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Amenable actions of amalgamated free products of free groups over a cyclic subgroup and generic property

Soyoung Moon

Abstract

We show that the amalgamated free products of two free groups over a cyclic subgroup admit amenable, faithful and transitive actions on infinite countable sets. This work generalizes the results on such actions for doubles of free group on two generators.

Actions moyennables de produits amalgamés de groupes libres sur un sous-groupe cyclique et propriétés génériques

Résumé

On montre que les produits amalgamés de groupes libres sur un sous-groupe cyclique admettent des actions moyennables, fidèles et transitives sur un ensemble dénombrable infini. Ce travail généralise le résultat concernant de telles actions pour les produits amalgamés de groupes libres sur deux générateurs.

1. Introduction

An action of a countable group G on a set X is *amenable* if there exists a sequence $\{A_n\}_{n\geq 1}$ of finite non-empty subsets of X such that for every $g \in G$, one has

$$\lim_{n\to\infty} \frac{|A_n \vartriangle g \cdot A_n|}{|A_n|} = 0.$$

Such a sequence is called a *Følner sequence* for the action of G on X. Thanks to a result of Følner [3], this definition is equivalent to the existence of a G-invariant mean on subsets of X.

Keywords: Amenable actions, amalgamated free products, Baire's property.
Math. classification: 43A07, 20E06, 57M07.

Definition 1.1. We say that a countable group G is in the class $\mathcal{A}$ if it admits an amenable, faithful and transitive action on an infinite countable set.

The question of understanding which groups are contained in $\mathcal{A}$ was raised by von Neumann and recently studied in a few papers ([1], [4], [5], [6]). In this note we add the following:

Theorem 1.2. *Let n, $m \geq 1$. Let $G = \mathbb{F}_{m+1} *_{\mathbb{Z}} \mathbb{F}_{n+1}$ be an amalgamated free product of two free groups over a cyclic subgroup such that the image of the generator of $\mathbb{Z}$ is cyclically reduced in both free groups. Then any finite index subgroup of G is in $\mathcal{A}$.*

The methods used in this work are analogous to those used in [6] to obtain Theorem 1.2 in case of $m = n = 1$. The role of the generic permutation α in [6] is now played by a n-tuple of permutations $(\alpha_1, ..., \alpha_n)$ and, for a cyclically reduced word $c = c(\alpha_1, ..., \alpha_n)$, we now prove genericity of the set of such n-tuples for which the permutation c has infinitely many orbits of size $k \in \mathbb{N}^*$, and all orbits finite. This new result allows us to apply the method of [6] in our new setting.

For X an infinite countable set, recall that $Sym(X)$ with the topology of pointwise convergence is a Baire space, i.e. every intersection of countably many dense open subsets is dense in $Sym(X)$. So for every $n \geq 1$, the product space $(Sym(X))^n$ is a Baire space. A subset of a Baire space is called *meagre* if it is a union of countably many closed subsets with empty interior; and *generic* or *dense* G_δ if its complement is meagre.

Remark 1.3. The amalgamated products appearing in Theorem 1.2 are known in combinatorial group theory as "*cyclically pinched one-relator groups*" (see [2]). These are exactly the groups admitting a presentation of the form $G = \langle a_1, \ldots, a_n, b_1, \ldots, b_m | c = d \rangle$ where $1 \neq c = c(a_1, \ldots, a_n)$ is a cyclically reduced non-primitive word (not part of a basis) in the free group $\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$, and $1 \neq d = d(b_1, \ldots, b_m)$ is a cyclically reduced non-primitive word in the free group $\mathbb{F}_m = \langle b_1, \ldots, b_m \rangle$. The most important examples of such groups are the surface groups, i.e. the fundamental group of a compact surface. The fundamental group of the closed orientable surface of genus g has the presentation $\langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$. By letting $c = [a_1, b_1] \cdots [a_{g-1}, b_{g-1}]$ and $d = [a_g, b_g]^{-1}$, the group decomposes as the free product of the free group $\mathbb{F}_{2(g-1)}$ on a_1, b_1, $\ldots$,

a_{g-1}, b_{g-1} and the free group $\mathbb{F}_2$ on a_g, b_g amalgamated over the cyclic subgroup generated by c in $\mathbb{F}_{2(g-1)}$ and d in $\mathbb{F}_2$, hence it is a cyclically pinched one-relator group.

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2. Graph extensions

A graph G consists of the set of vertices $V(G)$ and the set of edges $E(G)$, and two applications $E(G) \to E(G)$; $e \mapsto \bar{e}$ such that $\bar{\bar{e}} = e$ and $\bar{e} \neq e$, and $E(G) \to V(G) \times V(G)$; $e \mapsto (i(e), t(e))$ such that $i(e) = t(\bar{e})$. An element $e \in E(G)$ is a *directed edge* of G and $\bar{e}$ is the *inverse edge* of e. For all $e \in E(G)$, $i(e)$ is the *initial vertex* of e and $t(e)$ is the *terminal vertex* of e.

Let S be a set. A *labeling* of a graph $G = (V(G), E(G))$ on the set $S^{\pm 1} = S \cup S^{-1}$ is an application

$$l : E(G) \to S^{\pm 1}; e \mapsto l(e)$$

such that $l(\bar{e}) = l(e)^{-1}$. A *labeled graph* $G = (V(G), E(G), S, l)$ is a graph with a labeling l on the set $S^{\pm 1}$. A labeled graph is *well-labeled* if for any edges e, $e' \in E(G)$, $\big[i(e) = i(e')$ and $l(e) = l(e')\big]$ implies that $e = e'$.

A word $w = w_m \cdots w_1$ on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ is called *reduced* if $w_{k+1} \neq w_k^{-1}$, $\forall 1 \leq k \leq m-1$. A word $w = w_m \cdots w_1$ on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ is called *weakly cyclically reduced* if w is reduced and $w_m \neq w_1^{-1}$; this definition allows w_m and w_1 to be equal. Given a reduced word, we define two finite graphs labeled on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ as follows:

Definition 2.1. Let $w = w_m \cdots w_1$ be a reduced word on $\{\alpha_k^{\pm 1}, \alpha_{k-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$. The *path* of w (Figure 2.1) is a finite labeled graph $P(w, v_0)$ labeled on $\{\alpha_k^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta\}$ consisting of $m+1$ vertices and m directed edges $\{e_1, \ldots, e_m\}$ such that

- $i(e_{j+1}) = t(e_j)$, $\forall 1 \leq j \leq m-1$;
- $v_0 = i(e_1) \neq t(e_m)$;
- $l(e_j) = w_j$, $\forall 1 \leq j \leq m$.

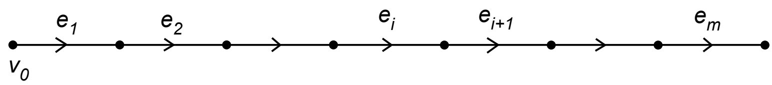

Figure 2.1. A path

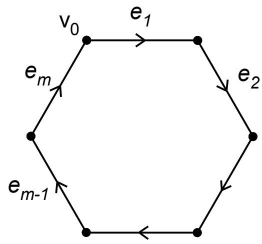

Figure 2.2. A cycle

The point v_0 is the *startpoint* and the point $t(e_m)$ is the *endpoint* of the path $P(w, v_0)$. The two points are the *extreme points* of the path.

Definition 2.2. Let $w = w_m \cdots w_1$ be a reduced word on $\{\alpha_k^{\pm 1}, \alpha_{k-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$. The *cycle* of w (Figure 2.2) is a finite labeled graph $C(w, v_0)$ labeled on $\{\alpha_k^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta\}$ consisting of m vertices and m directed edges $\{e_1, \ldots, e_m\}$ such that

- $i(e_{j+1}) = t(e_j)$, $\forall 1 \leq j \leq m-1$;
- $v_0 = i(e_1) = t(e_m)$;
- $l(e_j) = w_j$, $\forall 1 \leq j \leq m$.

The point v_0 is the *startpoint* of the cycle $C(w, v_0)$.

Notice that since w is a reduced word, the graph $P(w, v_0)$ is well-labeled. If w is weakly cyclically reduced, then $C(w, v_0)$ is also well-labeled.

Conversely, if $P = \{e_1, e_2, \ldots, e_n\}$ is a well-labeled path with $i(e_1) = v_0$, labeled by $l(e_i) = g_i$, $\forall i$, then there exists a unique reduced word $w = g_n \cdots g_1$ such that $P(w, v_0)$ is P. If $C = \{e_1, e_2, \ldots, e_n\}$ is a well-labeled cycle with $t(e_n) = i(e_1) = v_0$, labeled by $l(e_i) = g_i$, $\forall i$, then there exists a unique weakly cyclically reduced word $w_1 = g_n \cdots g_1$ such that $C(w, v_0)$ is C.

Let X be an infinite countable set. Let β be a transitive permutation of X. The *pre-graph* G_0 is a labeled graph consisting of the set of vertices $V(G_0) = X$ and the set of directed edges all labeled by β such that every vertex has exactly one entering edge and one outgoing edge, and $t(e) = \beta(i(e))$. One can imagine G_0 as the Cayley graph of $\mathbb{Z}$ with 1 as a generator.

Definition 2.3. An *extension* of G_0 is a well-labeled graph G labeled by $\{\alpha_k^{\pm 1}, \alpha_{k-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$, containing G_0, with $V(G) = V(G_0) = X$. We will denote it by $G_0 \subset G$.

In order to have a transitive action with some additional properties of the $\langle \alpha_k, \ldots, \alpha_1, \beta\rangle$-action on X, we shall extend inductively G_0 on $1 \leq i \leq k$ by adding finitely many directed edges labeled by α_i on G_0 where the edges labeled by β are already prescribed. In order that the added edges represent an action on X, we put the edges in such a way that the extended graph is well-labeled, and moreover we put an additional edge labeled by α_i on every endpoint of the extended edges by α_i; more precisely, if we have added n edges labeled by α_i between $x_0, x_1, \ldots, x_n$ successively, we put an α_i-edge from x_n to x_0 to have a cycle consisting of $n+1$ edges, which corresponds to a α_i-orbit of size $n+1$. On the points where no α_i-edges are involved, we can put any α_i-edge in a way that the extended graph is well-labeled and every point has an entering edge and an outgoing edge labeled by α_i (for example we can put a loop labeled by α_i, corresponding to the fixed points). In the end, the graph represents an $\langle \alpha_k, \ldots, \alpha_1, \beta\rangle$-action on X, i.e. G will be a Schreier graph.

Definition 2.4. Let G, G' be graphs labeled on a set $S^{\pm 1}$. A *homomorphism* $f : G \to G'$ is a map sending vertices to vertices, edges to edges, such that

- $f(i(e)) = i(f(e))$ and $f(t(e)) = t(f(e))$;
- $l(e) = l(f(e))$,

for all $e \in E(G)$.

If there exists an injective homomorphism $f : G \to G'$, we say that f is an *embedding*, and G *embeds* in G'.

Lemma 2.5. *Let $k \geq 1$. Let $w_k = w_k(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1, \beta)$ be a reduced word on $\{\alpha_k^{\pm 1}, \alpha_{k-1}^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$. For every finite subset F of G_0,* *there is an extension G of G_0 on which the path $P(w_k, v_0)$ embeds in G, the image of $P(w_k, v_0)$ in G does not intersect with F, and $G \setminus G_0$ is finite.*

Proof. Let us show this by induction on k. If $k = 1$, it follows from Proposition 6 in [6]. Indeed, in the proof of Proposition 6 in [6], we start by choosing any element $z_0 \in X$ to construct a path. Since the set X is infinite and there is no assumption on the starting point z_0 of the path, there are infinitely many choices for z_0.

For the proof of the induction step, consider the case

$$w_k = \alpha_k^{a_{2m}} w_{k-1}^{2m-1} \alpha_k^{a_{2m-2}} \cdots \alpha_k^{a_4} w_{k-1}^{3} \alpha_k^{a_2} w_{k-1}^{1}.$$

with $w_{k-1}^i = w_{k-1}^i(\alpha_{k-1}, \dots, \alpha_1, \beta)$ a reduced word on $\{\alpha_{k-1}^{\pm 1}, \dots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$, for all i. To simplify the notation, we assume that a_j is positive, $\forall j$.

Let $F \subset X$ be a finite subset of X. By hypothesis of induction, there is an extension G_1 of G_0 and an embedding f^1 such that $f^1 : P(w_{k-1}^1, v_0) \hookrightarrow G_1$ and the image of $P(w_{k-1}^1, v_0)$ in G_1 does not intersect with F. Let

$$f^1(v_0) = f^1\big(i(P(w_{k-1}^1, v_0))\big) =: z_0$$

and

$$f^1\big(t(P(w_{k-1}^1, v_0))\big) =: z_1.$$

Inductively on each $2 \leq i \leq m$, we apply the following algorithm:

Algorithm

(1) Take an extension G_{2i-2} of G_0 such that
   - $P(w_{k-1}^{2i-1}, v_{2i-2})$ embeds in G_{2i-2} such that the image does not intersect with F;
   - $G_{2i-2} \cap G_{2i-3} = G_0$ (this is possible since there are infinitely many extensions G'_{2i-2} of G_0 by hypothesis of induction and $G_{2i-3} \setminus G_0$ is finite).

(2) Let $f^{2i-1} : P(w_{k-1}^{2i-1}, v_{2i-2}) \hookrightarrow G_{2i-2} \cup G_{2i-3} =: G'_{2i-1}$ with
   - $f^{2i-1}\big(i(P(w_{k-1}^{2i-1}, v_{2i-2}))\big) = f^{2i-1}(v_{2i-2}) =: z_{2i-2}$;
   - $f^{2i-1}\big(t(P(w_{k-1}^{2i-1}, v_{2i-2}))\big) =: z_{2i-1}$.

(3) Choose $a_{2i-2} - 1$ points $\{p_1^{(a_{2i-2})}, \dots, p_{a_{2i-2}-1}^{(a_{2i-2})}\}$ outside of the finite set of all points appeared until now, and put the directed edges labeled by α_k from

- z_{2i-3} to $p_1^{(a_{2i-2})}$;
- $p_j^{(a_{2i-2})}$ to $p_{j+1}^{(a_{2i-2})}$, $\forall 1 \leq j \leq a_{2i-2} - 2$;
- $p_{a_{2i-2}-1}^{(a_{2i-2})}$ to z_{2i-2},

and let $G_{2i-1} := G'_{2i-1} \cup \{$the additional α_k-edges between z_{2i-3} and $z_{2i-2}\}$.

In the ends, we choose new a_{2m} points $\{p_1^{(a_{2m})}, \ldots, p_{a_{2m}}^{(a_{2m})}\}$ and put the directed edges labeled by α_k from z_{2m-1} to $p_1^{(a_{2m})}$, and from $p_j^{(a_{2m})}$ to $p_{j+1}^{(a_{2m})}$, $\forall 1 \leq j \leq a_{2m} - 1$.

By construction, the resulting graph $G_{2m-1} \cup P(\alpha^{a_{2m}}, z_{2m-1}) =: G$ is an extension of G_0 satisfying $P(w_k, v_0) \hookrightarrow G$ such that the image of $P(w_k, v_0)$ does not intersect with F. $\square$

Lemma 2.6. *Let $w = w(\alpha_n, \ldots, \alpha_1, \beta)$ be a weakly cyclically reduced word on $\{\alpha_n^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ such that α_i appears in the word w for some i (i.e. $w \notin \langle \beta \rangle$). For every finite subset F of G_0, there exists an extension G_{n+1} of G_0 such that the cycle $C(w, v_0)$ embeds in G_{n+1} and the image of $C(w, v_0)$ in G_0 does not intersect with F.*

Proof. Let us consider the case

$$w = \alpha_i^{a_{2m}} w_{2m-1} \alpha_i^{a_{2m-2}} \cdots \alpha_i^{a_4} w_3 \alpha_i^{a_2} w_1$$

written as the normal form of $\langle \alpha_n, \ldots, \alpha_{i+1}, \alpha_{i-1}, \ldots, \alpha_1, \beta \rangle * \langle \alpha_i \rangle$.

Since $w' = w_{2m-1} \alpha_i^{a_{2m-2}} \cdots \alpha_i^{a_4} w_3 \alpha_i^{a_2} w_1$ is reduced, by Lemma 2.5, there is an extension G'_{n+1} of G_0 and a homomorphism $f : P(w', v_0) \to G'_{n+1}$ such that $f(P(w', v_0))$ is a path in G'_{n+1} outside of F. Let $f(v_0) =: z_0$ be the startpoint of $f(P(w', v_0))$ and $f(w'(z_0)) =: z_{2m-1}$ be the endpoint of $f(P(w', v_0))$. To simplify the notation, we assume that a_j is positive, $\forall j$.

Choose $a_{2m} - 1$ new points $\{p_{a_m}, \ldots, p_{a_{2m}-1}\}$ and put the directed edges labeled by α_i from

- z_{2m-1} to p_1;
- p_j to p_{j+1}, $\forall 1 \leq j \leq a_{2m} - 2$;
- $p_{a_{2m}-1}$ to z_0.

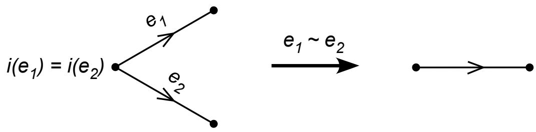

Figure 2.3

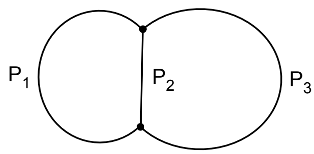

Figure 2.4

By construction, the resulting graph $G_{n+1} := G'_{n+1} \cup P(\alpha^{a_{2m}}, z_{2m-1})$ is an extension of G_0 and $C(w, v_0)$ embeds in G_{n+1} outside of F. $\square$

Let $c = c(\alpha_n, \ldots, \alpha_1, \beta)$ be a weakly cyclically reduced word on $\{\alpha_n^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ such that $c \notin \langle \beta \rangle$ and let $w = w(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1, \beta)$ be a reduced word on $\{\alpha_n^{\pm 1}, \ldots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ such that $w \notin \langle c \rangle$. Let $C(c, v_0)$ be the cycle of c with startpoint at v_0, and let $P(w, v_0)$ be the path of w with the same startpoint v_0 as $C(c, v_0)$ such that every vertex of $P(w, v_0)$ (other than v_0) is distinct from every vertex in $C(c, v_0)$. Let wv_0 be the endpoint of $P(w, v_0)$. Let $C(c, wv_0)$ be the cycle of c with startpoint at wv_0 such that every vertex of $C(c, wv_0)$ (other than wv_0) is distinct from every vertex in $P(w, v_0) \cup C(c, v_0)$. Let us denote by $Q_0(c, w)$ the union of $C(c, v_0)$, $P(w, v_0)$ and $C(c, wv_0)$. Let $Q(c, w)$ be the well-labeled graph obtained from $Q_0(c, w)$ by identifying the successive edges with the same initial vertex and the same label. That is, Q is the quotient graph $Q_0/[e_1 \sim e_2]$ where $e_1 \sim e_2$ if $i(e_1) = i(e_2)$ and $l(e_1) = l(e_2)$ (see Figure 2.3).

Notice that the well-labeled graph $Q(c, w)$ can have one, two or three cycles, and in each type of $Q(c, w)$, the quotient map $Q_0(c, w) \twoheadrightarrow Q(c, w)$ restricted to $C(c, v_0)$ and to $C(c, wv_0)$ is injective (each one separately).

Lemma 2.7. *There is an extension G_{n+1} of G_0 such that $Q(c, w)$ embeds in G_{n+1}.*

Proof. By Lemma 2.5 and 2.6, it is enough to show that every cycle in Q contains edges labeled by $\alpha_i^{\pm 1}$ for some i. For the cases where Q has one or two cycles, it is clear since the cycles in Q represent $C(c, v_0)$ and $C(c, wv_0)$, and $c \notin \langle\beta\rangle$. In the case where $Q(c, w)$ has three cycles, $Q(c, w)$ has three paths P_1, P_2 and P_3 such that $P_1 \cap P_2 \cap P_3$ are exactly two extreme points of P_i's, and $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_1 \cup P_3$ are the three cycles in $Q(c, w)$ (see Figure 2.4). So we need to prove that, if one of the three paths has edges labeled only on $\{\beta^{\pm 1}\}$, then the other two paths both contains edges labeled by $\alpha_i^{\pm 1}$ for some i. For this, it is enough to prove:

Claim. If the reduced word $c = \gamma\lambda$ is conjugate to the reduced word $\gamma\lambda'$ via a reduced word w, where $\gamma \in \langle\alpha_n, \alpha_{n-1}, \ldots, \beta\rangle \setminus \langle\beta\rangle$ and $\lambda \in \langle\beta\rangle$, then $wc = cw$. Furthermore, the word c can not be conjugate to the reduced word $\gamma^{-1}\lambda'$ with $\lambda' \in \langle\beta\rangle$.

Let us see how we can conclude Lemma 2.7 using the Claim. First of all, notice that c does not commute with w since we are treating the case where Q has three cycles. More precisely, in a free group, two elements commute if and only if they are both powers of the same word. So if $cw = wc$, then $c = \gamma^k$ and $w = \gamma^l$ with $k \neq l$, where γ is a non-trivial word, so that Q has one cycle. Suppose that P_1 consists of edges labeled only on $\{\beta^{\pm 1}\}$. One of the cycles among $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_1 \cup P_3$ consists of edges labeled by the letters of c up to cyclic permutation, let us say $P_1 \cup P_2$ (i.e. if $c = c_1 \cdots c_m$, given any startpoint v_0 in $P_1 \cup P_2$, the directed edges of the cycle $C(c, v_0)$ are labeled on a cyclic permutation of the sequence $\{c_m, \ldots, c_1\}$). Another cycle among $P_2 \cup P_3$ and $P_1 \cup P_3$ consists of edges labeled by the letters of the reduced form of $w^{-1}cw$ up to cyclic permutation. Since $c \notin \langle\beta\rangle$, the path P_2 has edges labeled by $\alpha_i^{\pm 1}$ for some i. Now, if the cycle representing $w^{-1}cw$ is $P_1 \cup P_3$, then the path P_3 has edges labeled by $\alpha_i^{\pm 1}$ since $w^{-1}cw \notin \langle\beta\rangle$ and P_1 has only edges labeled on $\{\beta^{\pm 1}\}$(this is because two words in the free group $\mathbb{F}$ define conjugate elements of $\mathbb{F}$ if and only if their cyclic reduction in $\mathbb{F}$ are cyclic permutations of one another). Suppose now that the cycle representing $w^{-1}cw$ is $P_2 \cup P_3$ and P_3 has edges labeled only on $\{\beta^{\pm 1}\}$. Then, c would be the form $\gamma\lambda$ up to cyclic permutation where $\gamma \in \langle\alpha_n, \alpha_{n-1}, \ldots, \beta\rangle \setminus \langle\beta\rangle$ (representing P_2) and $\lambda \in \langle\beta\rangle$ (representing P_1); and $w^{-1}cw$ would be the form $\gamma^{\pm 1}\lambda'$ up to cyclic permutation where $\lambda' \in \mathbb{F}_n$(representing P_3); but the Claim tells us that this is not possible, therefore P_3 contains edges labeled by $\alpha_i^{\pm 1}$ for some i.

Now we prove the Claim. Let $c = \gamma\lambda$ and $w^{-1}cw = \gamma\lambda'$ such that $\gamma \in \langle \alpha_n, \alpha_{n-1}, \dots, \beta\rangle \setminus \langle\beta\rangle$ and $\lambda, \lambda' \in \langle\beta\rangle$. Without loss of generality, we can suppose that $\gamma = \gamma_m\lambda_{m-1}\cdots\lambda_1\gamma_1$, with $\gamma_i \in \langle \alpha_n, \alpha_{n-1}, \dots, \beta\rangle \setminus \langle\beta\rangle$ and $\lambda_i \in \langle\beta\rangle$. Since $\gamma\lambda$ and $\gamma\lambda'$ are conjugate in a free group, there exists $1 \le k \le m$ such that

$$\gamma_k\lambda_{k-1}\cdots\lambda_1\gamma_1\lambda\gamma_m\lambda_{m-1}\cdots\gamma_{k+1}\lambda_k = \gamma\lambda' = \gamma_m\lambda_{m-1}\cdots\lambda_1\gamma_1\lambda'.$$

By identification of each letter, one deduces that $\lambda' = \lambda_k = \lambda_j$, for every j multiple of k in $\mathbb{Z}/m\mathbb{Z}$, and $\lambda = \lambda_{m-k}$. In particular, $\lambda = \lambda'$ so that $c = \gamma\lambda = \gamma\lambda' = w^{-1}cw$ and thus $cw = wc$. For the seconde statement, suppose by contradiction that there exists w such that $w^{-1}cw = \gamma^{-1}\lambda'$. Then by the similar identification as above we deduce that $\lambda^{-1} = \lambda'$, so $w^{-1}cw$ would be a cyclic permutation of c^{-1}, which is clearly not possible. $\square$

3. Construction of generic actions of free groups

Let X be an infinite countable set. We identify $X = \mathbb{Z}$. Let β be a transitive permutation of X (which is identified to the translation $x \mapsto x + 1$).

Let c be a non trivial weakly cyclically reduced word on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \dots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ such that the sum $S_c(\beta)$ of the exponents of β in the word c is zero. Thus necessarily c contains α_i for some i.

Let us denote by $S_c^+(\beta)$ the sum of positive exponents of β in the word c; by denoting $S_c^-(\beta)$ the sum of negative exponents of β in the word c, we have $0 = S_c(\beta) = S_c^+(\beta) + S_c^-(\beta)$ (for example, if $c = \alpha_1\beta^{-1}\alpha_2\beta^{-1}\alpha_n^2\beta^2$, then $S_c^+(\beta) = 2$). If c does not contain β, we set $S_c^+(\beta) = 0$.

Let $\{A_m\}_{m\ge 1}$ be a sequence of pairwise disjoint intervals of X such that $|A_m| \ge m + 2S_c^+(\beta)$, $\forall m \ge 1$. Clearly this sequence is a pairwise disjoint Følner sequence for β.

Proposition 3.1. *Let c be a weakly cyclically reduced word as above. There exists $\alpha = (\alpha_1, \dots, \alpha_n) \in (Sym(X))^n$ such that $\langle \alpha_n, \alpha_{n-1}, \dots, \alpha_1, \beta\rangle$ is free of rank $n+1$, and*

(1) *the action of $\langle \alpha_n, \alpha_{n-1}, \dots, \alpha_1, \beta\rangle$ on X is transitive and faithful;*

(2) *for all non trivial word w on $\{\alpha_n^{\pm 1}, \alpha_{n-1}^{\pm 1}, \dots, \alpha_1^{\pm 1}, \beta^{\pm 1}\}$ with $w \notin \langle c \rangle$, there exist infinitely many $x \in X$ such that $cx = x$, $cwx = wx$ and $wx \neq x$;*

(3) *there exists a pairwise disjoint Følner sequence $\{A_k\}_{k \geq 1}$ for $\langle \alpha_n, \alpha_{n-1}, \dots, \alpha_1, \beta \rangle$ which is fixed by c, and $|A_k| = k$, $\forall k \geq 1$;*

(4) *for all $k \geq 1$, there are infinitely many $\langle c \rangle$-orbits of size k;*

(5) *every $\langle c \rangle$-orbit is finite;*

(6) *for every finite index subgroup H of $\langle \alpha_n, \alpha_{n-1}, \dots, \alpha_1, \beta \rangle$, the H-action on X is transitive.*

With the notion of the permutation type, the conditions (4) and (5) mean that the word c has the permutation type $(\infty, \infty, \dots, ; 0)$.

Proof. For the proof, we are going to exhibit six generic subsets of $(Sym(X))^n$ that will do the job.

We start by claiming that the set $\mathcal{U}_1 =$

$$\left\{\alpha = (\alpha_1, \dots, \alpha_n) \in (Sym(X))^n \,\middle|\, \forall_{k \in \mathbb{Z} \setminus \{0\}}, \exists x \in X \text{ such that } c^k x \neq x\right\}$$

is generic in $(Sym(X))^n$. Indeed, for every $k \in \mathbb{Z} \setminus \{0\}$, let $\mathcal{V}_k = \{\alpha \in (Sym(X))^n | \forall x \in X,\ c^k x = x\}$. The set $\mathcal{V}_k$ is closed since if $\{\gamma_m\}_{m \geq 1}$ is a sequence in $\mathcal{V}_k$ converging to γ, then $c^k(\gamma_m)$ converges to $c^k(\gamma)$. To see the interior of $\mathcal{V}_k$ is empty, let $\alpha \in \mathcal{V}_k$ and let $F \subset X$ be a finite subset. There is an extension G_{n+1} of G_0 such that $P(c^k(\alpha'), v_0)$ embeds in G_{n+1} outside of F by Lemma 2.5. So in particular there is $x \in X \setminus F$ such that $c^k(\alpha')x \neq x$, so $\alpha' \notin \mathcal{V}_k$. By defining $\alpha'|_F = \alpha|_F$, we have shown that $\mathcal{U}_1$ is generic in $(Sym(X))^n$.

Let us show that the set

$$\mathcal{U}_2 = \Big\{\alpha = (\alpha_1, \dots, \alpha_n) \in (Sym(X))^n \,\Big|\, \forall w \neq 1 \in \langle \alpha_n, \dots, \alpha_1, \beta \rangle \setminus \langle c \rangle, \text{ there exist infinitely many } x \in X \text{such that } cx = x,\ cwx = wx \text{ and } wx \neq x\Big\}$$

is generic in $(Sym(X))^n$.

Indeed, for every non trivial word w in $\langle \alpha_n, \dots, \alpha_1, \beta \rangle \setminus \langle c \rangle$, let $\mathcal{V}_w = \{\alpha \in (Sym(X))^n |$ there exists a finite subset $K \subset X$ such that $(\mathrm{Fix}(c) \cap w^{-1}\mathrm{Fix}(c) \cap \mathrm{supp}(w)) \subset K\} = \bigcup_{K \mathrm{finite} \subset X} \{\alpha \in (Sym(X))^n | (\mathrm{Fix}(c) \cap w^{-1}\mathrm{Fix}(c) \cap \mathrm{supp}(w)) \subset K\}$. We shall show that the set $\mathcal{V}_w$ is meagre. It is an easy exercise to show that the set

$$\mathcal{V}_{w,K} = \{\alpha \in (Sym(X))^n | (\mathrm{Fix}(c) \cap w^{-1}\mathrm{Fix}(c) \cap \mathrm{supp}(w)) \subset K\}$$

is closed. To show that the interior of $\mathcal{V}_{w,K}$ is empty, let $\alpha \in \mathcal{V}_{w,K}$, and $F \subset X$ be a finite subset. We need to prove that for some α' defined as $\alpha'|_F = \alpha|_F$, we can extend the definition of α' outside of the finite subset such that $\alpha' \notin \mathcal{V}_{w,K}$. By Lemma 2.7, we can take an extension G_{n+1} of G_0 such that $Q(c(\alpha'), w)$ embeds in G_{n+1} outside of $F \cup \alpha(F) \cup K$, which proves the genericity of $\mathcal{U}_2$.

Now let us show that the set

$$\begin{aligned} \mathcal{U}_3 = \quad \{ \quad & \alpha = (\alpha_1, \dots, \alpha_n) \in (Sym(X))^n | \exists \{A_{m_k}\}_{k \geq 1} \text{ a subsequence of} \\ & \{A_m\}_{m \geq 1} \text{ such that } A_{m_k} \subset \mathrm{Fix}(\alpha_i),\ \forall k \geq 1,\ \forall 1 \leq i \leq n\} \end{aligned}$$

is generic in $(Sym(X))^n$.

Indeed, the set $\mathcal{U}_3$ can be written as $\mathcal{U}_3 = \bigcap_{N \geq 1} \{\alpha = (\alpha_1, \dots, \alpha_n) \in (Sym(X))^n | \exists k \geq N$ such that $A_k \subset \mathrm{Fix}(\alpha_i), \forall i\}$. We claim that for every $N \geq 1$, the set $\mathcal{V}_N = \{\alpha \in (Sym(X))^n | \forall k \geq N, A_k \subsetneq \cap_i \mathrm{Fix}(\alpha_i)\}$ is closed and of empty interior. It is closed since $\mathcal{V}_N = \bigcap_{k \geq N} \{\alpha \in (Sym(X))^n | A_k \subsetneq \cap_i \mathrm{Fix}(\alpha_i)\}$ and the set $\{\alpha \in (Sym(X))^n | A_k \subsetneq \cap_i \mathrm{Fix}(\alpha_i)\}$ is clearly closed. For the emptiness of its interior, let $\alpha \in \mathcal{V}_N$ and let $F \subset X$ be a finite subset. Let $k \geq N$ such that $A_k \cap (F \cup \alpha(F)) = \emptyset$. We can then take $\alpha' \in (Sym(X))^n$ fixing A_k and satisfying $\alpha'|_F = \alpha|_F$.

For (4), we show that the set

$$\begin{aligned} \mathcal{U}_4 = \quad \{ \quad & \alpha = (\alpha_1, \dots, \alpha_n) \in (Sym(X))^n | \text{ for every } m, \text{ there exist} \\ & \text{infinitely many } \langle c \rangle\text{-orbits of size } m\} \end{aligned}$$

is generic in $(Sym(X))^n$.

For all $m \geq 1$, let $\mathcal{V}_m = \{\alpha \in (Sym(X))^n |$ there exists a finite subset $K \subset X$ such that every $\langle c \rangle$-orbit of size m is contained in $K\} = \bigcup_{K \text{ finite} \subset X} \mathcal{V}_{m,K}$, where

$$\mathcal{V}_{m,K} = \{\alpha \in (Sym(X))^n | \text{ if } |\langle c \rangle \cdot x| = m, \text{ then } \langle c \rangle \cdot x \subset K\}.$$

$\cdot$ $\mathcal{V}_{m,K}$ *is of empty interior.* Let $F \subset X$ be a finite subset. Let $\alpha \in \mathcal{V}_{m,K}$. Take $x \notin (F \cup \alpha(F)) \cup K$. Since c contains α_i for some i, we can construct a cycle $c^m(\alpha')$ outside of $F \cup \alpha(F) \cup K$ such that $\alpha'|_F = \alpha|_F$ (Lemma 2.6), so that the orbit of x under α' is of size m and not contained in K.
$\cdot$ $\mathcal{V}_{m,K}$ *is closed.* Let $\{\gamma_l\}_{l\geq 1} \subset \mathcal{V}_{m,K}$ converging to $\gamma \in (Sym(X))^n$. Let $x \in X$ such that $|\langle c(\gamma)\rangle \cdot x| = m$. Since γ_l converges to γ, $c(\gamma_l)$ converges to $c(\gamma)$. Since $\langle c(\gamma)\rangle \cdot x$ is finite, there exists l_0 such that $\langle c(\gamma)\rangle \cdot x = \langle c(\gamma_l)\rangle \cdot x$, $\forall l \geq l_0$. Since $\gamma_l \in \mathcal{V}_{m,K}$ and $m = |\langle c(\gamma)\rangle \cdot x| = |\langle c(\gamma_l)\rangle \cdot x|$, we have $\langle c(\gamma_l)\rangle \cdot x \subset K$, $\forall l \geq l_0$. Therefore $\langle c(\gamma)\rangle \cdot x \subset K$, so that $\gamma \in \mathcal{V}_{m,K}$.

About (5), we prove that the set

$$\mathcal{U}_5 = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in (Sym(X))^n | \forall x \in X,\ \langle c\rangle \cdot x \text{ is finite } \}$$

is generic in $(Sym(X))^n$.

For all $x \in X$, let $\mathcal{V}_x = \{\alpha \in (Sym(X))^n | \langle c\rangle \cdot x \text{ is infinite }\}$. It is clear that the set $\mathcal{V}_x$ is closed. To see that the interior of $\mathcal{V}_x$ is empty, let $F \subset X$ be a finite subset and let $\alpha \in \mathcal{V}_x$. We shall show that there exists $\alpha' \notin \mathcal{V}_x$ such that $\alpha|_F = \alpha'|_F$. Denote $c = c(\alpha)$ and $c' = c(\alpha')$. We choose $p >> 1$ large enough so that

$$\begin{cases} (B(c^{-p-1}x, |c|) \cup B(c^{p+1}x, |c|)) \cap (F \cup \alpha(F)) = \emptyset; \\ (F \cup \alpha(F)) \subset B(x, |c^p|), \end{cases}$$

where $|c|$ is the length of c and $B(x, r)$ is the ball centered on x with the radius r.

We construct a path of c' outside of $B(x, |c^p|)$ starting from $c^{p+1}x$ which ends on $c^{-p-1}x$, i.e. $c'(c^{p+1}x) = c^{-p-1}x$. This is possible since c' contains α_i for some i (Lemma 2.5). On the points in $B(x, |c^{p+1}|)$, we define

$$\alpha'|_{B(x,|c^{p+1}|)} = \alpha|_{B(x,|c^{p+1}|)}.$$

In particular, $\alpha'|_F = \alpha|_F$, and $|\langle c'\rangle \cdot x|$ is finite.

Finally for (6), let

$$\mathcal{U}_6 = \{ \alpha = (\alpha_n, \ldots, \alpha_1) \in (Sym(X))^n | \text{ for every finite index subgroup } H \text{ of } \langle \alpha_1, \beta\rangle, \text{ the } H\text{-action on } X \text{ is transitive } \}.$$

By Proposition 4 in [6], the set $\mathcal{W} = \{\alpha_1 \in Sym(X)|$ for every finite index subgroup H of $\langle \alpha_1, \beta\rangle$, the H-action on X is transitive $\}$ is generic in $Sym(X)$. Thus $\mathcal{U}_6$ is generic in $(Sym(X))^n$ since $\mathcal{U}_6 = \mathcal{W} \times (Sym(X))^{n-1}$.

Now let $\alpha = (\alpha_1, \dots, \alpha_n) \in \cap_{i=1}^{6} \mathcal{U}_i$. It remains us to prove (3) and (6) in the Proposition. To simplify the notation, let $A_m := A_{m_k}$ be the subsequence of A_m fixed by α_i, $\forall 1 \leq i \leq n$ (genericity of $\mathcal{U}_3$).

Without loss of generality, let $c = w_1\beta^{b_1}w_2\beta^{b_2}\cdots w_l\beta^{b_l}$, where w_j are reduced words on $\{\alpha_n^{\pm 1}, \dots, \alpha_1^{\pm 1}\}$, $\forall 1 \leq j \leq l$. Recall that $\{A_m\}_{m\geq 1}$ is a sequence of pairwise disjoint intervals such that $|A_m| \geq m + 2S_c^+(\beta)$. If c does not contain β, then we can take the subinterval A'_m of A_m such that $|A'_m| = m$ for the Følner sequence which is fixed by c. If not, for all $m > S_c^+(\beta)$, let

$$E_m = \beta^{b_1}(A_m) \cap \beta^{b_2+b_1}(A_m) \cap \cdots \\ \cap \beta^{b_{l-1}+b_{l-2}+\cdots+b_1}(A_m) \cap \beta^{b_l+b_{l-1}+\cdots+b_1}(A_m).$$

Notice that $\beta^{b_l+b_{l-1}+\cdots+b_1}(A_m) = A_m$. We claim that the set E_m is not empty. Indeed, for every $1 \leq i \leq l$, the set

$$\beta^{b_i+b_{i-1}+\cdots+b_1}(A_m) \cap \beta^{b_p+b_{p-1}+\cdots+b_1}(A_m)$$

is not empty, $\forall 1 \leq p \leq i-1$ since $|b_i + b_{i-1} + \cdots + b_{p+1}| \leq S_c^+(\beta) < |A_m|$. Moreover, a family of intervals which meet pairwise, has non-empty intersection so that $E_m \neq \emptyset$.

In addition, let us show that c fixes the elements of E_m. Let $x \in E_m$ and let $1 \leq p \leq l-1$. There exists $a_{l-p+1} \in A_m$ such that $x = \beta^{b_{l-p}+b_{l-p-1}+\cdots+b_1}(a_{l-p+1})$. Then

$$\begin{aligned} \beta^{b_{l-p+1}+\cdots+b_{l-1}+b_l}(x) &= \beta^{b_l+b_{l-1}+\cdots+b_{l-p+1}}(x) \\ &= \beta^{b_l+b_{l-1}+\cdots+b_{l-p+1}} \cdot \beta^{b_{l-p}+b_{l-p-1}+\cdots+b_1}(a_{l-p+1}) \\ &= a_{l-p+1} \in A_m. \end{aligned}$$

Since w_j fixes every element in A_m, and the element $\beta^{b_{l-p+1}+\cdots+b_{l-1}+b_l}(x)$ is in A_m for every $1 \leq p \leq l-1$, the word c fixes x, $\forall x \in E_m$. Clearly the set E_m is a Følner sequence for $\langle \alpha_n, \alpha_{n-1}, \dots, \alpha_1, \beta \rangle$.

Furthermore, we have

$$A_m \cap \beta^{S_c^+(\beta)} A_m \cap \beta^{S_c^-(\beta)} A_m \subseteq E_m,$$

and

$$|A_m \cap \beta^{S_c^+(\beta)} A_m \cap \beta^{S_c^-(\beta)} A_m| = |A_m| - 2S_c^+(\beta) \geq m.$$

So $|E_m| \geq m$, and upon replacing E_m by a subinterval E'_m of E_m such that $|E'_m| = m$, we can suppose that $|E_m| = m$, $\forall m \geq 1$. Thus the sequence $\{E_m\}_{m\geq 1}$ is a Følner sequence satisfying the condition in (3) in the Proposition 3.1.

Furthermore, if H is a finite index subgroup of $\langle \alpha_n, \ldots, \alpha_1, \beta\rangle$, then $Q = H \cap \langle \alpha_1, \beta\rangle$ is a finite index subgroup of $\langle \alpha_1, \beta\rangle$, so by the genericity of $\mathcal{U}_6$ the Q-action is transitive and therefore the H-action on X is transitive. $\square$

4. Construction of $\mathbb{F}_{n+1} *_{\mathbb{Z}} \mathbb{F}_{m+1}$-actions, $n, m \geq 1$

Let X be an infinite countable set. Let $G = \langle \alpha_n, \alpha_{n-1}, \ldots, \alpha_1, \beta\rangle \curvearrowright X$ be the group action constructed as in Proposition 3.1 with the pairwise disjoint Følner sequence $\{A_k\}_{k\geq 1}$. Let $H = \langle \alpha_m, \alpha_{m-1}, \ldots, \alpha_1, \beta\rangle \curvearrowright X$ be the group action constructed as in Proposition 3.1 with the pairwise disjoint Følner sequence $\{B_k\}_{k\geq 1}$ and let d be a weakly cyclically reduced word on $\{\alpha_m, \alpha_{m-1}, \ldots, \alpha_1, \beta\}$ that does the role of c in Proposition 3.1. Let $Z = \{\sigma \in Sym(X) | \sigma c = d\sigma\}$. By virtue of the points (4) and (5) of Proposition 3.1, the set Z is not empty. Let

$$H^\sigma = \sigma^{-1} H \sigma = \langle \sigma^{-1}\alpha_m\sigma, \sigma^{-1}\alpha_{m-1}\sigma, \ldots, \sigma^{-1}\alpha_1\sigma, \sigma^{-1}\beta\sigma\rangle.$$

For $\sigma \in Z$, consider the amalgamated free product $G *_{\langle c=d\rangle} H^\sigma$ of G and H^σ along $\langle c = d\rangle$. The action of $G *_{\langle c=d\rangle} H^\sigma$ on X is given by $g \cdot x = gx$, and $h \cdot x = \sigma^{-1}h\sigma x$, $\forall g \in G$ and $\forall h \in H$.

Notice that the set Z is closed in $Sym(X)$. In particular, Z is a Baire space.

Proposition 4.1. *The set*

$$\mathcal{O}_1 = \{\sigma \in Z \mid \text{ the action of } G *_{\langle c=d\rangle} H^\sigma \text{ on } X \text{ is faithful } \}$$

is generic in Z.

Proof. For every non trivial word $w \in G *_{\langle c=d\rangle} H^\sigma$, let us show that the set

$$\mathcal{V}_w = \{\sigma \in Z | \forall x \in X, w^\sigma x = x\}$$

is closed and of empty interior. It is obvious that the set $\mathcal{V}_w$ is closed. To prove that the set $\mathcal{V}_w$ is of empty interior, let us treat the case where $w = a g_n h_n \cdots g_1 h_1$ with $a \in \langle c\rangle$, $g_i \in G \setminus \langle c\rangle$, and $h_i \in H \setminus \langle d\rangle$, $n \geq 1$. The corresponding element of $Sym(X)$ given by the action is $w^\sigma = a g_n \sigma^{-1} h_n \sigma \cdots g_1 \sigma^{-1} h_1 \sigma$. Let $\sigma \in \mathcal{V}_w$. Let $F \subset X$ be a finite subset. We shall show that there exists $\sigma' \in Z \setminus \mathcal{V}_w$ such that $\sigma'|_F = \sigma|_F$. For all $g \in G \setminus \langle c \rangle$ and $h \in H \setminus \langle d \rangle$, let

$$\widehat{g} = \{x \in X \mid cx = x,\ cgx = gx \text{ and } gx \neq x\ \},$$

$$\widehat{h} = \{x \in X \mid dx = x,\ dhx = hx \text{ and } hx \neq x\ \}.$$

By (2) of Proposition 3.1, these sets are infinite.

Choose any $x_0 \in \mathrm{Fix}(c) \setminus (F \cup \sigma(F))$. By induction on $1 \leq i \leq n$, we choose $x_{4i-3} \in \widehat{h_i}$ such that x_{4i-3}, $h_i x_{4i-3} \notin (F \cup \sigma(F))$ are new points. This is possible since $\widehat{h_i}$ is infinite. Then we define

$$\sigma'(x_{4i-4}) := x_{4i-3} \text{ and } \sigma'(\sigma^{-1}(x_{4i-3})) := \sigma(x_{4i-4}).$$

We set $x_{4i-2} := h_i x_{4i-3}$, which is different from x_{4i-3} and which is fixed by d, by definition of $\widehat{h_i}$. We choose $x_{4i-1} \in \widehat{g_i}$ such that x_{4i-1}, $g_i x_{4i-1} \notin (F \cup \sigma(F))$ are again new points. This is again possible since $\widehat{g_i}$ is infinite. Then we define

$$\sigma'(x_{4i-1}) := x_{4i-2} \text{ and } \sigma'(\sigma^{-1}(x_{4i-2})) := \sigma(x_{4i-1}).$$

We finally set $x_{4i} := g_i x_{4i-1}$. Then every point x on which σ' is defined verifies $\sigma' c(x) = d\sigma'(x)$. Indeed,

- $\sigma' c(x_{4i-4}) = \sigma'(x_{4i-4}) = x_{4i-3} = d(x_{4i-3}) = d\sigma'(x_{4i-4})$ since $x_{4i-4} \in \mathrm{Fix}(c)$ and $x_{4i-3} \in \mathrm{Fix}(d)$;

- $\sigma' c(\sigma^{-1}(x_{4i-3})) = \sigma'(\sigma^{-1}(x_{4i-3})) = \sigma(x_{4i-4}) = d\sigma(x_{4i-4}) = d\sigma'(\sigma^{-1}(x_{4i-3}))$ since $\sigma^{-1}(x_{4i-3}) \in \mathrm{Fix}(c)$ and $\sigma(x_{4i-4}) \in \mathrm{Fix}(d)$ because $\sigma \in Z$;

- $\sigma' c(x_{4i-1}) = \sigma'(x_{4i-1}) = x_{4i-2} = d(x_{4i-2}) = d\sigma'(x_{4i-1})$ since $x_{4i-2} \in \mathrm{Fix}(d)$ and $x_{4i-1} \in \mathrm{Fix}(c)$;

- $\sigma' c(\sigma^{-1}(x_{4i-2})) = \sigma'(\sigma^{-1}(x_{4i-2})) = \sigma(x_{4i-1}) = d\sigma(x_{4i-1}) = d\sigma'(\sigma^{-1}(x_{4i-2}))$ since $\sigma^{-1}(x_{4i-2}) \in \mathrm{Fix}(c)$ and $\sigma(x_{4i-1}) \in \mathrm{Fix}(d)$ because $\sigma \in Z$.

By construction, the $4n$ points defined by the subwords on the right of $w^{\sigma'}$ are all distinct. In particular, $w^{\sigma'} x_0 = x_{4n} \neq x_0$. If $w = h \in H \setminus \{\mathrm{Id}\}$, choose $x_0 \in \mathrm{Fix}(c) \setminus (F \cup \sigma(F))$, $x_1 \in \hat{h} \setminus (F \cup \sigma(F) \cup \{x_0\})$, $x_2 \in \mathrm{Fix}(c) \setminus (F \cup \sigma(F) \cup \{x_0, x_1\})$ and define $\sigma'(x_0) = x_1$, $\sigma'(x_2) = hx_1$, $\sigma'(\sigma^{-1}(x_1)) = \sigma(x_0)$, $\sigma'(\sigma^{-1}(hx_1)) = \sigma(x_2)$ so that $w^{\sigma'} x_0 = x_2 \neq x_0$. At last, if $w = g \in G \setminus \{\mathrm{Id}\}$, then there exists $x \in X$ such that $gx \neq x$ since G acts faithfully on X. For all other points, we define σ' to be equal to σ. Therefore, σ' constructed in this way is in $Z \setminus \mathcal{V}_w$ and $\sigma'|_F = \sigma|_F$. $\square$

Proposition 4.2. *The set*

$$\mathcal{O}_2 = \{\sigma \in Z \quad | \quad \exists\ \{k_l\}_{l\geq 1} \text{ a subsequence of } k \text{ such that } \sigma(A_{k_l}) = B_{k_l},\ \forall l \geq 1\}$$

is generic in Z.

Proof. Let us write

$$\mathcal{O}_2 = \bigcap_{N\in\mathbb{N}} \{\sigma \in Z | \text{ there exists } n \geq N \text{ such that } \sigma(A_n) = B_n\}.$$

We need to show that for all $N \in \mathbb{N}$, the set $\mathcal{V}_N = \{\sigma \in Z | \forall n \geq N, \sigma(A_n) \neq B_n\}$ is closed and of empty interior.

$\cdot$ *$\mathcal{V}_N$ is of empty interior.* Let $\sigma \in \mathcal{V}_N$. Let $F \subset X$ be a finite subset. Let $n \geq N$ large enough so that $A_n \cap (F \cup \sigma(F)) = \emptyset$ and $B_n \cap (F \cup \sigma(F)) = \emptyset$. This is possible since the sets $\{A_n\}$ (respectively the sets $\{B_n\}$) are pairwise disjoint. Let $A_n = \{a_1, \ldots, a_n\}$ and $B_n = \{b_1, \ldots, b_n\}$. We define $\sigma'(a_i) = b_i$ and $\sigma'(\sigma^{-1}(b_i)) = \sigma(a_i), \forall i$, which is well defined because $a_i \in \mathrm{Fix}(c)$ and $b_i \in \mathrm{Fix}(d)$. For all other points, we define σ' to be equal to σ. Therefore, $\sigma' \in Z \setminus \mathcal{V}_N$ and $\sigma'|_F = \sigma|_F$.

$\cdot$ *$\mathcal{V}_N$ is closed.* We have $\mathcal{V}_N = \bigcap_{n\geq N} \mathcal{W}_n$, where $\mathcal{W}_n = \{\sigma \in Z | \sigma(A_n) \neq B_n\}$. So the set $\mathcal{V}_N$ is closed being the intersection of closed sets. $\square$

Let $\sigma \in \mathcal{O}_1 \cap \mathcal{O}_2$. We claim that $\{A_{k_l}\}_{l\geq 1}$ is a Følner sequence for $G *_{\langle c=d\rangle} H^\sigma$. Indeed, $\{A_{k_l}\}$ is Følner for G, and for all $h \in H$, we have

$$\begin{aligned}
\lim_{l\to\infty} \frac{|A_{k_l} \triangle h \cdot A_{k_l}|}{|A_{k_l}|} &= \lim_{l\to\infty} \frac{|A_{k_l} \triangle \sigma^{-1} h \sigma A_{k_l}|}{|A_{k_l}|} = \lim_{l\to\infty} \frac{|\sigma A_{k_l} \triangle h\sigma A_{k_l}|}{|A_{k_l}|} \\
&= \lim_{l\to\infty} \frac{|B_{k_l} \triangle hB_{k_l}|}{|B_{k_l}|} = 0,
\end{aligned}$$

since $\{B_{k_l}\}$ is Følner for H, $\sigma(A_{k_l}) = B_{k_l}$ and $|A_{k_l}| = |B_{k_l}|$, for all $l \geq 1$.

Furthermore, if H is a finite index subgroup of $\mathbb{F}_{n+1} *_{\langle c=d\rangle} \mathbb{F}_{m+1}$, since every finite index subgroup of $\mathbb{F}_{n+1}$ acts transitively on X, *a fortiori* the H-action on X is transitive.

Therefore, we have:

Theorem 4.3. (1) *There exists a transitive, faithful and amenable action of the group $\mathbb{F}_{n+1} *_{\langle c=d \rangle} \mathbb{F}_{m+1}$ on X, where $c \in \mathbb{F}_{n+1}$ (respectively $d \in \mathbb{F}_{m+1}$) is a cyclically reduced non-primitive word such that the exponent sum of some generator occurring in c (respectively d) is zero.*

(2) *Every finite index subgroup of such a group admits transitive, faithful and amenable action on X.*

The complete proof of Theorem 1.2 is achieved from the following Lemma:

Lemma 4.4. *If c is a reduced word in $\mathbb{F}_n$, then there exists an automorphism ϕ of $\mathbb{F}_n$ such that the exponent sum of some generator occurring in $\phi(c)$ is zero.*

Proof. Since there is an epimorphism $\pi : \mathrm{Aut}(\mathbb{F}_n) \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}^n) \simeq GL_n(\mathbb{Z})$, it is enough to find a matrix $M \in GL_n(\mathbb{Z})$ such that the exponent sum $S_{\phi(c)}(t)$ of exponents of some generator t in the word $\phi(c)$ is zero, where $\phi \in \mathrm{Aut}(\mathbb{F}_n)$ is such that $\pi(\phi) = M \in GL_n(\mathbb{Z})$. Denote by $t_1, \ldots, t_n$ the generators of $\mathbb{F}_n$ such that $S_c(t_i) \neq 0$, $\forall 1 \leq i \leq n$. Let $m := lcm(S_c(t_1), S_c(t_2))$ be the least common multiple of $S_c(t_1)$ and $S_c(t_2)$. Then there exist m_1 and m_2 such that $m = m_1 S_c(t_1)$ and $m = m_2 S_c(t_2)$ so that $m_1 S_c(t_1) - m_2 S_c(t_2) = 0$. Moreover, the greatest common divisor $gcd(m_1, m_2)$ of m_1 and m_2 is 1, so by Bézout's identity, there exist a and b such that $m_1 a + m_2 b = 1$. So by letting $s := bS_c(t_1) + aS_c(t_2)$, the matrix

$$\begin{pmatrix} m_1 & -m_2 & & & \\ b & a & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$$

is in $GL_n(\mathbb{Z})$ and it sends $(S_c(t_1), S_c(t_2), \ldots, S_c(t_n))^t$ to $(0, s, \ldots, S_c(t_n))^t$. □

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