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A survey on symplectic singularities and symplectic resolutions

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Baohua Fu

Abstract

This is a survey written in an expositional style on the topic of symplectic singularities and symplectic resolutions.

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Introduction

This is a survey written in an expositional style on the topic of symplectic singularities and symplectic resolutions, which could also serve as an introduction to this subject.

We work over the complex number field. A normal variety W is called a *symplectic variety* if its smooth part admits a holomorphic symplectic form ω whose pull-back to any resolution $\pi : Z \to W$ extends to a holomorphic 2-form Ω on Z. If furthermore the extended 2-form Ω is a symplectic form, then π is called a *symplectic resolution*.

The existence and non-existence of symplectic resolutions are difficult to decide. However one hopes that a symplectic variety admits at most finitely many non-isomorphic symplectic resolutions (section 7.1).

Symplectic resolutions behave much like hyperKähler manifolds. Motivated by the work of D. Huybrechts ([29]), one expects that two symplectic resolutions are deformation equivalent (section 7.2). This would imply the invariance of the cohomology of the resolution, which is expected to be recovered by the Poisson cohomology of the symplectic variety (section 7.3). As a special case of Bondal-Orlov's conjecture, one expects that two symplectic resolutions are derived equivalent (section 7.4). Finally motivated by the results in dimension 4, one expects some simple birational geometry in codimension 2 for symplectic resolutions (section 7.5).

Examples of symplectic varieties include quotients $\mathbb{C}^{2n}/G$ with G a finite subgroup of $Sp(2n)$ and normalizations of nilpotent orbit closures in semi-simple Lie algebras.

For symplectic resolutions of nilpotent orbit closures, our understanding is more or less complete. However, our knowledge of symplectic resolutions of quotient singularities is rather limited (only cohomology and derived equivalence have been fully understood).

A funny observation is that all known examples of symplectic resolutions are modeled locally on Hilbert schemes or on Springer's resolutions. For O'Grady's symplectic resolution of moduli space of sheaves ([**?**], [**?**]), Kaledin and Lehn ([35]) proved that it is modeled locally on a Springer's resolution of a nilpotent orbit closure in $\mathfrak{sp}(4)$. It would be very interesting to find out more local models.

Acknowledgement: This paper is by no means intended to be a complete account of the topic. Therefore I apologize for the omission of any relevant work and references. I would like to take this opportunity to thank J. Alev and T. Lambre for their invitation to the journey "Algèbre et Géométrie de Poisson" (Clermond-Ferrand 2004), where they encouraged me to write this survey. I would like to thank D. Kaledin and J. Sawon for corrections and suggestions to a preliminary version of this paper.

1. Basic definitions and properties

1.1. Symplectic singularities

Since A. Beauville's pioneering paper [4], symplectic singularities have received a particular attention by many mathematicians. As explained in [4], the motivation of this notion comes from the analogy between rational Gorenstein singularities and Calabi-Yau manifolds.

By a *resolution* we mean a proper surjective morphism $\pi : Z \to W$ such that: (i). Z is smooth; (ii). $\pi^{-1}(W_{reg}) \to W_{reg}$ is an isomorphism. If furthermore π is a projective morphism, then π is called a *projective resolution*.

Recall that a compact Kähler manifold of dimension m is a Calabi-Yau manifold if it admits a nowhere vanishing holomorphic m-form. Its singular counterpart is varieties with rational Gorenstein singularities, i.e. normal varieties W of dimension m whose smooth part admits a holomorphic nowhere vanishing m-form such that its pull-back to any resolution $Z \to W$ extends to a holomorphic form on the whole of Z.

A holomorphic 2-form on a smooth variety is called *symplectic* if it is closed and non-degenerate at every point. Among Calabi-Yau manifolds, there are symplectic manifolds, i.e. smooth varieties which admit a holomorphic symplectic form. By analogy, the singular counterpart of symplectic manifolds is varieties with symplectic singularities (also called *symplectic varieties*), i.e. normal varieties W whose smooth part admits a holomorphic symplectic form ω such that its pull-back to any resolution $Z \to W$ extends to a holomorphic 2-form Ω on Z.

One should bear in mind that the 2-form Ω is closed but may be degenerated at some points. The following proposition follows from a theorem of Flenner on extendability of differential forms.

Proposition 1.1. *A normal variety with singular part having codimension ≥ 4 is symplectic if and only if its smooth part admits a holomorphic symplectic form.*

More generally, one has the following characterization of symplectic varieties:

Theorem 1.2 (Namikawa [42]). *A normal variety is symplectic if and only if it has only rational Gorenstein singularities and its smooth part admits a holomorphic symplectic form.*

1.2. Stratification theorem

In differential geometry, it is well-known that every Poisson structure on a real smooth manifolds gives rise to a foliation by symplectic leaves. The following stratification theorem extends this to the case of symplectic varieties.

Theorem 1.3 (Kaledin [30]). *Let W be a symplectic variety. Then there exists a canonical stratification $W = W_0 \supset W_1 \supset W_2 \cdots$ such that:*
(i). W_{i+1} is the singular part of W_i;
(ii). the normalization of every irreducible component of W_i is a symplectic variety.

One shows easily that every W_i is a Poisson subvariety in W. The difficult part is to show that the normalization of any Poisson subscheme in W is still a symplectic variety (Theorem 2.5 [30]). An immediate corollary is

Corollary 1.4. *Every irreducible component of the singular part of a symplectic variety has even codimension.*

It has been previously proved by Y. Namikawa ([43]) that the singular part of a symplectic variety has no codimension 3 irreducible components.

Corollary 1.5. *Let W be a symplectic variety which is locally a complete intersection, then the singular locus of W is either empty or of pure codimension 2.*

In fact it is proved in [4] (Proposition 1.4) that $Sing(W)$ is of codimension ≤ 3. Now Corollary 1.4 excludes the case of codimension 3. This corollary gives rise to the following:

Problem 1. Classify symplectic varieties which are of complete intersection, and those which admit a symplectic resolution.

Such examples include nilpotent cones in semi-simple Lie algebras and rational double points (ADE singularities).

1.3. **Symplectic resolutions**

By Hironaka's big theorem, any complex variety admits a resolution, but there may exist many different resolutions. One would like to find out some "good" resolutions. In dimension 1, the resolution is unique, which is given by the normalization. In dimension 2, one also finds a "good" resolution, the so-called minimal resolution, i.e. any other resolution factorizes through this resolution. When the dimension is higher, one finds the following class of preferred resolutions (crepant resolutions). However this notion is defined only for varieties with a $\mathbb{Q}$-Cartier canonical divisor.

Let W be a normal variety. A Weil divisor D on W is called $\mathbb{Q}$-*Cartier* if some non-zero multiple of D is a Cartier divisor. W is called $\mathbb{Q}$-*factorial* if every Weil divisor on W is $\mathbb{Q}$-*Cartier*. The quotient of a smooth variety by a finite group is $\mathbb{Q}$-factorial.

For a normal variety W, the closure of a canonical divisor of W_{reg} in W is called a *canonical divisor* of W, denoted by K_W. In general it is only a Weil divisor. Suppose that K_W is $\mathbb{Q}$-Cartier, i.e. there exists some non-zero integer n such that $\mathcal{O}_W(nK_W)$ is an invertible sheaf. Then for any resolution $\pi : Z \to W$, the pull-back $\pi^*(K_W) := \frac{1}{n}\pi^*(nK_W)$ is well-defined. The resolution π is called *crepant* if $K_Z \equiv \pi^*(K_W)$, i.e. if π preserves the canonical divisor.

One should bear in mind that a crepant resolution does not exist always (as we will show soon). However whenever such a resolution exists, there is a close relationship between the geometry of the resolution and the geometry of the singular variety.

For a symplectic variety W, a resolution $\pi : Z \to W$ is called *symplectic* if the extended 2-form Ω on Z is non-degenerate, i.e. if Ω defines a symplectic structure on Z. One might wonder if this definition depends on the special choice of the symplectic structure on the smooth part. However we have

Proposition 1.6. *Let W be a symplectic variety and $\pi : Z \to W$ a resolution. Then the following statements are equivalent:*

(i). π is crepant;

(ii). π is symplectic;

(iii). the canonical divisor K_Z is trivial;

(iv). for any symplectic form ω' on W_{reg}, the pull-back $\pi^*(\omega')$ extends to a symplectic form on Z.

Proof. The only implication to be proved is $(i) \Rightarrow (iv)$: Since W is symplectic, it has only rational Gorenstein singularities. Now by [42], any symplectic form ω' on W_{reg} extends to a holomorphic 2-form Ω' on Z. Notice that $\wedge^{top}\omega'$ gives a trivialization of K_W, $\wedge^{top}\Omega'$ also trivializes K_Z, since π is crepant, thus Ω' is symplectic. □

1.4. **Namikawa's work**

Some important results on projective symplectic varieties have been obtained by Y. Namikawa. In [42] (Theorem 7 and 8), he proved a stability theorem and a local Torelli theorem. The deformation theory of such a variety W is studied in [41] (Theorem 2.5), where he proved that if $codim\ Sing(W) \geq 4$, then the Kuranishi space $Def(W)$ is smooth.

In [43], it is shown that a symplectic variety has terminal singularities if and only if the codimension of the singular part is at least 4. Any flat deformation of a projective $\mathbb{Q}$-factorial terminal symplectic variety is locally trivial, i.e. it is not smoothable by flat deformations (see [47]). Notice that such a variety admis no symplectic resolutions (see the proof of Proposition 3.4).

It is conjectured in [47] that a projective symplectic variety is smoothable by flat deformations if and only if it admits a symplectic resolution.

Recently a local version of [47] is obtained by Y. Namikawa himself in [46]. A symplectic variety is called *convex* if there exists a projective birational morphism to an affine normal variety. For convex symplectic varieties, there exists a theory of Poisson deformations (see for example [22]). The main theorem of [46] says that for a convex symplectic variety with terminal singularities W such that W^{an} is $\mathbb{Q}$-factorial, any Poisson deformation of X is locally trivial (forgetting the Poisson structure).

2. **Examples**

2.1. **Quotient singularities**

Let W be a quasi-projective symplectic variety and G a finite subgroup of $Aut(W)$ preserving a symplectic form on W_{reg}. The symplectic form on $W^0 := W_{reg} - \cup_{g\neq 1} Fix(g)$ descends to a symplectic form on W^0/G, which extends to a symplectic form on $(W/G)_{reg}$, since the complement of W^0/G in W/G has codimension ≥ 2. Now it is shown in [4] (Proposition 2.4) that this symplectic form extends to a holomorphic 2-form in any resolution. In conclusion the quotient W/G is a symplectic variety. Here are some special cases:

Example 2.1. Let G be a finite sub-group of $SL(2,\mathbb{C})$. The quotient $\mathbb{C}^2/G$ is a symplectic variety with rational double points. It admits a unique symplectic resolution, given by the minimal resolution.

Example 2.2. Let V be a finite-dimensional symplectic vector space and $G < Sp(V)$ a finite sub-group. Then the quotient V/G is a symplectic variety. However, it is difficult to decide when V/G admits a symplectic resolution.

Example 2.3. Let Y be a smooth quasi-projective variety and $G < Aut(Y)$ a finite group. Then G acts on T^*Y preserving the natural symplectic structure, thus the quotient $(T^*Y)/G$ is a symplectic variety.

Example 2.4. Let W be a symplectic variety. Then the symmetric product $W^{(n)}$ is a symplectic variety. When W is smooth and $dim(W) \geq 4$, $W^{(n)}$ does not admit any symplectic resolution (see Proposition 3.4). However when W is a smooth symplectic surface S (for example an Abelian surface, a K3 surface or the cotangent bundle of a curve), $S^{(n)}$ admits a symplectic resolution given by $Hilb^n(S) \to S^{(n)}$. This is also the unique projective symplectic resolution of $S^{(n)}$ (see [21]).

Example 2.5. Let G be a finite subgroup of $SL(2)$ and $S \to \mathbb{C}^2/G$ the minimal resolution. Then the symmetric product $(\mathbb{C}^2/G)^{(n)}$ is naturally identified with $\mathbb{C}^{2n}/G'$, where G' is the wreath product of S_n with G. Now a symplectic resolution of $\mathbb{C}^{2n}/G'$ is given by the composition

$$Hilb^n(S) \to S^{(n)} \to Sym^n(\mathbb{C}^2/G).$$

Consider the case $G = \pm 1$. Then $S = T^*\mathbb{P}^1$. The central fiber of the symplectic resolution $\pi : Hilb^n(S) \to (\mathbb{C}^2/G)^{(n)}$ contains a component isomorphic to $\mathbb{P}^n$. By performing a Mukai flop (for details see section 7.5) along this component, one obtains another symplectic resolution which is non-isomorphic to π. More discussions on this example can be found in [20].

2.2. **Nilpotent orbit closures**

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra, i.e. the bilinear form $\kappa(u, v) := trace(ad_u \circ ad_v)$ is non-degenerate, where $ad_u : \mathfrak{g} \to \mathfrak{g}$ is the linear map given by $x \mapsto [u, x]$. Let $Aut(\mathfrak{g}) = \{\phi \in GL(\mathfrak{g}) | [\phi(u), \phi(v)] = [u, v], \forall u, v \in \mathfrak{g}\}$, which is a Lie group but may be disconnected, whose identity connected component is the adjoint group G of $\mathfrak{g}$.

An element $v \in \mathfrak{g}$ is called *semi-simple* (resp. *nilpotent*) if the linear map ad_v is semi-simple (resp. nilpotent), whose orbit under the natural action of G is denoted by $\mathcal{O}_v$, which is called a semi-simple orbit (resp. nilpotent orbit).

Semi-simple orbits in $\mathfrak{g}$ are parameterized by $\mathfrak{h}/W$, where $\mathfrak{h}$ is a Cartan sub-algebra in $\mathfrak{g}$ and W is the Weyl group. In particular there are infinitely many semi-simple orbits in $\mathfrak{g}$. Semi-simple orbits possed a rather simple geometry, for example they are closed and simply-connected.

To the contrary, nilpotent orbits have a much richer geometry. The classification of nilpotent orbits has been carried out by Kostant, Dynkin, Bala-Carter et. al. via either weighted Dynkin diagrams or partitions in the case of classical simple Lie algebras.

Example 2.6. An element in $\mathfrak{sl}_{n+1}$ is nilpotent if and only if it is conjugate to some matrix $diag(J_{d_1}, \dots, J_{d_k})$, where J_{d_i} is a $d_i \times d_i$ Jordan bloc with zeros on the diagonal, and $d_1 \geq d_2 \geq \cdots \geq d_k \geq 1$ are integers such that $\sum_{i=1}^{k} d_i = n+1$, i.e. $[d_1, \dots, d_k]$ is a partition of $n+1$. This gives a one-one correspondence between nilpotent orbits in $\mathfrak{sl}_{n+1}$ and partitions of $n+1$.

For other classical simple Lie algebras, a similar description of nilpotent orbits exists (see [13]). The following theorem is fundamental in the study of nilpotent orbits.

Theorem 2.7 (Jacobson-Morozov)**.** *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $v \in \mathfrak{g}$ a nilpotent element. Then there exist two elements $H, u \in \mathfrak{g}$ such that $[H, v] = 2v, [H, u] = -2u, [v, u] = H$.*

The triple $\{H, v, u\}$ is called a *standard triple*, which provides an isomorphism $\phi : \mathfrak{sl}_2 \to \mathbb{C}\langle H, v, u\rangle$. Now $\mathfrak{g}$ becomes an $\mathfrak{sl}_2$-module via ϕ and the adjoint action. Thus $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$, where $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [H, x] = ix\}$. Let $\mathfrak{p} = \oplus_{i\geq 0}\mathfrak{g}_i$ and P a connected subgroup of G with Lie algebra $\mathfrak{p}$. Then $\mathfrak{p}$ is a parabolic sub-algebra of $\mathfrak{g}$ and P is a parabolic subgroup of G. Let $\mathfrak{n} = \oplus_{i\geq 2}\mathfrak{g}_i$ and $\mathfrak{u} = \oplus_{i\geq 1}\mathfrak{g}_i$. The nilpotent orbit $\mathcal{O}_v$ is called *even* if $\mathfrak{g}_1 = 0$ or equivalently if $\mathfrak{g}_{2k+1} = 0$ for all $k \in \mathbb{Z}$. In this case, one has $\mathfrak{u} = \mathfrak{n} \simeq (\mathfrak{g}/\mathfrak{p})^*$.

The nilpotent orbit $\mathcal{O}_v$ is not closed in $\mathfrak{g}$. Its closure $\overline{\mathcal{O}_v}$ is a singular (in general non-normal) variety. There exists a natural proper resolution of $\overline{\mathcal{O}_v}$ as follows: $G \times^P \mathfrak{n} \xrightarrow{\pi} \overline{\mathcal{O}_v}$, where $G \times^P \mathfrak{n}$ is the quotient of $G \times \mathfrak{n}$ by P acting as $p(g, n) = (gp^{-1}, Ad_p(n))$. The group G acts on $G \times^P \mathfrak{n}$ by $g(h, n) = (gh, n)$. Then the resolution is G-equivariant and maps the orbit $G \cdot (1, v)$ isomorphically to $\mathcal{O}_v$.

For any $g \in G$, the tangent space $T_{Ad(g)v}\mathcal{O}_v$ is isomorphic to $[\mathfrak{g}, Ad(g)v]$. Now we define a 2-form ω on $\mathcal{O}_v$ as follows:

$$\omega_{Ad(g)v}([u_1, Ad(g)v], [u_2, Ad(g)v]) = \kappa([u_1, u_2], Ad(g)v).$$

The 2-form ω is in fact a closed non-degenerate 2-form, i.e. a holomorphic symplectic form on $\mathcal{O}_v$ (the so-called Kostant-Kirillov-Souriau form).

Proposition 2.8. *(*[3], [50]*) The symplectic form $\pi^*\omega$ on $G \cdot (1, v)$ can be extended to a global 2-form Ω on $G \times^P \mathfrak{n}$.*

Proof. Take an element $(g, n) \in G \times^P \mathfrak{n}$, then the tangent space of $G \times^P \mathfrak{n}$ at (g, n) is canonically isomorphic to $\mathfrak{g} \times \mathfrak{n}/\{(x, [n, x]) | x \in \mathfrak{p}\}$. We define a 2-form β on $\mathfrak{g} \times \mathfrak{n}$ as follows:

$$\beta_{(g,n)}((u, m), (u', m')) = \kappa([u, u'], n) + \kappa(m', u) - \kappa(m, u').$$

The kernel of $\beta_{(g,n)}$ is $\{(u, [n, u]) \mid u \in \oplus_{i \geq -1}\mathfrak{g}_i\}$, which contains the subspace $\{(x, [n, x]) | x \in \mathfrak{p}\}$, thus this 2-form descends to a 2-form Ω on $G \times^P \mathfrak{n}$. The 2-forms Ω and $\pi^*\omega$ coincide at the point $(1, v)$, thus they coincide on $G(1, v)$ since both are G-equivariant. $\square$

Corollary 2.9. *The 2-form Ω is symplectic if and only if $\mathfrak{g}_{-1} = 0$, i.e. if and only if $\mathcal{O}_v$ is an even nilpotent orbit.*

Proof. The kernel of Ω at $(g, 0)$ is isomorphic to $\mathfrak{g}_{-1}$, thus if Ω is symplectic, then $\mathfrak{g}_{-1} = 0$. Conversely if $\mathfrak{g}_{-1} = 0$, then $G \times^P \mathfrak{n} = G \times^P \mathfrak{u} \simeq T^*(G/P)$, which implies that the canonical bundle K of $G \times^P \mathfrak{n}$ is trivial. Notice that Ω^{top} gives a trivialization of K, thus Ω^{top} is non-zero everywhere, i.e. Ω is symplectic. $\square$

Notice that the resolution $G \times^P \mathfrak{n} \xrightarrow{\pi} \overline{\mathcal{O}}$ factorizes through the normalization $\widetilde{\mathcal{O}} \to \overline{\mathcal{O}}$, which gives a resolution of $\widetilde{\mathcal{O}}$.

Corollary 2.10. *The normalization $\widetilde{\mathcal{O}}$ of a nilpotent orbit closure in a complex semi-simple Lie algebra is a symplectic variety. The resolution $G \times^P \mathfrak{n} \to \widetilde{\mathcal{O}}$ is symplectic if and only if $\mathcal{O}$ is an even nilpotent orbit.*

One should remember that even for an even nilpotent orbit closure, there can exist some symplectic resolutions not of the above form. When $\mathcal{O}$ is not an even orbit, we can obtain a symplectic resolution by extremal contractions of the natural resolution given by $G \times^P \mathfrak{n}$, except for some particular orbits in D_n and one orbit in E_8 (see [19]).

2.3. **Isolated singularities**

Let V be a finite-dimensional symplectic vector space and $G < Sp(V)$ a finite sub-group. Suppose furthermore that the non trivial elements in G have all their eigenvalues different to 1, then the quotient G/V is a symplectic variety with an isolated singularity, which admits a symplectic resolution if and only if $dim(V) = 2$ (see Corollary 3.5).

For example, let ξ be the primitive cubic unit root, which acts on $\mathbb{C}^{2n}$ by the multiplication of ξ on the first n coordinates and by the multiplication of ξ^2 on the last n coordinates. Then the quotient $\mathbb{C}^{2n}/\langle\xi\rangle$ has an isolated symplectic singularity. A characterization of this singularity has been given in [14].

Another type of isolated symplectic singularities comes from minimal nilpotent orbits $\mathcal{O}_{min}$, i.e. the unique non-zero nilpotent orbit which is contained in the closure of all non-zero nilpotent orbits. Then $\overline{\mathcal{O}}_{min} = \mathcal{O}_{min} \cup \{0\}$ is normal with an isolated symplectic singularity. Conversely an isolated symplectic singularity with smooth projective tangent cone is analytically isomorphic to $\overline{\mathcal{O}}_{min}$ (see [4]). It is suggested in [4] to classify isolated symplectic singularities with trivial local fundamental group.

Among minimal nilpotent orbit closures, only those in $sl(n+1)$ admit a symplectic resolution (see Proposition 5.2). In this case, $\mathcal{O}_{min}$ consists of matrices of trace zero and rank 1. A symplectic resolution is given by $T^*\mathbb{P}^n \to \overline{\mathcal{O}}_{min}$. It is believed in that a projective isolated symplectic singularity (of dimension ≥ 4) admitting a symplectic resolution is isomorphic to a such singularity (see Corollary 3.5 for the case of quotient isolated singularities). Some discussions are given in [9].

Here is a deformation of this symplectic resolution. In the following, a point in $\mathbb{P}^n$ will also be thought of a line in $\mathbb{C}^{n+1}$. Let

$$\mathcal{Z} = \{(l, A, a) \in \mathbb{P}^n \times \mathfrak{gl}_{n+1} \times \mathbb{C} | Im(A) \subset l; Av = av, \ \forall v \in l\}$$

and $\mathcal{W}' = \{(A, a) \in \mathfrak{gl}_{n+1} \times \mathbb{C} | A^2 = aA; rk(A) = 1\}$. We denote by $\mathcal{W}$ the closure of $\mathcal{W}'$ in $\mathfrak{gl}_{n+1} \times \mathbb{C}$. Then the natural map $\mathcal{Z} \xrightarrow{f} \mathcal{W}$ is a deformation of the symplectic resolution $T^*\mathbb{P}^n \to \overline{\mathcal{O}}_{min}$. Notice that for $a \neq 0$ the map between fibers $\mathcal{Z}_a \xrightarrow{f_a} \mathcal{W}_a$ is an isomorphism.

3. Semi-smallness

Recall that a morphism $\pi : Z \to W$ is called *semi-small* if for every closed subvariety F in Z, we have $2 \cdot codim\ F \geq codim\ (\pi(F))$. This is a remarkable property, which enables us, for example, to use the intersection cohomology theory.

Example 3.1. Let S be a normal surface, then any resolution of S is semi-small. However this is not the case in higher dimension. In fact, the blowup of a point in the exceptional locus of a resolution gives a resolution which is never semi-small.

As discovered partially by J. Wierzba ([52]), Y. Namikawa ([42]), and Hu-Yau ([28]), then in full generality by D. Kaledin ([31], [30]), symplectic resolutions enjoy the semi-small property.

Theorem 3.2. *Let W be a normal algebraic variety with only rational singularities and $\pi : Z \to W$ a proper resolution. Suppose that Z admits a symplectic structure Ω. Then the resolution π is semi-small. In particular, a symplectic resolution is semi-small.*

Remark 3.3. When π is projective, this theorem has been proved by D. Kaledin ([30], Lemma 2.7). With minor changes, his proof works also for the proper case.

Proof. Let $Y \subset W$ be an irreducible closed subvariety and F an irreducible component of $\pi^{-1}(Y)$. One needs to show that $2{\cdot}codim\ (F) \geq codim\ \pi(F)$.

By Chow's lemma (see for example [25], Chap. II, exercise 4.10), there exists an algebraic variety F' and a birational proper morphism $f : F' \to F$ such that the composition $\pi \circ f : F' \to Y$ is a projective morphism.

Now take a projective resolution $X \to F'$ and denote by σ the composition morphism from X to Y. Let $\eta : X \to Z$ be the composition. By shrinking W and Y if necessary, we can assume that

(i). W is affine;

(ii). Y is smooth and $Y = \pi(F) = \sigma(X)$;

(iii). σ is smooth (this is possible since X is smooth, see [25] Corollary 10.7,Chap. II).

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\sigma"'] & F' \arrow[r, "f"] \arrow[d, "\pi\circ f"] & F \arrow[r] \arrow[d, "\pi"'] & Z \arrow[d, "\pi"'] \\ & Y \arrow[r, "id"] & Y \arrow[r] & W \end{tikzcd}$$

For any $y \in Y$, we denote by X_y (resp. F'_y, F_y) the fiber over y of the morphism σ (resp. $\pi \circ f, \pi$).

The arguments in [30] show that for any $x \in X_y$, we have $T_x X_y \subset ker(\eta^*\Omega)_x$. Let $U \subset F$ be the open set such that $\eta^{-1}(U) \to U$ is an isomorphism. Then for any point $z \in U$, $T_z F_y$ and $T_z F$ are orthogonal with respect to Ω. By our assumption, Ω is non-degenerate everywhere on Z, thus

$$dim(T_z F_y) + dim(T_z F) \leq dim(Z),$$

which gives the inequality in the theorem. □

Proposition 3.4. *Let X be a smooth irreducible symplectic variety and G a finite group of symplectic automorphisms on X. Suppose that V/G admits a symplectic resolution, then the subvariety $F = \cup_{g\neq 1} Fix(g) \subset X$ is either empty or of pure codimension 2 in X.*

Proof. Being a quotient of a $\mathbb{Q}$-factorial normal variety by a finite group, V/G is again $\mathbb{Q}$-factorial and normal. This gives that any component E of the exceptional locus of a proper resolution $\pi : Z \to X/G$ is of codimension 1. On the other hand, if π is a symplectic resolution, then by the semi-smallness, we have $2 = 2 \cdot codim(E) \geq codim(\pi(E))$. Suppose that X/G is not smooth, then the singular locus of X/G is of codimension ≥ 2, hence $codim(\pi(E)) = 2$.

However, the singular locus of X/G is contained in $p(F)$, where $p : X \to X/G$ is the natural map, hence $codim(F) \geq 2$. Notice that for any $g \in G$, $Fix(g)$ is of even dimension since g is symplectic, thus F has no codimension 1 component, i.e. F is of pure codimension 2. □

Corollary 3.5. *Let X be a smooth irreducible symplectic variety and G a finite group of symplectic automorphisms on X. Suppose that X/G has only isolated singularities, then X/G admits a symplectic resolution if and only if $dim(X) = 2$.*

4. **Quotient case**

In this section, we study symplectic resolutions for quotient symplectic varieties. Let $V = \mathbb{C}^{2n}$ and $G < Sp(2n)$ a finite sub-group. For an element $g \in G$, we denote by V^g the linear subspace of points fixed by g. An element g is called a *symplectic reflection* if $codim\ V^g = 2$.

Theorem 4.1 (Verbitsky [51]). *Suppose that V/G admits a symplectic resolution $\pi : Z \to V/G$. Then G is generated by symplectic reflections.*

Proof. Let G_0 be the subgroup of G generated by symplectic reflections, then one has a natural map $\sigma : V/G_0 \to V/G$. Let $F = \cup_{g \notin G_0} p_0(V^g)$, where $p_0 : V \to V/G_0$ is the natural projection. Then $codim F \geq 4$ and $V/G_0 - F \to V/G - \sigma(F)$ is a non-ramified covering of degree $\sharp G/G_0$.

Let $Z_0 = V/G_0 \times_{V/G} Z$ be the fiber product and $\pi_0 : Z_0 \to V/G_0$ the projection to the first factor. Then $\sigma_0 : Z_0 - \pi_0^{-1}(F) \to Z - R$ is a non-ramified covering with degree $\sharp G/G_0$, where $R = \pi^{-1}(\sigma(F))$.

Now the semi-smallness of π implies $codim(R) \geq 2$. So for the fundamental groups, one has $\pi_1(Z - R) \simeq \pi_1(Z) = 1$, where the last equality follows from the fact that any resolution of V/G is simply-connected. This shows that the non-ramified covering σ_0 is of degree 1, thus $G = G_0$. $\square$

Remark 4.2. A classification of finite symplectic groups generated by symplectic reflections is obtained in [24] (Theorem 10.1 and Theorem 10.2) and also in [12].

Here are some examples of finite symplectic groups. One should bear in mind that there are finite symplectic groups which are not of this type. Let L be a complex vector space and $G < GL(L)$ a finite sub-group. Then G acts on $L \oplus L^*$ preserving the natural symplectic structure. Recall that an element $g \in G$ is a *complex reflection* if $codim(L^g) = 1$. An element $g \in G$ is a complex reflection if and only if it is a symplectic reflection when considered as an element in $Sp(L \oplus L^*)$. Now the precedent theorem implies:

Corollary 4.3 (Kaledin [33]). *Suppose that $(L\oplus L^*)/G$ admits a projective symplectic resolution $\pi : Z \to (L\oplus L^*)/G$, then G is generated by complex reflections.*

The proof of D. Kaledin is different to the one presented above. Here is an outline of his proof. One observes that there exists a natural $\mathbb{C}^*$ action on $(L\oplus L^*)/G \simeq (T^*L)/G$. For a symplectic resolution $\pi : Z \to (L\oplus L^*)/G$, one shows that the $\mathbb{C}^*$-action on $(T^*L)/G$ lifts to Z in such a way that π is $\mathbb{C}^*$-equivariant. In fact, this is a general fact for symplectic resolutions (see Lemma 5.12 [22]).

The key point is to show that if π is furthermore projective, then for every $x \in L/G \subset (T^*L)/G$, there exists at most finitely many points in $\pi^{-1}(x)$ which are fixed by the $\mathbb{C}^*$-action on Z. The proof is based on the equation $\lambda^*\Omega = \lambda\Omega$ and the semi-smallness of the map π (see Proposition 6.3 [33]).

Now since a generic point on L/G is smooth, the map $\pi : \pi^{-1}(L/G) \to L/G$ is generically one-to-one and surjective, thus there exists a connected component Y of fixed points $Z^{\mathbb{C}^*}$ such that $\pi : Y \to L/G$ is dominant and generically one-to-one, which is also finite by the key point. Now that L/G is normal implies that $\pi : Y \to L/G$ is in fact an isomorphism. Since Z is smooth, $Z^{\mathbb{C}^*}$ is a union of smooth components, so Y is smooth, thus L/G is smooth. Now a classical result then implies that G is generated by complex reflections.

This geometric approach can be developed further to obtain the following theorem, which holds also for the more general case $(T^*X)/G$ with X a smooth variety and $G < Aut(X)$ a finite sub-group.

Theorem 4.4 (Fu [18]). *Let L be a vector space and $G < GL(L)$ a finite sub-group. Suppose we have a projective symplectic resolution $\pi : Z \to (T^*L)/G$. Then:*

(i). Z contains a Zariski open set U which is isomorphic to $T^*(L/G)$;

(ii). the restricted morphism $\pi : T^*(L/G) \to (T^*L)/G$ is the natural one, which is independent of the resolution.

From what we have discussed above, there exists a connected component Y of $Z^{\mathbb{C}^*}$ which is isomorphic to L/G. Notice that the symplectic structure Ω on Z satisfies $\lambda^*\Omega = \lambda\Omega$ and Y is of half dimension. One deduces that Y is in fact a Lagrangian sub-manifold of Z. Now using a classical result of A. Bialynicki-Birula ([6]), one can prove that the attraction variety of Y with respect to the $\mathbb{C}^*$-action is isomorphic to $T^*(L/G)$. Then one shows that under this isomorphism, the morphism $T^*(L/G) \xrightarrow{\pi} (T^*L)/G$ is the following natural one: take a point $[x] \in L/G$ and a co-vector $\alpha \in T^*_{[x]}(L/G)$. We define a co-vector $\beta \in T^*_x L$ by $< \beta, v >=< \alpha, p_*(v) >$ for all $v \in T_x L$, where $p : L \to L/G$ is the natural projection. Then $\pi([x], \alpha) = [x, \beta]$.

A crucial question is how big the open set U is in $Exc(\pi)$ and how the rest part looks like. Some partial answer to this question is obtained in [18], where one needs the following version of McKay correspondence:

Theorem 4.5 (Kaledin [32])**.** *Let V be a symplectic vector space and $G < Sp(V)$ a finite sub-group. Suppose we have a symplectic resolution $\pi : Z \to V/G$. Then there exists a basis (represented by maximal cycles of π) of $H_{2k}(Z, \mathbb{Q})$ indexed by the conjugacy classes of elements $g \in G$ such that codim $V^g = 2k$.*

Example 4.6. One special case is the following: let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Let G be the Weyl group acting on $\mathfrak{h}$. Then $W := (\mathfrak{h} \oplus \mathfrak{h}^*)/G$ is a symplectic variety. In the case of simple Lie algebras, it is proved in [22] (Theorem 1.1) that W admits a symplectic resolution if and only if $\mathfrak{g}$ is of type A, B or C.

The case of type A can be constructed as follows: let $Hilb^n(\mathbb{C}^2) \xrightarrow{\pi} (\mathbb{C}^2)^{(n)}$ be the Hilbert-Chow morphism and $\Sigma : (\mathbb{C}^2)^{(n)} \to \mathbb{C}^2$ the sum map. Then $\pi_0 : (\Sigma \circ \pi)^{-1}(0) \to \Sigma^{-1}(0)$ is a symplectic resolution of $\Sigma^{-1}(0)$, and $\Sigma^{-1}(0)$ is nothing but $(\mathfrak{h} \oplus \mathfrak{h}^*)/\mathcal{S}_n$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}_n$.

However it is difficult to decide for which $G < Sp(V)$ the quotient V/G admits a symplectic resolution. The following problem is open even when $dim(V) = 4$.

Problem 2. (i). Classify finite sub-groups $G < Sp(2n, \mathbb{C})$ such that $\mathbb{C}^{2n}/G$ admits a symplectic resolution. (ii). Parameterizes all symplectic resolutions of V/G.

Another obstruction to the existence of a symplectic resolution makes use of the so-called Calogero-Moser deformation of V/G , which is a canonical deformation of V/G constructed by Etingof-Ginzburg (see also [22]).

Theorem 4.7 (Ginzburg-Kaledin [22])**.** *Suppose that V/G admits a symplectic resolution, then a generic fiber of the Calogero-Moser deformation of V/G is smooth.*

In [23], I. Gordon proved that a generic fiber of the Calogero-Moser deformation is singular for the quotient of some symplectic reflection groups(Coxeter groups of type D_{2n}, E, F, H etc.). One may expect that his method can be used to obtain more examples of V/G which do not admit any symplectic resolution.

5. Nilpotent orbit closures

5.1. Symplectic resolutions

As shown in Corollary 2.10, the normalization of a nilpotent orbit closure is a symplectic variety. We will now discuss their symplectic resolutions.

Proposition 5.1. *Every nilpotent orbit closure in* $\mathfrak{sl}(n+1)$ *admits a symplectic resolution.*

Proof. Let $\mathcal{O}$ be the nilpotent orbit corresponding to the partition $[d_1, \ldots, d_k]$. The dual partition is defined by $s_j = \sharp\{i | d_i \geq j\}$. The closure of $\mathcal{O}$ is

$$\overline{\mathcal{O}} = \{A \in \mathfrak{sl}(n+1) | \dim \ker A^j \geq \sum_{i=1}^{j} s_i\},$$

which is normal ([38]).

We define a flag variety F as follows:

$$F = \{(V_1, \ldots, V_l) | V_j \text{ vector space of dim} \sum_{i=1}^{j} s_i, V_j \subset V_{j+1} \text{ for all } j\},$$

whose cotangent bundle T^*F is isomorphic to the coincidence variety $Z := \{(A, V_\bullet) \in \mathfrak{sl}(n+1) \times F | AV_i \subseteq V_{i-1}\ \forall i\}$. The projection >from Z to the first factor gives a morphism $\pi : T^*F \to \overline{\mathcal{O}}$, which is in fact a resolution. Since T^*F has trivial canonical bundle, π is a symplectic resolution of $\overline{\mathcal{O}}$. □

One may wonder if every nilpotent orbit closure admits a symplectic resolution. Unfortunately this is not the case, as shown by the following:

Proposition 5.2. *Let* $\mathfrak{g}$ *be a simple Lie algebra. Then the closure* $\overline{\mathcal{O}}_{min}$ *admits a symplectic resolution if and only if* $\mathfrak{g}$ *is of type* A.

Proof. The Picard group of $\mathcal{O}_{min}$ is $\mathbb{Z}_2$ when $\mathfrak{g}$ is of type C, and is 0 if $\mathfrak{g}$ is not of type A or C. Thus $\overline{\mathcal{O}}_{min}$ is in fact a normal $\mathbb{Q}$-factorial variety.

Now by the argument in the proof of Proposition 3.4, one sees that $\overline{\mathcal{O}}_{min}$ admits no symplectic resolution if $\mathfrak{g}$ is not of type A. □

Now the question is how to decide whether a nilpotent orbit closure admits a symplectic resolution or not. If yes, can we find all of its symplectic resolutions? This question is answered by the following

Theorem 5.3. *(Fu [16]) Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and G its adjoint group. Let $\widetilde{\mathcal{O}}$ be the normalization of a nilpotent orbit closure. Suppose that we have a symplectic resolution $\pi : Z \to \widetilde{\mathcal{O}}$, then there exists a parabolic subgroup P of G such that Z is isomorphic to $T^*(G/P)$. Furthermore under this isomorphism, the map π becomes the moment map with respect the action of G (where $\mathfrak{g}$ is identified with its dual via the Killing form).*

Recall that a parabolic subgroup P is called a *polarization* of $\mathcal{O}$ if $\overline{\mathcal{O}}$ is the image of the map $T^*(G/P) \to \mathfrak{g}$. Every parabolic subgroup is a polarization of some nilpotent orbit, but not every nilpotent orbit admits a polarization and those admitting a polarization are called *Richardson orbits*.

Corollary 5.4. *The normalization $\widetilde{\mathcal{O}}$ of a nilpotent orbit closure in a semi-simple Lie algebra admits a symplectic resolution if and only if (i). $\mathcal{O}$ is a Richardson nilpotent orbit; (ii). there exists a polarization P such that the moment map $T^*(G/P) \to \widetilde{\mathcal{O}}$ is birational.*

The key observation is that there exists a $\mathbb{C}^*$ action on nilpotent orbits, which follows directly from the Jacobson-Morozov theorem (Theorem 2.7). This $\mathbb{C}^*$ (and G) action lifts not only to the normalization $\widetilde{\mathcal{O}}$, but also to the symplectic resolution Z. If we denote by Ω the symplectic form on Z, then one feature of the $\mathbb{C}^*$ action is $\lambda^*\Omega = \lambda\Omega$. Together with the results of [6], one shows that there exists an open set U in Z which is isomorphic to T^*Z_0, where Z_0 is a connected component of $Z^{\mathbb{C}^*}$.

Now the action of G on Z restricts to an action on Z_0, which is in fact transitive, thus Z_0 is isomorphic to G/P for some parabolic sub-group of G. Furthermore the restricted morphism of π to U is in fact the moment map, which is a proper morphism, thus U is the whole of Z.

Using results in [26] on polarizations of nilpotent orbits, one obtains (see [16]) a classification of nilpotent orbit closures of classical type whose normalization admits a symplectic resolution .

This result can be generalized to odd degree coverings of nilpotent orbits (see [15]), where an interesting phenomenon appears: there exist some nilpotent orbits which admit some symplectic resolutions, but not their coverings, and there exist some nilpotent orbits which do not admit any symplectic resolution, while some of their coverings do admit some symplectic resolutions. A similar phenomenon appears also in [35].

5.2. **Birational geometry**

As shown in the precedent section, every symplectic resolution of a nilpotent orbit closure is of the form $T^*(G/P) \to \widetilde{\mathcal{O}}$. However there can exist several polarizations which give different symplectic resolutions $T^*(G/P_i) \to \widetilde{\mathcal{O}}, (i = 1, 2)$. The birational geometry of the two resolutions is encoded in the rational map $T^*(G/P_1) - - \to T^*(G/P_2)$. This section is to present several results on these rational maps.

Consider the nilpotent orbit $\mathcal{O} = \mathcal{O}_{[2^k,1^{n-2k}]}$ in $\mathfrak{sl}_n$, where $2k < n$. The closure $\overline{\mathcal{O}}$ admits exactly two symplectic resolutions given by

$$T^*G(k, n) \xrightarrow{\pi} \overline{\mathcal{O}} \xleftarrow{\pi^+} T^*G(n-k, n),$$

where $G(k, n)$ (resp. $G(n-k, n)$) is the Grassmannian of k (resp $n-k$) dimensional subspaces in $\mathbb{C}^n$. Let ϕ be the induced birational map $T^*G(k, n) - - \to T^*G(n-k, n)$. Then π and π^+ are both small and ϕ is a flop, which is called a *stratified Mukai flop of type A*. These are the flops studied by E. Markman in [39].

Let $\mathcal{O}$ be the orbit $\mathcal{O}_{[2^{k-1},1^2]}$ in $\mathfrak{so}_{2k}$, where $k \geq 3$ is an odd integer. Let $G^+_{iso}(k), G^-_{iso}(k)$ be the two connected components of the orthogonal Grassmannian of k-dimensional isotropic subspace in $\mathbb{C}^{2k}$ (endowed with a fixed non-degenerate symmetric form). Then we have two symplectic resolutions $T^*G^+_{iso}(k) \to \overline{\mathcal{O}} \leftarrow T^*G^-_{iso}(k)$. This diagram is called a *stratified Mukai flop of type D*.

Let $Z \xrightarrow{\pi} W \xleftarrow{\pi'} Z'$ be two resolution of a variety W. Then the diagram is called a *locally trivial family of stratified Mukai flops of type A (resp. of type D)* if there exists a partial open covering $\{U_i\}$ of W which contains the singular part of W such that each diagram $\pi^{-1}(U_i) \to U_i \leftarrow (\pi')^{-1}(U_i)$ is a trivial family of a stratified Mukai flop of type A (resp. of type D).

Theorem 5.5 (Namikawa [44]). *Let $\mathcal{O}$ be a nilpotent orbit in a classical complex Lie algebra and $Z \to \widetilde{\mathcal{O}} \leftarrow Z'$ two symplectic resolutions. Then* *the birational map $Z - - \to Z'$ can be decomposed into finite number of diagrams $Z_i \to W_i \leftarrow Z_{i+1} (i = 1, \ldots, k-1)$ with $Z_1 = Z$ and $Z_k = Z'$ such that each diagram is locally a trivial family of stratified Mukai flops of type A or of type D.*

The proof in [44] consists of a case-by-case study, using the classification of polarizations of a nilpotent orbit of classical type in [26]. The drawback is that nilpotent orbits of exceptional type cannot be dealt with, since a classification of polarization is not known in this case. However, Y. Namikawa took another method in [45] by using Dynkin diagrams instead of partitions to prove that a similar result holds for nilpotent orbit closures in exceptional simple Lie algebras, where when $\mathfrak{g}$ is of type E_6, a new stratified Mukai flop appears, and these are all flop types we need. Recently we extend this result to Springer maps with the same degrees, with a completely different approach (see [19]).

6. Symplectic moduli spaces

Consider a $K3$ or abelian surface S endowed with an ample divisor H. Let M_v be the moduli space of rank $r > 0$ H-semi-stable torsion free sheaves on S with Chern class (c_1, c_2), where v is the Mukai vector associated to (r, c_1, c_2). The open sub-scheme M_v^s in M_v parameterizing H-stable sheaves is smooth, whose tangent space at a point $[E]$ is canonically isomorphic to $Ext^1_S(E,E)$. The Yoneda coupling composed with the trace map gives a bilinear form $Ext^1_S(E,E) \times Ext^1_S(E,E) \to H^2(S, \mathcal{O}_S) = \mathbb{C}$, which glues to a symplectic form on M_v^s ([40], [7]).

If v is primitive, then $M_v^s = M_v$ is a smooth projective symplectic variety. However for a multiple $v = mv_0$ of a primitive vector v_0 with $m \geq 2$, the moduli space M_v is singular. In the case of $m = 2$ and $\langle v_0, v_0 \rangle = 2$, M_v admits a unique symplectic resolution constructed by O'Grady ([**?**], [**?**]), where $\langle v_0, v_0 \rangle$ is the Mukai pairing. What happens for other singular moduli spaces?

Theorem 6.1 (Kaledin-Lehn-Sorger, [36])**.** *Suppose that H is mv_0-general, then the moduli space M_{mv_0} is a projective symplectic variety which does not admit any symplectic resolution if $m > 2$ and $\langle v_0, v_0 \rangle \geq 2$ or if $m \geq 2$ and $\langle v_0, v_0 \rangle > 2$*

In fact, they proved that under the hypothesis of the theorem, the moduli space M_{mv_0} is locally factorial. Then the argument in the proof of Proposition 3.4 (see also Corollary 1.3 [16]) shows that M_{mv_0} has no symplectic resolution, since the codimension of the singular part is of codimension ≥ 4.

For a $K3$ surface S, the case of $m = 2$ has been proven in [35], and the case of $v = (2, 0, 2n)$ with $n \geq 3$ is proved by Choy and Kiem in [10]. For abelian surfaces, the case of $v = (2, 0, 2n)$ with $n \geq 2$ is proved in [11]. The proof of Choy and Kiem is based on another obstruction to the existence of a symplectic resolution, which we present in the following.

The Hodge-Deligne polynomial of a variety X is defined as

$$E(X; u, v) = \sum_{p,q} \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(X, \mathbb{C})) u^p v^q,$$

where $h^{p,q}(H_c^k(X, \mathbb{C}))$ is the dimension of (p, q) Hodge-Deligne component in the kth cohomology group with compact supports.

Let W be a symplectic variety and $p : X \to W$ a resolution of singularities such that the exceptional locus of p is a divisor whose irreducible components $D_1, \cdots, D_k$ are smooth with only normal crossings. Then $K_X = \sum_i a_i D_i$ with $a_i \geq 0$, since W has only rational Gorenstein singularities. For any subset $J \subset I := \{1, \cdots, k\}$, one defines $D_J = \cap_{j \in J} D_j$, $D_\emptyset = X$ and $D_J^0 = D_J - \cup_{i \in I-J} D_i$.Then the stringy E-function of W is defined by:

$$E_{st}(W; u, v) = \sum_{J \subset I} E(D_J^0; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}.$$

Theorem 6.2 (Batyrev [1]). *The stringy function is independent of the choice of a resolution. For a symplectic resolution $Z \to W$, one has $E_{st}(W; u, v) = E(Z; u, v)$, in particular, the stringy function is a polynomial.*

In [10], they used Kirwan's resolution to calculate the stringy function of M_v and then proved that this function is not a polynomial, thus $M_{(2,0,2n)}$ admits no symplectic resolution for $n \geq 3$. Similar method is used in [11].

One may wonder if this method can be used to prove the non-existence of a symplectic resolution for some quotients $\mathbb{C}^{2n}/G$. Unfortunately, this does not work. In fact, it is shown in [2] that $E_{st}(\mathbb{C}^{2n}/G) = \sum_i C_i(G)(uv)^{2n-i}$, where $C_i(G)$ is the number of conjugacy classes in G whose fix point is of codimension $2i$. For the minimal nilpotent orbit closure $\overline{\mathcal{O}}_{min}$ in a simple complex Lie algebra of classical type, one calculates that only for type D, the stringy Euler function of $\overline{\mathcal{O}}_{min}$ is not a polynomial, thus $\overline{\mathcal{O}}_{min}$ admits no symplectic resolutions (compare Proposition 5.2).

7. Some conjectures

This section is to list some unsolved conjectures on symplectic resolutions.

7.1. Finiteness

Let W be a symplectic variety and $Z \xrightarrow{\pi} W \xleftarrow{\pi^+} Z^+$ two resolutions. Then π and π^+ are said *isomorphic* if the rational map $\pi^{-1} \circ \pi^+ : Z^+ - - \to Z$ is an isomorphism. π and π^+ are said *equivalent* if there exists an automorphism ψ of W such that $\psi \circ \pi$ and π^+ are isomorphic.

Conjecture 1. ([21]) A symplectic variety has at most finitely many non-isomorphic symplectic resolutions.

For nilpotent orbit closures, this conjecture is verified, since there are only finitely many conjugacy classes of parabolic subgroups in a semi-simple Lie group. It is proved in [21] that a symplectic variety of *dimension 4* has at most finitely many non-isomorphic *projective* symplectic resolutions. Some quotient varieties are shown to admit at most one *projective* symplectic resolution in [21]. Such examples include symmetric products of a smooth symplectic surface, the quotient $\mathbb{C}^{2n}/G$, where $G < Sp(2n)$ is a finite sub-group whose symplectic reflections form a single conjugacy class. The next case to be studied is $\mathbb{C}^{2n}/G$ for a general G.

The proof of this conjecture can be divided into two parts: (i). a symplectic variety can have at most finitely many non-equivalent symplectic resolutions; (ii). almost all automorphism of a symplectic variety can be lifted to any symplectic resolution.

7.2. Deformations

Recall that a deformation of a variety X is a flat morphism $\mathcal{X} \xrightarrow{p} S$ from a variety $\mathcal{X}$ to a pointed smooth connected curve $0 \in S$ such that $p^{-1}(0) \cong X$. Moreover, a deformation of a proper morphism $f : X \to Y$ is a proper S-morphism $F : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X} \to S$ is a deformation of X and $\mathcal{Y} \to S$ is a deformation of Y.

Let $X \xrightarrow{f} Y \xleftarrow{f^+} X^+$ be two proper morphisms. One says that f and f^+ are *deformation equivalent* if there exists deformations of f and f^+: $\mathcal{X} \xrightarrow{F} \mathcal{Y} \xleftarrow{F^+} \mathcal{X}^+$ such that for any general $s \in S$ the morphisms $\mathcal{X}_s \xrightarrow{F_s} \mathcal{Y}_s \xleftarrow{F_s^+} \mathcal{X}_s^+$ are isomorphisms.

As to the relation between two symplectic resolutions, we have the following:

Conjecture 2. ([21] [31]) Suppose that we have two symplectic resolutions $\pi_i : Z_i \to W, i = 1, 2$, then π_1 is deformation equivalent to π_2, and Z_1 is diffeomorphic to Z_2.

The motivation of this conjecture is the well-known theorem of D. Huybrechts ([29]), which says that two birational compact hyperkähler manifolds are deformation equivalent. This conjecture is true when W is projective ([41]). For nilpotent orbit closures in a simple Lie algebra, this conjecture is shown to be true in [44] (see [17] for a weaker version). Under a rather restrictive additional assumption, this conjecture is proved in [31]. We proved in [20] that any two projective symplectic resolutions of $\mathbb{C}^4/G$ are deformation equivalent, where $G < Sp(4)$ is a finite subgroup.

7.3. Cohomology

By Conjecture 2, the cohomology ring $H^*(Z, \mathbb{C})$ of a symplectic resolution $Z \to W$ is independent of the resolution, in particular this invariant can be regarded as an invariant of W, instead of Z. How can we recover this invariant from W?

In the case of quotient varieties V/G, there exists an orbifold cohomology $H^*_{orb}(V/G, \mathbb{C})$ which is isomorphic as an algebra to $H^*(Z, \mathbb{C})$ for a symplectic resolution $Z \to V/G$ (see [22]). However for a general variety, the orbifold cohomology is not defined.

There exists a natural cohomology on a symplectic variety W, the Poisson cohomology $HP^*(W)$ (see [22]). Our hope is

Conjecture 3. Let $Z \to W$ be a symplectic resolution, then $H^*(Z, \mathbb{C}) \simeq HP^*(W)$ as vector spaces. In particular, $HP^*(W)$ is finite-dimensional.

This is true if W is itself smooth. For a symplectic resolution $Z \to V/G$, it is proved in [22] that $H^i(Z, \mathbb{C}) \simeq HP^i(V/G)$ for $i = 0, 1, 2$.

7.4. **Derived equivalence**

As a special case of the Bondal-Orlov-Kawamata's (see [37]) conjecture, one has:

Conjecture 4. Suppose that we have two symplectic resolutions $Z_i \to W, i = 1, 2$, then there is an equivalence of derived categories of coherent sheaves $D^b(Z_1) \sim D^b(Z_2)$.

This is shown to be true for four dimensional symplectic varieties by Y. Kawamata and Y. Namikawa independently, using the work of J. Wierzba and J. Wisniewski ([53]). For a symplectic resolution of a quotient variety $Z \to V/G$, it is shown in [5] that there exists an equivalence of derived categories $D^b(Coh(Z)) \simeq D^b(Coh(V)^G)$. In particular, the conjecture is verified in this case.

The next case to be studied is nilpotent orbit closures in a classical simple Lie algebra. By Theorem 5.5, this is essentially reduced to prove the equivalence for the stratified Mukai flops of type A, D and E.

Very recently, D. Kaledin proved this conjecture locally on W in [34]. Furthermore, he showed that if W admits an expanding $\mathbb{C}^*$ action (such as nilpotent orbit closures), then the conjecture is true. However it is not easy to compute the equivalent functor in any explicit fashion, contrary to the case done by Y. Kawamata and Y. Namikawa.

7.5. **Birational geometry**

One way of constructing a symplectic resolution from another is to perform Mukai's elementary transformations ([40]), which can be described as follows. Let W be a symplectic variety and $\pi : Z \to W$ a symplectic resolution. Assume that W contains a smooth closed subvariety Y and π^{-1} contains a subvariety P such that the restriction of π to P makes P a $\mathbb{P}^n$-bundle over Y. If $codim(P) = n$, then we can blow up Z along P and then blow down along the other direction, which gives another (proper) symplectic resolution $\pi^+ : Z^+ \to W$, provided that Z^+ remains in our category of algebraic varieties. The diagram $Z \to W \leftarrow Z^+$ is called *Mukai's elementary transformation* (MET for short) over W with center Y. A *MET in codimension 2* is a diagram which becomes a MET after removing subvarieties of codimension greater than 2. The following conjecture is proposed in [28] (see also the survey [27]).

Conjecture 5. ([28]) Let W be a symplectic variety which admits two projective symplectic resolutions $\pi : Z \to W$ and $\pi^+ : Z^+ \to W$. Then the birational map $\phi = (\pi^+)^{-1} \circ \pi : Z - - \to Z^+$ is related by a sequence of METs over W in codimension 2.

Notice that since the two resolutions π, π^+ are both crepant, the birational map ϕ is isomorphic in codimension 1. This conjecture is true for four-dimensional symplectic varieties by the work of Wierzba and Wiśniewski ([53]) (while partial results have been obtained in [8], see also [9]). In [18], this conjecture is proved for nilpotent orbits in classical simple Lie algebras. For quotient varieties $\mathbb{C}^{2n}/G$, this conjecture is recently proved in [20]. If we pass to higher codimension, we expect that two symplectic resolutions are related by stratified Mukai flops of type A, D or E (see [19]).

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