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Abstract

In this paper, following the p-adic integration theory worked out by A. F. Monna and T. A. Springer [4, 5] and generalized by A. C. M. van Rooij and W. H. Schikhof [6, 7] for the spaces which are not σ-compacts, we study the class of integrable p-adic functions with respect to Bernoulli measures of rank 1. Among these measures, we characterize those which are invertible and we give their inverse in the form of series.

1. Preliminaries.

In what follows, we denote by p a prime number, $\mathbb{Q}$ the field of rational numbers provided with the p-adic absolute value, $\mathbb{Q}_p$ the field of p-adic numbers that is the completion of $\mathbb{Q}$ for the p-adic absolute value and by $\mathbb{Z}_p$ the ring of p-adic integers. We denote by v_p the normalized valuation of $\mathbb{Q}_p$.

Let X be a totally discontinuous compact space and $\Omega(X)$ the Boolean algebra of closed and open subsets of X. If U belongs to $\Omega(X)$, one denotes by χ_U the characteristic function of U which is a continuous function. For K a complete ultrametric valued field, $\mathcal{C}(X,K)$ is the Banach algebra of the continuous functions from X into K provided with the norm of uniform convergence, $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

Definition 1.1. A measure on X is an additive map $\mu\colon\ \Omega(X) \to K$ such that

$$\|\mu\| = \sup_{V\in\Omega(X)} |\mu(V)| < +\infty.$$

One denotes by $M(X,K)$ the space of measures on X. Provided with the norm $\|\mu\| = \sup_{U\in\Omega(X)} |\mu(U)|$, it is an ultrametric K-Banach space.

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Let μ be a measure on X; for any locally constant function

$$f = \sum_{j=1}^{n} \lambda_j \chi_{U_j},$$

putting $\varphi_\mu(f) = \sum_{j=1}^{n} \lambda_j \mu(U_j)$, one defines on the space $Loc(X, K)$ of the locally constant functions a continuous linear form such that

$$|\varphi_\mu(f)| \le \|\mu\| \|f\|_\infty,$$

then

$$\|\varphi_\mu\| = \sup_{f \neq 0} \frac{|\varphi_\mu(f)|}{\|f\|_\infty} \le \|\mu\|.$$

The linear form φ_μ on $Loc(X,\ K)$ associated to μ being continuous for the uniform norm on $Loc(X,\ K)$ and since this space is a dense subspace of $\mathcal{C}(X,\ K)$, one sees that φ_μ extends to an unique continuous linear form on $\mathcal{C}(X,\ K)$ with the same norm and also noted φ_μ.

On the other hand, if φ is a continuous linear form on the Banach space $\mathcal{C}(X, K)$, by setting for any closed and open subset U of X: $\mu_\varphi(U) = \varphi(\chi_U)$, one defines a measure μ_φ on X such that $\|\mu_\varphi\| \le \|\varphi\|$.

Therefore, a measure $\mu = \mu_\varphi$ on X which corresponds to some continuous linear form φ on $\mathcal{C}(X, K)$ is such that $\varphi = \varphi_\mu$ and $\|\varphi_\mu\| = \|\mu\|$. Hence one sees that $M(X,\ K)$ is isometrically isomorphic to the dual Banach space $\mathcal{C}(X,\ K)'$ of $\mathcal{C}(X,\ K)$.

Let μ be a measure on X and φ_μ the continuous linear form associated to μ. One defines an ultrametric seminorm on $\mathcal{C}(X,\ K)$ by setting, for $f \in \mathcal{C}(X,\ K)$:

$$\|f\|_\mu = \sup_{g \in \mathcal{C}(X,\ K),\ g \neq 0} \frac{|\mu(fg)|}{\|g\|_\infty},$$

where $\mu(f) = \varphi_\mu(f)$ for $f \in \mathcal{C}(X,\ K)$.

Let us remind some fundamental notions on p-adic integration theory.

Theorem 1.2 (Schikhof)**.** *For any $\mu \in M(X,\ K)$, there exists a unique upper semicontinuous function $N_\mu\colon\ X \to [0,\ \infty)$ such that*

$$\|f\|_\mu = \sup_{x \in X} |f(x)| N_\mu(x).$$

The function N_μ is given by the formula

$$N_\mu(x) = \inf_{U \in \Omega(X), x \in U} \|\chi_U\|_\mu.$$

Proof. See [7, page 278] or [6, Lemma 7.2] for a proof of the theorem. □

For any closed and open subset $U \in \Omega(X)$

$$\|\chi_U\|_\mu = \sup_{V \subset U, V \in \Omega(X)} |\mu(V)|.$$

This relation is important for many computations which will follow.

If μ is a measure on X and $f\colon X \to K$ a locally constant function, one sets $\varphi_\mu(f) = \int_X f(x)d\mu(x)$, called the integral of f with respect to μ.

Definition 1.3 (μ-integrable functions)**.** Let μ be a measure on X and $f\colon X \to K$ a function; one puts

$$\|f\|_s = \sup_{x \in X} |f(x)| N_\mu(x).$$

One says that :

- f is μ-negligible if $\|f\|_s = 0$ and a subset U of X is μ-negligible if $\|\chi_U\|_s = 0$.
- f is μ-integrable if there exists a sequence $(f_n)_n$ of locally constant functions such that $\lim_{n \to +\infty} \|f - f_n\|_s = 0$.

One sets $\int_X f(x)d\mu(x) = \lim_{n \to +\infty} \int_X f_n(x)d\mu(x)$, which is seen to be independent of the sequence $(f_n)_n$.

For $x \in X$ and μ be a measure on X, one has $N_\mu(x) \leq \|\mu\|$ and one can show that $\|f\|_s = \|f\|_\mu$ for any continuous functions $f\colon X \to K$.

In the sequel, one denotes by $\mathcal{L}^1(X,\ \mu)$ the spaces of μ-integrable functions and by $\mathrm{L}^1(\mathrm{X},\ \mu)$ the quotient space $\mathcal{L}^1(X,\ \mu)/\Re$, where $\Re$ is the equivalence relation defined by $f \Re g$ if $f - g$ is μ-negligible.

For any continuous function $f\colon X \to K$, one has $\|f\|_s \leq \|\mu\| \|f\|_\infty$.

Since the space of locally constant functions is uniformly dense in the space of continuous functions, one sees that any continuous function is μ-integrable, in other words : $\mathcal{C}(X,\ K) \subseteq \mathcal{L}^1(X,\ K)$.

Furthermore if $f \in \mathcal{C}(X,\ K)$, one has $\int_X f(x)d\mu(x) = \varphi_\mu(f)$.

2. Integrable functions for the Bernoulli measures of rank 1.

We assume now that the complete valued field K is a valued extension of $\mathbb{Q}_p$ and we let α be a p-adic unit. Let us remind that the Bernoulli polynomials $(B_n(x))_{n\geq 0}$ are defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{k\geq 0}B_k(x)\frac{t^k}{k!}.$$

Definition 2.1 (Koblitz [3], Proposition, page 35)**.** Let $k\geq 1$ be a fixed integer and $B_k(x)$ be the k-th Bernoulli polynomial. For any integer $n\geq 1$ and $a\in\{0,1,\cdots,p^n-1\}$, put

$$\mu_k(a+p^n\mathbb{Z}_p)=p^{n(k-1)}B_k(\frac{a}{p^n}).$$

If $U=\bigcup_{i=1}^N(a_i+p^{n_i}\mathbb{Z}_p)$ is a partition of the closed and open subset U of $\mathbb{Z}_p$, setting $\mu_k(U)=\sum_{i=1}^N\mu_k(a_i+p^{n_i}\mathbb{Z}_p)$, one can prove, with properties of Bernoulli polynomials, that this sum is independent of any such partition of U and one obtains an additive map $\mu_k\colon\ \Omega(\mathbb{Z}_p)\to K$ called the *Bernoulli distribution* of rank k.

Definition 2.2 (B. Mazur)**.** Let $k\geq 1$ be a fixed integer and α be a p-adic unit. The *Bernoulli measure* of rank k *normalized* by α is the measure defined by setting for any closed and open set $U\in\Omega(\mathbb{Z}_p)$

$$\mu_{k,\alpha}(U)=\mu_k(U)-\alpha^{-k}\mu_k(\alpha U). \tag{2.1}$$

Let a be a p-adic integer, whose Hensel expansion is $a=\sum_{i\geq 0}a_ip^i$. For an integer $n\geq 1$, one puts

$$(a)_n=\sum_{i<n}a_ip^i \text{ and } [a]_n=\sum_{i\geq 0}a_{n+i}p^i.$$

One has then

$$[a]_n=\frac{a}{p^n}-\frac{(a)_n}{p^n}\in\mathbb{Z}_p \text{ and } [a\alpha]_n=\frac{a\alpha}{p^n}-\frac{(a\alpha)_n}{p^n}\in\mathbb{Z}_p.$$

Setting $U=a+p^n\mathbb{Z}_p$ for any integer $a\in\{0,1,\ldots,p^n-1\}$ and $k=1$ in the relation (2.1), one has $\mu_{1,\alpha}(a+p^n\mathbb{Z}_p)=B_1(\frac{a}{p^n})-\alpha^{-1}B_1(\frac{(a\alpha)_n}{p^n})$.

As $B_1(x)=x-1/2$, one obtains

$$\mu_{1,\alpha}(a+p^n\mathbb{Z}_p)=(\frac{a}{p^n}-\frac{1}{2})-\alpha^{-1}(\frac{a\alpha}{p^n}-[a\alpha]_n-\frac{1}{2}).$$

Thus, for all integers $n\geq 1$ and $a\in\{0,1,\ldots,p^n-1\}$,

$$\mu_{1,\alpha}(a+p^n\mathbb{Z}_p)=\frac{1}{2\alpha}(1-\alpha+2[a\alpha]_n). \tag{2.2}$$

Proposition 2.3. *The measure $\mu_{1,-1}$ is equal to $-\delta_0$, where δ_0 is the Dirac measure at 0. The space of $\mu_{1,-1}$-integrable functions is equal to the space of all functions $f\colon \mathbb{Z}_p \to K$.*

Proof. Let $n \geq 1$ be an integer, and a be an integer such that $0 \leq a \leq p^n - 1$; according to the relation (2.2), one has : $\mu_{1,-1}(a + p^n\mathbb{Z}_p) = -1 - [-a]_n$.

- For $a = 0$, it follows that $\mu_{1,-1}(p^n\mathbb{Z}_p) = -1$.
- For $1 \leq a \leq p^n - 1$, one has $-(p^n - 1) \leq -a \leq -1$ and
$$1 \leq p^n - a \leq p^n - 1.$$

As $-1 = \sum_{i\geq 0}(p-1)p^i$, one has
$$-a = (p^n - a) - p^n = (p^n - a) + p^n \sum_{i\geq 0}(p-1)p^i.$$

Hence, one has $[-a]_n = \sum_{i\geq 0}(p-1)p^i = -1$ and $\mu_{1,-1}(a + p^n\mathbb{Z}_p) = 0$.

Let δ_0 be the Dirac measure at 0. It is readily seen that $\mu_{1,-1} = -\delta_0$ and $\mathrm{L}^1(\mathbb{Z}_p,\, \mu_{1,-1})$ is algebraically isomorphic to K. $\square$

We now assume that α is a p-adic unit different from 1 and of -1 and we set $\gamma_\alpha = \inf_{x\in\mathbb{Z}_p} N_{\mu_{1,\alpha}}(x)$.

Let $j \geq 1$ be an integer; for any integer $a \in \{0, 1, \cdots, p^j - 1\}$, one has
$$|\mu_{1,\alpha}(a + p^j\mathbb{Z}_p)| = |\frac{1}{2\alpha}(1 - \alpha + 2[a\alpha]_j)| \leq \max(|1-\alpha|,\, |2[a\alpha]_j|) \leq 1.$$
Let us remind that any closed and open subset V of $\mathbb{Z}_p$ can be written as disjoint union $V = \bigsqcup_{k=1}^{m}(a_k + p^{j_k}\mathbb{Z}_p)$. Hence, one has
$$|\mu_{1,\alpha}(V)| \leq \max_{1\leq k\leq m} |\mu_{1,\alpha}(a_k + p^{j_k}\mathbb{Z}_p)| \leq 1.$$
Thus, for all integer $n \geq 1$ and a such that $0 \leq a \leq p^n - 1$, one has
$$\|\chi_{a+p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} = \sup_{V\subset a+p^n\mathbb{Z}_p} |\mu_{1,\alpha}(V)| \leq 1$$
Moreover, we have $N_{\mu_{1,\alpha}}(x) \leq \|\mu_{1,\alpha}\| \leq 1$, for any p-adic integer x.

Lemma 2.4. *Let $\alpha = 1 + bp^r$ be a principal unit of the ring of p-adic integer, different from 1, with $r = v_p(\alpha - 1) \geq 1$. For any p-adic integer x, one has*

- $N_{\mu_{1,\alpha}}(x) \geq \frac{1}{p^r}$, *if p is odd;*

- $N_{\mu_{1,\alpha}}(x) \geq \frac{1}{2^{r-1}}$, *if* $p=2$ *and* $r \geq 2$.

Therefore $\gamma_\alpha \geq \frac{1}{p^r}$ *if* $p \neq 2$ *and* $\gamma_\alpha \geq \frac{1}{2^{r-1}}$ *if* $p=2$ *and* $r \geq 2$.

Proof. Let us remind that

$$\mu_{1,1+bp^r}(a+p^n\mathbb{Z}_p) = \frac{1}{\alpha}\Big([a\alpha]_n - \frac{1}{2}bp^r\Big),$$

where n and a be an integers such that $n \geq 1$ and $a \in \{0,1,\ldots,p^n-1\}$, and where $r = v_p(\alpha-1) \geq 1$. One has two cases :

First case : p odd.

- If $a=0$, one has $|\mu_{1,\alpha}(p^n\mathbb{Z}_p)| = \frac{1}{p^r}$; it follows that $\|\chi_{p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \geq \frac{1}{p^r}$.
- Now, assume that $1 \leq a \leq p^n-1$;
  (1) If $|[a\alpha]_n| < \frac{1}{p^r}$, one has $|\mu_{1,1+bp^r}(a+p^n\mathbb{Z}_p)| = |\frac{1}{2}bp^r| = \frac{1}{p^r}$;
  (2) If $|[a\alpha]_n| > \frac{1}{p^r}$, one has $|\mu_{1,1+bp^r}(a+p^n\mathbb{Z}_p)| = |[a\alpha]_n| > \frac{1}{p^r}$.
  In these two cases, one obtains $\|\chi_{a+p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \geq \frac{1}{p^r}$.
  (3) If $|[a\alpha]_n| = \frac{1}{p^r}$, consider $c_n + c_{n+1}p + \cdots + c_{n+r}p^r + \cdots$ the Hensel expansion of $[a\alpha]_n$. One then has $c_n = c_{n+1} = \cdots = c_{n+r-1} = 0$ and $c_{n+r} \neq 0$. It follows that $[a\alpha]_{n+1} = c_{n+r}p^{r-1} + c_{n+r+1}p^r + \cdots$; since $|2[a\alpha]_{n+1}| = |[a\alpha]_{n+1}|$, one has

  $$|\mu_{1,\alpha}(a+p^{n+1}\mathbb{Z}_p)| = \frac{1}{p^{r-1}} \geq \frac{1}{p^r}$$

  and $\|\chi_{a+p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \geq \frac{1}{p^r}$.

  Let V_x be an open and closed neighborhood of x. There exists an integer $j_0 \geq 1$ such that $x + p^{j_0}\mathbb{Z}_p \subset V_x$. Thus, one has $\|\chi_{x+p^{j_0}\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \leq \|\chi_{V_x}\|_{\mu_{1,\alpha}}$.

It follows that $\|\chi_{V_x}\|_{\mu_{1,\alpha}} \geq \frac{1}{p^r}$. Taking infimum, one obtains

$$N_{\mu_{1,\alpha}}(x) \geq \frac{1}{p^r}.$$

Second case : $p=2$ and $r \geq 2$. Putting $\alpha = 1 + 2^r b$, one has

$$\mu_{1,\alpha}(a+2^n\mathbb{Z}_2) = \frac{1}{\alpha}([a\alpha]_n - 2^{r-1}b).$$

- If $a = 0$, for any integer $n \geq 1$, one has $|\mu_{1,\alpha}(2^n\mathbb{Z}_2)| = \frac{1}{2^{r-1}}$.

  It follows that $\|\chi_{2^n\mathbb{Z}_2}\|_{\mu_{1,\alpha}} \geq \frac{1}{2^{r-1}}$.

- Let us suppose that $1 \leq a \leq 2^n - 1$.
  (1) If $|[a\alpha]_n| < \frac{1}{2^{r-1}}$, one has $|\mu_{1,\alpha}(a + 2^n\mathbb{Z}_2)| = \frac{1}{2^{r-1}}$;
  (2) If $|[a\alpha]_n| > \frac{1}{2^{r-1}}$, one has $|\mu_{1,\alpha}(a + 2^n\mathbb{Z}_2)| = |[a\alpha]_n| > \frac{1}{2^{r-1}}$.
  In these two cases, one obtains $\|\chi_{a+2^n\mathbb{Z}_2}\|_{\mu_{1,\alpha}} \geq \frac{1}{2^{r-1}}$.
  (3) If $|[a\alpha]_n| = \frac{1}{2^{r-1}}$, as in the First case (3), one shows that $\|\chi_{a+2^n\mathbb{Z}_2}\|_{\mu_{1,\alpha}} \geq \frac{1}{2^{r-1}}$.

  Let $x \in \mathbb{Z}_2$; one shows as in the First case that $N_{\mu_{1,\alpha}}(x) \geq \frac{1}{2^{r-1}}$ and again that $\gamma_\alpha \geq \frac{1}{2^{r-1}}$.

□

Lemma 2.5. *Let p be an odd prime number, $\alpha = \alpha_0 + bp^r$ be a p-adic unit, where α_0 is an integer such that $2 \leq \alpha_0 \leq p - 1$ and $r = v_p(\alpha - \alpha_0) \geq 2$. One has $N_{\mu_{1,\alpha}}(x) \geq \frac{1}{p^r}$ for any p-adic integer x and $\gamma_\alpha \geq \frac{1}{p^r}$.*

Proof. Let p be an odd prime number and $\alpha = \alpha_0 + bp^r$ be a p-adic unit, where α_0 is an integer such that $2 \leq \alpha_0 \leq p - 1$ and $r = v_p(\alpha - \alpha_0) \geq 2$. Let us remind that, for all integers $n \geq 1$ and a such that $0 \leq a \leq p^n - 1$, one has

$$\mu_{1,\alpha}(a + p^n\mathbb{Z}_p) = \frac{1}{\alpha}([a\alpha]_n + \frac{1-\alpha}{2}) = \frac{1}{\alpha}[([a\alpha]_n - \frac{\alpha_0 - 1}{2}) - \frac{1}{2}bp^r].$$

- If $a = 0$, one has $|\mu_{1,\alpha}(p^n\mathbb{Z}_p)| = |\frac{1-\alpha}{2\alpha}| = 1$;

- Let us suppose that $a \in \{1, 2, \cdots, p^n - 1\}$.
  (1) If $|[a\alpha]_n - \frac{\alpha_0-1}{2}| < \frac{1}{p^r}$, one has $|\mu_{1,\alpha}(a + p^n\mathbb{Z}_p)| = |\frac{1}{2}bp^r| = \frac{1}{p^r}$.
  (2) If $|[a\alpha]_n - \frac{\alpha_0-1}{2}| > \frac{1}{p^r}$, one has $|\mu_{1,\alpha}(a + p^n\mathbb{Z}_p)| > \frac{1}{p^r}$.
  In these two cases, one obtains $\|\chi_{a+p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \geq \frac{1}{p^r}$.
  (3) If $|[a\alpha]_n - \frac{\alpha_0-1}{2}| = \frac{1}{p^r}$, let $c_n + c_{n+1}p + \ldots$ be the Hensel expansion of $[a\alpha]_n$; there is two cases according to the parity of α_0:
  First case : α_0 odd.
  One has $c_n = \frac{\alpha_0-1}{2}$, $c_{n+1} = \cdots = c_{n+r-1} = 0$ and $c_{n+r} \neq 0$.

Hence $[a\alpha]_{n+1} = c_{n+r}p^{r-1} + c_{n+r+1}p^r + \dots$. It follows that

$$|[a\alpha]_{n+1} - \frac{\alpha_0 - 1}{2}| = |\frac{\alpha_0 - 1}{2}| = 1$$

and $|\mu_{1,\alpha}(a + p^{n+1}\mathbb{Z}_p)| = 1$.

Second case : α_0 even.

The Hensel expansion of $\frac{\alpha_0 - 1}{2}$ is

$$\frac{\alpha_0 - 1}{2} = \frac{p + \alpha_0 - 1}{2} + \sum_{i \geq 1} \frac{p-1}{2} p^i.$$

In this case, one has : $c_n = \frac{p+\alpha_0-1}{2}$ and for $j \in \{n+1, \dots, n+r-1\}$, $c_j = \frac{p-1}{2}$. Hence,

$$[a\alpha]_{n+1} = \sum_{i=0}^{r-2} \frac{p-1}{2} p^i + \sum_{i \geq r-1} c_{n+i+1} p^i.$$

Therefore

$$[a\alpha]_{n+1} - \frac{\alpha_0 - 1}{2} = -\frac{\alpha_0}{2} + \sum_{i \geq r-1} (c_{n+i+1} - \frac{p-1}{2}) p^i.$$

Thus, one has $|[a\alpha]_{n+1} - \frac{\alpha_0-1}{2}| = | - \frac{\alpha_0}{2}| = 1$ and

$$|\mu_{1,\alpha}(a + p^{n+1}\mathbb{Z}_p)| = 1.$$

Finally, in these two cases, we have proved that

$$\|\chi_{a+p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \geq \frac{1}{p^r}.$$

As in the proof of Lemma 2.4, one proves that $N_{\mu_{1,\alpha}}(x) \geq p^{-r}$, for any p-adic integer x and $\gamma_\alpha = \inf_{x \in \mathbb{Z}_p} N_{\mu_{1,\alpha}}(x) \geq p^{-r}$. □

Lemma 2.6.

- *Let p be an odd prime number and α be an integer ≥ 2 which is a p-adic unit not congruent to 1 modulo p, then $\gamma_\alpha = 1$.*
- *Let α be a negative integer < -1 which is a p-adic unit; one has then*

$$\gamma_\alpha \geq \min\Big(|\frac{1-\alpha}{2}|, |\frac{1+\alpha}{2}|\Big) > 0.$$

Proof. Let us remind that $\mu_{1,\alpha}(a+p^n\mathbb{Z}_p) = \frac{1}{2\alpha}(1-\alpha+2[a\alpha]_n)$, for all integers $n \geq 1$ and $a \in \{0, 1, \ldots, p^n - 1\}$.

- Let p be an odd prime number and $\alpha \geq 2$ be an integer which is a p-adic unit such that $\alpha \not\equiv 1 \pmod{p}$ and $n \geq 1$ be a fixed integer; let us consider an integer a such that $0 \leq a \leq p^n - 1$.

  If $a = 0$, one has $\|\chi_{p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \geq |\mu_{1,\alpha}(p^n\mathbb{Z}_p)| = |\alpha - 1| = 1$.

  Now, let us suppose that $1 \leq a \leq p^n - 1$ and let us consider an integer j such that $p^j \geq \alpha p^n - \alpha + 1$; one has $\alpha \leq a\alpha \leq \alpha p^n - \alpha < p^j$. It follows that $(a\alpha)_j = a\alpha$ and $[a\alpha]_j = 0$. In this case, one has $|\mu_{1,\alpha}(a + p^j\mathbb{Z}_p)| = 1$. Hence, one has $\|\chi_{a+p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \geq 1$, for all integers n and a such that $n \geq 1$ and $a \in \{0, 1, \cdots, p^n - 1\}$.

  Thus, as in the proof of Lemma 2.4, we have $N_{\mu_{1,\alpha}}(x) \geq 1$, for any p-adic integer x. Since $N_{\mu_{1,\alpha}}(x) \leq 1$, for any p-adic integer x, the function $N_{\mu_{1,\alpha}}$ is constant and $N_{\mu_{1,\alpha}}(x) = 1 = \gamma_\alpha$.

- Let α be a negative integer < -1 which is a p-adic unit, $n \geq 1$ be an integer and $a \in \{1, 2, \cdots, p^n - 1\}$. One obtains a strictly positive integer while setting $m = -a\alpha$; let us denote by $s(m)$ the highest power of p in the Hensel expansion of m. One has two cases:

  First case : $m = p^{s(m)}$. One has $a\alpha = -m = p^{s(m)} \sum_{k\geq 0}(p-1)p^k$. Thus $[a\alpha]_j = \sum_{i\geq 0}(p-1)p^i = -1$ and $\mu_{1,\alpha}(a+p^j\mathbb{Z}_p) = -\frac{\alpha+1}{2\alpha}$, for any integer $j > \max(s(m),\, n)$. It follows that

  $$|\mu_{1,\alpha}(a+p^j\mathbb{Z}_p)| = |\frac{\alpha+1}{2\alpha}| = |\frac{\alpha+1}{2}| \leq \|\chi_{a+p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}}.$$

  Second case : $m \neq p^{s(m)}$. One has $-m = (p^{s(m)+1} - m) - p^{s(m)+1}$; the Hensel expansion of $a\alpha = -m$ is given by

  $$a\alpha = \sum_{\ell=0}^{s(m)} \beta_\ell p^\ell + \sum_{j\geq 0}(p-1)p^{s(m)+1+j}.$$

  Thus, for any integer $j > \max(s(m)+1,\, n)$, one has $[a\alpha]_j = \sum_{i\geq 0}(p-1)p^i = -1$ and

  $$|\mu_{1,\alpha}(a+p^j\mathbb{Z}_p)| = |\frac{\alpha+1}{2}| \leq \|\chi_{a+p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}}.$$

On other hand, one has

$$\|\chi_{p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \geq |\mu_{1,\alpha}(p^n\mathbb{Z}_p)| = |\frac{\alpha-1}{2}|.$$

It follows that $\|\chi_{a+p^n\mathbb{Z}_p}\|_{\mu_{1,\alpha}} \geq \min\left(|\frac{\alpha+1}{2}|, |\frac{\alpha-1}{2}|\right)$, for any integer $n \geq 1$ and any integer $a \in \{0, 1, \cdots, p^n - 1\}$. One concludes that $\gamma_\alpha > 0$.

□

Theorem 2.7. *Let α be a p-adic unit of one of the following forms :*

- *$\alpha = 1 + bp^r$, where $r = v_p(\alpha - 1)$ is such that $r \geq 1$ if $p \neq 2$ and $r \geq 2$ if $p = 2$;*
- *$\alpha = \alpha_0 + bp^r$, if $p \neq 2$, $\alpha_0 \in \{2, \ldots, p-1\}$ and $r = v_p(\alpha - \alpha_0) \geq 2$;*
- *$\alpha \geq 2$ is an integer such that $\alpha \not\equiv 1 \pmod{p}$ (with p odd).*
- *α is a negative integer different from -1.*
  The space $\mathcal{L}^1(\mathbb{Z}_p, \mu_{1,\alpha})$ of $\mu_{1,\alpha}$-integrable functions is equal to the space of continuous functions $\mathcal{C}(\mathbb{Z}_p, K)$.

Furthermore, one has $\mathrm{L}^1(\mathbb{Z}_p, \mu_{1,\alpha}) = \mathcal{C}(\mathbb{Z}_p, K)$.

Proof. Let us suppose that the conditions on α (of Theorem 2.7) are satisfied. According to Lemmas 2.4, 2.5 and 2.6, one has $\gamma_\alpha > 0$. Hence, one has $\gamma_\alpha \leq N_{\mu_{1,\alpha}}(x) \leq 1$ for any p-adic integers x. Thus, for any function $f\colon \mathbb{Z}_p \to K$, one has

$$\gamma_\alpha \|f\|_\infty \leq \|f\|_s \leq \|f\|_\infty.$$

Let us assume that $f\colon \mathbb{Z}_p \to K$ is a $\mu_{1,\alpha}$-integrable function. There exists a sequence $(f_n)_{n\geq 0}$ of locally constant functions such that $\lim_{n\to+\infty} \|f - f_n\|_s = 0$. Since $\gamma_\alpha\|f - f_n\|_\infty \leq \|f - f_n\|_s$, f is a uniform limit of continuous functions. Hence f is continuous. It follows that $\mathcal{L}^1(\mathbb{Z}_p, \mu_{1,\alpha}) = C(\mathbb{Z}_p, K)$.

Moreover the null function is the only $\mu_{1,\alpha}$-negligible function and one has $\mathrm{L}^1(\mathbb{Z}_p, \mu_{1,\alpha}) = \mathcal{C}(\mathbb{Z}_p, K)$. □

Remark 2.8. It remains to characterize the $\mu_{1,\alpha}$-integrable functions, where $\alpha = \alpha_0 + bp$ is not an integer, $\alpha_0 \in \{2, \ldots, p-1\}$ and $v_p(\alpha - \alpha_0) = 1$ for $p \neq 2$ and $\alpha = 1 + 2b$ not a negative integer ≤ -1 with $v_2(\alpha - 1) = 1$ for $p = 2$.

3. **Inversibility of measures $\mu_{1,\alpha}$.**

In what follows, if $n \geq 1$ is an integer, we denote by $s(n)$ the highest power of p in the Hensel expansion of n.

Lemma 3.1. *Let p be a prime number and $\alpha = 1+bp^r$ be a principal unit of the ring of p-adic integer different of* 1 *(with $r = v_p(\alpha - 1) \geq 1$).*
There exists an integer $n \geq 1$ such that $|[n\alpha]_{s(n)+1}| = 1$.

Proof. Let $\alpha = 1 + bp^r$ be a principal unit of the ring of p-adic integer different from 1 (with $r = v_p(\alpha - 1) \geq 1$) and m be an integer such that $m \geq r$. Let us consider the positive integer $n = p^{m-r+1}(1+p+\cdots+p^{r-1})$; one has $s(n) = m$ and $n = p^{s(n)-r+1} + \cdots + p^{s(n)}$. Hence, one has

$$\begin{aligned} n\alpha &= [p^{s(n)-r+1} + \cdots + p^{s(n)}](1 + bp^r) \\ &= p^{s(n)-r+1} + \cdots + p^{s(n)} + bp^{s(n)+1}(1 + p + \cdots + p^{r-1}) \end{aligned}$$

It follows that $[n\alpha]_{s(n)+1} = b(1 + p + \cdots + p^{r-1})$ and $|[n\alpha]_{s(n)+1}| = 1$. $\square$

Definition 3.2. Let n be an integer ≥ 0, and x be a p-adic integer.
The integer n is called an initial part of x, and one notes $n \lhd x$ if $|x-n| < \frac{1}{n}$.

M. van der Put could showed that the sequence of functions $(e_n)_n$ defined by $e_n(x) = \begin{cases} 1 \text{ if } n \lhd x \\ 0 \text{ overwise} \end{cases}$. is an orthonormal base of $\mathcal{C}(\mathbb{Z}_p,\ K)$; $(e_n)_n$ is called the van der Put base.

Theorem 3.3. *Let p be a prime number and $\alpha \neq 1$ be a p-adic unit. Then* $\|\mu_{1,\alpha}\| = 1$.

Proof.

- If p be an odd prime number and α be a p-adic unit of the form $\alpha = \alpha_0 + bp^r$, where $\alpha_0 \in \{2, 3, \cdots, p-1\}$ and $r = v_p(\alpha - \alpha_0) \geq 1$, one has $1 = |\frac{1-\alpha}{2\alpha}| = |\langle\mu_{1,\alpha},\ e_0\rangle| \leq \|\mu_{1,\alpha}\| \leq 1$. Hence, one has $\|\mu_{1,\alpha}\| = 1$.

- If $\alpha = 1 + bp^r$ is a principal unit of the ring of p-adic integers, different from 1, one has two cases :
  (1) $p \neq 2$ or $r \geq 2$.
  According to Lemma 3.1, there exists an integer $n_0 \geq 1$ such that $|[n_0\alpha]_{s(n_0)+1}| = 1$. In this case, one has $1 = |\langle\mu_{1,\alpha}, e_{n_0}\rangle| \leq \|\mu_{1,\alpha}\| \leq 1$. It follows that $\|\mu_{1,\alpha}\| = 1$.

(2) $r = 1$ and $p = 2$.
One has $\alpha = 1 + 2b$ and $\langle \mu_{1,1+2b}, e_0 \rangle = \frac{-2b}{2(1+2b)} = \frac{-b}{\alpha}$.
Hence,

$$1 = |\frac{-b}{\alpha}| = |\langle \mu_{1,1+2b}, e_0 \rangle| \leq \|\mu_{1,1+2b}\| \leq 1$$

and $\|\mu_{1,1+2b}\| = 1$.

- If α is a p-adic unit such that $2 \leq \alpha \leq p - 1$ (with p odd), one has

$$1 = \left|\frac{1-\alpha}{2\alpha}\right| = |\langle \mu_{1,\alpha}, e_0 \rangle| \leq \|\mu_{1,\alpha}\| \leq 1.$$

It follows that $\|\mu_{1,\alpha}\| = 1$.

$\square$

Let δ_a be the Dirac measure associated to the p-adic integer a. Let us put $\omega = \delta_1 - \delta_0$. It is known that any measure $\mu \in M(\mathbb{Z}_p, K)$ can be written as a pointwise convergent series $\mu = \sum_{n \geq 0} \langle \mu, Q_n \rangle Q'_n$, where $(Q_n)_{n \geq 0}$ is an orthonormal base of $\mathcal{C}(\mathbb{Z}_p, K)$ called the Mahler basis, defined by $Q_n(x) = \binom{x}{n}$ and $(Q'_n)_{n \geq 0}$ is the dual family of $(Q_n)_{n \geq 0}$ defined by $\langle Q'_n, Q_m \rangle = \delta_{nm}$. The convolution product $Q'_n \star Q'_m$ gives Q'_{n+m}; one deduces that $Q'_n = Q'^n_1$. As $Q'_1 = \omega$, one has $Q'_n = \omega^n$; it follows that $\mu = \sum_{n \geq 0} \langle \mu, Q_n \rangle \omega^n$. Hence the measure μ corresponds to the formal power series of bounded coefficients $S_\mu = \sum_{n \geq 0} \langle \mu, Q_n \rangle X^n$. Therefore, the algebra $M(\mathbb{Z}_p, K)$, provided with the convolution product, is isometrically isomorphic to the algebra $K\langle X \rangle$ of bounded formal power series with bounded coefficients, the norm being the supremum of the coefficients.

Let us remind (see for instance [2] or [1]) that an element S of the Banach algebra $K\langle X \rangle$ is invertible if and only if $\|S\| = |S(0)| \neq 0$.

Theorem 3.4. *Let p be a prime number, and α be a p-adic unit.*

The measure $\mu_{1,\alpha}$ is invertible for the convolution product if and only if $\alpha \not\equiv 1 \pmod p$ if p is odd (resp. $\alpha \not\equiv 1 \pmod 4$ for $p = 2$).

Moreover its inverse ν_α is given by the formula

$$\nu_\alpha = \sum_{n \geq 0} d_n(\alpha) \omega^n,$$

where $d_0(\alpha) = \frac{2\alpha}{1-\alpha}$, $d_1(\alpha) = \frac{1+\alpha}{3(1-\alpha)}$ *and for* $n \geq 2$*:*

$$d_n(\alpha) = \alpha^n (\frac{2\alpha}{1-\alpha})^{n+1} \sum_{\substack{1\leq j\leq n \\ i_1+\cdots+i_j=n \\ i_1,\ldots,i_j\in\{1,\ldots,n\}}} (-1)^j \binom{\alpha^{-1}}{i_1+2} \cdots \binom{\alpha^{-1}}{i_j+2}.$$

Proof. Let p be a prime number; let us denote by $S_{1,\alpha}(X)$ the formal power series with bounded coefficients which corresponds to the measure $\mu_{1,\alpha}$. One then has $S_{1,\alpha}(0) = \langle \mu_{1,\alpha},\ Q_0 \rangle = \frac{1-\alpha}{2\alpha}$.

The measure $\mu_{1,\alpha}$ is invertible in $M(\mathbb{Z}_p,\ K)$ (for the convolution product) if and only if $S_{1,\alpha}$ is invertible in the Banach algebra $K\langle X \rangle$ (for the Cauchy product). According to Theorem 3.3, the norm of measure $\mu_{1,\alpha}$ is equal to 1. Hence, $S_{1,\alpha}$ is invertible in $K\langle X \rangle$ if and only if:

$$1 = \|S_{1,\alpha}\| = |S_{1,\alpha}(0)| = \Big|\frac{1-\alpha}{2\alpha}\Big| = \Big|\frac{1-\alpha}{2}\Big|.$$

Thus $\mu_{1,\alpha}$ is invertible if and only if $\alpha - 1 \not\equiv 0 \pmod{p}$ for $p \neq 2$ (respectively $\alpha \not\equiv 1 \pmod{4}$ for $p = 2$).

Since

$$(1+X)^{\alpha^{-1}} = \sum_{j\geq 0} \binom{\alpha^{-1}}{j} X^j,$$

one obtains

$$S_{1,\alpha}(X) = U_\alpha(X)[1 + XU_\alpha(X)]^{-1},$$

where

$$U_\alpha(X) = \alpha \sum_{j\geq 0} \binom{\alpha^{-1}}{j+2} X^j.$$

Moreover, if α is a p-adic unit such that $\alpha - 1 \not\equiv 0 \pmod{p}$ for $p \neq 2$ and $\alpha \not\equiv 1 \pmod{4}$ for $p = 2$, one has $1 = |\frac{1-\alpha}{2\alpha}| = |U_\alpha(0)| \leq \|U_\alpha\| \leq 1$. One has $\|U_\alpha\| = |U_\alpha(0)| \neq 0$; hence U_α is invertible. It is readily seen that $1 + XU_\alpha$ is invertible; one deduces that $\mu_{1,\alpha} = U_\alpha(\omega)[1 + \omega U_\alpha(\omega)]^{-1}$. Thus, the convolution inverse ν_α of the measure $\mu_{1,\alpha}$ is then given by $\nu_\alpha = U_\alpha(\omega)^{-1}[1 + \omega U_\alpha(\omega)] = \omega + U_\alpha(\omega)^{-1}$. Setting

$$c_j(\alpha) = \binom{\alpha^{-1}}{j+2}$$

for $j \geq 0$, $b_0(\alpha) = 0$ and $b_j(\alpha) = c_0(\alpha)^{-1}c_j(\alpha)$ for $j \geq 1$, one has

$$\begin{aligned} U_\alpha(\omega)^{-1} &= \alpha^{-1}c_0(\alpha)^{-1}\Big[1 + \sum_{n\geq 1} b_n(\alpha)\omega^n\Big]^{-1} \\ &= \alpha^{-1}c_0(\alpha)^{-1}\sum_{j\geq 0}(-1)^j\Big[\sum_{n\geq 1} b_n(\alpha)\omega^n\Big]^j \\ &= \frac{2\alpha}{1-\alpha} + \frac{2\alpha}{1-\alpha}\sum_{j\geq 1}\sum_{n\geq 1}(-1)^j b_n(j,\alpha)\omega^n, \end{aligned}$$

with $b_n(j,\alpha) = \sum_{i_1+\cdots+i_j=n} b_{i_1}(\alpha)\cdots b_{i_j}(\alpha)$.

Since $b_0(\alpha) = 0$,

$$b_n(j,\alpha) = \sum_{i_1+\cdots+i_j=n} b_{i_1}(\alpha)\cdots b_{i_j}(\alpha) = 0$$

for $j \geq n+1$, one has

$$b_n(j,\alpha) = \sum_{\substack{i_1,\ldots,i_j\geq 1 \\ i_1+\cdots+i_j=n}} b_{i_1}(\alpha)\cdots b_{i_j}(\alpha), \text{ for } j \leq n.$$

More precisely, one has

$$b_n(j,\alpha) = \Big(\frac{2\alpha^2}{1-\alpha}\Big)^n \sum_{\substack{i_1,\ldots,i_j\in\{1,\ldots,n\} \\ i_1+\cdots+i_j=n}} \binom{\alpha^{-1}}{i_1+2}\cdots\binom{\alpha^{-1}}{i_j+2}, \text{ for } j \leq n.$$

It follows that

$$U_\alpha(\omega)^{-1} = \frac{2\alpha}{1-\alpha} + \frac{2\alpha}{1-\alpha}\sum_{n\geq 1}\sum_{j=1}^{n}(-1)^j b_n(j,\alpha)\omega^n$$

and

$$\nu_\alpha = \frac{2\alpha}{1-\alpha}\delta_0 + \Big[1 - \frac{2\alpha}{1-\alpha}b_1(1,\alpha)\Big]\omega + \frac{2\alpha}{1-\alpha}\sum_{n\geq 2}\sum_{j=1}^{n}(-1)^j b_n(j,\alpha)\omega^n.$$

As $b_1(1,\alpha) = b_1(\alpha) = \binom{\alpha^{-1}}{2}^{-1}\binom{\alpha^{-1}}{3} = \frac{1-2\alpha}{3\alpha}$, one obtains

$$\begin{aligned} \nu_\alpha &= \frac{2\alpha}{1-\alpha}\delta_0 + \frac{1+\alpha}{3(1-\alpha)}\omega + \frac{2\alpha}{1-\alpha}\sum_{n\geq 2}\Big[\sum_{j=1}^{n}(-1)^j b_n(j,\alpha)\Big]\omega^n \\ &= \sum_{n\geq 0} d_n(\alpha)\omega^n \end{aligned}$$

where $d_0(\alpha) = \frac{2\alpha}{1-\alpha}$, $d_1(\alpha) = \frac{1+\alpha}{3(1-\alpha)}$ and for $n \geq 2$:

$$\begin{aligned} d_n(\alpha) &= \frac{2\alpha}{1-\alpha} \sum_{j=1}^{n} (-1)^j b_n(j,\alpha) \\ &= \alpha^n \Big(\frac{2\alpha}{1-\alpha}\Big)^{n+1} \sum_{\substack{1\leq j\leq n \\ i_1+\cdots+i_j=n \\ i_1,\ldots,i_j\in\{1,\ldots,n\}}} (-1)^j \binom{\alpha^{-1}}{i_1+2} \cdots \binom{\alpha^{-1}}{i_j+2} \end{aligned}$$

□

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