C_{j+2})$. It follows that for any natural number j , one has

$$(C_0, C_1)Q(\alpha_1) \dots Q(\alpha_j) = (C_j, C_{j+1}), \quad (2.7)$$

We conclude by using Eq.(2.3).

Suppose first that $i = 1$. Then, with Eq.(2.6),

$$\begin{aligned} q(\alpha_2, \dots, \alpha_n)C_0 + C_{n+1} \\ &= q(\alpha_2, \dots, \alpha_n)C_0 - q(\alpha_2, \dots, \alpha_n)C_0 + q(\alpha_1, \dots, \alpha_n)C_1 \\ &= q(\alpha_1, \dots, \alpha_n)C_1, \end{aligned}$$

which proves the identity for $i = 1$. Suppose now that $i > 1$. Then we have by Lemma 2.2 (with u the empty word, $a = \alpha_1$, $v = \alpha_2 \cdots \alpha_{i-1}$, $b = \alpha_i$, $w = \alpha_{i+1} \cdots \alpha_n$):

$$\begin{aligned} q(\alpha_1, \dots, \alpha_{i-1})q(\alpha_2, \dots, \alpha_n) \\ &= q(\alpha_{i+1}, \dots, \alpha_n) + q(\alpha_1, \dots, \alpha_n)q(\alpha_2, \dots, \alpha_{i-1}). \end{aligned}$$

Thus we obtain, with Eq.(2.6), the previous equality and Eq.(2.6) again,

$$\begin{aligned} q(\alpha_{i+1}, \dots, \alpha_n)C_0 + q(\alpha_1, \dots, \alpha_{i-1})C_{n+1} \\ &= q(\alpha_{i+1}, \dots, \alpha_n)C_0 + q(\alpha_1, \dots, \alpha_{i-1})(-q(\alpha_2, \dots, \alpha_n)C_0 + q(\alpha_1, \dots, \alpha_n)C_1) \\ &= (q(\alpha_{i+1}, \dots, \alpha_n) - q(\alpha_1, \dots, \alpha_{i-1})q(\alpha_2, \dots, \alpha_n))C_0 \\ &\quad + q(\alpha_1, \dots, \alpha_{i-1})q(\alpha_1, \dots, \alpha_n)C_1 \\ &= -q(\alpha_1, \dots, \alpha_n)q(\alpha_2, \dots, \alpha_{i-1})C_0 + q(\alpha_1, \dots, \alpha_{i-1})q(\alpha_1, \dots, \alpha_n)C_1 \\ &= q(\alpha_1, \dots, \alpha_n)(-q(\alpha_2, \dots, \alpha_{i-1})C_0 + q(\alpha_1, \dots, \alpha_{i-1})C_1) \\ &= q(\alpha_1, \dots, \alpha_n)C_i. \end{aligned}$$

□

It has been shown in [9] (in the more general case of SL_k -tilings) that tameness of SL_2 -tilings is characterized by the fact that the infinite matrix of 2 by 2 minors is of rank 1. This is important for the proof of the next result.

Corollary 2.6. *For $n, m \geq 0$, let $(a_{ij})_{0 \leq i \leq n+1, 0 \leq j \leq m+1}$ be a connected submatrix of a tame SL_2 -tiling t (i denotes the row number and j the column number). Let $\beta_0, \dots, \beta_{n+1}$ denote the linearization coefficients of the corresponding rows of t and $\alpha_0, \dots, \alpha_{m+1}$ denote those of the corresponding columns. Then*

$$\det\left(\begin{pmatrix} a_{00} & a_{0,m+1} \\ a_{n+1,0} & a_{n+1,m+1} \end{pmatrix}\right) = q(\beta_1, \dots, \beta_n)q(\alpha_1, \dots, \alpha_m).$$

Proof. We suppose first that $n = 0$. Denote by C_j the column vector of the first two elements of column j of matrix (a_{ij}) :

$$C_j = \begin{pmatrix} a_{0,j} \\ a_{1,j} \end{pmatrix}.$$

Then we have by Lemma 2.5, $q(\alpha_2, \dots, \alpha_m)C_0 + C_{m+1} = q(\alpha_1, \dots, \alpha_m)C_1$. Thus

$$\det(C_0, C_{m+1}) = q(\alpha_1, \dots, \alpha_m) \det(C_0, C_1) = q(\alpha_1, \dots, \alpha_m),$$

since t is an SL_2 -tiling and therefore $\det(C_0, C_1) = 1$.

Now, in the general case, it follows from [9], Prop.4, that the determinant of the corollary is equal to the product

$$\det\left(\begin{pmatrix} a_{00} & a_{0,m+1} \\ a_{1,0} & a_{1,m+1} \end{pmatrix}\right) \det\left(\begin{pmatrix} a_{00} & a_{0,1} \\ a_{n+1,0} & a_{n+1,1} \end{pmatrix}\right),$$

which proves the corollary, by the first part. □

2.6. Frontier

We call *frontier* a bi-infinite sequence

$$\dots \xi_{-2}x_{-2}\xi_{-1}x_{-1}\xi_0x_0\xi_1x_1\xi_2x_2\xi_3x_3\dots \quad (2.8)$$

where $\xi_i \in \{x, y\}$ and x_i are elements of K^* , for any $i \in \mathbb{Z}$. It is called *admissible* if there are arbitrarily large and positive, and arbitrarily large and negative i 's such that $\xi_i = x$, and similarly for y ; in other words, none of the two sequences $(\xi_n)_{n \geq 0}$ and $(\xi_n)_{n \leq 0}$ is ultimately constant. The x_i 's are called the *variables* of the frontier.

Each frontier may be embedded into the plane, though not uniquely: the variables label points in the plane, and the x (resp. y) determine a bi-infinite discrete path, in such a way that x (resp. y) corresponds to a segment of the form $[(a, b), (a + 1, b)]$ (resp $[(a, b), (a, b - 1)]$); recall the coordinate conventions, see Figure 2.4. For example, the path corresponding to the frontier $\dots x_{-4}xx_{-3}yx_{-2}yx_{-1}yx_0xx_1xx_2yx_3xx_4xx_5\dots$ is given in Figure 2.4.

We need the following notation of [6]. Let

$$M(a, x, b) = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \text{ and } M(a, y, b) = \begin{pmatrix} b & 0 \\ 1 & a \end{pmatrix}.$$

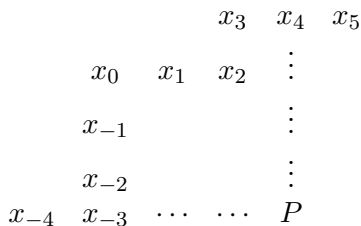


FIGURE 2.4. A frontier

Given an admissible frontier, embedded in the plane as explained previously, let $P \in \mathbb{Z}^2$. Then we obtain a finite word, which is a factor of the frontier, by projecting the point P horizontally and vertically onto the frontier. We call this word the *word* of P . It is illustrated in Figure 2.4, where the word of the point P is $x_{-3}yx_{-2}yx_{-1}yx_0xx_1xx_2yx_3xx_4$. We define the word of a point only for points below the frontier; for points above, the situation is symmetric and the definition is exactly similar.

Theorem 2.7. *Given an admissible frontier, there exists a unique tame SL_2 -tiling t of the plane over K , extending the embedding of the frontier into the plane. It is defined, for any point P below the frontier, with associated word $x_0\xi_1x_1\xi_2 \cdots \xi_{n+1}x_{n+1}$, where $n \geq 1$, $x_i \in K^*$ and $\xi_i \in \{x, y\}$, by the formula*

$$t(P) = \frac{1}{x_1x_2 \cdots x_n} (1, x_0)M(x_1, \xi_2, x_2) \cdots M(x_{n-1}, \xi_n, x_n)(1, x_{n+1})^t. \tag{2.9}$$

The existence of the tiling, together with the formula, are proven in [6]. The uniqueness and the tameness follow from [9] (uniqueness was proved in [6] without the hypothesis of tameness, but under some extra assumption on K). Note that $\xi_1 = y$ and $\xi_{n+1} = x$, by definition of the word associated to P .

For later use, we introduce the notation

$$M(x_1\xi_2x_2\cdots x_{n-1}\xi_nx_n) = M(x_1, \xi_2, x_2) \cdots M(x_{n-1}, \xi_n, x_n).$$

As a particular case of the previous construction, consider a frontier having period n ; this means that it is of the form ${}^\infty(x_1\xi_1 \dots x_n\xi_n)^\infty$. Then the associated tiling has period determined by the vector $(p, -q)$, where p (resp. q) is the number of x 's (resp. of y 's) among ξ_1, \dots, ξ_n . Note that $p + q = n$. Moreover, the sequence of linearization coefficients of the columns of the tiling has period p , and the sequence of linearization coefficients of the rows has period q .

3. Representative functions

Fix a commutative ring R . Let A, B be R -algebras. We say that a function $f : A \rightarrow B$ is *representative over R with values in B* if there exists a natural number n , an R -algebra homomorphism $\mu : A \rightarrow B^{n \times n}$, a row matrix $\lambda \in B^{1 \times n}$ and a column matrix $\gamma \in B^{n \times 1}$ such that for any $a \in A$

$$f(a) = \lambda\mu(a)\gamma.$$

This definition may seem too general¹, but it is justified by the following result.

Proposition 3.1. *The composition of two representative functions is representative.*

Proof. Let A, B, C be three R -algebras and $f : A \rightarrow B$, $g : B \rightarrow C$ be representative functions. We have for any $a \in A$ $f(a) = \lambda\mu(a)\gamma$, where the notations are as above; and for any $b \in B$, $g(b) = \kappa\nu(b)\delta$ where for some natural number p , ν is an R -algebra homomorphism $B \rightarrow C^{p \times p}$, $\kappa \in C^{1 \times p}$ and $\delta \in C^{p \times 1}$.

1. For a matrix $M \in B^{q \times r}$, denote by $\nu(M)$ the matrix in $(C^{p \times p})^{q \times r}$ by replacing each entry of M by its image under ν . Note that if M, N are matrices over B whose product is defined, then $\nu(MN) = \nu(M)\nu(N)$. Note also that, under $p \times p$ -block decomposition, the rings $(C^{p \times p})^{q \times r}$ and $C^{pq \times pr}$ are canonically isomorphic, and we identify them.

2. Define the mapping $\pi : A \rightarrow (C^{p \times p})^{n \times n}$ by $\pi(a) = \nu(\mu(a))$. Then by 1., π is a ring homomorphism and $\nu(\lambda)\pi(a)\nu(\gamma) = \nu(f(a))$. Thus $g \circ f(a) =$

¹It could even be more general, by replacing rings by semirings, with applications in Automata Theory and Tropical Geometry

$\kappa\nu(f(a))\delta = \kappa\nu(\lambda)\pi(a)\nu(\gamma)$. This shows that $g \circ f$ is a representative function $A \rightarrow C$, since $\kappa\nu(\lambda)$ is a row vector of size $1 \times np$, π a ring homomorphism $A \rightarrow C^{np \times np}$ and $\nu(\gamma)\delta$ a column vector of size $np \times 1$. \square

Given a group G and an R -algebra B , we say that a function from G into B is *representative with values in B* if the natural extension of this function to the group algebra RG is representative.

These definitions fit with the classical case. If A is a K -algebra, K a field, then a linear mapping $A \rightarrow K$ is representative, in the classical sense if and only if it is representative over K with values in K , in the above meaning. In particular, a function from a group G (or a semigroup) into K is representative (see [1] p.72) if and only if the linear mapping from the group algebra KG into K that it defines is representative over K with values in K (see [20] Ex.1.5.11 p. 41).

By diagonal sum of matrices, it is easily seen that the sum of two representative functions is representative. Moreover, if f is a representative function defined in a group G and a is an element of the group, the function $h(g) = f(ag)$ is also representative; we denote $h = f.a$.

Lemma 3.2. *Let R be a commutative ring and B an R -algebra, G a group, H a subgroup of finite index and f a representative function of H over R with values in B . Define for any $g \in G$,*

$$\phi(g) = \begin{cases} f(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Then ϕ is a representative function of G with values in B .

Proof. This is proved by mimicking the matrix construction of an induced character. We know the existence of a group homomorphism μ from H into the group $GL_n(B)$, a row matrix $\lambda \in B^{1 \times n}$, a column matrix $\gamma \in B^{n \times 1}$ such that for $h \in H$, one has $f(h) = \lambda\mu(h)\gamma$. Let x_1, \dots, x_d be representatives of the left cosets gH of $G \bmod H$. For $j = 1, \dots, d$ and g in G , let $i = g.j$ and $h_j \in H$ be such that $gx_j = x_i h_j$; they are uniquely defined by this equation. Define the square matrix $R(g)$ of size nd , with $d \times d$ blocks of size n as follows: the (i, j) -block is $\mu(h_j)$; all other blocks are zero.

It is then classical that R is a group homomorphism from G into $GL_{nd}(B)$. We may assume that $x_1 = 1$. Then the $(1, 1)$ -block of $R(g)$ is nonzero if and only if g is in H , in which case it is equal to $\mu(g)$. Define the $1 \times nd$ -row matrix $L = (\lambda, 0, \dots, 0)$ and the the $nd \times 1$ -column

matrix $C = (\gamma^t, 0, \dots, 0)^t$. Then $\phi(g) = LR(g)C$ and ϕ is a representative function of G with values in B . \square

Corollary 3.3. *Let R be a commutative ring and B an R -algebra, G a group, and H a subgroup of finite index. For each left coset C of $G \bmod H$, with representative a_C , let f_C be a representative function of the group H with values in B . Define the function f on G by $f(g) = f_C(h)$ if $g \in C$, $g = a_C h$. Then f is a representative function of G with values in B .*

Proof. Let the function h_C be equal to f_C on H and 0 elsewhere; it is representative by the lemma. Now f is the sum over all cosets C of the functions $h_C \cdot a_C^{-1}$, which proves the corollary. \square

Proposition 3.4. *Suppose that K is a field. Let t be the SL_2 -tiling of the plane associated to a periodic frontier. Let R be the subring of K generated by the variables of the frontier and their inverses. Then the function $\mathbb{Z}^2 \rightarrow R$, $(x, y) \mapsto t(x, y)$ is a representative function of the group \mathbb{Z}^2 with values in R .*

Lemma 3.5. *Consider a function $\mathbb{Z}^2 \rightarrow R$, $(x, y) \mapsto s(x, y)$ which is of the following form: $s(x, y)$ is the $(1, 1)$ -coefficient of the matrix $A^y B C^x$, where A, B, C are fixed square matrices of the same size with A, C invertible over the commutative ring R . Then s is a representative function of the group \mathbb{Z}^2 with values in R .*

Proof. Consider the free R -module \mathcal{H} of square matrices over R of the same size as A . Then the group \mathbb{Z}^2 acts on it by $(x, y) \cdot M = A^y M C^x$. Then $s(x, y) = \phi((x, y) \cdot B)$, where ϕ is the linear form on \mathcal{H} which maps M onto its $(1, 1)$ -coefficient. Taking a basis of \mathcal{H} , we obtain that t is representative. \square

Lemma 3.6. *Let n, p be positive integers and $s_{i,j}$, with $0 \leq i \leq n-1$, $0 \leq j \leq p-1$ be representative functions of the group \mathbb{Z}^2 into R . Define $t : \mathbb{Z}^2 \rightarrow R$, $t(x, y) = s_{r_1, r_2}(q_1, q_2)$ if $x = nq_1 + r_1$ and $y = q_2p + r_2$ (euclidean division of x by n and of y by p). Then t is representative.*

Proof. Let H be the subgroup of \mathbb{Z}^2 generated by the vectors $(n, 0)$ and $(0, p)$. Then we obtain the lemma by applying Corollary 3.3. \square

Proof. (Proposition 3.4) We know by Th.2.7 that the tiling is tame; moreover the bi-infinite sequence of column (resp. row) linearization coefficients

is periodic; denote by $(\alpha_j)_{j \in \mathbb{Z}}$ (resp. $(\beta_i)_{i \in \mathbb{Z}}$) this sequence and denote its period by n (resp. p).

These coefficients belong to R ; this fact follows indeed from Th.2.7 and from the fact that for each three adjacent columns, one may find three points A, B, C on them, on the same horizontal line, and such that A, B are on the frontier; then the linearization coefficient of the middle column is $\frac{t(A)+t(C)}{t(B)}$, which is in R since $t(A)$ and $t(B)$ are variables of the frontier and $t(C)$ is given by Eq.(2.9). It follows from Eq.(2.7) that one has $(C_0, C_1)Q(\alpha_1 \dots \alpha_j) = (C_j, C_{j+1})$, for any natural number j , if we denote $Q(\alpha_1 \dots \alpha_j) = Q(\alpha_1) \cdots Q(\alpha_j)$.

Now, we have $\alpha_{j+n} = \alpha_j$. It follows that for any natural numbers q, r , we have

$$(C_0, C_1)[Q(\alpha_1) \cdots Q(\alpha_n)]^q Q(\alpha_1) \cdots Q(\alpha_r) = (C_{nq+r}, C_{nq+r+1}). \quad (3.1)$$

We claim that this is even true for any integer q . Indeed, an equality similar to Eq.(2.7) holds for negative indices: for $j > 0$,

$$(C_0, C_1)Q(\alpha_0)^{-1} \cdots Q(\alpha_{-i+1})^{-1} = (C_{-j}, C_{-j+1}).$$

From this the claim follows by induction on negative q .

Eq.(3.1) implies that if $x = nq + r$, then

$$\begin{pmatrix} t(0, 0) & t(0, 1) \\ t(1, 0) & t(1, 1) \end{pmatrix} [Q(\alpha_1 \dots \alpha_n)]^q Q(\alpha_1 \dots \alpha_r) = \begin{pmatrix} t(x, 0) & t(x, 1) \\ t(x+1, 0) & t(x+1, 1) \end{pmatrix}$$

Now, similar calculation apply to rows. Putting this together, we obtain that for any integers x, y with $x = nq_1 + r_1$, $y = pq_2 + r_2$, one obtains that the matrix

$$\begin{pmatrix} t(x, y) & t(x, y+1) \\ t(x+1, y) & t(x+1, y+1) \end{pmatrix}$$

is equal to

$$Q(\beta_{r_2} \dots \beta_1)[Q(\beta_p \dots \beta_1)]^{q_2} \begin{pmatrix} t(0, 0) & t(0, 1) \\ t(1, 0) & t(1, 1) \end{pmatrix} [Q(\alpha_1 \dots \alpha_n)]^{q_1} Q(\alpha_1 \dots \alpha_{r_1}).$$

Note that

$$[Q(\alpha_1 \dots \alpha_n)]^{q_1} Q(\alpha_1 \dots \alpha_{r_2}) = Q(\alpha_1 \dots \alpha_{r_2})[Q(\alpha_{r_2+1} \alpha_{r_2+2} \dots \alpha_{r_2})]^{q_1},$$

since both sides are equal to

$$Q(\alpha_1, \dots, \alpha_n, \dots, \alpha_1, \dots, \alpha_n, \alpha_1, \dots, \alpha_{r_2})$$

(with the sequence $\alpha_1, \dots, \alpha_n$ repeated q_1 times), and similarly for the β 's.

Thus we have, for the given x, y , $t(x, y) = s_{r_1, r_2}(q_1, q_2)$, where s_{r_1, r_2} is of the form given in Lemma 3.5, hence is a representative function on \mathbb{Z}^2 with values in R .

This implies the proposition, by Lemma 3.6. □

4. Case $\tilde{\mathbb{A}}$

We call $\tilde{\mathbb{A}}_{n-1}$ -quiver an acyclic quiver of type $\tilde{\mathbb{A}}_{n-1}$; that is, an acyclic directed graph such that the underlying undirected graph is an n -gon.

4.1. The mutated quivers

A description of the mutated quivers of type $\tilde{\mathbb{A}}$ is known, see [7] or [3]. Our description below is equivalent to it.

A *decorated $\tilde{\mathbb{A}}$ -quiver* is a quiver G defined as follows. Let Q be a quiver of type $\tilde{\mathbb{A}}_r$ for some r with $1 \leq r \leq n - 1$. Choose a subset of the set of arrows of Q and, for each arrow $x \rightarrow y$ in this subset, associate a pre-continuant tree, connected to Q via its root r and the two additional arrows $y \rightarrow r \rightarrow x$. This yields a planar quiver G having Q as a full subquiver; the latter is referred to as the *cyclic part* of G .

See Figure 4.1 for an example of a decorated $\tilde{\mathbb{A}}$ -quiver: the arrows of Q are boldfaced.

Lemma 4.1. *The class of decorated $\tilde{\mathbb{A}}$ -quivers is closed under mutation.*

Proof. Let j be a vertex of a decorated $\tilde{\mathbb{A}}$ -quiver G , and denote by G' its mutation at j . We claim that G' is a decorated $\tilde{\mathbb{A}}$ -quiver with cyclic part Q' .

- a) Suppose first that j is a vertex in Q . Let i, k be the vertices adjacent to j in Q . If in Q these three vertices form a path of length 2, $i \rightarrow j \rightarrow k$ say, then Q' is obtained by suppressing j in Q and replacing these two arrows by an arrow $i \rightarrow k$; the pre-continuant tree corresponding to the new arrow $i \rightarrow k$ of Q' is obtained by taking the new root j , and putting the tree of $i \rightarrow j$ as the left subtree and the tree of $j \rightarrow k$ as the right subtree;

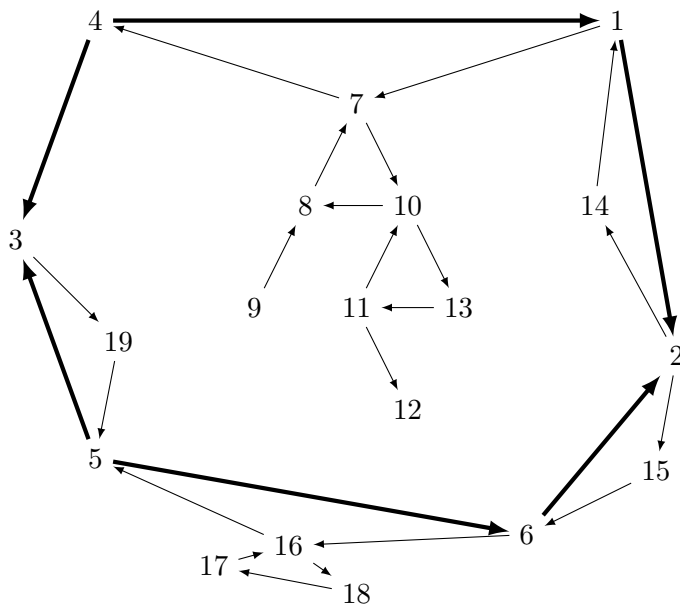


FIGURE 4.1. A decorated $\tilde{\mathbb{A}}$ -quiver with $n = 19$ vertices

- b) If in Q one has $i \rightarrow j \leftarrow k$ or $i \leftarrow j \rightarrow k$, then these two arrows are reversed in Q' ; the corresponding pre-continuant trees are exchanged: more precisely, if there is a pre-continuant tree with root l such that $i \leftarrow l \leftarrow j$ (resp. $j \rightarrow l \rightarrow k$), then, after mutation, it becomes so that we have $j \leftarrow l \leftarrow k$ (resp. $i \rightarrow l \rightarrow j$);
- c) Suppose now that j is a vertex in $G \setminus Q$; that is, j is a vertex on one of the continuant trees, T say. If j is not the root of T , we apply the construction of Subsection 2.3, disregarding the labels;
- d) If j is a root, then denote by i, k the adjacent vertices in Q , with $i \leftarrow j \leftarrow k$. Then Q is replaced by Q' , which has the new vertex j , with

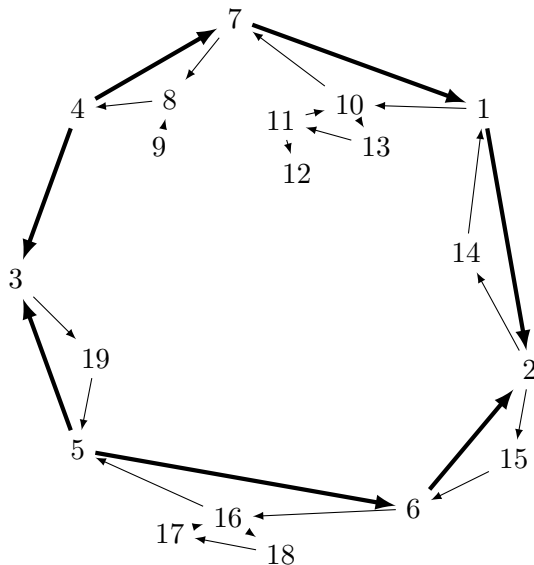


FIGURE 4.2. Mutation at 7

new arrows $i \rightarrow j \rightarrow k$. The pre-continuant tree corresponding to the arrow $i \rightarrow j$ (resp. $j \rightarrow k$) in Q' is the left (resp. right) subtree of T .

□

Note that the mutations of type a) are inverse of those of type d), and that mutations of type b) and c) are their own inverses. As an example of case d), see Figure 4.2 which is obtained by mutation at vertex 7 of Figure 4.1. Case a) is obtained by reversing this mutation. An example of case b) is seen in Figure 4.3, which is obtained by mutation at vertex 5 of Figure 4.1.

4.2. Elementary properties of decorated $\tilde{\mathbb{A}}$ -quivers

Consider a decorated $\tilde{\mathbb{A}}$ -quiver with n vertices and cyclic part Q . We associate to each arrow of Q a positive natural number that we call its *length*: it is 1 if this arrow has no associated pre-continuant tree, and it is l if this arrow has a pre-continuant tree with $l - 1$ vertices. For example, in

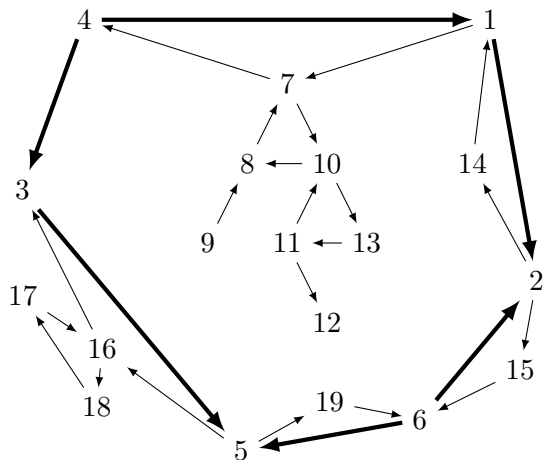


FIGURE 4.3. Mutation at 5

Figure 4.1, the arrow $4 \rightarrow 3$ has length 1 and the arrow $4 \rightarrow 1$ has length 8.

Each arrow of length l is naturally isometric to an euclidean segment of length l . For later use, we need to consider other points than the vertices in G ; we call them *points*: a point is either a vertex of Q , or a point on an arrow of Q located at an integer distance from each of the vertices of this arrow, considering the previous isometry. Thus an arrow of length l has $l + 1$ points on it: its two vertices (tail and head) and the $l - 1$ additional points, which are not vertices of the graph. For example the arrow $4 \rightarrow 1$ in Figure 4.1, of length 8, has 9 points on it, including the two vertices.

We denote by p (resp. q) the sum of the lengths of the clockwise oriented (counter-clockwise) arrows. Then by definition of the length, $p + q = n$. It is easy to verify that p and q are invariant under mutation. We call p, q the *parameters* of G .

Note also that, as it was observed in Subsection 2.2, that each pre-continuant tree of G will become naturally a continuant tree once we

associate to the corresponding arrow of Q , of length l , a sequence of length $l - 1$ on the ring R . This will be done in the next subsection.

For this we fix a planar $\tilde{\mathbb{A}}_{n-1}$ -quiver, denoted by G_0 and called the *initial quiver*, with set of vertices $\{1, \dots, n\}$ and p clockwise oriented arrows of the form $i \rightarrow i + 1$ (in which case we define $\xi_i = x$) and q counterclockwise oriented arrows of the form $i \leftarrow i + 1$ (in which case we define $\xi_i = y$), with $i + 1$ taken $\pmod n$; then $p + q = n$. Note that this is a particular decorated $\tilde{\mathbb{A}}$ -quiver, with no attached pre-continuant trees, and therefore with all arrows of length 1.

We associate to G_0 the SL_2 -tiling t as in [6]. In other words we consider the frontier ${}^\infty(x_1\xi_1 \dots x_n\xi_n)^\infty$ where x_i is the *initial variable* attached to vertex i .

Note that this tiling t is periodic modulo the vector $(p, -q)$, since its frontier is the infinite power of a word having p times the letter x and q times the letter y . Note also that the sequence of linearization coefficients of the columns (resp. rows) of t is periodic of period p (resp. q). See Subsection 2.6.

Given two points on the same horizontal line in \mathbb{Z}^2 , call *linearization sequence between them* the word (the finite sequence) formed by the column linearization coefficients of t for the columns lying strictly between them, scanning in increasing order of the x -coordinate. Similarly, for two points lying on the same vertical line.

4.3. Embedding of a decorated $\tilde{\mathbb{A}}$ -quiver into \mathbb{Z}^2

An *embedding* of a decorated $\tilde{\mathbb{A}}$ -quiver G into \mathbb{Z}^2 is a universal covering of its cyclic part Q , contained in the euclidean plane, which respects the length of arrows, and which respects the orientation; that is, in such a way that clockwise (resp. counter-clockwise) oriented arrows of length l of Q correspond to horizontal (resp. vertical) segments of the form $[(u, v), (u + l, v)]$ (resp. $[(u, v), (u, v + l)]$) with u, v integers.

We denote by $\pi(i)$ the set of points in \mathbb{Z}^2 which correspond to the vertex $i \in G$. This set of points is by construction of the form $A + \mathbb{Z}(p, -q)$, for some $A \in \mathbb{Z}^2$.

For example, Figure 4.4 shows an embedding of the quiver of Figure 4.1. In this figure, we have represented the pre-continuant trees of G , which however are not formally part of the embedding and which are represented for a better understanding.

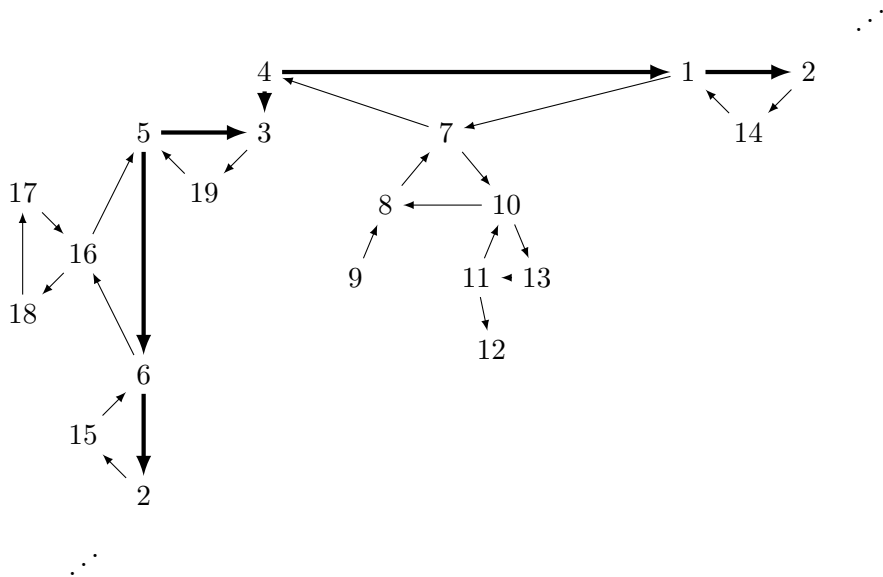


FIGURE 4.4. Embedding of Figure 4.1

Now, we see that in an embedded decorated $\tilde{\mathbb{A}}$ -quiver G , all pre-continuant trees of G become continuant trees: indeed, to each clockwise (resp. counter-clockwise) arrow, associate to it the sequence of column (resp. row) linearization coefficients between A and B , where $A \rightarrow B$ is one of the corresponding arrows in the embedding; note that the length of this sequence is equal to $l - 1$, where l is the length of the arrow, since the embedding respects the length. There are infinitely many arrows, but by periodicity, this is well-defined. We call this sequence the *linearization sequence of the arrow*; it depends on the embedding. For later use, note that this dependence is modulo the subgroup of \mathbb{Z}^2 generated by the vectors $(p, 0)$ and $(0, q)$; indeed, the column (resp. row) linearization coefficients have period p (resp. q). Hence, if we translate correspondingly the embedding, the linearization sequences of arrows do not change.

Recall the definition of points of G given in Subsection 4.2. Clearly, the mapping π may naturally be extended to the arrows of Q , hence to the points of G , by respecting the length.

For technical reasons, we need to introduce the following notions. Given an embedding π of G , there is a unique point $P = P(G, \pi)$ in \mathbb{Z}^2 defined as follows: it is the intersection of the bi-infinite path defined by π and the x -axis, and which has the smallest x -coordinate (the intersection may be a horizontal segment). We denote by $\xi(G, \pi)$ the x -coordinate of P . We note that P corresponds to a unique point on the cyclic part Q of G , that we denote by $u(G, \pi)$; this point is called the *distinguished point* associated to the pair (G, π) . It depends on the embedding.

Lemma 4.2. *Let u be a point on G . Then for each vertex k on the cyclic part of G , there is a vector $(i, j) \in \mathbb{Z}^2$ such that $\pi(k) = \pi(u) + (i, j) + \mathbb{Z}(p, -q)$ for any embedding π of G .*

Proof. This follows because the covering respects the lengths of arrows and their orientations. \square

We describe now how the embeddings are modified by mutations. We refer to the cases a) to d) in Subsection 4.1:

- a) the new embedding is obtained by suppressing the vertices corresponding to j in the covering and by gluing the arrows $i \rightarrow j$ and $j \rightarrow k$ into a unique one $i \rightarrow k$;
- b) in the embedding of G , the points corresponding to i, j, k form three consecutive corners of a rectangle; then j is replaced by the fourth corner;
- c) the new embedding is the same as the old one;
- d) this is the reversal of case a).

As examples, see Figure 4.5, which is the embedding of Figure 4.2 and which is obtained from Figure 4.4 by a type d) mutation on vertex 7; and Figure 4.6, which is the embedding of Figure 4.3 and which is obtained from Figure 4.4 by a type b) mutation on vertex 5.

We have defined mutations on the pairs (G, π) . They induce mutations on the distinguished points $u(G, \pi)$. We show below that if one considers only the pair $(G, u(G, \pi))$, the mutations may be defined independently of π . This will be useful when we want to build representative functions and finite group actions of the mutation group; indeed, the pairs (G, π) are infinitely many, but the pairs $(G, u(G, \pi))$ are finite in number.

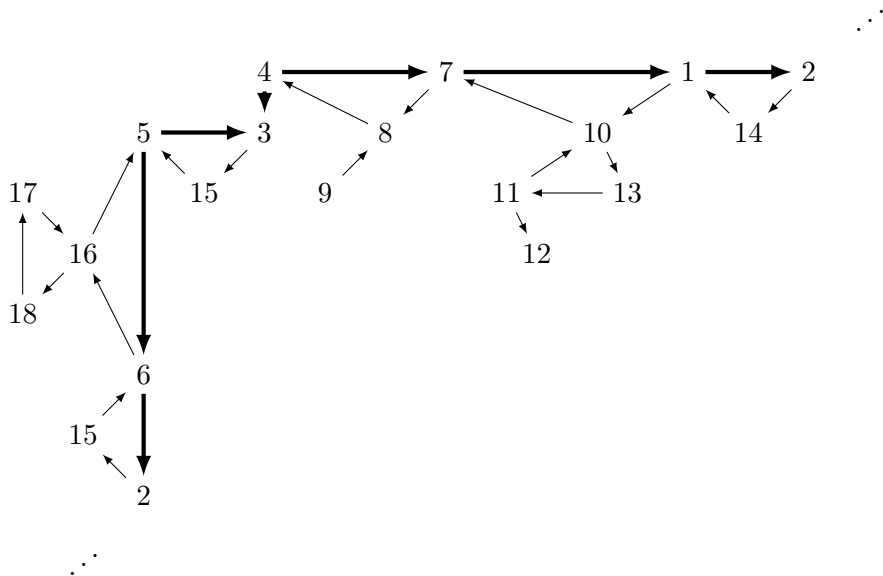


FIGURE 4.5. Mutation at 7: embedding

Lemma 4.3. *Given a decorated $\tilde{\mathbb{A}}$ -quiver G with distinguished point u , one may define the mutation at vertex j of the pair (G, u) in such a way that the mutated pair (G', u') satisfies: for each embedding π of G with $u(G, \pi) = u$, one has $u(G', \pi') = u'$.*

In other words, the distinguished point of G corresponding to an embedding π is mutated independently of π itself.

Proof. This is verified as follows: the point $u(G, \pi)$ is invariant except if the mutation is of type b) and if the point $P = P(G, \pi)$ is on the counterclockwise arrow a of Q (the cyclic part of G) incident to j ; in this case the x -coordinate $\xi(G, \pi)$ of P is increased or decreased by the length of the clockwise arrow incident to j , depending whether j is the tail or the head of these two arrows; moreover, if u is at distance l_1 of j , with $l = l_1 + l_2$ equal to the length of the arrow a , then u' is on the counterclockwise arrow incident to j in the mutated graph G' , at distance l_2 of j . \square

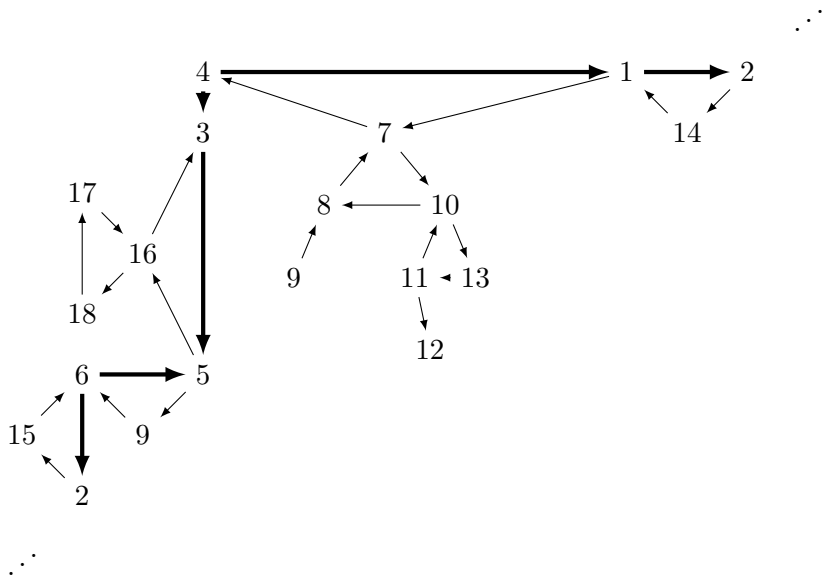


FIGURE 4.6. Mutation at 5: embedding

See for example Figure 4.4 and Figure 4.6 (mutation at 5 of Figure 4.4), with P on the arrow $5 \rightarrow 6$ in Figure 4.4 (thus the x -axis intersects this arrow); then after mutation, P is in Figure 4.6 on the arrow $3 \rightarrow 5$.

4.4. The mutated seeds in type $\tilde{\mathbb{A}}$

Using the notations of Subsection 4.2, we start with the *initial quiver* G_0 and we let t be the corresponding SL_2 -tiling. Recall that G_0 has p clockwise arrows and q counter-clockwise arrows, with $p + q = n$ which is the number of vertices of G_0 . We let S_0 denote the *initial seed*, that is, $S_0 = (G_0, \{x_1, \dots, x_n\})$.

Theorem 4.4. *Each seed $S = (G, \{y_1, \dots, y_n\})$ in the mutation class of S_0 is obtained as follows: for some decorated $\tilde{\mathbb{A}}$ -quiver G with parameters p, q and some embedding π of G , the i -th cluster variable y_i of S is:*

- if i is on the cyclic part of G , then y_i is equal to $t(\pi(i))$, where t is the SL_2 -tiling associated to S_0 ;

- if i is a vertex of one of the continuant trees of G , then y_i is equal to the variable associated to this vertex.

In order to understand this result, note that the tiling t is periodic modulo the vector $(p, -q)$, since so is its frontier; and the set $\pi(i)$ is of the form $A + \mathbb{Z}(p, -q)$ for some point A , by definition of an embedding; hence $t(\pi(i))$ is well-defined. Moreover, as seen previously, each pre-continuant tree of G becomes naturally a continuant tree, once an embedding of G is given.

To illustrate the theorem, consider Figure 4.4: the vertices of G that appear as vertices in the covering (that is, 2,6,5,3,4,1) have as variable their image under t . The other ones correspond to the second case in the theorem: for example, let a, b, c, d, e, f, g be the sequence of column linearization coefficients corresponding to the column lying strictly between the columns of 4 and that of 1; then the vertex 7 gets the variable $q(abcdefg)$, the vertex 10 gets $q(defg)$, 8 gets $q(ab)$ and 12 gets $q(e)$.

Proof. It is enough to prove that: (i) the initial seed is obtained in this way and that: (ii) the statement is compatible with mutations. For (i), this follows by choosing the initial embedding π_0 in such a way that the corresponding covering of $G_0 = Q_0$ (G_0 is equal to its cyclic part; there are no pre-continuant trees in G_0) is the frontier of the tiling. The latter has been constructed in such a way that this is possible.

We prove now (ii). Suppose that the mutation at j is of type d) (of which we take the notations). By symmetry, we may assume that $i \rightarrow k$ is a clockwise oriented arrow. Let A, C denote two points in the embedding π on the same horizontal which correspond to i and k respectively (the reader may use Figure 4.4 with $i = 4, j = 7, k = 1$, and A, C corresponding to 4 and 1 in the figure). Let $\alpha_1, \dots, \alpha_r, \beta, \gamma_1, \dots, \gamma_s$ be the column linearization coefficients for the columns strictly lying between A and C , with r (resp. s) equal to the number of vertices of the left (resp. right) subtree of the continuant tree corresponding to this arrow (in the figure $r = 2, s = 4$); denote by a (resp. b) the roots of these subtrees (in the figure, a, b correspond to 8,10). Then the variables at i, k, j, a, b are respectively: $t(A), t(C), q(\alpha_1, \dots, \alpha_r, \beta, \gamma_1, \dots, \gamma_s), q(\alpha_1, \dots, \alpha_r), q(\gamma_1, \dots, \gamma_s)$. Moreover the arrows incident to j are $j \rightarrow i, j \rightarrow b, k \rightarrow j, a \rightarrow j$. After mutation, j is on the cyclic part Q' of the mutated quiver G' and corresponds to a point B in the new embedding located between A and C at distance $r + 1$ of A (see Figure 4.5, with B corresponding to 7); hence the

mutated variable at j is $t(B)$. Thus we must verify that

$$\begin{aligned} & t(B)q(\alpha_1, \dots, \alpha_r, \beta, \gamma_1, \dots, \gamma_s) \\ &= t(A)q(\gamma_1, \dots, \gamma_s) + t(C)q(\alpha_1, \dots, \alpha_r, \beta, \gamma_1, \dots, \gamma_s). \end{aligned}$$

This is Lemma 2.5.

Suppose now that the mutation is of type b). We may by symmetry assume that j is such that the arrows incident to j are of the form $j \rightarrow k$ and $j \rightarrow i$. Let A, B, C be three consecutive points in the embedding π corresponding to i, j, k . See Figure 4.4 with i, j, k equal to 6, 5, 3 and A, B, C the corresponding points. Denote by $\alpha_1, \dots, \alpha_r$ (resp. β_1, \dots, β_s) the column (resp. row) linearization coefficients of the columns (resp. rows) strictly lying between B and C (resp. B and A). Let D be the fourth point of the rectangle on A, B, C (in Figure 4.6, D corresponds to 5). Let a (resp. b) be the root of the tree of the arrow $j \rightarrow k$ (resp. $j \rightarrow i$) (in Figure 4.4, a, b correspond to 19, 16). Then we have the arrows $a \rightarrow j$ and $b \rightarrow j$. The variables in G at i, j, k, a, b are respectively $t(A), t(B), t(C), q(\alpha_1, \dots, \alpha_r), q(\beta_1, \dots, \beta_s)$; after mutation, the variable at j becomes $t(D)$. Thus we have to verify that

$$t(B)t(D) = t(A)t(C) + q(\alpha_1, \dots, \alpha_r)q(\beta_1, \dots, \beta_s).$$

This is a consequence of Corollary 2.6.

A type a) mutation is the reverse of a type d) mutation and is treated similarly. The case of a type c) mutation follows from Lemma 2.4. \square

4.5. Transjective/Nontransjective variables

It follows from the previous theorem that the cluster variables either appear as elements of the SL_2 -tiling, or as continuant polynomials of the linearization coefficients of the tiling; actually, only finitely many of them are of the latter form, since the pre-continuant trees appearing on decorated $\tilde{\mathbb{A}}$ -quivers are finite in number and since the sequence of linearization coefficients of the tiling are periodic. We shall see below that the two cases are mutually exclusive.

Let Q be a finite acyclic quiver and K an algebraically closed field. We denote by KQ the path algebra of Q , by $\mathrm{mod}KQ$ the category of finitely generated right KQ -modules and by $\mathcal{D}^b(\mathrm{mod}KQ)$ the bounded derived category over $\mathrm{mod}KQ$. Let τ denote the Auslander-Reiten translation and $[1]$ the shift in $\mathcal{D}^b(\mathrm{mod}KQ)$. The *cluster category* \mathcal{C}_Q of Q is defined to be

the orbit category of $\mathcal{D}^b(\text{mod}KQ)$ under the action of the automorphism $\tau^{-1}[1]$, see [11]. The Auslander-Reiten quiver $\Gamma(\mathcal{C}_Q)$ of \mathcal{C}_Q has a unique component containing all the objects in $KQ[1]$, that is, the shifts of the indecomposable projective KQ -modules. This component is the *transjective component* Γ_{tr} of $\Gamma(\mathcal{C}_Q)$ and its objects are called *transjective*. If Q is a Dynkin quiver, then $\Gamma(\mathcal{C}_Q) = \Gamma_{tr}$. Otherwise, Γ_{tr} is isomorphic to the repetitive quiver $\mathbb{Z}Q$ of Q , and $\Gamma(\mathcal{C}_Q)$ has infinitely many additional, so-called *regular*, components which are either stable tubes (if Q is euclidean), or of type $\mathbb{Z}\mathbb{A}_\infty$ (if Q is wild). Now, it is shown in [15] that there exists a bijection X_γ (called *canonical cluster character*) between the isomorphism classes of indecomposable objects M in \mathcal{C}_Q which have no self-extensions and the cluster variables X_M . A cluster variable which is the image of a transjective object in \mathcal{C}_Q under the canonical cluster character is called *transjective*. The others will be called *nontransjective*.

Lemma 4.5. *Let Q be a quiver of type $\tilde{\mathbb{A}}$ with p clockwise oriented arrows and q counterclockwise oriented arrows. Then there are exactly $p(p-1) + q(q-1)$ nontransjective cluster variables.*

Proof. According to the description of the cluster category \mathcal{C}_Q , the non-transjective cluster variables are in bijection with the regular indecomposable objects in \mathcal{C}_Q without self-extensions lying in the stable tubes. Now such an indecomposable lies necessarily in one of the two exceptional tubes of ranks p and q . Using the fact that the tubes are standard, it is easily seen that the tube of rank p (resp. q) contains exactly $p(p-1)$ (resp. $q(q-1)$) objects without self-extensions. \square

Theorem 4.6. *The transjective cluster variables are exactly those appearing on the SL_2 -tiling. The nontransjective variables are exactly those appearing on the continuant trees of the decorated $\tilde{\mathbb{A}}$ -quivers.*

Proof. It is known [4] that the transjective variables are exactly those that are obtained by mutating only on sources or on sinks of the quivers. Now, as already observed in [6], the variables appearing on the SL_2 -tiling are obtained by this kind of mutations, hence are all transjective. Thus the nontransjective variables all appear on the continuant trees. Hence it suffices to show that the number of variables on the continuant trees is at most $p(p-1) + q(q-1)$. Now, the sequence of column linearization coefficients is periodic of period p ; and the variables on the continuant trees corresponding to columns are of the form $q(\alpha_1, \dots, \alpha_k)$ with $1 \leq k \leq p-1$,

since p is the maximum length of a clockwise oriented arrow in the cyclic part of a decorated $\tilde{\mathbb{A}}$ -quiver in the mutation class of G_0 , because it has necessarily parameters p, q . Therefore, the number of such variables is at most the number of factors of length k , $1 \leq k \leq p - 1$, of the sequence of column linearization coefficients; hence there are at most $p(p - 1)$ such variables. This ends the proof, by symmetry. \square

As in the Introduction, let \mathcal{M} denote the group generated by the set $\{\mu_1, \dots, \mu_n\}$ of mutations, subject to the relations $\mu_i^2 = 1$.

Theorem 4.7. *Let $i \in \{1, \dots, n\}$. Let y be a nontransjective cluster variable. The set of $m \in \mathcal{M}$ such that $y_i(S_0^m) = y$ is a union of cosets of a subgroup of finite index of \mathcal{M} .*

Proof. Consider the set of decorated $\tilde{\mathbb{A}}$ -quivers in the mutation class of G_0 , with embedding considered modulo the subgroup H of \mathbb{Z}^2 generated by the vectors $(p, 0)$ and $(0, q)$. This set is finite. By a remark made in Subsection 4.3, the continuant trees attached to G using π depend only on $\pi \bmod H$. Now \mathcal{M} acts on this finite set. Moreover, $y_i(S_0^m) = y$ is equivalent to the fact that $G_0.m = G$ satisfies: i is a vertex of one of the continuant trees of G and to this vertex is associated the variable y . These two conditions depend only on the previous action. \square

Observe that the transjective variables are given, following [6], by Formula (2.9) in Theorem 2.7; this formula gives at the same time positivity and the Laurent phenomenon. We conclude this subsection by giving a similar formula for nontransjective variables. We limit ourselves to the case where these variables are obtained as continuant polynomials of column linearization coefficients, the case of rows being symmetric.

Given a finite set of consecutive columns of the SL_2 -tiling associated to a frontier, we call *word* of this set the word that codes the intersection of the frontier with this set of columns, augmented with the first step to its left and the first to its right. For example the word of the set of columns containing the variables from x_0 to x_4 in Figure 2.4 is

$$x_{-4}xx_{-3}yx_{-2}yx_{-1}yx_0xx_1xx_2yx_3xx_4xx_5.$$

Note that the first and last steps of the path are always horizontal.

Theorem 4.8. *Consider an SL_2 -tiling t associated to some frontier with variables in K . Let C_1, \dots, C_k be k successive columns of t , with linearization coefficients $\alpha_1, \dots, \alpha_k$. Let $w = x_0\xi_1 \cdots \xi_n x_{n+1}$ be the word associated*

to this set of columns. Then the continuant polynomial $q(\alpha_1, \dots, \alpha_k)$ is equal to

$$\frac{1}{x_1 x_2 \cdots x_n} (x_0, 1) M(x_1, \xi_2, x_2) \cdots M(x_{n-1}, \xi_{n-1}, x_n) (1, x_{n+1})^t.$$

Lemma 4.9. *Let $x_1, \dots, x_n, y_1, \dots, y_p$ be nonzero elements of K . Then*

$$\begin{aligned} & \frac{1}{x_1 \cdots x_n y_1 \cdots y_p} M(x_1 y x_2 \cdots y x_n x y_1 y y_2 \cdots y y_p) \\ &= \frac{1}{x_1 \cdots x_n} M(x_1 y x_2 \cdots y x_n) \begin{pmatrix} 1 & \\ & y_1 \end{pmatrix} \frac{1}{y_1 \cdots y_p} (x_n, 1) M(y_1 y y_2 \cdots y y_p) \\ & \qquad \qquad \qquad - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Proof. We have

$$\begin{aligned} & M(x_1 y x_2 \cdots y x_n x y_1 y y_2 \cdots y y_p) \\ & \qquad \qquad \qquad = M(x_1 y x_2 \cdots y x_n) M(x_n x y_1) M(y_1 y y_2 \cdots y y_p). \end{aligned}$$

Now

$$\begin{aligned} M(x_n x y_1) &= \begin{pmatrix} x_n & 1 \\ 0 & y_1 \end{pmatrix} = \begin{pmatrix} x_n & 1 \\ x_n y_1 & y_1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ x_n y_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \\ & y_1 \end{pmatrix} (x_n, 1) - x_n y_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Moreover

$$\begin{aligned} & M(x_1 y x_2 \cdots y x_n) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M(y_1 y y_2 \cdots y y_p) \\ &= \begin{pmatrix} x_2 & 0 \\ 1 & x_1 \end{pmatrix} \cdots \begin{pmatrix} x_n & 0 \\ 1 & x_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_2 & 0 \\ 1 & y_1 \end{pmatrix} \cdots \begin{pmatrix} y_p & 0 \\ 1 & y_{p-1} \end{pmatrix} \\ & \qquad \qquad \qquad = \begin{pmatrix} 0 & 0 \\ x_1 \cdots x_{n-1} y_2 \cdots y_p & 0 \end{pmatrix}. \end{aligned}$$

Putting all this together, we obtain the lemma. \square

Proof. (Theorem 4.8) Consider first the case $k = 1$, that is, there is only the column C_1 . Then

$$w = x_0 x x_1 y x_2 \dots y x_n x x_{n+1}.$$

Note that x_1 is in column C_1 . Let P be the point in the discrete plane at the right of the point on the frontier which is labelled by the variable x_1 . Then

the linearization coefficient α_1 of column C_1 is equal to $\frac{x_0+t(P)}{x_1}$. Now, the word associated to P (as in Subsection 2.6) is $x_1yx_2\dots yx_nxx_{n+1}$. Thus by Th. 2.7, we have

$$t(P) = \frac{1}{x_2 \cdots x_n} (1, x_1) M(x_2, y, x_3) \cdots M(x_{n-1}, y, x_n) (1, x_{n+1})^T.$$

Let

$$A = \begin{pmatrix} b \\ d \end{pmatrix} = M(x_1, y, x_2) \cdots M(x_{n-1}, y, x_n) (1, x_{n+1})^T.$$

Then, since the second row of $M(x_1, y, x_2)$ is $(1, x_1)$, we have $t(P) = \frac{d}{x_2 \cdots x_n}$. Since $M(x_1, y, x_2) \cdots M(x_{n-1}, y, x_n)$ is a product of lower triangular matrices, its $(1,1)$ -entry is by definition of $M(a, y, b)$ equal to $x_2 \cdots x_n$. Thus $b = x_2 \cdots x_n$. Now $(x_0, 1)A$ is equal to $x_0b + d$. Divided by $x_1 \cdots x_n$, this gives

$$\frac{x_0b + d}{x_1 \cdots x_n} = \frac{x_0x_2 \cdots x_n + d}{x_1 \cdots x_n} = \frac{x_0 + \frac{d}{x_2 \cdots x_n}}{x_1} = \alpha_1.$$

This proves the result for $k = 1$.

We now verify that the expression $r_k = r(\alpha_1, \dots, \alpha_k)$ in the theorem satisfies the recursion (2.2), first for $k = 2$ then for $k \geq 3$. This will end the proof.

Let C_1, C_2 be two successive columns, with α_1, α_2 as respective linearization coefficients. Then the words associated to the sets of columns $\{C_1\}$, $\{C_2\}$ and $\{C_1, C_2\}$ are respectively

$$x_0x_1yx_2 \cdots yx_nxy_1, x_nxy_1yy_2 \cdots yy_pxz, x_0xx_1yx_2 \cdots yx_nxy_1yy_2 \cdots yy_pxz$$

for some integers n, p . Thus

$$r_1 = (1/x_1 \cdots x_n)(x_0, 1)M(x_1yx_2 \cdots x_n)(1, y_1)^T$$

and $r_2 = (1/x_1 \cdots x_ny_1 \cdots y_p)(x_0, 1)M(x_1yx_2 \cdots x_nxy_1 \cdots y_p)(1, z)^T$. Now multiply the identity of the lemma on the left by $(x_0, 1)$ and on the right by $(1, z)^T$. We obtain r_2 on the left-hand side. On the right-hand side we have $r_1\alpha_2 - 1$, since by the first part of the proof

$$\alpha_2 = \frac{1}{y_1 \cdots y_p} (x_n, 1)M(y_1yy_2 \cdots yy_p)(1, z)^T$$

and since

$$(x_0, 1) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (1, z)^T = 1.$$

Thus $r_2 = r_1\alpha_2 - 1$.

Let C_1, C_2, \dots, C_k be k successive columns. Then the words associated to the three sets of columns $\{C_1, \dots, C_{k-2}\}$, $\{C_1, \dots, C_{k-1}\}$, $\{C_1, \dots, C_k\}$ are respectively of the form

$$wx x_1, wx x_1 y x_2 \cdots y x_n x y_1, wx x_1 y x_2 \cdots y x_n x y_1 y y_2 \cdots y y_p x z,$$

for some word w and some natural numbers n, p . Moreover, the word associated to $\{C_k\}$ is

$$x_n x y_1 y y_2 \cdots y y_p x z.$$

Thus

$$\begin{aligned} r_{k-2} &= (1/X)(x_0, 1)M(w)(1, x_1)^T, \\ r_{k-1} &= (1/X x_1 \cdots x_n)(x_0, 1)M(wx x_1 y x_2 \cdots x_n)(1, y_1)^T \end{aligned}$$

and

$$r_k = (1/X x_1 \cdots x_n y_1 \cdots y_p)(x_0, 1)M(wx x_1 y x_2 \cdots x_n x y_1 \cdots y_p)(1, z)^T,$$

where x_0 is the first letter of w and where X is the product of all the variables in w , except x_0 . We multiply the identity of the lemma on the left by $(1/X)(x_0, 1)M(wx x_1)$ and on the right by $(1, z)^T$. We obtain an identity whose left-hand side is

$$\begin{aligned} &(1/X)(x_0, 1)M(wx x_1) \frac{1}{x_1 \cdots x_n y_1 \cdots y_p} M(x_1 y x_2 \cdots y x_n x y_1 y y_2 \cdots y y_p)(1, z)^T \\ &= \frac{1}{X x_1 \cdots x_n y_1 \cdots y_p} (x_0, 1)M(wx x_1 \cdots x_n x y_1 \cdots y_p)(1, z)^T, \end{aligned}$$

that is, r_k ; its right-hand side is equal to

$$\begin{aligned} &(1/X)(x_0, 1)M(wx x_1) \left[\frac{1}{x_1 \cdots x_n} M(x_1 y x_2 \cdots y x_n) \begin{pmatrix} 1 \\ y_1 \end{pmatrix} \frac{1}{y_1 \cdots y_p} (x_n, 1) \right. \\ &\quad \left. M(y_1 y y_2 \cdots y y_p) - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] (1, z)^T \\ &= \frac{1}{X x_1 \cdots x_n} (x_0, 1)M(wx x_1 y x_2 \cdots y x_n) \begin{pmatrix} 1 \\ y_1 \end{pmatrix} \frac{1}{y_1 \cdots y_p} (x_n, 1) \\ &\quad M(y_1 y y_2 \cdots y y_p)(1, z)^T - (1/X)(x_0, 1)M(wx x_1) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (1, z)^T. \end{aligned}$$

This is equal to $r_{k-1}\alpha_k - r_{k-2}$, since by the first part of the proof

$$\alpha_k = \frac{1}{y_1 \cdots y_p} (x_n, 1)M(y_1 y y_2 \cdots y y_p)(1, z)^T,$$

since $M(wx_1) = M(w)M(z_0xx_1)$ (where z_0 is the last letter of w), and since

$$\begin{pmatrix} z_0 & 1 \\ 0 & x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (1, z)^T = (1, x_1)^T.$$

□

4.6. A linear representation of the mutation group

Recall that \mathcal{M} denotes the group of mutations. We have a right action of \mathcal{M} on the finite set \mathcal{G} of decorated $\tilde{\mathbb{A}}$ -quivers with n vertices and parameters p, q , defined on the generators (the mutations) in Subsection 4.1. We denote by $G.m$ this action, for $G \in \mathcal{G}$ and $m \in \mathcal{M}$. Likewise (see Subsection 4.3) \mathcal{M} acts on the right on the set of pairs (G, π) , where π is an embedding of G into \mathbb{Z}^2 . Note that, by definition of the mutations, we have a compatibility condition between both actions: $(G, \pi).m = (G.m, \pi')$.

Denote by \mathcal{G}_0 the finite set of pairs (G, u) , where u is a point of G . There is a natural action of \mathcal{M} on the pairs $(G, u) \in \mathcal{G}_0$, that we denote by $(G, u).m$ for $m \in M$. This action has been defined on the generators at the end of Subsection 4.3.

To G and π as above, we have associated in Subsection 4.3 a point $u = u(G, \pi)$ of G . The following lemma is a consequence of Lemma 4.3.

Lemma 4.10. *For any $m \in M$, one has $u((G, \pi).m) = (G, u).m$.*

We now define a function $\delta : \mathcal{G}_0 \times \mathcal{M} \rightarrow \mathbb{Z}[x, x^{-1}]$, where \mathcal{M} is the group of mutations. This mapping is defined as follows: let $(G, u) \in \mathcal{G}_0$, $m \in M$; let π be some embedding of $G \in \mathcal{G}$ into \mathbb{Z}^2 such that $u = u(G, \pi)$; and define $(G', \pi') = (G, \pi).m$; let $i = \xi(G', \pi') - \xi(G, \pi)$; then $\delta((G, u), m)$ is the Laurent monomial x^i . This is well-defined, that is, does not depend on the chosen embedding π satisfying $u = u(G, \pi)$.

From this construction follows

Lemma 4.11. *One has for any $m, m' \in M$,*

$$\delta((G, u), mm') = \delta((G, u), m)\delta((G, u).m, m').$$

We can now define a linear representation of the group of mutations.

Lemma 4.12. *For $m \in M$, define a matrix $\mu(m)$, indexed by \mathcal{G}_0 as follows: for any $(G, u) \in \mathcal{G}_0$, the $((G, u), (G, u).m)$ -entry is equal to the element $\delta((G, u), m)$. The other entries are 0. Then μ is a homomorphism from M into the group $GL_N(\mathbb{Z}[x, x^{-1}])$, where N is the cardinality of \mathcal{G}_0 .*

Proof. The matrix $\mu(m)$ has exactly one nonzero entry in each row and each column, and this entry is a Laurent monomial. Hence it is an element of $GL_N(\mathbb{Z}[x, x^{-1}])$. The fact that it is a homomorphism follows from Lemma 4.11. \square

Theorem 4.13. *Let $k \in \{1, \dots, n\}$. Then the function*

$$\mathcal{M} \rightarrow R = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}], m \mapsto y_k(S_0^m)$$

is a representative function of the group of mutations \mathcal{M} with values in R .

Proof. We define the initial embedding π_0 of the initial seed S_0 as at the beginning of the proof of Th. 4.4. Let $u_0 = u(G, \pi)$. Define $\lambda \in R^{1 \times \mathcal{G}_0}$ by $\lambda_{(Q_0, u_0)} = x^{i_0}$, where $i_0 = \xi(Q_0, \pi_0)$, while the other components of λ are 0. Define now $\gamma \in R^{\mathcal{G}_0 \times 1}$ by: if k is not on the cyclic part of G , then $\gamma_{(G, u)} = 0$; if k is on the cyclic part of G , choose some vector $(i, j) \in \mathbb{Z}^2$ such that for any embedding π of G , $\pi(k) = u(G, \pi) + (i, j) + \mathbb{Z}(p, -q)$ (see Lemma 4.2); then let $\gamma_{(G, u)} = x^i y^j$.

It is seen that then one has $\lambda \mu(m) \gamma = x^i y^j$ where $(Q_0, \pi_0).m = (G, \pi)$, with $\pi(k) = (i, j) + \mathbb{Z}(p, -q)$ if k is on the cyclic part of G , and $= 0$ otherwise. Call ϕ the representative function $\phi(m) = \lambda \mu(m) \gamma$ of M with value in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$.

Now, by Prop.3.1 and Prop.3.4, the composition $\psi = t \circ \phi$ is a representative function of \mathcal{M} with value in R such that $\psi(m)$ is equal, by Th.4.4 to $y_k(S_0^m)$ if k lies in the cyclic part of $Q_0.m$, and to 0 otherwise. Moreover the set of $m \in \mathcal{M}$ such that k does not lie in the cyclic part of $Q_0.m$ and has as associated variable a fixed variable y is by Th.4.7 a finite union of cosets of a normal subgroup of \mathcal{M} ; thus the theorem follows from Corollary 3.3 and the additivity of representative functions. \square

4.7. A noncommutative rational series

Consider the free monoid \mathcal{M} generated by the set $\{\mu_1, \dots, \mu_n\}$ of mutations. In this monoid consider the subset L of words m that do not contain two successive occurrences of the same letter.

Theorem 4.14. *Let $k \in \{1, \dots, n\}$. The series $\sum_{m \in L} y_k(S_0^m)$ is rational over the ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.*

Proof. This follows by the Kleene-Schützenberger theorem ([10, Th. 7.1]): a series is rational if and only if it is a representative function of the monoid \mathcal{M} . It implies, using Th. 4.13, that the series $\sum y_k(S_0^m)$, where the sum is over all elements m of \mathcal{M} , is rational over the ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Now, by Hadamard product with the language L (which is a rational language), the series of the theorem is rational (Cor. III.2.3 in [10]). \square

In view of the positivity conjecture of [25] and the rationality over \mathbb{N} of the sequences considered in [6], it is legitimate to ask if the series of the theorem is also rational over the semiring $\mathbb{N}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. We show by a counterexample that this is not true in general.

Indeed, consider the $\tilde{\mathbb{A}}_2$ -quiver with the arrows $1 \rightarrow 2 \rightarrow 3$ and $1 \rightarrow 3$. The corresponding function δ is shown in Figure 4.7, with the following conventions: ijk denotes the decorated $\tilde{\mathbb{A}}_2$ -quiver with arrows $i \rightarrow j \rightarrow k$ and $i \rightarrow k$ with no attached pre-continuant tree; ik denotes the decorated $\tilde{\mathbb{A}}_1$ quiver with a double arrow $i \rightarrow k$ and with a one node pre-continuant tree j attached by the arrows $i \leftarrow j \leftarrow k$. Note that, due to the special form of these quivers, the distinguished point $u(G, \pi)$ is always equal to i . On the figure, an arrow from vertex a to vertex b labelled μ_i/x^l means that $a.\mu_i = b$ (action of the mutations on the set \mathcal{G}) and $\delta(a, \mu_i) = x^l$ (we use here the formalism of input/output automata); we have represented only one half of the arrows: to the previous arrow is associated the reverse arrow $b \rightarrow a$ with label μ_i/x^{-l} .

All this allows to compute the function δ . For example, let $w(p, q) = (\mu_1\mu_2\mu_3)^p\mu_2\mu_1\mu_2(\mu_3\mu_2\mu_1)^q$. Then we have

$$123.w(p, q) = 321, \delta(123, w(p, q)) = x^{3p}x^2x^{3q} = x^{3p+2+3q}.$$

This implies that starting from the initial seed $(123, \{x_1, x_2, x_3\})$ and initial embedding π_0 , and applying the sequence of mutations $w(p, q)$ gives the seed $((321, \{y_1, y_2, y_3\}),$ and the embedding π with $\pi(1) = \pi_0(1) + (3p + 3q, 0)$.

Suppose now that the series $S(x_1, x_2, x_3)$ of the theorem is, for $k = 1$, rational over the semiring $\mathbb{N}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then replacing each variable x_i by 1, we obtain a series $T = S(1, 1, 1)$ which is rational over \mathbb{N} . Since the set of $w(p, q)$, $p, q \in \mathbb{N}$ is a rational language, the series $W = \sum(T, w(p, q))w(p, q)$ is also rational over \mathbb{N} , see [10] Cor. III.2.3. Hence the set of words $w(p, q)$ whose coefficient in W is 1 must be a rational language, by [10] Cor. III.2.7. Now, the SL_2 -tiling over \mathbb{N} obtained

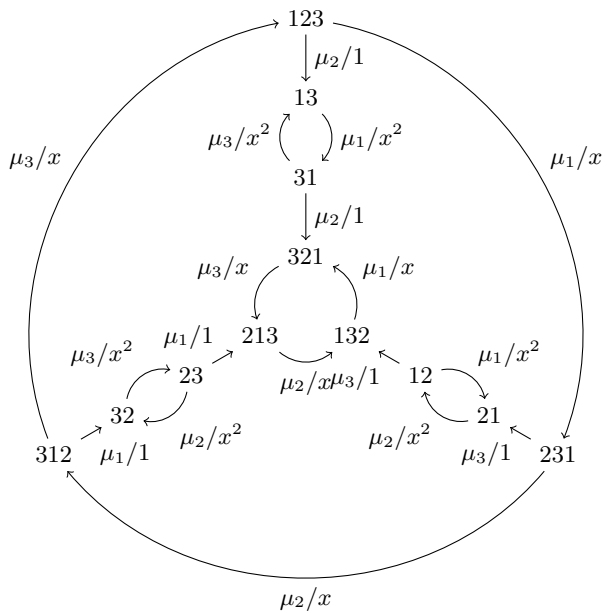


FIGURE 4.7. The function δ

by replacing the variables by 1 has 1's only on its frontier (compare with the SL_2 -tiling given on p.3152 in [6]); moreover the only linearization coefficient which come into play here is 2. Hence this language is the set of words $w(p, p)$: this set is well-known to be not a rational language.

5. Case \mathbb{A}

5.1. Mutation of a continuant tree at the root

We show that, under suitable hypothesis (which are satisfied in the case \mathbb{A}), each continuant tree has at least two structures of continuant tree, if one changes the parameters.

Lemma 5.1. *Let a_1, \dots, a_{n+3} be a sequence of elements of R such that*

$$Q(a_1)Q(a_2) \cdots Q(a_{n+3}) = -1. \quad (5.1)$$

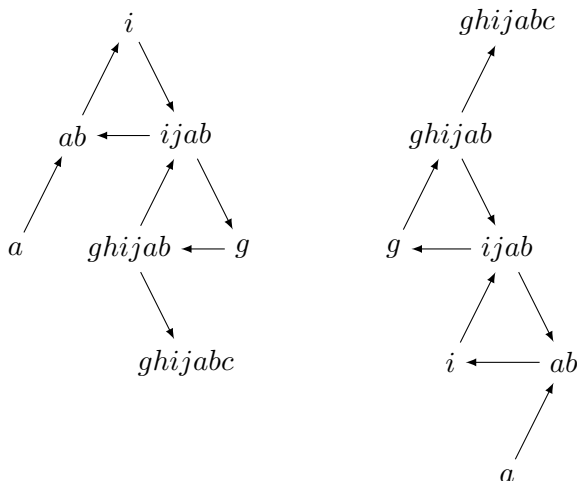


FIGURE 5.1. The tree of Figure 1 viewed differently

Let G be a continuant tree with root equal to the word $a_1 \cdots a_n$. Then for some k with $1 \leq k \leq n$, G has a leaf labelled k and there is a continuant tree G' isomorphic to G as labelled quiver, with the root of G' corresponding to k in G .

The lemma is illustrated by Figure 5.1, which shows the same quiver as in Figure 2.1: at the left the nodes get new words, which give the same continuant polynomials thanks to lemma 2.3 (under the hypothesis $Q(abcdefghij) = -1$; for example, $q(i) = q(abcdefg)$, and also $q(efg) = q(ijab)$ since we have also $Q(defghijabc) = -1$); at the right, this quiver is shown to be a continuant tree.

Proof. In order to prove the lemma, we associate to the continuant tree a triangulation of an $n + 3$ -gon whose vertices are labelled by a_1, \dots, a_{n+3} in this order: to each node $a_i \dots a_j$ of the continuant tree, associate the diagonal joining the vertices a_{i-1} and a_{j+1} , with the indices taken modulo $n + 3$; in particular, the root will give the diagonal from a_{n+1} to a_{n+3} . The construction is illustrated in Figure 5.2: this triangulation corresponds to the tree of Figure 2.1, but also to the tree of Figure 5.1,

right part. Inspection of this example shows that for a given triangulation, one obtains a corresponding tree for each isolated vertex (that is, a vertex without incident diagonal), which corresponds to the root of that tree. Since each triangulation has at least two isolated vertices, the lemma follows if one takes into account the identity on continuant polynomials given by Lemma 2.3: indeed, the hypothesis of this lemma implies that $Q(a_i)Q(a_{i+1}) \cdots Q(a_{n+3})Q(a_1) \cdots Q(a_{i-1}) = -1$, so that $q(a_i \cdots a_j) = q(a_{j+2} \cdots a_{n+3} a_1 \cdots a_{i-2})$. \square

5.2. The mutated seeds in type \mathbb{A}

Consider the Dynkin diagram \mathbb{A}_n , with vertex set $\{1, \dots, n\}$, edge set $\{\{i, i+1\}, i = 1, \dots, n-1\}$. We give to this diagram some orientation, obtaining the initial quiver Q_0 , which determines the initial seed $S_0 = (Q_0, \{x_1, \dots, x_n\})$.

The following result is essentially due to Conway and Coxeter, see [17, (18) p. 91] and [18, p. 177-178].

Theorem 5.2. *There exists a sequence a_1, \dots, a_{n+3} of Laurent polynomials in the initial variables, with coefficients in \mathbb{N} , such that Eq. (5.1) holds and that each mutated seed is a continuant tree with vertex set $\{1, \dots, n\}$ and with root $(a_{i+1}, \dots, a_{i+n})$, for some $i = 1, \dots, n+3$, with indices taken mod $n+3$.*

Proof. It is enough by Lemma 2.4 and Lemma 5.1 to show that there exists a sequence as in the statement such that the initial seed S_0 is a continuant tree of the form described in the statement. Consider the frieze with variables associated to the quiver Q_0 , see [6] 8.2, [14] Section 5. Its period is $n+3$. The lemma follows using the same method as in [9] 8.1, with \mathbb{N} replaced by the semiring $\mathbb{N}[x_i^{\pm 1}]$, and the formula of [9] which gives the value of each entry of the tiling using a continuant polynomial. \square

Note that mutation of a continuant tree corresponds to the classical flip of a triangulation: in some quadrilateral, replace one diagonal by the opposite one. This is illustrated in Figure 5.3: the triangulation on the right part corresponds (this correspondance is explained in the proof of Lemma 5.1) to the continuant tree on the left part, which, when mutated at vertex $abcde$ gives the continuant tree of Figure 2.1, corresponding to the triangulation of Figure 5.2: the two triangulations are obtained by exchanging the diagonals hc and jf . This is a particular case of a general

MUTATING SEEDS: TYPES \mathbb{A} AND $\tilde{\mathbb{A}}$.

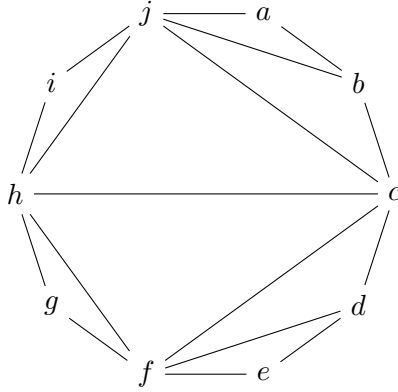


FIGURE 5.2. The triangulation corresponding to the continuant trees of Figures 1 and 12

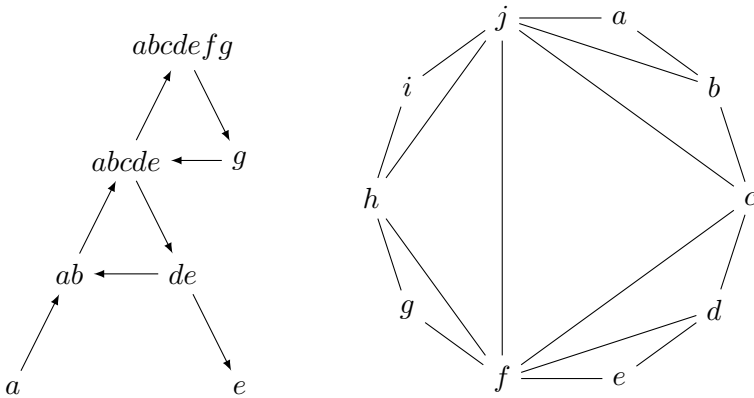


FIGURE 5.3. Another continuant tree and its associated triangulation

construction indicated in [24] 12.2, see also [14] Section 5. Our approach however is different and quite elementary.

6. Conjectures

Theorem 4.4 describes each seed in the mutation class of the initial seed S_0 ; it is the seed which is naturally associated (as explained in the theorem) to some decorated \tilde{A} -quiver G and some embedding π of G . Conversely, it seems likely that each seed of this form is in the mutation class of the initial seed.

Moreover, we conjecture that Th. 4.7, Th. 4.13 and Th. 4.14 extend to all euclidean diagrams. Note that for Dynkin diagrams, these extensions are immediate since the mutation classes of seeds are finite, by Fomin and Zelevinsky's finite type classification [26].

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