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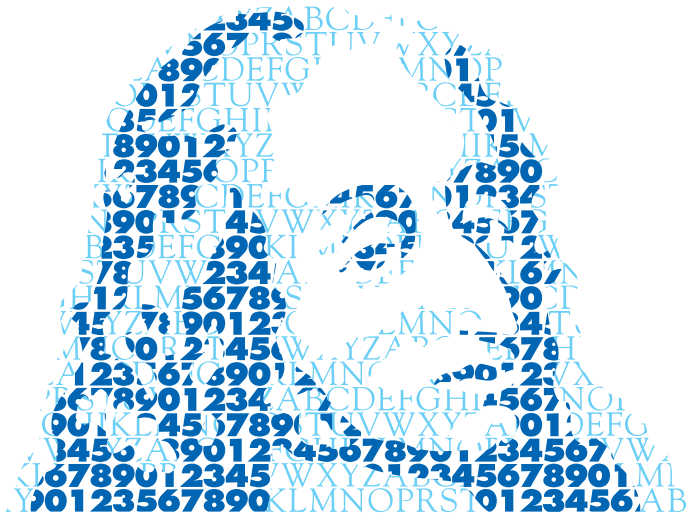

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Fully Invariant Subgroups of n-Summable Primary Abelian Groups

Peter Danchev

Abstract

We present a number of results concerning fully invariant subgroups of n-summable groups.

1. Introduction

Throughout this paper suppose all groups are Abelian, p-primary for some arbitrary but a fixed prime, written additively as is the custom when exploring Abelian groups. All notions and notations are standard and follow those from [8] and [9]. For instance, for any element g from a group G, the letter $|g|_G$ will always denote the height of g in G, that is, the maximal ordinal δ such that $g \in p^\delta G$; whence $g \in p^\delta G \setminus p^{\delta+1}G$.

A problem of interest is to find suitable properties of groups that are inherited by special subgroups called *fully invariant*. Recall that a subgroup F of a group G is said to be fully invariant if each endomorphism of G sends F into itself. For some classes of groups (e.g., fully transitive groups or, in particular, totally projective groups) these subgroups are completely described - see, for instance, [9] and [11]. Specifically, they have the form $G(\alpha) = \{x \in G : |p^i x|_G \geq \alpha_i, i < \omega\}$, where $\alpha = \{\alpha_i\}_{i<\omega}$ is an increasing sequence of ordinals and symbols ∞. In addition, for any ordinal γ, $p^\gamma G$ is fully invariant in G.

Moreover, it was shown in [10] and [12] that a group G is totally projective if, and only if, $G(\alpha)$ and $G/G(\alpha)$ are totally projective; however whether or not this remains true for some arbitrary fully invariant subgroup F is still unknown. It is worthwhile noticing that this extends the classical result of Nunke [13] which says that G is totally projective if, and only if, both $p^\alpha G$ and $G/p^\alpha G$ are totally projective.

Keywords: n-summable groups, fully invariant subgroups, quotients, σ-summable groups.
Math. classification: 20K10.

The purpose that motivates the writing of the present article is to try to modify to what extent the quoted above results for totally projective groups can be proved for summable groups and their generalizations, named n-summable groups. This continues our recent results in this directions - cf. [5, 2, 4, 3].

2. n-Summable groups and their fully invariant subgroups

Imitating [7], the valuated p^n-socle V is said to be *n-summable* if it is isometric to the valuated direct sum $\oplus_{i\in I}V_i$ of a collection of countable valuated p^n-socles V_i where $i \in I$ and I is an index set. A group G is said to be n-summable if $G[p^n]$, as a valuated p^n-socle under the valuation induced by the height function, is n-summable. In particular, a group is 1-summable exactly when it is summable in the usual sense of the term. For more detailed information about summable and n-summable groups, we refer the interested reader to [8] and [7].

Due to the importance of Nunke-like theorems, referred to above [13], we prove the following (see also [1]).

Proposition 2.1. *Suppose G is a group and λ is an ordinal.*

(a) *If G is n-summable, then $p^\lambda G$ and $G/p^{\lambda+n}G$ are n-summable.*

(b) *If λ is countable and both $p^\lambda G$ and $G/p^\lambda G$ are n-summable, then G is n-summable.*

Proof. (a): If $G[p^n]$ is the valuated direct sum of countable valuated groups, then the same can clearly be said of $(p^\lambda G)[p^n]$. Next, let H be a $p^{\lambda+n}$-high subgroup of G, and let $H' = [H + p^{\lambda+n}G]/p^{\lambda+n}G$, so that $H \cong H'$ are isotype in G and $G' = G/p^{\lambda+n}G$, respectively. It follows that there are valuated direct sum decompositions

$$G[p^n] = H[p^n] \oplus X \text{ and } G'[p^n] = H'[p^n] \oplus X'$$

where $p^{\lambda+n}G \subseteq X \subseteq p^\lambda G$ and $X' \subseteq p^\lambda G'$. Note that X' is, in fact, n-summable (see, e.g., [7]). Therefore, since G is n-summable, we can conclude $H[p^n]$ and X are n-summable. This, in turn, implies that $H'[p^n]$ and $p^{\lambda+n}G$ are n-summable, so that G' and $p^{\lambda+n}G$ are n-summable. The fact that $p^{\lambda+n}G$ is n-summable readily implies that $p^\lambda G$ shares this property.

(b): Let H and X be as in part (a). Since $p^\lambda G$ is n-summable, we can infer that X has this property. We need, therefore, to show that $H[p^n]$ is also n-summable. Since $G/p^\lambda G$ is n-summable and λ is countable, we can conclude that $(G/p^\lambda G)[p^n]$ is n-Honda, i.e., it is the ascending union of subgroups Y_m, for $m < \omega$, such that $\mathcal{S}_n = \{|y|_{G/p^\lambda G} : y \in Y_m\}$ is finite. If we let $Y'_m = \{x \in H : x + p^\lambda G \in Y_m\}$, then clearly $H[p^n]$ is the ascending union of the Y'_m and $\{|x|_G : x \in Y'_m\} \subseteq \mathcal{S}_n \cup [\lambda, \lambda + n - 1]$ is finite. Therefore, $H[p^n]$ is also n-Honda, and hence n-summable. □

There are lots of examples of summable groups G such that $G/p^\omega G$ is not summable; in fact suppose B is an unbounded direct sum of cyclics and $\overline{B}$ is its torsion-completion. Then $G = \overline{B}/B[p^n]$ can easily be seen to be n-summable, but on the other hand

$$G/p^\omega G = \overline{B}/B[p^n]/\overline{B}[p^n]/B[p^n] \cong \overline{B}/\overline{B}[p^n] \cong p^n\overline{B}$$

is not even summable. So Proposition 2.1(a) does not hold if $\lambda + n$ is replaced by λ.

In addition, if G is a totally projective group of length λ, where $\omega_1 < \lambda < \omega_1 \cdot 2$, then $p^{\omega_1}G$ and $G/p^{\omega_1}G$ are both n-summable, but G is not; so Proposition 2.1(b) does not hold for uncountable λ.

As an immediate consequence, we yield the following (see [1] too):

Corollary 2.2. *A group G is summable iff $p^{\omega+1}G$ and $G/p^{\omega+1}G$ are summable.*

The last statement can slightly be extended to the following.

Proposition 2.3. *A group G is summable if, and only if, $p^{\omega+k}G$ and $G/p^{\omega+k}G$ are summable for some $k < \omega$.*

Proof. First, observe that

$$G/p^{\omega+1}G \cong G/p^{\omega+k}G/p^{\omega+1}G/p^{\omega+k}G = G/p^{\omega+k}G/p^{\omega+1}(G/p^{\omega+k}G).$$

Moreover, since a group A is summable uniquely when so is p^mA for some natural m, we derive that $p^{\omega+k}G$ is summable precisely when the same holds for $p^{\omega+1}G$. Henceforth, we apply the above corollary to infer the wanted claim. □

Lemma 2.4. *If $L = (p^{\omega+1}G)[p]$ and G/L is summable, then G is summable.*

Proof. First note that

$$p^{\omega+2}G = p(p^{\omega+1}G) \cong (p^{\omega+1}G)/(p^{\omega+1}G)[p] = p^{\omega+1}G/L = p^{\omega+1}(G/L)$$

and, hence, $p^{\omega+2}G$ is summable. But this plainly implies that $p^{\omega+1}G$ is summable, and in view of Corollary 2.2 it remains only to show that $G/p^{\omega+1}G$ is also summable. Indeed, since $L \subseteq p^{\omega+1}G$, we have

$$G/p^{\omega+1}G \cong G/L/p^{\omega+1}G/L = G/L/p^{\omega+1}(G/L)$$

and the latter group is really summable because G/L is so by hypothesis. □

We will hereafter assume that all fully invariant subgroups of G are of the type

$$G(\alpha) = \{x \in G : |p^i x|_G \geq \alpha_i, i < \omega\}$$

where $\alpha = \{\alpha_i\}_{i<\omega}$ is an increasing sequence of ordinals and symbols ∞.

Corollary 2.5. *If $L = G(\alpha)$ and G/L is summable, then G/pL is summable.*

Proof. Observe that $L/pL = (p^\alpha(G/pL))[p]$ where $\alpha = \alpha_0 = \omega + 1$. Therefore, $(G/pL)/(L/pL) \cong G/L$ is summable and we need only employ the previous lemma to G/pL. □

So, we come to the following sufficient condition for summability.

Theorem 2.6. *Let $p^{\omega\cdot 2}G = 0$. If α is an increasing sequence of ordinals and symbols ∞ such that both $G(\alpha)$ and $G/G(\alpha)$ are summable, then G itself is summable.*

Proof. Let $L = G(\alpha)$. If the sequence α contains any symbol ∞, then there is a positive integer j such that $p^j L = 0$. But then repeated applications of Corollary 2.5 yield the desired conclusion that $G/p^j L \cong G$ is summable. Thus we may assume that α is an increasing sequence of ordinals that does not contain symbols of the type ∞, and take $\lambda = sup(\alpha)$ where $\alpha = \{\alpha_i\}_{i<\omega}$. It is not hard to see that $p^\omega L = p^\lambda G$ where $\lambda \leq \omega \cdot 2$.

Foremost, suppose for a moment $\lambda = \omega \cdot 2$, whence $p^\omega L = 0$ so that L is separable summable and thus a direct sum of cyclic groups. In view of [12], we obtain that G is σ-summable. Furthermore, appealing to [6, Proposition 6.5], we deduce that $p^\omega G$ is σ-summable, i.e., a direct sum of cyclic groups and hence summable. Therefore $p^{\omega+i}G$ is summable for any natural number i.

Next, suppose that $\lambda < \omega \cdot 2$. Consequently, $p^\lambda G$ is summable. But $\lambda = \omega + m$ for some $m < \omega$, so that $p^{\omega+m}G$ being summable immediately forces that $p^{\omega+i}G$ is summable for all $i < \omega$.

Furthermore, in virtue of Proposition 2.3, we need only verify in the both cases for λ alluded to above that $G/p^{\alpha_i}G$ is summable for some $i \geq 1$, where $\alpha_i = \omega + i$. But since $p^i L \subseteq p^{\alpha_i}G$, we have again with this proposition at hand that

$$G/p^{\alpha_i}G \cong (G/p^i L)/p^{\alpha_i}(G/p^i L)$$

is summable since each $G/p^i L$ is summable by the subsequent application i times of Corollary 2.5. □

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