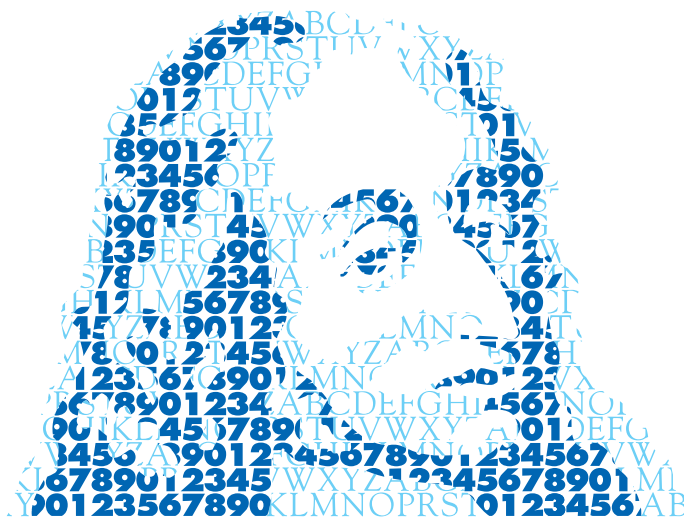


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P-adic Spaces of Continuous Functions I

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P-adic Spaces of Continuous Functions I

ATHANASIOS KATSARAS

Abstract

Properties of the so called θ_o -complete topological spaces are investigated. Also, necessary and sufficient conditions are given so that the space $C(X, E)$ of all continuous functions, from a zero-dimensional topological space X to a non-Archimedean locally convex space E , equipped with the topology of uniform convergence on the compact subsets of X to be polarly barrelled or polarly quasi-barrelled.

Introduction

Let \mathbb{K} be a complete non-Archimedean valued field and let $C(X, E)$ be the space of all continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E . We will denote by $C_b(X, E)$ (resp. by $C_{rc}(X, E)$) the space of all $f \in C(X, E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E . The dual space of $C_{rc}(X, E)$, under the topology t_u of uniform convergence, is a space $M(X, E')$ of finitely-additive E' -valued measures on the algebra $K(X)$ of all clopen, i.e. both closed and open, subsets of X . Some subspaces of $M(X, E')$ turn out to be the duals of $C(X, E)$ or of $C_b(X, E)$ under certain locally convex topologies.

In section 2 of this paper, we give some results about the space $M(X, E')$. The notion of a θ_o -complete topological space was given in [2]. In section 3 we study some of the properties of θ_o -complete spaces. Among other results, we prove that a Hausdorff zero-dimensional space is θ_o -complete iff it is homeomorphic to a closed subspace of a product of ultrametric spaces. In section 4, we give necessary and sufficient conditions for the space $C(X, E)$, equipped with the topology of uniform convergence on the compact subsets of X , to be polarly barrelled or polarly quasi-barrelled,

Keywords: Non-Archimedean fields, zero-dimensional spaces, locally convex spaces.

Math. classification: 46S10, 46G10.

1. Preliminaries

Throughout this paper, \mathbb{K} will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over \mathbb{K} , we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over \mathbb{K} (see [9]). Unless it is stated explicitly otherwise, X will be a Hausdorff zero-dimensional topological space, E a Hausdorff locally convex space and $cs(E)$ the set of all continuous seminorms on E . The space of all \mathbb{K} -valued linear maps on E is denoted by E^* , while E' denotes the topological dual of E . A seminorm p , on a vector space G over \mathbb{K} , is called polar if $p = \sup\{|f| : f \in G^*, |f| \leq p\}$. A locally convex space G is called polar if its topology is generated by a family of polar seminorms. A subset A of G is called absolutely convex if $\lambda x + \mu y \in A$ whenever $x, y \in A$ and $\lambda, \mu \in \mathbb{K}$, with $|\lambda|, |\mu| \leq 1$. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [3]) and by $\nu_o X$ the \mathbf{N} -repletion of X , where \mathbf{N} is the set of natural numbers. We will let $C(X, E)$ denote the space of all continuous E -valued functions on X and $C_b(X, E)$ (resp. $C_{rc}(X, E)$) the space of all $f \in C(X, E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E . In case $E = \mathbb{K}$, we will simply write $C(X), C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the \mathbb{K} -valued characteristic function of A . Also, for $X \subset Y \subset \beta_o X$, we denote by \bar{B}^Y the closure of B in Y . If $f \in E^X, p$ a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology β_o on $C_b(X, E)$ (see [4]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and h is in the space $B_o(X)$ of all bounded \mathbb{K} -valued functions on X which vanish at infinity, i.e. for every $\epsilon > 0$ there exists a compact subset Y of X such that $|h(x)| < \epsilon$ if $x \notin Y$.

Let $\Omega = \Omega(X)$ be the family of all compact subsets of $\beta_o X \setminus X$. For $H \in \Omega$, let C_H be the space of all $h \in C_{rc}(X)$ for which the continuous extension h^{β_o} to all of $\beta_o X$ vanishes on H . For $p \in cs(E)$, let $\beta_{H,p}$ be the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p$, $h \in C_H$. For $H \in \Omega, \beta_H$ is the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p$, $h \in C_H, p \in cs(E)$.

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The inductive limit of the topologies $\beta_H, H \in \Omega$, is the topology β . Replacing Ω by the family Ω_1 of all \mathbb{K} -zero subsets of $\beta_o X$, which are disjoint from X , we get the topology β_1 . Recall that a \mathbb{K} -zero subset of $\beta_o X$ is a set of the form $\{x \in \beta_o X : g(x) = 0\}$, for some $g \in C(\beta_o X)$. We get the topologies β_u and β'_u replacing Ω by the family Ω_u of all $Q \in \Omega$ with the following property: There exists a clopen partition $(A_i)_{i \in I}$ of X such that Q is disjoint from each $\overline{A_i}^{\beta_o X}$. Now β_u is the inductive limit of the topologies $\beta_Q, Q \in \Omega_u$. The inductive limit of the topologies $\beta_{H,p}$, as H ranges over Ω_u , is denoted by $\beta_{u,p}$, while β'_u is the projective limit of the topologies $\beta_{u,p}, p \in cs(E)$. For the definition of the topology β_e on $C_b(X)$ we refer to [7].

Let now $K(X)$ be the algebra of all clopen subsets of X . We denote by $M(X, E')$ the space of all finitely-additive E' -additive measures m on $K(X)$ for which the set $m(K(X))$ is an equicontinuous subset of E' . For each such m , there exists a $p \in cs(E)$ such that $\|m\|_p = m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0, A \supset B \in K(X)\}.$$

The space of all $m \in M(X, E')$ for which $m_p(X) < \infty$ is denoted by $M_p(X, E')$. If $m \in M_p(X, E')$, then for $x \in X$ we define

$$N_{m,p}(x) = \inf\{m_p(V) : x \in V \in K(X)\}.$$

In case $E = \mathbb{K}$, we denote by $M(X)$ the space of all finitely-additive bounded \mathbb{K} -valued measures on $K(X)$. An element m of $M(X)$ is called τ -additive if $m(V_\delta) \rightarrow 0$ for each decreasing net (V_δ) of clopen subsets of X with $\bigcap V_\delta = \emptyset$. In this case we write $V_\delta \downarrow \emptyset$. We denote by $M_\tau(X)$ the space of all τ -additive members of $M(X)$. Analogously, we denote by $M_\sigma(X)$ the space of all σ -additive m , i.e. those m with $m(V_n) \rightarrow 0$ when $V_n \downarrow \emptyset$. For an $m \in M(X, E')$ and $s \in E$, we denote by ms the element of $M(X)$ defined by $(ms)(V) = m(V)s$.

Next we recall the definition of the integral of an $f \in E^X$ with respect to an $m \in M(X, E')$. For a non-empty clopen subset A of X , let \mathcal{D}_A be the family of all $\alpha = \{A_1, A_2, \dots, A_n; x_1, x_2, \dots, x_n\}$, where $\{A_1, \dots, A_n\}$ is a clopen partition of A and $x_k \in A_k$. We make \mathcal{D}_A into a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2 . For an $\alpha = \{A_1, A_2, \dots, A_n; x_1, x_2, \dots, x_n\} \in \mathcal{D}_A$ and $m \in M(X, E')$,

we define

$$\omega_\alpha(f, m) = \sum_{k=1}^n m(A_k) f(x_k).$$

If the limit $\lim \omega_\alpha(f, m)$ exists in \mathbb{K} , we will say that f is m -integrable over A and denote this limit by $\int_A f dm$. We define the integral over the empty set to be 0. For $A = X$, we write simply $\int f dm$. It is easy to see that if f is m -integrable over X , then it is m -integrable over every clopen subset A of X and $\int_A f dm = \int \chi_A f dm$. If τ_u is the topology of uniform convergence, then every $m \in M(X, E')$ defines a τ_u -continuous linear functional ϕ_m on $C_{rc}(X, E)$, $\phi_m(f) = \int f dm$. Also every $\phi \in (C_{rc}(X, E), \tau_u)'$ is given in this way by some $m \in M(X, E')$.

For $p \in cs(E)$, we denote by $M_{t,p}(X, E')$ the space of all $m \in M_p(X, E')$ for which m_p is tight, i.e. for each $\epsilon > 0$, there exists a compact subset Y of X such that $m_p(A) < \epsilon$ if the clopen set A is disjoint from Y . Let

$$M_t(X, E') = \bigcup_{p \in cs(E)} M_{t,p}(X, E').$$

Every $m \in M_{t,p}(X, E')$ defines a β_0 -continuous linear functional u_m on $C_b(X, E)$,

$u_m(f) = \int f dm$. The map $m \mapsto u_m$, from $M_t(X, E')$ to $(C_b(X, E), \beta_o)'$, is an algebraic isomorphism. For $m \in M_\tau(X)$ and $f \in \mathbb{K}^X$, we will denote by $(VR) \int f dm$ the integral of f , with respect to m , as it is defined in [9]. We will call $(VR) \int f dm$ the (VR) -integral of f .

For all unexplained terms on locally convex spaces, we refer to [8] and [9].

2. Some results on $M(X, E')$

Theorem 2.1. *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$, for all $s \in E$, and let $p \in cs(E)$ with $\|m\|_p < \infty$. Then :*

(1) $m_p(V) = \sup_{x \in V} N_{m,p}(x)$ for every $V \in K(X)$.

(2) The set

$$\text{supp}(m) = \bigcap \{V \in K(X) : m_p(V^c) = 0\}$$

is the smallest of all closed support sets for m .

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(3) $\text{supp}(m) = \overline{\{x : N_{m,p}(x) \neq 0\}}$.

(4) If V is a clopen set contained in the union of a family $(V_i)_{i \in I}$ of clopen sets, then

$$m_p(V) \leq \sup\{m_p(V_i) : i \in I\}.$$

Proof: (1). If $x \in V$, then $N_{m,p}(x) \leq m_p(V)$ and so

$$m_p(V) \geq \alpha = \sup_{x \in V} N_{m,p}(x).$$

On the other hand, let $m_p(V) > d$. There exists a clopen set W , contained in V , and $s \in E$ with $|m(W)s|/p(s) > d$. Let $\mu = ms \in M_\tau(X)$. Then

$$|\mu|(W) = \sup_{x \in W} N_\mu(x).$$

Let $x \in W$ be such that $N_\mu(x) > d \cdot p(s)$. Now $N_{m,p}(x) \geq d$. In fact, assume the contrary and let Z be a clopen neighborhood of x contained in W and such that $m_p(Z) < d$. Now

$$N_\mu(x) \leq |\mu|(Z) = \sup\{|m(Y)s| : Z \supset Y \in K(X)\} \leq p(s) \cdot m_p(Z) \leq d \cdot p(s).$$

This contradiction proves (1).

(2).

$$X \setminus \text{supp}(m) = \bigcup\{W \in K(X) : m_p(W) = 0\}.$$

Let $V \in K(X)$ be disjoint from $\text{supp}(m)$. For each $x \in V$, there exists $W \in K(X)$, with $x \in W$ and $m_p(W) = 0$ and so $N_{m,p}(x) = 0$. It follows that

$$m_p(V) = \sup_{x \in V} N_{m,p}(x) = 0,$$

which proves that $\text{supp}(m)$ is a support set for m . On the other hand, let Y be a closed support set for m . There exists a decreasing net (V_δ) of clopen sets with $Y = \bigcap V_\delta$. Let $W \in K(X)$ be disjoint from Y . For each clopen set V contained in W and each $s \in E$, we have $V \cap V_\delta \downarrow \emptyset$ and so $\lim_\delta (ms)(V \cap V_\delta) = 0$. Since V_δ^c is disjoint from Y , we have $m(V_\delta^c) = 0$ and so $m(V) = m(V_\delta \cap V)$, which implies that $m(V)s = 0$, for all $s \in E$, i.e. $m(V) = 0$, and hence $m_p(W) = 0$. Therefore $\text{supp}(m) \subset W^c$. Taking V_δ^c in place of W , we get that $\text{supp}(m) \subset \bigcap V_\delta = Y$, which proves (2).

(3) Let $G = \overline{\{x : N_{m,p}(x) \neq 0\}}$. If $V \in K(X)$ is disjoint from G , then

$$m_p(V) = \sup_{x \in V} N_{m,p}(x) = 0,$$

and so $\text{supp}(m) \subset V^c$, which implies that $\text{supp}(m) \subset G$. On the other hand, let $x \notin \text{supp}(m)$. There exists a clopen neighborhood W of x disjoint from $\text{supp}(m)$. Since $\text{supp}(m)$ is a support set for m , we have that $m_p(W) = 0$ and thus $N_{m,p} = 0$ on W , which proves that $x \notin G$. Thus G is contained in $\text{supp}(m)$ and (3) follows.

(4). Let $m_p(V) > \alpha > 0$. There exists a clopen set A contained in V and $s \in E$ such that $|m(A)s|/p(s) > \alpha$. If $\mu = ms \in M_\tau(X)$, then $|\mu|(V) \geq |m(A)s| > \alpha \cdot p(s)$. In view of [9, p. 250] there exists an i such that $m_p(V_i) \geq |\mu|(V_i)/p(s) > \alpha$, which clearly completes the proof.

Theorem 2.2. *Let $m \in M(X, E')$ be such that $ms \in M_\sigma(X)$ for all $s \in E$ (this in particular holds if $m \in M_\sigma(X, E')$). Let $p \in cs(E)$ be such that $m_p(X) < \infty$. If a clopen set V is contained in the union of a sequence (V_n) of clopen sets, then $m_p(V) \leq \sup_n m_p(V_n)$.*

Proof : We show first that, for $\mu \in M_\sigma(X)$, then there exists an n with $|\mu|(V) \leq |\mu|(V_n)$. In fact, this is clearly true if $|\mu|(V) = 0$. Assume that $|\mu|(V) > 0$ and let $W_n = \bigcup_1^n V_k$. Since $W_n^c \cap V \downarrow \emptyset$, there exists n such that $|\mu|(V \cap W_n^c) < |\mu|(V)$. Since $V \subset (V \cap W_n^c) \cup W_n$, it follows that

$$|\mu|(V) \leq |\mu|(W_n) = \max_{1 \leq k \leq n} |\mu|(V_k),$$

and the claim follows for μ . Suppose now that $m_p(V) > r > 0$. There exists a clopen subset W of V and $s \in E$ such that $|m(W)s| > r \cdot p(s)$. Let $\mu = ms$. Then $\mu \in M_\sigma(X)$ and $|\mu|(V) \geq |m(W)s| > r \cdot p(s)$. By the first part of the proof, there exists an n such that $|\mu|(V_n) > r \cdot p(s)$. Hence, there exists a clopen subset D of V_n such that $|\mu|(D) > r \cdot p(s)$. Now $|m|_p(V_n) \geq |m(D)s|/p(s) > r$, which completes the proof.

For $X \subset Y \subset \beta_o X$, and $m \in M(X)$, we denote by m^Y the element of $M(Y)$ defined by $m^Y(V) = m(V \cap X)$. We denote by m^{v_o} and m^{β_o} the m^Y for $Y = v_o X$ and $Y = \beta_o X$, respectively.

We have the following easily established

Theorem 2.3. *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$. Then :*

$$(1) \text{supp}(m^{\beta_o}) = \overline{\text{supp}(m)}^{\beta_o X}.$$

$$(2) \text{supp}(m) = \text{supp}(m^{\beta_o}) \cap X.$$

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(3) If m has compact support, then $\text{supp}(m) = \text{supp}(m^{\beta_o})$.

Theorem 2.4. Let $m \in M_p(X, E')$ and $\mu = m^{\beta_o}$. The following are equivalent:

- (1) $\text{supp}(\mu) \subset v_o X$.
- (2) If $V_n \downarrow \emptyset$, then there exists an n_o such that $m(V_n) = 0$ for every $n \geq n_o$.
- (3) If $V_n \downarrow \emptyset$, then there exists an n such that $m(V) = 0$ for every clopen set V contained in V_n .
- (4) For every $Z \in \Omega_1$ there exists a clopen subset A on $\beta_o X$ disjoint from Z and such that $\text{supp}(\mu) \subset A$.
- (5) If $V_n \downarrow \emptyset$, then there exists an n such that $m_p(V_n) = 0$.

Proof: (1) \Rightarrow (2). If $V_n \downarrow \emptyset$, then the set $\bigcap \overline{V_n}^{\beta_o X}$ is disjoint from $v_o X$ and so $\text{supp}(\mu) \subset \bigcup_n \overline{V_n}^{c \beta_o X}$. In view of the compactness of $\text{supp}(\mu)$, there exists an n_o with $\text{supp}(\mu) \subset \overline{V_{n_o}}^{c \beta_o X}$. If now $n \geq n_o$, then $m(V_n) = 0$.

(2) \Rightarrow (3). Let $V_n \downarrow \emptyset$ and suppose that, for each n , there exists a clopen subset A of V_n such that $m(A) \neq 0$.

Claim. For each n , there exists $k > n$ and a clopen set B with $V_k \subset B \subset V_n$ and $m(B) \neq 0$. Indeed there exists a clopen subset A of V_n such that $m(A) \neq 0$. For each k , let $B_k = V_k \cap A$, $D_k = V_k \setminus B_k$. Then $D_k \downarrow \emptyset$. By our hypothesis, there exists $k > n$ such that $m(D_k) = 0$. Let $B = A \cup D_k$. Then $V_k \subset B \subset V_n$. Since A and D_k are disjoint, we have that $m(B) = m(A) \neq 0$ and the claim follows. By induction, we choose $n_1 = 1 < n_2 < \dots$ and clopen sets B_k such that $V_{n_{k+1}} \subset B_k \subset V_{n_k}$ and $m(B_k) \neq 0$. Since $B_k \downarrow \emptyset$ and $m(B_k) \neq 0$ for every k , we arrived at a contradiction.

(3) \Rightarrow (4). Let $Z \in \Omega_1$. There exists a decreasing sequence (V_n) of clopen sets with $Z = \bigcap \overline{V_n}^{\beta_o X}$. By our hypothesis, there exists an n such that $m(V) = 0$ for each clopen subset V of V_n . Now it suffices to take $A = \overline{V_n}^{c \beta_o X}$.

(4) \Rightarrow (1). Let $z \in \beta_o X \setminus v_o X$. There exists a decreasing sequence (V_n) of clopen sets with $z \in Z = \bigcap \overline{V_n}^{\beta_o X}$. Clearly $Z \in \Omega_1$. Thus, there exists a

clopen subset A of $\beta_o X$ disjoint from Z and containing $\text{supp}(\mu)$. Hence z is not in $\text{supp}(\mu)$.

(3) \Rightarrow (5). It is trivial.

(5) \Rightarrow (1). Let $z \in \beta_o X \setminus v_o X$. There exists a decreasing sequence (V_n) of clopen sets with $z \in Z = \bigcap \overline{V_n}^{\beta_o X}$. Let n be such that $m_p(V_n) = 0$. If $G = \overline{V_n}^{\beta_o X}$, then $\mu_p(G) = 0$ and so $\text{supp}(\mu) \subset \beta_o X \setminus G$, which implies that $z \notin \text{supp}(\mu)$. This completes the proof.

Theorem 2.5. *For an $m \in M_p(X, E')$, the following are equivalent :*

- (1) m has a compact support, i.e. $m \in M_c(X, E')$.
- (2) $\text{supp}(m^{\beta_o}) \subset X$.
- (3) If $V_\delta \downarrow \emptyset$, then there exists a δ_o such that $m(V_\delta) = 0$ for all $\delta \geq \delta_o$.
- (4) If $V_\delta \downarrow \emptyset$, then there exists a δ such $m(V) = 0$ for each clopen subset V of V_δ .
- (5) If $H \in \Omega$, then there exists a clopen subset A of $\beta_o X$, disjoint from H and containing $\text{supp}(m^{\beta_o})$.
- (6) If $V_\delta \downarrow \emptyset$, then there exists a δ such that $m_p(V_\delta) = 0$.

Proof : In view of Theorem 2.3, (1) implies (2).

(2) \Rightarrow (3). Let $V_\delta \downarrow \emptyset$. By the compactness of $\text{supp}(m^{\beta_o})$, there exists δ_o such that $\text{supp}(m^{\beta_o}) \subset V_{\delta_o}^c$ and so $m(V_\delta) = 0$ for $\delta \geq \delta_o$.

(3) \Rightarrow (4). Let $V_\delta \downarrow \emptyset$ and suppose that, for each δ , there exists a clopen subset V of V_δ with $m(V) \neq 0$.

Claim: For each δ there exist $\gamma \geq \delta$ and a clopen set A such that $V_\gamma \subset A \subset V_\delta$ and $m(A) \neq 0$. In fact, there exists a clopen subset G of V_δ with $m(G) \neq 0$. For each γ , let $Z_\gamma = V_\gamma \cap G$, $W_\gamma = V_\gamma \setminus Z_\gamma$. Then $W_\gamma \downarrow \emptyset$. By our hypothesis, there exists $\gamma \geq \delta$ with $m(V_\gamma) = 0$. Let $A = G \cup W_\gamma$. Since the sets G and W_γ are disjoint, we have that $m(A) = m(G) \neq 0$. Since $V_\gamma \subset A \subset V_\delta$, the claim follows.

Let now \mathcal{F} be the family of all clopen subsets A of X with the following property: There are γ, δ , with $\gamma \geq \delta$, $V_\gamma \subset A \subset V_\delta$ and $m(A) \neq 0$. Since $\mathcal{F} \downarrow \emptyset$, we got a contradiction.

(4) \Rightarrow (5). If $H \in \Omega$, then there exists a decreasing net (V_δ) of clopen

subsets of X with $\bigcap \overline{V_\delta}^{\beta_o X} = H$. Since $V_\delta \downarrow \emptyset$, there exists δ such that $m(V) = 0$ for each clopen subset V of V_δ . Now it suffices to take $A = \overline{V_\delta}^{\beta_o X}$.

(5) \Rightarrow (1). Let $z \in \beta_o X \setminus X$. By (5), there exists a clopen subset A of $\beta_o X$ containing $\text{supp}(m^{\beta_o})$ and not containing z .

(4) \Rightarrow (6). It is trivial.

(6) \Rightarrow (2). Let $z \in \beta_o X \setminus X$. There exists a decreasing net (V_δ) of clopen sets with $\{z\} = \bigcap \overline{V_\delta}^{\beta_o X}$. Let δ be such that $m_p(V_\delta) = 0$. If $\mu = m^{\beta_o}$, then $\mu_p(\overline{V_\delta}^{\beta_o X}) = m_p(V_\delta) = 0$ and so $\text{supp}(\mu)$ is disjoint from the closure of V_δ in $\beta_o X$, which implies that $z \notin \text{supp}(\mu)$.

This completes the proof.

3. θ_o -Complete Spaces

Recall that $\theta_o X$ is the set of all $z \in \beta_o X$ with the following property: For each clopen partition (V_i) of X there exists i such that $z \in \overline{V_i}^{\beta_o X}$ (see [2]). By [2, Lemma 4.1] we have $X \subset \theta_o X \subset v_o X$. For each clopen partition $\alpha = (V_i)_{i \in I}$ of X , let

$$W_\alpha = \bigcup_{i \in I} V_i \times V_i.$$

Then the family of all W_α , α a clopen partition of X , is a base for a uniformity $\mathcal{U}_c = \mathcal{U}_c^X$, compatible with the topology of X , and $(\theta_o X, \mathcal{U}_c^{\theta_o X})$ coincides with the completion of (X, \mathcal{U}_c) . We will say that X is θ_o -complete iff $X = \theta_o X$. As it is shown in [2], if Y is a θ_o -complete and $f : X \rightarrow Y$ is a continuous function, then f has a continuous extension $f^{\theta_o} : \theta_o X \rightarrow Y$. A subset A of X is called bounding if every $f \in C(X)$ is bounded on A . Note that several authors use the term bounded set instead of bounding. But in this paper we will use the term bounding to distinguish from the notion of a bounded set in a topological vector space. A set $A \subset X$ is bounding iff $\overline{A}^{v_o X}$ is compact. In this case (as it is shown in [2, Theorem 4.6]) we have that $\overline{A}^{\theta_o X} = \overline{A}^{v_o X} = \overline{A}^{\beta_o X}$. Clearly a continuous image of a bounding set is bounding. Let us say that a family \mathcal{F} of subsets of X is finite on a subset A of X if the family $\{f \in \mathcal{F} : F \cap A \neq \emptyset\}$ is finite. We have the following easily established

Lemma 3.1. *For a subset A of X , the following are equivalent :*

- (1) A is bounding.
- (2) Every continuous real-valued function on X is bounded on A .
- (3) Every locally finite family of open subsets of X is finite on A .
- (4) Every locally finite family of clopen subsets of X is finite on A .

By [1, Theorem 4.6] every ultraparacompact space (and hence every ultrametrizable space) is θ_o -complete.

Theorem 3.2. *Every complete Hausdorff locally convex space E is θ_o -complete.*

Proof: Let \mathcal{U} be the usual uniformity on E , i.e. the uniformity having as a base the family of all sets of the form

$$W_{p,\epsilon} = \{(x, y) : p(x - y) \leq \epsilon\}, p \in cs(E), \epsilon > 0.$$

Given $W_{p,\epsilon}$, we consider the clopen partition $\alpha = (V_i)_{i \in I}$ of E generated by the equivalence relation $x \sim y$ iff $p(x - y) \leq \epsilon$. Then $W_{p,\epsilon} = W_\alpha$ and so \mathcal{U} is coarser than \mathcal{U}_c . Since (E, \mathcal{U}) is complete and \mathcal{U}_c is compatible with the topology of E , it follows that (E, \mathcal{U}_c) is complete and the result follows.

Corollary 3.3. *A subset B , of a complete Hausdorff locally convex space E , is bounding iff it is totally bounded.*

Proof: If B is bounding, then $\overline{B} = \overline{B}^{\theta_o E}$ is compact and hence totally bounded, which implies that B is totally bounded. Conversely, if B is totally bounded, then \overline{B} is totally bounded. Thus \overline{B} is compact and hence B is bounding.

Theorem 3.4. *If G is a locally convex space (not necessarily Hausdorff), then every bounding subset A of G is totally bounded.*

Proof: Assume first that G is Hausdorff. Let \hat{G} be the completion of G . The closure \hat{B} of A in \hat{G} is bounding and hence \hat{B} is totally bounded, which implies that A is totally bounded. If G is not Hausdorff, we consider the quotient space $F = G/\overline{\{0\}}$ and let $u : G \rightarrow F$ be the quotient map. Since u is continuous, the set $u(A)$ is bounding, and hence totally bounded, in F . Let now V be a convex neighborhood of zero in G . Then, $u(V)$

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is a neighborhood of zero in F . Let S be finite subset of A such that $u(A) \subset u(S) + u(V)$. But then

$$A \subset S + V + \overline{\{0\}} \subset S + V + V = S + V,$$

which proves that A is totally bounded.

Theorem 3.5. *We have:*

- (1) *Closed subspaces of θ_o -complete spaces are θ_o -complete.*
- (2) *If $X = \prod X_i$, with $X_i \neq \emptyset$ for all i , then X is θ_o -complete iff each X_i is θ_o -complete .*
- (3) *If $(Y_i)_{i \in I}$ is a family of θ_o -complete subspaces of X , then $Y = \bigcap Y_i$ is θ_o -complete.*
- (4) *$\theta_o X$ is the smallest of all θ_o -complete subspaces of $\beta_o X$ which contain X .*

Proof: (1). Let Z be a closed subspace of a θ_o -complete space X and let (x_δ) be a \mathcal{U}_c^Z -Cauchy net in Z . Then (x_δ) is \mathcal{U}_c^X -Cauchy and hence $x_\delta \rightarrow x \in X$. Moreover, $x \in Z$ since Z is closed.

(2). Each X_i is homeomorphic to a closed subspace of X . Thus X_i is θ_o -complete if X is θ_o -complete. Conversely, suppose that each X_i is θ_o -complete. If (x^δ) is a \mathcal{U}_c^X -Cauchy net, then (x_i^δ) is a $\mathcal{U}_c^{X_i}$ -Cauchy net in X_i and hence $x_i^\delta \rightarrow x_i \in X_i$. If $x = (x_i)$, then $x^\delta \rightarrow x$, which proves that (X, \mathcal{U}_c) is complete.

(3). Let $X = \prod Y_i$ and consider the map $f : Y \rightarrow X$, $f(x)_i = x$ for all i . Then $f : Y \rightarrow f(Y) = D$ is a homeomorphism. Also D is a closed subspace of X . Since X is θ_o -complete, it follows that D is θ_o -complete and hence Y is θ_o -complete.

(4). Since $\theta_o X$ is θ_o -complete (by [2, Theorem 4.9]) and $X \subset \theta_o X \subset \beta_o X$, the result follows from (3).

Theorem 3.6. *For a point $z \in \beta_o X$, the following are equivalent :*

- (1) $z \in \theta_o X$.
- (2) *If Y is a Hausdorff ultraparacompact space and $f : X \rightarrow Y$ continuous, then $f^{\beta_o}(z) \in Y$, where $f^{\beta_o} : \beta_o X \rightarrow \beta_o Y$ is the continuous extension of f .*

- (3) For every ultrametric space Y and every $f : X \rightarrow Y$ continuous, we have that $f^{\beta_o}(z) \in Y$.

Proof: (1) \Rightarrow (2). Since $\theta_o Y = Y$, the result follows from [2, Theorem 4.4].

(2) \Rightarrow (3). It is trivial.

(3) \Rightarrow (1). Assume that $z \notin \theta_o X$. Then, there exists a clopen partition (A_i) of X such that $z \notin \bigcup_i \overline{A_i}^{\beta_o X}$. Let $f_i = \chi_{A_i}$ and define

$$d : X \times X \rightarrow \mathbf{R}, \quad d(x, y) = \sup_i |f_i(x) - f_i(y)|.$$

Then d is a continuous ultrapseudometric on X . Let $Y = X_d$ be the corresponding ultrametric space and let $\pi : X \rightarrow Y_d$ be the quotient map, $x \mapsto \tilde{x}_d = \tilde{x}$. Since π is continuous, there exists (by (3)) an $x \in X$ such that $\pi^{\beta_o}(z) = \tilde{x}_d$. Let (x_δ) be a net in X converging to z . Then $\tilde{x}_\delta = \pi^{\beta_o}(x_\delta) \rightarrow \pi^{\beta_o}(z) = \tilde{x}$, and so $d(x_\delta, x) \rightarrow 0$. If $x \in A_i$, then $|f_i(x_\delta) - 1| \rightarrow 0$, and so there exists δ_o such that $x_\delta \in A_i$ when $\delta \geq \delta_o$. But then $z \in \overline{A_i}^{\beta_o X}$, a contradiction. This completes the proof.

Theorem 3.7. *Let X be a dense subspace of a Hausdorff zero-dimensional space Y . The following are equivalent :*

- (1) $Y \subset \theta_o X$ (more precisely, Y is homeomorphic to a subspace of $\theta_o X$).
- (2) Each continuous function, from X to any ultrametric space Z , has a continuous extension to all of Y .

Proof: (1) implies (2) by the preceding Theorem.

(2) \Rightarrow (1). We will prove first that, for each clopen subset V of X , we have that $\overline{V}^Y \cap \overline{V^c}^Y = \emptyset$, and so \overline{V}^Y is clopen in Y . Indeed, define

$$d : X \times X \rightarrow \mathbf{R}, \quad d(x, y) = \max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|\},$$

where $f_1 = \chi_V, f_2 = \chi_{V^c}$. Then d is a continuous ultrapseudometric on X . Let $\pi : X \rightarrow X_d$ be the quotient map. By our hypothesis, there exists a continuous extension $h : Y \rightarrow X_d$ of π . Suppose that $z \in \overline{V}^Y \cap \overline{V^c}^Y$. There are nets $(x_\delta), (y_\gamma)$, in V, V^c respectively, such that $x_\delta \rightarrow z$, and $y_\gamma \rightarrow z$. Let \tilde{d} be the ultrametric of X_d and let δ_o, γ_o be such that

$$\tilde{d}(\pi(x_\delta), h(z)) < 1 \quad \text{and} \quad \tilde{d}(\pi(y_\gamma), h(z)) < 1$$

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when $\delta \geq \delta_o, \gamma \geq \gamma_o$. Now

$$d(x_{\delta_o}, y_{\gamma_o}) = \tilde{d}(\pi(x_{\delta_o}), \pi(y_{\delta_o})) < 1,$$

a contradiction. Thus \overline{V}^Y is clopen in Y . If $A = \overline{V}^Y, B = \overline{V}^{cY}$, then

$$\overline{A}^{\beta_o Y} \cap \overline{B}^{\beta_o Y} = \overline{V}^{\beta_o Y} \cap \overline{V}^{c\beta_o Y} = \emptyset.$$

This, being true for each clopen subset V of X , implies that $\beta_o X = \beta_o Y$ and so $X \subset Y \subset \beta_o Y = \beta_o X$. Now our hypothesis (2) and the preceding Theorem imply that $Y \subset \theta_o X$, and the result follows.

Theorem 3.8. *For each continuous ultrapseudometric d on X , there exists a continuous ultrapseudometric d^{θ_o} on $\theta_o X$ which is an extension of d . Moreover, d^{θ_o} is the unique continuous extension of d .*

Proof: Consider the ultrametric space X_d and let \tilde{d} be its ultrametric. Let h be the continuous extension of the quotient map $\pi : X \rightarrow X_d$ to all of $\theta_o X$. Define

$$d^{\theta_o} : \theta_o X \times \theta_o X \rightarrow \mathbf{R}, \quad d^{\theta_o}(y, z) = \tilde{d}(h(y), h(z)).$$

It is easy to see that d^{θ_o} is a continuous ultrapseudometric which is an extension of d . Finally, let ϱ be any continuous ultrapseudometric on $\theta_o X$, which is an extension of d , and let $y, z \in \theta_o X$. There are nets $(y_\delta)_{\delta \in \Delta}, (z_\gamma)_{\gamma \in \Gamma}$ in X which converge to y, z , respectively. Let $\Phi = \Delta \times \Gamma$ and consider on Φ the order $(\delta_1, \gamma_1) \geq (\delta_2, \gamma_2)$ iff $\delta_1 \geq \delta_2$ and $\gamma_1 \geq \gamma_2$. For $\phi = (\delta, \gamma) \in \Phi$, we let $a_\phi = y_\delta, b_\phi = z_\gamma$. Then $a_\phi \rightarrow y, b_\phi \rightarrow z$. Thus

$$\varrho(y, z) = \lim \varrho(a_\phi, b_\phi) = \lim \tilde{d}(h(a_\phi), h(b_\phi)) \tag{3.1}$$

$$= \lim d^{\theta_o}(a_\phi, b_\phi) = d^{\theta_o}(y, z) \tag{3.2}$$

and hence $\varrho = d^{\theta_o}$, which completes the proof.

Theorem 3.9. *Let (H_n) be a sequence of equicontinuous subsets of $C(X)$. If $z \in \theta_o X$, then there exists $x \in X$ such that $f^{\theta_o}(z) = f(x)$ for all $f \in \bigcup H_n = H$.*

Proof: Define

$$d : X^2 \rightarrow \mathbf{R}, \quad d(x, y) = \max_n \min \{1/n, \sup_{f \in H_n} |f(x) - f(y)|\}.$$

Then d is a continuous ultrapseudometric on X . Take $Y = X_d$ and let $\pi : X \rightarrow Y$ be the quotient map. Then $\pi^{\beta_o}(z) = u \in Y$. Choose $x \in$

X with $\pi(x) = u$, and let (x_δ) be a net in X converging to z in $\beta_o X$. Now $f(x_\delta) \rightarrow f^{\beta_o}(z)$ for all $f \in H$. Since $\pi(x_\delta) \rightarrow \pi(x)$, we have that $d(x_\delta, x) \rightarrow 0$, and so $|f(x_\delta) - f(x)| \rightarrow 0$ for all $f \in H$. Thus, for $f \in H$, we have $f(x) = \lim f(x_\delta) = f^{\beta_o}(z)$, and the result follows.

Theorem 3.10. *If $H \subset C(X)$ is equicontinuous, then the family*

$$H^{\theta_o} = \{f^{\theta_o} : f \in H\}$$

is equicontinuous on $\theta_o X$. Moreover, if H is pointwise bounded, then the same holds for H^{θ_o}

Proof: Define

$$d : X^2 \rightarrow \mathbf{R}, \quad d(x, y) = \min\{1, \sup_{f \in H} |f(x) - f(y)|\}.$$

Let $\pi^{\theta_o} : \theta_o X \rightarrow X_d$ be the continuous extension of the quotient map $\pi : X \rightarrow X_d$. Let $z \in \theta_o X$ and $\epsilon > 0$. There exists $x \in X$ such that $\pi^{\theta_o}(z) = \pi(x)$. Let (x_δ) be a net in X converging to z . Then $\pi(x_\delta) \rightarrow \pi^{\theta_o}(z) = \pi(x)$ and so $d(x_\delta, x) \rightarrow 0$. Thus, for $f \in H$, we have $f^{\theta_o}(z) = \lim f(x_\delta) = f(x)$. The set $W = \{y \in X : d(x, y) \leq \epsilon\}$ is d -clopen (hence clopen) in X and so $\overline{W}^{\theta_o X} = V$ is clopen in $\theta_o X$. Since $x_\delta \in W$ eventually, it follows that $z \in V$. Now, for $f \in H$ and $a \in V$, we have that $|f^{\theta_o}(a) - f^{\theta_o}(z)| \leq \epsilon$. In fact, there exists a net (y_γ) in W converging to a . Thus

$$|f^{\theta_o}(a) - f^{\theta_o}(z)| = |f(x) - f^{\theta_o}(a)| = \lim_{\gamma} |f(x) - f(y_\gamma)| \leq \epsilon.$$

This proves that H^{θ_o} is equicontinuous on $\theta_o X$. The last assertion follows from the preceding Theorem.

Theorem 3.11. $\mathcal{U}_c = \mathcal{U}_c^X$ is the uniformity \mathcal{U} generated by the family of all continuous ultrapseudometrics on X .

Proof: Let (A_i) be a clopen partition of X and let $W = \bigcup A_i \times A_i$. Define

$$d(x, y) = \sup_i |f_i(x) - f_i(y)|,$$

where $f_i = \chi_{A_i}$. Then d is a continuous ultrapseudometric on X . Since

$$W = \{(x, y) : d(x, y) < 1/2\},$$

it follows that \mathcal{U}_c is coarser than \mathcal{U} . Conversely, let d be a continuous ultrapseudometric on X , $\epsilon > 0$ and $D = \{(x, y) : d(x, y) \leq \epsilon\}$. If α is the

clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $d(x, y) \leq \epsilon$, then $D = W_\alpha$ and the result follows.

Theorem 3.12. *Let $(Y_i, f_i)_{i \in I}$ be the family of all pairs (Y, f) , where Y is an ultrametric space and $f : X \rightarrow Y$ a continuous map. Then*

$$\theta_o X = \bigcap_{i \in I} (f_i^{\beta_o})^{-1}(Y_i).$$

Proof: It follows from Theorem 3.6.

Theorem 3.13. *A Hausdorff zero-dimensional space X is θ_o -complete iff it is homeomorphic to a closed subspace of a product of ultrametric spaces.*

Proof: Every ultrametric space is θ_o -complete. Thus the sufficiency follows from Theorem 3.5. Conversely, assume that X is θ_o -complete and let $(Y_i, f_i)_{i \in I}$ be as in the preceding Theorem. Then $X = \bigcap_i Z_i$, $Z_i = (f_i^{\beta_o})^{-1}(Y_i)$. Let $Y = \prod Y_i$ with its product topology. The map $u : X \rightarrow Y$, $u(x)_i = f_i(x)$, is one-to-one. Indeed, let $x \neq y$ and choose a clopen neighborhood V of x not containing y . Let $f = \chi_V$ and

$$d : X \times Y \rightarrow \mathbf{R}, \quad d(a, b) = |f(a) - f(b)|.$$

The quotient map $\pi : X \rightarrow X_d$ is continuous and $\pi(x) \neq \pi(y)$, which implies that $u(x) \neq u(y)$. Clearly u is continuous. Also $u^{-1} : u(X) \rightarrow X$ is continuous. Indeed, let V be a clopen subset of X containing x_o and consider the pseudometric $d(x, y) = |\chi_V(x) - \chi_V(y)|$. Let $\pi : X \rightarrow X_d$ be the quotient map. There exists a $i \in I$ such that $Y_i = X_d$ and $f_i = \pi$. Then

$$f_i(V) = \pi(V) = \{\pi(x) : \tilde{d}(\pi(x) - \pi(x_o)) < 1\}.$$

The set $\pi(V)$ is open in $Y_i = X_d$. Let $\pi_i : Y \rightarrow Y_i$ be the i th-projection map and $G = \pi_i^{-1}(\pi(V))$. If $x \in X$ is such that $u(x) \in G$, then $f_i(x) = u(x)_i \in \pi(V)$ and so $d(x, x_o) < 1$, which implies that $x \in V$ since $x_o \in V$. This proves that $u : X \rightarrow u(X)$ is a homeomorphism. Finally, $u(X)$ is a closed subspace of Y . In fact, let (x_δ) be a net in X with $u(x_\delta) \rightarrow y \in Y$. Then $f_i(x_\delta) \rightarrow y_i$ for all i . Going to a subnet if necessary, we may assume that $x_\delta \rightarrow z \in \beta_o X$. Now $f_i(x_\delta) \rightarrow f_i^{\beta_o}(z)$ in $\beta_o Y_i$. But then $f_i^{\beta_o}(z) = y_i \in Y_i$, for all i , and hence $z \in \theta_o X = X$, by the preceding Theorem. Thus $y_i = f_i(z)$, for all i , and hence $y = u(z)$. This proves that X is homeomorphic to a closed subspace of Y and the result follows.

Corollary 3.14. *Every Hausdorff ultraparacompact space is homeomorphic to a closed subspace of a product of ultrametric spaces.*

Theorem 3.15. *For a subset A of X , the following are equivalent :*

- (1) A is bounding.
- (2) A is \mathcal{U}_c -totally bounded.
- (3) For each continuous ultrapseudometric d on X , A is d -totally bounded.

Proof: In view of Theorem 3.11, (2) is equivalent to (3). Also, by [2, Theorem 4.6], (1) implies (2).

(2) \Rightarrow (1). Let $f \in C(X)$,

$$A_1 = \{x : |f(x)| \leq 1\}, \quad A_{n+1} = \{x : n < |f(x)| \leq n + 1\}$$

for $n \geq 1$. Then (A_n) is a clopen partition of X . Let $W = \bigcup_n A_n \times A_n$. By our hypothesis, there are x_1, \dots, x_N in A such that $A \subset \bigcup_1^N W[x_k]$. For each $1 \leq k \leq N$, there exists n_k such that $x_k \in A_{n_k}$. Then $A \subset \bigcup_1^N A_{n_k}$ and so

$$\|f\|_A \leq \max_{1 \leq k \leq N} n_k,$$

which proves that A is bounding.

4. Polarly Barrelled Spaces of Continuous Functions

Definition 4.1. A Hausdorff locally convex space E is called :

- (1) polarly barrelled if every bounded subset of $E'_\sigma = (E', \sigma(E', E))$ is equicontinuous.
- (2) polarly quasi-barrelled if every strongly bounded subset of E' is equicontinuous.

We will denote by $C_c(X, E)$ the space $C(X, E)$ equipped with the topology of uniform convergence on compact subsets of X . By $M_c(X, E')$ we will denote the space of all $m \in M(X, E')$ with compact support. The dual space of $C_c(X, E)$ coincides with $M_c(X, E')$.

Recall that a zero-dimensional Hausdorff topological space X is called a μ_σ -space (see [2]) if every bounding subset of X is relatively compact. We

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denote by $\mu_o X$ the smallest of all μ_o -subspaces of $\beta_o X$ which contain X . Then $X \subset \mu_o X \subset \theta_o X$ and, for each bounding subset A of X , the set $\overline{A}^{\beta_o X}$ is contained in $\mu_o X$ (see [2]). Moreover, if Y is another Hausdorff zero-dimensional space and $f : X \rightarrow Y$, then $f^{\beta_o}(\mu_o X) \subset \mu_o Y$ and so there exists a continuous extension $f^{\mu_o} : \mu_o X \rightarrow \mu_o Y$ of f .

Theorem 4.2. *Assume that $E' \neq \{0\}$ and let $G = C_c(X, E)$. Then G is polarly barrelled iff X is a μ_o -space and E polarly barrelled.*

Proof: Assume that G is polarly barrelled.

I. E is polarly barrelled. Indeed, let Φ be a w^* -bounded subset of E' and let $x \in X$. For $u \in E'$, let

$$u_x : G \rightarrow \mathbb{K}, \quad u_x(f) = u(f(x)).$$

Let $H = \{u_x : u \in \Phi\}$. For $f \in C(X, E)$, we have

$$\sup_{u \in \Phi} |u_x(f)| = \sup_{u \in \Phi} |u(f(x))| < \infty$$

and so H is a w^* -bounded subset of G' . By our hypothesis, there exists $p \in cs(E)$ and Y a compact subset of X such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset H^o.$$

But then $\{s \in E : p(s) \leq 1\} \subset \Phi^o$ and so Φ is equicontinuous.

II. X is a μ_o -space. In fact, let A be a bounding subset of X and let $x' \in E'$, $x' \neq 0$. Define p on E by $p(x) = |x'(s)|$. Then $p \in cs(E)$. The set

$$D = \{f \in G : \|f\|_{A,p} \leq 1\}$$

is a polar barrel in G and so D is a neighborhood of zero in G . Let Y a compact subset of X and $q \in cs(E)$ be such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset D.$$

But then $A \subset Y$ and so \overline{A} is compact.

Conversely, suppose that E is polarly barrelled and X a μ_o -space. Let H be a w^* -bounded subset of the dual space $M_c(X, E')$ of G . Let $s \in E$ and

$$D = \{ms : m \in H\} \subset M(X).$$

For $h \in C_{rc}(X)$, we have that

$$\sup_{m \in H} | \langle ms, h \rangle | = \sup_{m \in H} | \langle m, hs \rangle | < \infty.$$

Thus, considering $M(X)$ as the dual of the Banach space $F = (C_{rc}(X), \tau_u)$, D is w^* -bounded of F' and so $\sup_{m \in H} \|ms\| = d_s < \infty$. Hence, $|m(V)s| \leq d_s$ for all $V \in K(X)$. It follows that the set

$$M = \bigcup_{m \in H} m(K(X))$$

is a w^* -bounded subset of E' . Since E is polarly barrelled, there exists $p \in cs(E)$ such that $|u(s)| \leq 1$ for all $u \in M$ and all $s \in E$ with $p(s) \leq 1$. Hence $\sup_{m \in H} \|m\|_p < \infty$. We may choose p so that $\|m\|_p \leq 1$ for all $m \in H$. Let

$$Z = S(H) = \overline{\bigcup_{m \in H} \text{supp}(m)}.$$

Then Z is bounding. In fact, assume that Z is not bounding. Then, by [6, Proposition 6.6], there exists a sequence (m_n) in H and $f \in C(X, E)$ such that $\langle m_n, f \rangle = \lambda^n$, for all n , where $|\lambda| > 1$, which contradicts the fact that H is w^* -bounded. By our hypothesis now, Z is compact. Since

$$\{f \in G : \|f\|_{Z,p} \leq 1\} \subset H^o,$$

the result follows.

Corollary 4.3. *$C_c(X)$ is polarly barrelled iff X is a μ_o -space.*

Let now G, E be Hausdorff locally convex spaces. We denote by $L_s(G, E)$ the space $L(G, E)$ of all continuous linear maps, from G to E , equipped with the topology of simple convergence.

Theorem 4.4. *Assume that E is polar and let G be polarly barrelled. If E is a μ_o -space (e.g. when E is metrizable or complete), then $L_s(G, E)$ is a μ_o -space.*

Proof: Let Φ be a bounding subset of $L_s(G, E)$. For $x \in G$, the set

$$\Phi(x) = \{\phi(x) : \phi \in \Phi\}$$

is a bounding subset of E and hence its closure M_x in E is compact. Φ is a topological subspace of E^G and it is contained in the compact set $M = \prod_{x \in G} M_x$. Since the closure of Φ in E^G is compact, it suffices to show that this closure is contained in $L(G, E)$. To this end, we prove first that, given a polar neighborhood W of zero in E , there exists a neighborhood

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U of zero in G such that $\phi(U) \subset W$ for all $\phi \in \Phi$. In fact, for $\phi \in \Phi$, let ϕ' be the adjoint map. Let

$$Z = \bigcup_{\phi \in \Phi} \phi'(H),$$

where H is the polar of W in E' . If $x \in G$, then $\Phi(x)$ is a bounded subset of E and hence $\Phi(x) \subset \alpha W$, for some $\alpha \in \mathbb{K}$. If now $\phi \in \Phi$ and $u \in H$, then

$$| \langle \phi'(u), x \rangle | = | \langle u, \phi(x) \rangle | \leq |\alpha|,$$

which proves that Z is a w^* -bounded subset of G' . As G is polarly barrelled, the polar $U = Z^\circ$, of Z in G , is a neighborhood of zero and $\phi(U) \subset H^\circ = W$, for all $\phi \in \Phi$, which proves our claim. Let now $\phi \in E^G$ be in the closure of Φ . Then ϕ is linear. There exists a net (ϕ_δ) in Φ converging to ϕ in E^G . If $x \in U$, then $\phi(x) = \lim \phi_\delta(x) \in W$, which proves that ϕ is continuous. Hence the result follows.

Corollary 4.5. *If E is polarly barrelled, then the weak dual E'_σ of E is a μ_o -space.*

Theorem 4.6. *Suppose that E is polar and G polarly barrelled. For $f \in C(X, E)$, let $f^{\mu_o} : \mu_o X \rightarrow \hat{E}$ be its continuous extension. If $T : G \rightarrow C_c(X, E)$ is a continuous linear map, then the map*

$$\tilde{T} : G \rightarrow C_c(\mu_o X, \hat{E}), \quad s \mapsto (Ts)^{\mu_o},$$

is continuous

Proof: Note that \hat{E} is θ_o -complete and hence a μ_o -space. Let

$$\phi : X \rightarrow L_s(G, E), \quad \langle \phi(x), s \rangle = (Ts)(x).$$

Then ϕ is continuous. Since $L_s(G, \hat{E})$ is a μ_o -space, there exists a continuous extension

$$\phi^{\mu_o} : \mu_o X \rightarrow L_s(G, \hat{E}).$$

Let now A be a compact subset of $\mu_o X$ and p a polar continuous seminorm on E . We denote also by p the continuous extension of p to all of \hat{E} . Let

$$V = \{g \in C(\mu_o X, \hat{E}) : \|g\|_{A,p} \leq 1\}.$$

The set $\Phi = \phi^{\mu_o}(A)$ is compact in $L_s(G, \hat{E})$. As in the proof of Theorem 4.4, there exists a neighborhood U of zero in G such that

$$\psi(U) \subset W = \{s \in \hat{E} : p(s) \leq 1\},$$

for all $\psi \in \Phi$. Now, for $y \in A$ and $s \in U$, we have

$$p((\tilde{T}s)(y)) = p(\langle \phi^{\mu_o}(y), s \rangle) \leq 1$$

and so $\tilde{T}s \in V$. This proves that \tilde{T} is continuous and the result follows.

Theorem 4.7. *Assume that E is polar and polarly barrelled and let τ_o be the locally convex topology on $C(X, E)$ generated by the seminorms $f \mapsto \|f^{\mu_o}\|_{A,p}$, where A ranges over the family of all compact subsets of $\mu_o X$ and $p \in cs(E)$. Then :*

- (1) $(C(X, E), \tau_o)$ is polarly barrelled and τ_o is finer than τ_b (and hence finer than τ_c).
- (2) If τ is any polarly barrelled topology on $C(X, E)$ which is finer than τ_c , then τ is finer than τ_o . Hence τ_o is the polarly barrelled topology associated with each of the topologies τ_b and τ_c .

Proof: (1). Since E is polarly barrelled, the same is true for \hat{E} . The space

$F = C_c(\mu_o X, \hat{E})$ is polarly barrelled and the map

$$S : (C(X, E), \tau_o) \rightarrow F, \quad f \mapsto f^{\mu_o},$$

is a linear homeomorphism. Thus τ_o is polarly barrelled. Also, since for each bounding subset B of X , its closure $\overline{B}^{\mu_o X}$ is compact, it follows that τ_o is finer than τ_b .

(2). Let τ be a polarly barrelled topology on $C(X, E)$, which is finer than τ_c , and let $G = (C(X, E), \tau)$. The identity map

$$T : G \rightarrow C_c(X, E)$$

is continuous and hence the map

$$\tilde{T} : G \rightarrow C_c(\mu_o X, \hat{E}), \quad f \mapsto f^{\mu_o},$$

is continuous. This proves that τ_o is coarser than τ and the Theorem follows.

Theorem 4.8. *Suppose that E is polar. Then $G = (C(X, E), \tau_b)$ is polarly barrelled iff E is polarly barrelled and, for each compact subset A of $\mu_o X$, there exists a bounding subset B of X such that $A \subset \overline{B}^{\mu_o X}$.*

Proof: Assume that G is polarly barrelled. It is easy to see that E is polarly barrelled. In view of the preceding Theorem, $\tau_b = \tau_o$. Thus, for each compact subset A of $\mu_o X$ and each non-zero $p \in cs(E)$, there exist a bounding subset B of X and $q \in cs(E)$ such that

$$\{f \in C(X, E) : \|f\|_{B,q} \leq 1\} \subset \{f : \|f^{\mu_o}\|_{A,p} \leq 1\}.$$

It follows easily that $A \subset \overline{B}^{\mu_o X}$. Conversely, suppose that the condition is satisfied. The condition clearly implies that τ_o is coarser than τ_b and hence $\tau_b = \tau_o$, which implies that G is polarly barrelled by the preceding Theorem.

Let us say that a family \mathcal{F} of subsets of a set Z is finite on a subset F of Z if the family of all members of \mathcal{F} which meet F is finite.

Definition 4.9. A subset D , of a topological space Z , is said to be w -bounded if every family \mathcal{F} of open subsets of Z , which is finite on each compact subset of Z , is also finite on D . If this happens for families of clopen sets, then D is said to be w_o -bounded. We say that Z is a w -space (resp. a w_o -space) if every w -bounded (resp. w_o -bounded) subset is relatively compact.

Lemma 4.10. *A subset D , of a zero-dimensional topological space Z , is w -bounded iff it is w_o -bounded.*

Proof: Assume that D is not w -bounded. Then, there exists an infinite sequence (x_n) of distinct elements of D and a sequence (V_n) of open sets such that $x_n \in V_n$ and (V_n) is finite on each compact subset of X . By [5, Lemma 2.5], there exists a subsequence (x_{n_k}) and pairwise disjoint clopen sets W_k with $x_{n_k} \in W_k$. We may choose $W_k \subset V_{n_k}$. Now (W_k) is clearly finite on each compact subset of X , which implies that D is not w_o -bounded. Hence the Lemma follows.

We easily get the following

Lemma 4.11. *Every w_o -bounded subset of X is bounding.*

Theorem 4.12. *Assume that $E' \neq \{0\}$. Then $G = C_c(X, E)$ is polarly quasi-barrelled iff E is polarly quasi-barrelled and X a w_o -space.*

Proof: Suppose that E is polarly quasi barrelled and X a w_o -space. Let H be a strongly bounded subset of the dual space $M_c(X, E)$ of G . We

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show first that there exists $p \in cs(E)$ such that $\sup_{m \in H} \|m\|_p < \infty$. In fact, let B be a bounded subset of E and consider the set

$$D = \{ms : m \in H, s \in B\}.$$

If $h \in C_{rc}(X)$, then the set $\{hs : s \in B\}$ is a bounded subset of G and so

$$\sup_{m \in H} \left| \int hs \, dm \right| = \sup_{m \in H} \left| \int h \, d(ms) \right| < \infty.$$

Considering D a subset of the dual of the Banach space $F = (C_{rc}(X), \tau_u)$, we see that D is a w^* -bounded subset of F' and hence equicontinuous. Thus

$$d = \sup_{m \in H, s \in B} \|ms\| < \infty.$$

Let

$$\Phi = \bigcup_{m \in H} m(K(X)).$$

Then for $A \in K(X)$, $s \in B$, $m \in H$, we have $|m(A)s| \leq \|ms\| \leq d$. Hence Φ is a strongly bounded subset of E' . By our hypothesis, Φ is an equicontinuous subset of E' . Thus, there exists $p \in cs(E)$ such that $|m(A)s| \leq 1$ for all $m \in H$ and all $s \in E$ with $p(s) \leq 1$. It follows from this that $\sup_{m \in H} \|m\|_p = r < \infty$. We may choose p so that $r \leq 1$. Let now

$$Y = S(H) = \overline{\bigcup_{m \in H} \text{supp}(m)}.$$

Then Y is w_o -bounded. Assume the contrary. Then, there exists a sequence (V_n) of distinct clopen subsets of X , such that $V_n \cap Y \neq \emptyset$ for all n and (V_n) is finite on each compact subset of X . For each n there exists $m_n \in H$ with $V_n \cap \text{supp}(m_n) \neq \emptyset$. Then $(m_n)_p(V_n) > 0$. There are a clopen subset W_n of V_n and $s_n \in E$, with $p(s_n) \leq 1$, such that $m(W_n)s_n = \gamma_n \neq 0$. Let $|\lambda| > 1$ and take

$$M = \{\gamma_n^{-1} \lambda^n \chi_{W_n} s_n : n \in \mathbf{N}\}.$$

Since (W_n) is finite on each compact subset of X , it follows that M is a bounded subset of G and so M is absorbed by H^o . Let $\lambda_o \neq 0$ be such that $M \subset \lambda_o H^o$. But then

$$1 \geq |\lambda_o^{-1} \gamma_n^{-1} \lambda^n m_n(W_n) s_n| = |\lambda_o^{-1} \lambda^n|$$

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for all n , which is a contradiction. So Y is w_o -bounded and hence compact by our hypothesis. Moreover

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset H^o.$$

Indeed, let $\|f\|_{Y,p} \leq 1$. The set $V = \{x : p(f(x)) > 1\}$ is disjoint from Y and hence $m_p(V) = 0$ for all $m \in H$. Thus, for $m \in H$, we have

$$\left| \int_V f \, dm \right| \leq \|f\|_p \cdot m_p(V) = 0$$

and so

$$\left| \int f \, dm \right| = \left| \int_{V^c} f \, dm \right| \leq m_p(V^c) \leq 1.$$

Conversely, suppose that G is polarly quasi-barrelled. Let Φ be a strongly bounded subset of E' and let $x \in X$. For $u \in E'$, define u_x on G by $u_x(f) = u(f(x))$. Then $u_x \in G'$. The set $H = \{u_x : u \in \Phi\}$ is a strongly bounded subset of G' . Indeed, let D be a bounded subset of G . Since the set $\{f(x) : f \in D\}$ is a bounded subset of E , we have that

$$\sup_{f \in D, u \in \Phi} |u_x(f)| = \sup_{f \in D, u \in \Phi} |u(f(x))| < \infty.$$

By our hypothesis, H is an equicontinuous subset of G' . Thus, there exists a compact subset Y of X and $p \in cs(E)$ such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\}.$$

But then $\{s \in E : p(s) \leq 1\} \subset \Phi^o$ and so Φ is an equicontinuous subset of E' , which proves that E is polarly quasi-barrelled. Finally, let A be a w_o -bounded subset of X and choose a non-zero element x' of E' . Let $p(s) = |x'(s)|$ and consider the set

$$Z = \{f \in G : \|f\|_{A,p} \leq 1\}.$$

Then Z is a polar set. We will show that Z is bornivorous. So, suppose that there exists a bounded subset M of G which is not absorbed by Z . Then, there exists a sequence (f_n) in M , $\|f_n\|_{A,p} > n$. Let

$$V_n = \{x : p(f_n(x)) > n\}.$$

Then V_n intersects A . Since A is w_o -bounded, there exists a compact subset Y of X such that (V_n) is not finite on Y , which is a contradiction since $\sup_{f \in M} \|f\|_{Y,p} < \infty$. This contradiction shows that Z absorbs bounded

subsets of G . In view of our hypothesis, there exist a compact subset Y of X and $q \in cs((E))$ such that

$$\{f \in G : \|f\|_{Y,q} \leq 1\},$$

which implies that $A \subset Y$ and so A is relatively compact. This clearly completes the proof.

Corollary 4.13. (1) $C_c(X)$ is polarly quasi-barrelled iff X is a w_o -space.

(2) If $E' \neq \{0\}$, then $C_c(X, E)$ is polarly quasi-barrelled iff both E and $C_c(X)$ are polarly quasi-barrelled.

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