ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 16, n° 2 (2009), p. 321-338.

<http://ambp.cedram.org/item?id=AMBP_2009__16_2_321_0>

Publication éditée par le laboratoire de mathématiques de l'université Blaise-Pascal, UMR 6620 du CNRS Clermont-Ferrand — France

Annales mathématiques Blaise Pascal **16**, 321-338 (2009)

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Abstract

We extend some results by Gol′dshtein, Kuz′minov, and Shvedov about the L_p-cohomology of warped cylinders to $L_{p,q}$-cohomology for $p \neq q$. As an application, we establish some sufficient conditions for the nontriviality of the $L_{p,q}$-torsion of a surface of revolution.

Cohomologie $L_{p,q}$ des cylindres tordus

Résumé

On généralise quelques résultats par Gol′dshtein, Kuz′minov et Shvedov sur la cohomologie L_p des cylindres tordus à cohomologie $L_{p,q}$ pour $p \neq q$. Comme application, on établit des conditions suffisantes pour la non-nullité de la torsion $L_{p,q}$ d'une surface de révolution.

1. Introduction

Let M be a Riemannian manifold. For $1 \leq p \leq \infty$ and a positive continuous function $\sigma : M \to \mathbb{R}$, denote by $L^j_p(M, \sigma)$ the Banach space of measurable forms of degree j on M with the finite norm

$$\|\omega\|_{L^j_p(M,\sigma)} = \begin{cases} \left\{ \int_M |\omega(x)|^p \sigma^p(x)\, dx \right\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x \in M} |\omega(x)|\sigma(x) & \text{if } p = \infty. \end{cases}$$

Here dx stands for the volume element of M and $|\omega(x)|$ is the modulus of the exterior form $\omega(x)$. In the usual way, we also define the spaces $L_{p,\mathrm{loc}}(M)$.

Denote by $D^j(M) = C_0^{\infty,j}(M)$ the space of smooth forms of degree j on M having compact support included in $\operatorname{Int} M$. A form $\psi \in L^{j+1}_{1,\mathrm{loc}}(M)$

Keywords: Differential form, $L_{p,q}$-cohomology, $L_{p,q}$-torsion, warped cylinder.
Math. classification: 58A12, 46E30.

is called the (*weak*) *differential* $d\omega$ of $\omega \in L^j_{1,\mathrm{loc}}(M)$ if

$$\int\limits_U \omega \wedge du = (-1)^{j+1} \int\limits_U \psi \wedge u$$

for every orientable domain $U \subset \operatorname{Int} M$ and every form $u \in D^{\dim M - j - 1}(M)$ having support in U.

For two weights σ_j, σ_{j+1} on M, put

$$W^j_{p,q}(M, \sigma_j, \sigma_{j+1}) = \{\omega \in L^j_p(M, \sigma_j) \mid d\omega \in L^{j+1}_q(M, \sigma_{j+1})\}.$$

The space $W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ is endowed with the norm

$$\|\omega\|_{W^j_{p,q}(M,\sigma_j,\sigma_{j+1})} = \|\omega\|_{L^j_p(M,\sigma_j)} + \|d\omega\|_{L^{j+1}_q(M,\sigma_{j+1})}.$$

If $p = q$ then it is often more convenient to consider the equivalent norm

$$\|\omega\|_{W^j_p(M,\sigma_j,\sigma_{j+1})} = \Big(\|\omega\|^p_{L^j_p(M,\sigma_j)} + \|d\omega\|^p_{L^{j+1}_p(M,\sigma_{j+1})}\Big)^{1/p}.$$

In the sequel we let $V^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ denote the closure of $D^j(M)$ in the norm of $W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$.

Given an arbitrary subset $A \subset M$, let $W^j_{p,q}(M, A, \sigma_j, \sigma_{j+1})$ be the closure in $W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ of the subspace spanned by all forms $\omega \in W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ which vanish on some neighborhood of A (depending on ω).

Let $Z^j_q(M, \sigma_j)$ be the subspace in $W^j_{q,q}(M, \sigma_j, \sigma_j)$ that consists of all forms ω such that $d\omega = 0$ and let

$$\begin{aligned} B^j_{p,q}(M, \sigma_{j-1}, \sigma_j) = \{&\theta \in W^j_{q,q}(M, \sigma_j, \sigma_j) \\ &\mid \theta = d\psi \quad \text{for some} \quad \psi \in W^{j-1}_{p,q}(M, \sigma_{j-1}, \sigma_j)\}. \end{aligned}$$

The spaces

$$H^j_{p,q}(M, \sigma_{j-1}, \sigma_j) = Z^j_q(M, \sigma_j)/B^j_{p,q}(M, \sigma_{j-1}, \sigma_j)$$

and

$$\overline{H}^j_{p,q}(M, \sigma_{j-1}, \sigma_j) = Z^j_q(M, \sigma_j)/\overline{B}^j_{p,q}(M, \sigma_{j-1}, \sigma_j),$$

where $\overline{B}^j_{p,q}(M, \sigma_{j-1}, \sigma_j)$ is the closure of $B^j_{p,q}(M, \sigma_{j-1}, \sigma_j)$ in $L^j_q(M, \sigma_j)$ (equivalently, in $W^j_{q,q}(M, \sigma_j, \sigma_j)$) are called the jth $L_{p,q}$-*cohomology* and the jth *reduced* $L_{p,q}$*-cohomology* of the Riemannian manifold M *with weights* σ_{j-1} *and* σ_j. The quotient space

$$T^j_{p,q}(M,\sigma_{j-1},\sigma_j) = \overline{B}^j_{p,q}(M,\sigma_{j-1},\sigma_j)/B^j_{p,q}(M,\sigma_{j-1},\sigma_j)$$

will be referred to as the $L_{p,q}$*-torsion* of M with the given weights. Clearly, the space $T^j_{p,q}(M,\sigma_{j-1},\sigma_j)$ is isomorphic to the closure of the zero in $H^j_{p,q}(M,\sigma_{j-1},\sigma_j)$.

Given a subset $A \subset M$, the *relative nonreduced and reduced* $L_{p,q}$*-cohomology spaces* $H^j_{p,q}(M,A,\sigma_{j-1},\sigma_j)$ and $\overline{H}^j_{p,q}(M,A,\sigma_{j-1},\sigma_j)$ are defined as

$$H^j_{p,q}(M,A,\sigma_{j-1},\sigma_j) = Z^j_q(M,A,\sigma_j)/B^j_{p,q}(M,A,\sigma_{j-1},\sigma_j)$$

and

$$\overline{H}^j_{p,q}(M,A,\sigma_{j-1},\sigma_j) = Z^j_q(M,A,\sigma_j)/\overline{B}^j_{p,q}(M,A,\sigma_{j-1},\sigma_j),$$

where the relative spaces $Z^j_q(M,A,\sigma_j)$ and $B^j_{p,q}(M,A,\sigma_{j-1},\sigma_j)$ are defined as their absolute analogs above with the spaces $W^j_{p,q}(M,\sigma_j,\sigma_j)$ and $W^{j-1}_{p,q}(M,\sigma_{j-1},\sigma_j)$ replaced by the spaces

$$W^j_{p,q}(M,A,\sigma_j,\sigma_j) \text{ and } W^{j-1}_{p,q}(M,A,\sigma_{j-1},\sigma_j).$$

For $p = q$, we write the subscript p instead of p,p throughout. If the weights involved in the definition of the corresponding space are equal to 1 then they will be omitted.

The spaces $W_{p,q}$ and $L_{p,q}$-cohomology were introduced at the beginning of the 1980's by Gol′dshtein, Kuz′minov, and Shvedov [3, 4, 5, 6, 7, 8], who obtained many results concerning $W_{p,q}$-forms and especially L_p-cohomology. Later $L_{p,q}$-cohomology was considered in [11, 12, 13, 14, 15, 17, 22].

In this paper, we, following [9, 10], look for conditions of the nontriviality of the $L_{p,q}$-cohomology and $L_{p,q}$-torsion on warped cylinders, a class of warped products of Riemannian manifolds. By the *warped product* $X \times_f Y$ of two Riemannian manifolds (X,g_X) and (Y,g_Y) *with the warping function* $f : X \to \mathbb{R}_+$ we mean the product manifold $X \times Y$ endowed with the metric $g_X + f^2(x)g_Y$. If $X = [a,b[$ is a half-interval on the real line then $X \times_f Y$ is referred to as the *warped cylinder*. The study of the L_2-cohomology of warped cylinders was initiated by Cheeger [2].

The structure of the article is as follows. In Section 2, we adapt the results of [9] about the L_p-cohomology of a half-interval to the case $p \neq q$.

After that, using these $L_{p,q}$-results, in Section 3, we prove a partial $L_{p,q}$-generalization of Theorem 1 of [9] about the L_p-cohomology of a warped cylinder $[a,b[\times_f Y$ depending on the analytic properties of the function f. As an application, we obtain an extension of the necessary condition for the triviality of the $L_{p,q}$-torsion of a surface of revolution in $\mathbb{R}^{n+2}$ [16] from the case $p=q$ to arbitrary p,q such that $\frac{1}{q}-\frac{1}{p}<\frac{1}{n+1}$.

2. Weighted $L_{p,q}$-cohomology of a half-interval

Consider a half-interval $[a,b[$, $-\infty<a<b\leq\infty$ and positive continuous functions $v_0,v_1:[a,b[\to\mathbb{R}$. For $1<p,q<\infty$, the space $W^0_{p,q}([a,b[,v_0,v_1)$ can be identified with the space of the functions $g\in L_p([a,b[,v_0)$ whose weak derivative $g'\in L_q([a,b[,v_1)$. As above, endow $W^0_{p,q}([a,b[,v_0,v_1)$ with the norm

$$\|g\|_{W^0_{p,q}([a,b[,v_0,v_1)}=\left(\int_a^b|g(t)|^pv_0^pdt\right)^{1/p}+\left(\int_a^b|g'(t)|^qv_1^qdt\right)^{1/q}.$$

From the classical Sobolev Embedding Theorem it follows that the functions of the class $W^0_{p,q}([a,b[,v_0,v_1)$ are continuous on $[a,b[$. Consider also the space

$$W^0_{p,q}([a,b[,\{a\},v_0,v_1)=\{f\in W^0_{p,q}([a,b[,\{a\},v_0,v_1)\mid f(a)=0\}.$$

We have

$$H^1_{p,q}([a,b[,v_0,v_1)=W^1_q([a,b[,v_1,v_1)/dW^0_{p,q}([a,b[,v_0,v_1);$$
$$H^1_{p,q}([a,b[,\{a\},v_0,v_1)=W^1_q([a,b[,\{a\},v_1,v_1)/dW^0_{p,q}([a,b[,\{a\},v_0,v_1).$$

The spaces $\overline{H}^1_{p,q}([a,b[,v_0,v_1)$ and $\overline{H}^1_{p,q}([a,b[,\{a\},v_0,v_1)$ are described similarly.

We call the following assertion the lemma about the Hardy inequality [1, 10, 21]:

Lemma 2.1. *Suppose that* $1\leq p,q\leq\infty$, $\frac{1}{q}+\frac{1}{q'}=1$, $\alpha,\beta\in[-\infty,\infty]$, $I_{\alpha,\beta}$ *is the interval with endpoints* α *and* β, v_0 *and* v_1 *are continuous positive functions on* $I_{\alpha,\beta}$. *Then for the existence of a global constant* C *such that*

$$\left|\int_\alpha^\beta\left|v_0(t)\int_\alpha^\tau g(t)dt\right|^pd\tau\right|^{1/p}\leq C\left|\int_\alpha^\beta|v_1(t)g(t)|^qdt\right|^{1/q}$$

for every $g \in L_q(I_{\alpha,\beta}, v_1)$*, it is necessary and sufficient that*

$$\chi_{p,q}(\alpha, \beta, v_0, v_1) < \infty\,.$$

Here

$$\chi_{p,q}(\alpha, \beta, v_0, v_1) = \sup_{\tau \in I_{\alpha,\beta}} \left\{ \left| \int_\tau^\beta |v_0(t)|^p dt \right|^{1/p} \left| \int_\alpha^\tau |v_1(t)|^{-q'} dt \right|^{1/q'} \right\}$$

if $p \geq q$;

$$\chi_{p,q}(\alpha, \beta, v_0, v_1) = \left| \int_\alpha^\beta \left(\left| \int_\alpha^\tau |v_1(t)|^{-q'} dt \right|^{p-1} \left| \int_\tau^\beta |v_0(t)|^p dt \right| \right)^{\frac{q}{q-p}} |v_1(\tau)|^{-q'} d\tau \right|^{\frac{q-p}{pq}}$$

if $p < q$.

If $p = 1$ $(q' = \infty)$ *then the corresponding integral must be replaced by* $\operatorname{ess\,sup}$.

The constant $\chi_{p,q}(\alpha, \beta, v_0, v_1)$ will be referred to as the *Hardy constant.*

The following lemma was proved in [9] for $p = q$ and $v_0 = v_1$. The proof given in [9] holds for different p and q and different v_0 and v_1.

Lemma 2.2. *Suppose that* $\alpha, \beta \in [-\infty, \infty]$, $v_0, v_1 : I_{\alpha,\beta} \to \mathbb{R}$ *are positive continuous functions, and* $\chi_{p,q}(\alpha, \beta, v_0, v_1) = \infty$*. Then there exists a nonnegative function* h *such that*

$$\left| \int_\alpha^\beta v_1^q(t) h^q(t) dt \right| < \infty, \qquad \left| \int_\alpha^\beta v_0^p(\tau) \left| \int_\alpha^\tau h(t) dt \right|^p d\tau \right| = \infty.$$

As in [9], Lemma 2 yields the following assertion.

Theorem 2.3. *If* v_0, v_1 *are positive continuous functions on* $[a, b[$ *and* $1 < p, q < \infty$ *then*

(1) $H^1_{p,q}([a, b[, \{a\}, v_0, v_1) = 0 \iff \chi_{p,q}(a, b, v_0, v_1) < \infty$;

(2) $H^1_{p,q}([a, b[, v_0, v_1) = 0 \iff \chi_{p,q}(a, b, v_0, v_1) < \infty$ *or* $\chi_{p,q}(b, a, v_0, v_1) < \infty$.

Let

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0 \tag{2.1}$$

be an exact sequence of Banach complexes, i.e., complexes in the category of Banach spaces and bounded linear operators. Sequence (2.1) yields an exact sequence of the cohomology spaces

$$\dots \to H^{k-1}(C) \xrightarrow{\partial} H^k(A) \xrightarrow{\varphi^*} H^k(B) \xrightarrow{\psi^*} H^k(C) \to \dots$$

with continuous operators ∂^*, φ^*, ψ^* and a semi-exact sequence of the reduced cohomology spaces

$$\dots \to \overline{H}^{k-1}(C) \xrightarrow{\overline{\partial}} \overline{H}^k(A) \xrightarrow{\overline{\varphi}^*} \overline{H}^k(B) \xrightarrow{\overline{\psi}^*} \overline{H}^k(C) \to \dots \tag{2.2}$$

Under certain conditions, sequence (2.2) is exact at some terms (see [10, 18, 20]). In particular, Gol′dshtein, Kuz′minov, and Shvedov proved the following assertion in [10, Theorem 1(1)]:

Lemma 2.4. *If $H^k(C)$ is separated and $\dim \partial(H^{k-1}(C)) < \infty$ then the sequence $\overline{H}^{k-1}(C) \xrightarrow{\overline{\partial}} \overline{H}^k(A) \xrightarrow{\overline{\varphi}^*} \overline{H}^k(B) \xrightarrow{\overline{\psi}^*} \overline{H}^k(C)$ is exact.*

As was explained in [12], we can describe the jth weighted $L_{p,q}$-cohomology of an n-dimensional Riemannian manifold M with given weights σ_{j-1} and σ_j in terms of Banach complexes. To this end, consider an arbitrary sequence $\pi = \{p_0, p_1, \dots, p_n\} \subset [1, \infty]$ with $p_{j-1} = p$ and $p_j = q$ and a sequence of positive continuous weights $\sigma = \{\sigma_k\}_{k=0}^n$ with the given σ_{j-1} and σ_j. Given a subset $A \subset M$, put

$$W_\pi^k(M, A, \sigma) = W_{p_k, p_{k+1}}(M, A, \sigma_k, \sigma_{k+1}).$$

Here we have assumed that $p_{n+1} = p_n$ and $\sigma_{n+1} = \sigma_n$.

Since the exterior differential is a bounded operator

$$d^{k-1} : W_\pi^{k-1}(M, A, \sigma) \to W_\pi^k(M, A, \sigma),$$

we obtain a Banach complex

$$0 \to W_\pi^0(M, A, \sigma) \xrightarrow{d^0} W_\pi^1(M, A, \sigma) \to \dots \xrightarrow{d^{n-1}} W_\pi^n(M, A, \sigma) \to 0. \tag{2.3}$$

By the k-th *L_π-cohomology $H_\pi^k(M, A, \sigma)$ (reduced k-th L_π-cohomology $\overline{H}_\pi^k(M, A, \sigma)$) of the Riemannian manifold M with respect to A with weight* σ we mean the cohomology (reduced cohomology) of (2.3). Thus,

$$H_\pi^k(M, A, \sigma) = H_{p_{k-1}, p_k}^k(M, A, \sigma_{k-1}, \sigma_k)$$

and

$$\overline{H}_\pi^k(M, A, \sigma) = \overline{H}_{p_{k-1}, p_k}^k(M, A, \sigma_{k-1}, \sigma_k)$$

for all k. In particular,

$$H_\pi^j(M, A, \sigma) = H_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j),$$
$$\overline{H}_\pi^j(M, A, \sigma) = \overline{H}_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j).$$

Take $M = [a, b[$, $A = \{a\}$, $1 < p, q < \infty$, $\pi = \{p, q\}$, and a pair of weights $v = \{v_0, v_1\}$. We have the following exact sequence of Banach complexes:

$$0 \to W^*_\pi([a, b[, \{a\}, v) \xrightarrow{\mathfrak{j}} W^*_\pi([a, b[, v) \xrightarrow{\mathfrak{i}} H^*(\{a\}) \to 0,$$

where $H^*(\{a\})$ is the complex with the only nontrivial term $H^0(\{a\}) = \mathbb{R}$. Here the mappings $\mathfrak{i}$ and $\mathfrak{j}$ are defined as follows: $\mathfrak{j}$ is the inclusion mapping; if $g \in W^0_\pi([a, b[, v)$ then $\mathfrak{i}g = g(a)$ (recall that g is continuous) and in dimension one $\mathfrak{j}$ is zero. Lemma 2.4 yields the exact sequence

$$\mathbb{R} = H^0(\{a\}) \xrightarrow{\overline{\partial}} \overline{H}^1_{p,q}([a, b[, \{a\}, v_0, v_1) \xrightarrow{\overline{\mathfrak{j}}^*} \overline{H}^1_{p,q}([a, b[, v_0, v_1).$$

Thus, we infer the following assertion, proved for $p = q$ in [9]. With what has been said above, the proof of [9] extends to the case of $p \neq q$ without change.

Theorem 2.5. *If v_0, v_1 are positive continuous functions on $[a, b[$, $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$ then*

(1) $\overline{H}^1_{p,q}([a, b[, v_0, v_1) = 0$;

(2) $\overline{H}^1_{p,q}([a, b[, \{a\}, v_0, v_1) = 0$ *if and only if* $\int\limits_a^b v_1^{-q'}(t)dt = \infty$ *or* $\int\limits_a^b v_0^p(t)dt < \infty$;

(3) *If $\overline{H}_{p,q}([a, b[, \{a\}, v_0, v_1) \neq 0$ then*

$$\overline{\partial} : \mathbb{R} = H^0(\{a\}) \to \overline{H}^1_{p,q}([a, b[, \{a\}, v_0, v_1)$$

is an isomorphism.

3. $L_{p,q}$-cohomology of the warped cylinder $C^f_{a,b}$

Let Y be an orientable manifold of dimension n, $C^f_{a,b}Y = [a, b[\times_f Y$. Put $Y_a = \{a\} \times Y$. Generally speaking, $C^f_{a,b}$ is a Lipschitz Riemannian manifold in the sense of [3] but we will assume throughout for simplicity that $\partial Y = \varnothing$ to make $C^f_{a,b}$ smooth, which will be enough for our purposes.

Suppose that $1 < p < \infty$ and $1 < q < \infty$.

In [9], Gol′dshtein, Kuz′minov, and Shvedov introduced the bilinear mapping

$$\nu : L_p^{j-1}(Y) \times L_p^1([a,b[, f^{\frac{n}{p}-j+1}) \to L_p^j(C_{a,b}^f Y),$$

$\nu(\varphi, gdt) = gdt \wedge \varphi$. In [9] it was proved that ν is continuous and if $\varphi \in Z_p^{j-1}(Y)$ then $\nu_\varphi = \nu(\varphi,\cdot) : L_p^1([a,b[, f^{\frac{n}{p}-j+1}) \to L_p^j(C_{a,b}^f Y)$ induces continuous mappings

$$\nu_\varphi^* : H_p^1([a,b[, f^{\frac{n}{p}-j+1}) \to H_p^j(C_{a,b}^f Y);$$
$$\tilde{\nu}_\varphi^* : H_p^1([a,b[, \{a\}, f^{\frac{n}{p}-j+1}) \to H_p^j(C_{a,b}^f Y, Y_a).$$

Supposing that $\varphi \in Z_p^{j-1}(Y) \cap Z_q^{j-1}(Y)$, we similarly become convinced that the mapping $\nu_\varphi = \nu(\varphi,\cdot)$ induces continuous mappings

$$\nu_\varphi^* : H_{p,q}^1([a,b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \to H_{p,q}^j(C_{a,b}^f Y);$$
$$\tilde{\nu}_\varphi^* : H_{p,q}^1([a,b[, \{a\}, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \to H_{p,q}^j(C_{a,b}^f Y, Y_a).$$

Now, assume that $\psi \in L_{p'}^{n+1-j}(Y)$ ($p' = \frac{p}{p-1}$) and $\omega \in L_p^j(C_{a,b}Y)$. Write ω in the form $\omega = \omega_A + dt \wedge \omega_B$, where ω_A, ω_B do not contain dt [10]. Following [9], introduce the continuous operator

$$\mu_\psi \colon\ L_p^j(C_{a,b}^f Y) \to L_p^1([a,b[, f^{\frac{n}{p}-j+1})$$

by the formula

$$\mu_\psi \omega = \Big(\int_Y \omega_B(t) \wedge \psi\Big) dt.$$

The following lemma was proved in [9] for $p = q$ and $\psi \in V_{p'}^{n-j+1}(Y)$. The proof in [9] easily extends to $p \neq q$:

Lemma 3.1. *If $\psi \in D^{n-j+1}(Y)$ and $d\psi = 0$ then μ_ψ induces continuous mappings*

$$\mu_\psi^* : H_{p,q}^j(C_{a,b}^f Y) \to H_{p,q}^1([a,b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1});$$
$$\tilde{\mu}_\psi^* : H_{p,q}^j(C_{a,b}^f Y, Y_a) \to H_{p,q}^1([a,b[, \{a\}, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1})$$

We have the following theorem partially generalizing item 7 of Theorem 1 in [9]:

Theorem 3.2. *Suppose that Y is an orientable n-dimensional Riemannian manifold, $\infty < a < b \leq \infty$, $f : [a,b[\to \mathbb{R}$ is a positive continuous function, $1 < p < \infty$, $1 < q < \infty$. Assume that there exists $\varphi \in$*

$Z_p^{j-1}(Y) \cap Z_q^{j-1}(Y)$ *such that* $\int_Y \varphi \wedge \gamma \neq 0$ *for some form* $\gamma \in D^{n-j+1}(Y)$, $d\gamma = 0$.

The following hold:

(1) *if* $\chi_{p,q}(a, b, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$ *then* $H^j_{p,q}(C^f_{a,b}Y, Y_a) \neq 0$;

(2) *if* $\chi_{p,q}(a, b, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$ *and* $\chi_{p,q}(b, a, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$ *then* $T^j_{p,q}(C^f_{a,b}Y) \neq 0$ *and, hence,* $\dim H^j_{p,q}(C^f_{a,b}Y) = \infty$.

Proof. Let $\varphi \in Z_p^{j-1}(Y) \cap Z_q^{j-1}(Y)$ be a cocycle having the property mentioned in the theorem and let $\gamma \in D^{n-j+1}(M)$ be a form such that $\int_Y \varphi \wedge \gamma = 1$. Then $\mu^*_\gamma \circ \nu^*_\varphi = \mathrm{id}$, $\tilde{\mu}^*_\gamma \circ \tilde{\nu}^*_\varphi = \mathrm{id}$ [9]. Consequently, the mappings

$$\nu^*_\varphi : H^1_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \to H^j_{p,q}(C^f_{a,b}Y)$$

and

$$\tilde{\nu}^*_\varphi : H^1_{p,q}([a, b[, \{a\}, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \to H^j_{p,q}(C^f_{a,b}Y, Y_a)$$

are injective.

Suppose that $\chi_{p,q}(a, b, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$. Then, by Theorem 2.3, $H^1_{p,q}([a, b[, \{a\}, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \neq 0$. Therefore, $H^j_{p,q}(C^f_{a,b}Y, Y_a) \neq 0$.

Assume now that

$$\chi_{p,q}(a, b, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$$

and

$$\chi_{p,q}(b, a, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty.$$

Then, by Theorem 2.3, $H^1_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \neq 0$. Since, by Theorem 2.5, $\overline{H}_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = 0$, we have

$$T^1_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \neq 0.$$

Now, if we had $T^j_{p,q}(C^f_{a,b}Y) = 0$, ν^*_φ would be a continuous injective mapping with values in the Hausdorff space $H^j_{p,q}(C^f_{a,b}Y)$, and so the cohomology space $H^1_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1})$ would also be Hausdorff, i.e., without torsion. Thus, $T^j_{p,q}(C^f_{a,b}Y) \neq 0$. The theorem is proved. □

$L_{p,q}$-torsion of a surface of revolution

Let M be a surface of revolution in $\mathbb{R}^{n+2}$, i.e., the $(n+1)$-dimensional surface defined by the equation

$$f^2(x_1) = x_2^2 + \cdots + x_{n+2}^2, \quad (x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2},\ x_1 \geq 0, \qquad (3.1)$$

where $f : [0, \infty[\to \mathbb{R}$ is a positive smooth function. The manifold M is the product $[0, \infty[\times \mathbb{S}^n$ endowed with the metric

$$g_M = (1 + f'^2(x_1))dx_1^2 + f^2(x_1)dy^2$$

induced from $\mathbb{R}^{n+2}$, where dx_1^2 and dy^2 are the conventional Riemannian metrics on $[0, \infty[$ and the sphere $\mathbb{S}^n$. In other words, M may be considered as the warped product $[0, \infty[\times_F \mathbb{S}^n$, where $F = f \circ G^{-1}$, $G(x) = \int_0^x \sqrt{1 + f'^2(t)}dt$.

In [17], we have proved the following fact:

Theorem 3.3. *Suppose that f is unbounded, $p, q \in [1, \infty[$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}$, $1 \leq j \leq n+1$. Then $T_{p,q}^j(M) \neq 0$.*

Kuz′minov and Shvedov [19] established that when f is bounded from above, $T_p^j(M)$ is zero for all j, $2 \leq j \leq n$ and that, for $j = 1, n+1$, the triviality of $T_p^j(M)$ depends on the finiteness of some Hardy constants. This is due to the connection between the L_p-cohomology of the warped product $C_{a,b}^f Y$ and the weighted L_p-cohomology of $[a, b[$ given in the mentioned papers [9, 10]. Above we have shown that there is a connection of this type for $L_{p,q}$-cohomology. Namely, by Theorem 3.2, since $\mathbb{S}^n$ is compact and the de Rham cohomology $H^{j-1}(\mathbb{S}^n)$ of $\mathbb{S}^n$ is nontrivial if $j = 1, n+1$, for $T_{p,q}^j(M)$ $(j = 1, n+1)$ to be zero, it is necessary that $\chi_{p,q}(0, \infty, F^{\frac{n}{p}-j+1}, F^{\frac{n}{q}-j+1}) < \infty$ or $\chi_{p,q}(\infty, 0, F^{\frac{n}{p}-j+1}, F^{\frac{n}{q}-j+1}) < \infty$.

The main result of this section is a generalization of Theorems 2 and 2′ of [16] and is formulated as follows:

Theorem 3.4. *Let M be the surface of revolution* (3.1)*. Suppose that $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}$, $j \in \{1, n+1\}$. If $T_{p,q}^j(M) = 0$ then $\lim\limits_{x \to \infty} f(x) = 0$ and* $\operatorname{vol} M < \infty$.

Proof. Put $k = j - 1$, $q' = \frac{q}{q-1}$.

We have the following equalities:

$$\chi^0_{p,q} \equiv \chi_{p,q}(0,\infty,F^{\frac{n}{p}-k},F^{\frac{n}{q}-k})$$

$$= \sup_{\tau>0}\Big\{\Big(\int\limits_{\tau}^{\infty} f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/p}\Big(\int\limits_{0}^{\tau} f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/q'}\Big\};$$

$$\chi^\infty_{p,q} \equiv \chi_{p,q}(\infty,0,F^{\frac{n}{p}-k},F^{\frac{n}{q}-k})$$

$$= \sup_{\tau>0}\Big\{\Big(\int\limits_{0}^{\tau} f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/p}\Big(\int\limits_{\tau}^{\infty} f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/q'}\Big\}$$

if $p \geq q$;

$$\chi^0_{p,q} \equiv \chi_{p,q}(0,\infty,F^{\frac{n}{p}-k},F^{\frac{n}{q}-k})$$

$$= \Bigg(\int\limits_0^\infty \Big[\Big(\int\limits_0^{H(x)} f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{p-1}\int\limits_{H(x)}^{\infty} f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big]^{\frac{q}{q-p}} \times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx\Bigg)^{\frac{q-p}{qp}}; \quad (3.2)$$

$$\chi^\infty_{p,q} \equiv \chi_{p,q}(\infty,0,F^{\frac{n}{p}-k},F^{\frac{n}{q}-k})$$

$$= \Bigg(\int\limits_0^\infty \Big[\Big(\int\limits_{H(x)}^{\infty} f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{p-1}\int\limits_{0}^{H(x)} f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big]^{\frac{q}{q-p}} \times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx\Bigg)^{\frac{q-p}{qp}}$$

if $p < q$. Here $H(x)$ is the function inverse to the arc length function $G(x) = \int_0^x \sqrt{1+f'^2(t)}dt$.

The main element in the proof of Theorem 3.4 is the following lemma which has some independent interest.

Lemma 3.5. *If $\frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}$, $1 < p < \infty$, $1 < q < \infty$, $0 \leq k \leq n$, then the following hold:*

(1) *if $\chi^0_{p,q} < \infty$ or $\chi^\infty_{p,q} < \infty$ then $\lim\limits_{t\to\infty} f(t) = 0$;*

(2) *if* $\frac{n}{p} - k \le 0$ *then* $\chi^0_{p,q} = \infty$;
(3) *if* $\frac{n}{q} - k \ge 0$ *then* $\chi^\infty_{p,q} = \infty$.

Proof. Suppose first that $p \ge q$.

Assume that $\chi^0_{p,q} < \infty$. Then

$$\int_0^\infty f^{n-kp}(t)\sqrt{1+f'^2(t)}dt < \infty \tag{3.3}$$

Since

$$f^{n-kp}(t)\sqrt{1+f'^2(t)} \ge f^{n-kp}(t)|f'(t)|,$$

it follows that the integral

$$\int_0^\infty f^{n-kp}(t)f'(t)dt$$
$$= \begin{cases} \frac{1}{n-kp+1}\lim\limits_{t\to\infty}(f^{n-kp+1}(t) - f^{n-kp+1}(0)) & \text{if } n-kp \ne -1, \\ \lim\limits_{t\to\infty}\log\frac{f(t)}{f(0)} & \text{if } n-kp = -1 \end{cases} \tag{3.4}$$

is finite.

There appear several possibilities:

(a) $\frac{n}{p} - k > 0$. The above implies that there exists a finite limit $\lim\limits_{t\to\infty} f(t)$, which is zero by (3.3).

(b) $\frac{n}{p} - k = 0$. This is impossible in view of (3.3).

(c) $-\frac{1}{p} < \frac{n}{p} - k < 0$. Then $n - kp + 1 > 0$ and $f(t)$ has a finite limit as $t \to \infty$, which contradicts (3.3).

(d) $\frac{n}{p} - k = -\frac{1}{p}$. A contradiction to (3.3).

(e) $\frac{n}{p} - k < -\frac{1}{p}$. In this case, $n - kp < -1$. Hence, $\lim\limits_{t\to\infty} f(t) = \infty$.

Note that, since $\frac{1}{q}-\frac{1}{p}<\frac{1}{n+1}$, we have $k+1>\frac{n+1}{p}+1>\frac{n+1}{q}$, whence $-(\frac{n}{q}-k)q'+1>0$. We infer

$$\begin{aligned}
&\Big(\int_\tau^\infty f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/p}\Big(\int_0^\tau f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/q'}\\
&\quad\geq\Big(\int_\tau^\infty f^{n-kp}(t)|f'(t)|dt\Big)^{1/p}\Big(\int_0^\tau f^{-(\frac{n}{q}-k)q'}(t)|f'(t)|dt\Big)^{1/q'}\\
&\quad\geq\Big|\int_\tau^\infty f^{n-kp}(t)f'(t)dt\Big|^{1/p}\Big|\int_0^\tau f^{-(\frac{n}{q}-k)q'}(t)f'(t)dt\Big|^{1/q'}\\
&\quad\geq\Big(\frac{f^{n-kp+1}(\tau)}{|n-kp+1|}\Big)^{1/p}\Big|\frac{f^{-\frac{n}{q}-k)q'+1}(\tau)-f^{-(\frac{n}{q}-k)q'+1}(0)}{-(\frac{n}{q}-k)q'+1}\Big|^{1/q'}\\
&\quad=C\cdot f^{\frac{n+1}{p}-\frac{n+1}{q}+1}(\tau)|1-f^{-(\frac{n}{q}-k)q'+1}(0)f^{(\frac{n}{q}-k)q'-1}(\tau)|^{1/q'}. \qquad (3.5)
\end{aligned}$$

The last quantity in (3.5) is equivalent to $Cf^{\frac{n+1}{p}-\frac{n+1}{q}+1}(\tau)$ as $\tau\to\infty$ and, hence, tends to infinity. Therefore, $\chi^0_{p,q}=\infty$, and we obtain a contradiction.

Thus, if $\chi^0_{p,q}<\infty$ then $\lim\limits_{t\to 0}f(t)=0$ and $\frac{n}{p}-k>0$.

Suppose now that $\chi^\infty_{p,q}<\infty$. Then

$$\int_0^\infty f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt<\infty \qquad (3.6)$$

and, hence, there exists a finite integral

$$\begin{aligned}
&\int_0^\infty f^{-(\frac{n}{q}-k)q'}(t)f'(t)dt\\
&\quad=\begin{cases}\frac{\lim\limits_{t\to\infty}f^{-(\frac{n}{q}-k)q'+1}(t)-f^{-(\frac{n}{q}-k)q'+1}(0)}{-(\frac{n}{q}-k)q'+1} & \text{if } -(\frac{n}{q}-k)q'\neq-1,\\ \lim\limits_{t\to\infty}\log\frac{f(t)}{f(0)} & \text{if } -(\frac{n}{q}-k)q'=-1.\end{cases} \qquad (3.7)
\end{aligned}$$

As in the case $\chi^0_{p,q} < \infty$, we infer that either $\frac{n}{q} - k < 0$ and $\lim\limits_{t\to\infty} f(t) = 0$ or $(\frac{n}{q} - k)q' > 1$ and $\lim\limits_{t\to\infty} f(t) = \infty$. In the latter case we have:

$$\chi^\infty_{p,q} \geq \sup_{\tau>0}\left\{\left(\frac{f^{-(\frac{n}{q}-k)q'+1}(\tau)}{1-(\frac{n}{q}-k)q'}\right)^{1/q'}\left|\frac{f^{n-kp+1}(\tau) - f^{n-kp+1}(0)}{n-kp+1}\right|^{1/p}\right\}$$
$$= C\sup_{\tau>0}\{f^{\frac{n+1}{p}-\frac{n+1}{q}+1}(\tau)|1 - f^{n-kp+1}(0)f^{-(n-kp+1)}(\tau)|^{1/p}\}, \quad (3.8)$$

where $C = \text{const} > 0$. Since $k < \frac{n+1}{q} - 1 < \frac{n+1}{p}$, we have $n - kp + 1 > 0$, and, hence, the last quantity in (3.8) behaves like $Cf^{\frac{n+1}{p}-\frac{n+1}{q}+1}(\tau)$ and, consequently, tends to infinity as $\tau \to \infty$. Hence, $\chi^\infty_{p,q} = \infty$; a contradiction.

Thus, if $\chi^\infty_{p,q} < \infty$ then $\lim\limits_{t\to 0} f(t) = 0$ and $\frac{n}{q} - k < 0$.

We now pass to the case $p < q$.

Suppose that $\chi^0_{p,q} < \infty$. Then, as above, we have (3.3) and (3.4) and conclude that either $\frac{n}{p} - k > 0$ and $\lim\limits_{t\to\infty} f(t) = 0$ or $\frac{n}{p} - k < -\frac{1}{p}$ and $\lim\limits_{t\to\infty} f(t) = \infty$. Show that the latter case is impossible. By (3.2), we infer

$$(\chi^0_{p,q})^{\frac{pq}{q-p}} \geq \int_0^\infty \left[\left|\int_0^{H(x)} f^{-(\frac{n}{q}-k)q'}(t)f'(t)dt\right|^{p-1}\left|\int_{H(x)}^\infty f^{n-kp}(t)f'(t)dt\right|\right]^{\frac{q}{q-p}}$$
$$\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx$$
$$= \int_0^\infty\left[\left|\frac{f^{-(\frac{n}{q}-k)q'+1}(H(x)) - f^{-(\frac{n}{q}-k)q'+1}(0)}{-(\frac{n}{q}-k)q'+1}\right|^{p-1}\left|\frac{f^{n-kp+1}(H(x))}{n-kp+1}\right|\right]^{\frac{q}{q-p}}$$
$$\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx$$
$$= \int_0^\infty\left[\left|\frac{F^{-(\frac{n}{q}-k)q'+1}(s) - F^{-(\frac{n}{q}-k)q'+1}(0)}{-(\frac{n}{q}-k)q'+1}\right|^{p-1}\left|\frac{F^{n-kp+1}(s)}{n-kp+1}\right|\right]^{\frac{q}{q-p}} F^{-(\frac{n}{q}-k)q'}(s)ds$$
$$= C\int_0^\infty F^N(s)|1 - F^{-(\frac{n}{q}-k)q'+1}(0)F^{(\frac{n}{q}-k)q'-1}(s)|ds. \quad (3.9)$$

Here $C = \text{const} > 0$ and

$$N = \left(\left(-\left(\frac{n}{q}-k\right)q'+1\right)(p-1)+n-kp+1\right)\frac{q}{q-p}-\left(\frac{n}{q}-k\right)q'$$
$$=\left[\left(1-\frac{p-1}{q-1}\right)\frac{q}{q-p}-\frac{1}{q-1}\right]n-\left[\left(\frac{q(p-1)}{q-1}-p\right)\frac{q}{q-p}+\frac{1}{q-1}\right]k+\frac{pq}{q-p}$$
$$=n-k+\frac{pq}{q-p}>0.$$

Moreover, $(\frac{n}{q}-k)q'-1<0$, since $\frac{n}{q}<\frac{n+1}{q}<\frac{n+1}{p}<k$. Consequently, the expression under the last integral in (3.9) is equivalent to $CF^{n-k+\frac{pq}{q-p}}(s)$, i.e., tends to ∞ as $s\to\infty$ and, thus, the integral does not exist. A contradiction.

Suppose now that $\chi^{\infty}_{p,q}<\infty$. Then we have (3.6) and (3.7) and infer that, in this case, either $\frac{n}{q}-k<0$ and $\lim\limits_{t\to\infty}f(t)=0$ or $q'(\frac{n}{q}-k)>1$ and $\lim\limits_{t\to\infty}f(t)=\infty$. In the latter case, we infer

$$(\chi^{\infty}_{p,q})^{\frac{pq}{q-p}}\geq\int\limits_0^{\infty}\Big[\Big|\int\limits_0^{H(x)}f^{n-kp}(t)f'(t)dt\Big|\Big|\int\limits_{H(x)}^{\infty}f^{-(\frac{n}{q}-k)q'}(t)f'(t)dt\Big|^{p-1}\Big]^{\frac{q}{q-p}}$$
$$\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx$$
$$=\int\limits_0^{\infty}\Big[\Big|\frac{f^{n-kp+1}(H(x))-f^{n-kp+1}(0)}{n-kp+1}\Big|\Big|\frac{f^{-(\frac{n}{q}-k)q'+1}(H(x))}{-(\frac{n}{q}-k)q'+1}\Big|^{p-1}\Big]^{\frac{q}{q-p}}$$
$$\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx$$
$$=\int\limits_0^{\infty}\Big[\Big|\frac{F^{n-kp+1}(s)-F^{n-kp+1}(0)}{n-kp+1}\Big|\Big|\frac{F^{-(\frac{n}{q}-k)q'+1}(s)}{-(\frac{n}{q}-k)q'+1}\Big|^{p-1}\Big]^{\frac{q}{q-p}}F^{-(\frac{n}{q}-k)q'}(s)ds$$
$$=C\int\limits_0^{\infty}F^{N}(s)|1-F^{n-kp+1}(0)F^{-(n-kp+1)}(s)|ds. \quad (3.10)$$

Here, as above, $C=\text{const}>0$, $N=n-k+\frac{pq}{q-p}>0$, and $-(n-kp+1)<0$. Thus, the expression under the integral is equivalent to $CF^{n-k+\frac{pq}{q-p}}(s)$, i.e., tends to infinity as $s\to\infty$.

Lemma 3.5 is completely proved. □

Now, return to the proof of Theorem 3.4. Suppose that $T^j_{p,q}(M) = 0$ for $j = 1$ or $j = n + 1$. Then, by Theorem 3.2,

$$\chi_{p,q}(0, \infty, F^{\frac{n}{p}-j+1}, F^{\frac{n}{q}-j+1}) < \infty$$

(and, hence, $\int_0^\infty f^{(\frac{n}{p}-j+1)p}(t)\sqrt{1+f'^2(t)}dt < \infty$) or

$$\chi_{p,q}(\infty, 0, F^{\frac{n}{p}-j+1}, F^{\frac{n}{q}-j+1}) < \infty$$

(and, hence, $\int_0^\infty f^{-(\frac{n}{q}-j+1)q'}(t)\sqrt{1+f'^2(t)}dt < \infty$). By Lemma 3.5, this implies that $\lim\limits_{t\to\infty} f(t) = 0$ and, in both cases,

$$\operatorname{vol} M = s_n \int_0^\infty f^n(t)\sqrt{1+f'^2(t)}dt < \infty.$$

Here s_n stands for the volume of the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.

The theorem is proved. □

Acknowledgments. The author was supported by INTAS (Grant 03–51–3251), the Specific Targeted Project GALA within the NEST Activities of the Commission of the European Communities (Contract No. 028766), the State Maintenance Program for the Leading Scientific Schools and Junior Scientists of the Russian Federation (Grants NSh 8526.2006.1, NSh 5682.2008.1), and a grant of the President of the Russian Federation (Grant MK-2137.2008.1).

The paper was begun in January 2007 during the author's research stay at the Institut des Hautes Études Scientifiques (IHÉS) in Bures-sur-Yvette. The author would like to thank the Director Professor Jean-Pierre Bourguignon for the invitation to visit the IHÉS and all people at the Institute for their warm hospitality. The author also expresses his gratitude to Vladimir Gol′dshtein, Vladimir Kuz′minov, Aleksandr Romanov, and Igor Shvedov for useful discussions.

The author is also indebted to the anonymous referee for extremely valuable remarks and suggestions which substantially improved the exposition.

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