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Abstract

Let N be an H-type group and $S \simeq N \times \mathbb{R}^+$ be its harmonic extension. We study a left invariant Hardy–Littlewood maximal operator $M_\rho^{\mathcal{R}}$ on S, obtained by taking maximal averages with respect to the right Haar measure over left-translates of a family $\mathcal{R}$ of neighbourhoods of the identity. We prove that the maximal operator $M_\rho^{\mathcal{R}}$ is of weak type $(1,1)$.

1. Introduction

Let $\mathfrak{n}$ be a Heisenberg type Lie algebra (briefly, an H-type Lie algebra) with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $|\cdot|$. We denote by N the connected and simply connected Lie group associated to $\mathfrak{n}$; N is called an H-type group. Let S be the one-dimensional extension of N obtained by letting $A = \mathbb{R}^+$ act on N by homogeneous dilations. Let H denote a vector in $\mathfrak{a}$ acting on $\mathfrak{n}$ with eigenvalues $1/2$ and (possibly) 1; we extend the inner product on $\mathfrak{n}$ to the algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ by requiring $\mathfrak{n}$ and $\mathfrak{a}$ to be orthogonal and H unitary. The algebra $\mathfrak{s}$ is a solvable Lie algebra. It is natural to endow S with the *left invariant* Riemannian metric d which agrees with the inner product of $\mathfrak{s}$ at the identity. The group S is a one-dimensional harmonic extension of the H-type group N. Let ρ denote a fixed right invariant measure on S. The metric measured space (S, d, ρ) is of exponential growth.

Harmonic extensions of H-type groups were introduced by Kaplan [15]. As Riemannian manifolds, these solvable Lie groups include properly all rank one symmetric spaces of the noncompact type. In fact, most of them are nonsymmetric harmonic manifolds, which provide counterexamples to

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Lichnerowicz conjecture. The geometry of these extensions has been studied by E. Damek and F. Ricci in [6, 5, 7, 8] and M. Cowling, A.H. Dooley, A. Korányi and Ricci in [2, 3].

In this paper we study the *left invariant* Hardy–Littlewood type maximal operator $M_\rho^{\mathcal{F}}$, defined by

$$M_\rho^{\mathcal{F}} f(x) = \sup_{F \in \mathcal{F}} \frac{1}{\rho(xF)} \int_{xF} |f| \, \mathrm{d}\rho \qquad \forall f \in L^1_{\mathrm{loc}}(\rho),$$

where $\mathcal{F}$ is a family of open subsets of S which contain the identity.

We shall focus on a particular family $\mathcal{R}$, which is described in detail at the beginning of Section 3. It may be worth observing that the family $\mathcal{R}$ contains "small sets", which are balls with respect to the left invariant Riemannian metric d, and "big sets", which are "rectangles". We shall see that sets in $\mathcal{R}$ are a generalization of "admissible parallelopipeds" in [14] and "rectangles" in [13].

Our main result is that $M_\rho^{\mathcal{R}}$ is bounded from $L^1(\rho)$ to the Lorentz space $L^{1,\infty}(\rho)$. This extends previous results of S. Giulini and P. Sjögren [13] (see the discussion below).

The main motivation to prove that the maximal operator $M_\rho^{\mathcal{R}}$ is of weak type $(1,1)$ is that this is a key step to prove that the metric measured space (S, d, ρ) is a Calderón–Zygmund space in the sense of [14, Definition 1.1]. More precisely, Hebisch and Steger [14] gave an axiomatic definition of Calderón–Zygmund space and proved that NA groups associated to real hyperbolic spaces, which fall in the class of groups we consider in this paper, are Calderón–Zygmund spaces. To prove this result they introduced a suitable family $\mathcal{R}$ of sets and proved that the maximal operator $M_\rho^{\mathcal{R}}$ is of weak type $(1,1)$. In our paper we extend and generalize their family $\mathcal{R}$ to the context of harmonic extensions of H-type groups. This will be a key to show that S is a Calderón–Zygmund space. The Calderón–Zygmund decomposition of integrable functions on S and its applications to the study of multipliers of a distinguished left invariant Laplacian on S, which will appear in a forthcoming paper [16], improve previous results in [1, 4, 12, 14].

The maximal operator $M_\rho^{\mathcal{F}}$ is also related to a number of results in the literature concerning maximal operators on solvable Lie groups, in particular on the affine group of the real line, which we now briefly summarize.

Let Γ be the affine group of the real line, i.e. $\Gamma = \mathbb{R} \times \mathbb{R}^+$, endowed with the product

$$(b, a)\,(b', a') = (b + a^{1/2} b', a\,a') \qquad \forall (b, a),\ (b', a') \in \mathbb{R} \times \mathbb{R}^+ .$$

Note the factor $a^{1/2}$ in the product rule above, instead of the more common factor a [10, 11, 13]. Our definition is consistent with that usually adopted for harmonic extensions of H-type groups (see [2]).

Let ρ and λ denote right and left Haar measures of Γ, respectively. Given a family $\mathcal{F}$ of open subsets of Γ containing the identity, let $\mathcal{F}^{-1}$ denote the family $\{F^{-1} :\ F \in \mathcal{F}\}$. The *right invariant* maximal operator $M_\lambda^{\mathcal{F}}$ defined, for every f in $L^1_{\mathrm{loc}}(\lambda)$, by

$$M_\lambda^{\mathcal{F}} f(x) = \sup_{F \in \mathcal{F}} \frac{1}{\lambda(Fx)} \int_{Fx} |f| \,\mathrm{d}\lambda,$$

has been considered by several authors [10, 11, 13].

It is straighforward to check that for all f in $L^1_{\mathrm{loc}}(\rho)$

$$M_\rho^{\mathcal{F}} f = \big(M_\lambda^{\mathcal{F}^{-1}} f^\vee\big)^\vee,$$

where $f^\vee(x) = f(x^{-1})$. Thus $M_\rho^{\mathcal{F}}$ is bounded on $L^p(\rho)$ (respectively of weak type $(1,1)$ with respect to ρ) if and only if $M_\lambda^{\mathcal{F}^{-1}}$ is bounded on $L^p(\lambda)$ (respectively of weak type $(1,1)$ with respect to λ).

In the literature many authors studied the maximal operator $M_\lambda^{\mathcal{F}^{-1}}$. Since our result will be relevant to study $L^p(\rho)$ multipliers of a left invariant Laplacian, we find it more natural to consider the maximal operator $M_\rho^{\mathcal{F}}$, which is left invariant and defined with respect to the measure ρ. Then we restate the results proved in the literature for the maximal operator $M_\lambda^{\mathcal{F}^{-1}}$ in terms of $M_\rho^{\mathcal{F}}$, for a particular family of sets $\mathcal{F}$ which we now introduce.

Let $\beta \geq 1/2$, $F_{r,\beta} = \{(b, a) \in \Gamma :\ |b| < r^\beta,\ a \in (1/r, r)\}$, for all $r > 1$, and $\mathcal{F} = \{F_{r,\beta} :\ r > 1\}$. Note that we consider $\beta \geq 1/2$, instead of $\beta \geq 1$ as in [10, 11, 13], because of the factor $a^{1/2}$ instead of the factor a in the product rule on Γ.

If $\beta = 1/2$, then G. Gaudry, Giulini, A.M. Mantero [10] proved that $M_\rho^{\mathcal{F}^{-1}}$ is not bounded on $L^p(\rho)$, for all $p \in (1, \infty)$. Moreover, Giulini [11] proved that $M_\rho^{\mathcal{F}}$ is bounded on $L^p(\rho)$, for $p \in (1, \infty)$, but it is not of weak type $(1,1)$ with respect to ρ [13].

If $\beta > 1/2$, then Giulini and Sjögren [13] proved that $M_\rho^{\mathcal{F}}$ is of weak type $(1,1)$ with respect to the measure ρ.

In this paper we generalize to the context of harmonic extensions of H-type groups the aforementioned result in [13] corresponding to the case $\beta > 1/2$.

Our paper is organized as follows: in Section 2 we recall the definition of an H-type group N and its harmonic extension S. In Section 3 we introduce a family $\mathcal{R}$ of open subsets of S and the left invariant maximal operator $M_\rho^{\mathcal{R}}$ associated to this family; we prove that $M_\rho^{\mathcal{R}}$ is bounded from $L^1(\rho)$ to $L^{1,\infty}(\rho)$.

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2. Harmonic extensions of H-type groups

In this section we recall the definition of H-type groups and we describe their harmonic extensions. For details see [2] and [3].

Let $\mathfrak{n}$ be a Lie algebra equipped with an inner product $\langle\cdot,\cdot\rangle$ and denote by $|\cdot|$ the corresponding norm. Let $\mathfrak{v}$ and $\mathfrak{z}$ be complementary orthogonal subspaces of $\mathfrak{n}$ such that $[\mathfrak{n},\mathfrak{z}] = \{0\}$ and $[\mathfrak{n},\mathfrak{n}] \subseteq \mathfrak{z}$. According to Kaplan [15] the algebra $\mathfrak{n}$ is of H-type if for every unitary Z in $\mathfrak{z}$ the map $J_Z : \mathfrak{v} \to \mathfrak{v}$ defined by

$$\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle \qquad \forall X, Y \in \mathfrak{v}$$

is orthogonal. In this case the connected and simply connected Lie group N associated to $\mathfrak{n}$ is called an H-type group. We identify N with its Lie algebra $\mathfrak{n}$ by the exponential map

$$\begin{aligned} \mathfrak{v}\times\mathfrak{z} &\to N \\ (X,Z) &\mapsto \exp(X+Z)\,. \end{aligned}$$

The product law in N is

$$(X,Z)(X',Z') = \Big(X+X', Z+Z'+(1/2)\,[X,X']\Big) \qquad \forall X,\ X' \in \mathfrak{v}\,\forall Z,\ Z' \in \mathfrak{z}\,.$$

The group N is a two-step nilpotent group, hence unimodular, with Haar measure $\mathrm{d}X\,\mathrm{d}Z$. We define on N the dilation

$$\delta_a(X,Z) \;=\; (a^{1/2}X, a\,Z) \qquad \forall (X,Z)\in N \quad \forall a \in \mathbb{R}^+\,.$$

The group N is an homogeneous group with homogeneous norm

$$\mathcal{N}(X,Z)=\left(\frac{|X|^4}{16}+|Z|^2\right)^{1/4}\qquad\forall (X,Z)\in N\,.$$

Note that, for all a in $\mathbb{R}^+$, $\mathcal{N}\big(\delta_a(X,Z)\big)=a^{1/2}\,\mathcal{N}(X,Z)$. We denote by d_N the corresponding homogeneous norm on N given by

$$d_N(n_0,n)=\mathcal{N}(n_0^{-1}n)\qquad\forall n_0,n\in N\,.$$

The homogeneous dimension of N is $Q=(m_{\mathfrak{v}}+2m_{\mathfrak{z}})/2$, where $m_{\mathfrak{v}}$ and $m_{\mathfrak{z}}$ denote the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$ respectively. For all n_0 in N and $r>0$ the homogeneous ball $B_N(n_0,r)$ centred at n_0 of radius r has measure $r^{2Q}|B_N(0_N,1)|$.

Given an H-type group N, let S be the one-dimensional extension of N obtained by letting $A=\mathbb{R}^+$ act on N by homogeneous dilations. Let H denote a vector in $\mathfrak{a}$ acting on $\mathfrak{n}$ with eigenvalues $1/2$ and (possibly) 1; we extend the inner product on $\mathfrak{n}$ to the algebra $\mathfrak{s}=\mathfrak{n}\oplus\mathfrak{a}$ by requiring $\mathfrak{n}$ and $\mathfrak{a}$ to be orthogonal and H to be unitary. The algebra $\mathfrak{s}$ is a solvable Lie algebra. The group S is called the harmonic extension of the H-type group N. The map

$$\begin{array}{rcl}\mathfrak{v}\times\mathfrak{z}\times\mathbb{R}^+ & \to & S\\ (X,Z,a) & \mapsto & \exp(X+Z)\exp(\log aH)\end{array}$$

gives global coordinates on S. The product in S is given by the rule

$$(X,Z,a)(X',Z',a')=\Big(X+a^{1/2}X',Z+a\,Z'+(1/2)\,a^{1/2}[X,X'],a\,a'\Big)$$

for all (X,Z,a), (X',Z',a') in S. We shall denote by $n=m_{\mathfrak{v}}+m_{\mathfrak{z}}+1$ the dimension of S. The group S is nonunimodular: the right and left Haar measures on S are given by $\mathrm{d}\rho(X,Z,a)=a^{-1}\,\mathrm{d}X\,\mathrm{d}Z\,\mathrm{d}a$ and $\mathrm{d}\lambda(X,Z,a)=a^{-Q-1}\,\mathrm{d}X\,\mathrm{d}Z\,\mathrm{d}a$ respectively. We denote by $L^1(\rho)$ the space of integrable functions with respect to the measure ρ and by $L^{1,\infty}(\rho)$ the Lorentz space of all measurable functions f such that

$$\sup_{t>0}\,t\,\rho\big(\{x\in S:\ |f(x)|>t\}\big)<\infty\,.$$

We equip S with the left invariant Riemannian metric d which agrees with the inner product on $\mathfrak{s}$ at the identity e and denote by $B\big((n_0,a_0),r\big)$ the ball in S centred at (n_0, a_0) of radius r. Note that

$$\rho(B(e,r)) \asymp \begin{cases} r^n & \text{if } r \in (0,1) \\ \mathrm{e}^{Qr} & \text{if } r \in [1,+\infty)\,. \end{cases} \tag{2.1}$$

This shows that S, equipped with right Haar measure, is a group of exponential growth.

3. The maximal operator $M_\rho^{\mathcal{R}}$

In this section we introduce a family $\mathcal{R}$ of sets in S and study the weak type $(1,1)$ boundedness of the associated left invariant maximal operator $M_\rho^{\mathcal{R}}$.

We first define a subsidiary family of sets centred at the identity as follows:

$$E_{r,\beta} = \begin{cases} B(e, \log r) & \text{if } 1 < r < \mathrm{e} \\ B_N(0_N, r^\beta) \times (\frac{1}{r}, r) & \text{if } r \geq \mathrm{e}\,, \end{cases} \tag{3.1}$$

where $1/2 < b < \beta < B$, and b, B are constants.

Note that "small sets" $E_{r,\beta}$, $1 < r < \mathrm{e}$, are balls with respect to the left invariant metric d, while "big sets" $E_{r,\beta}$, $r \geq \mathrm{e}$, are "rectangles". We shall now give a motivation to definition (3.1).

In the particular case of NA groups associated to real hyperbolic spaces Hebisch and Steger [14] define a family of "admissible parallelopipeds". They give a different definition of small parallelopipeds and big parallelopipeds and use these sets in the Calderón–Zygmund decomposition of integrable functions.

Our purpose is to replace "admissible parallelopipeds" with sets $E_{r,\beta}$ in the Calderón–Zygmund decomposition of integrable functions on harmonic extensions of H-type groups. Then we give a definition of small and big sets $E_{r,\beta}$ which generalize the "admissible parallelopipeds". In particular, according to [14, Definition 1.1], it is easy to check that there exists a constant C indipendent on r such that

$$E_{r,\beta} \subseteq B(e, C\,\log r) \qquad \forall r > 1\,. \tag{3.2}$$

Note that we consider $A = \mathbb{R}^+$, as in the notation usually adopted in harmonic extensions of H-type groups, while in [14] $A = \mathbb{R}$, so that definition (3.1) and condition (3.2) could appear different from [14] but are not (it suffices to use an exponential notation on $\mathbb{R}^+$).

Another motivation to definition (3.1) is that we are interested in a generalization of the result obtained by Giulini and Sjögren in the particular case of the affine group of the real line [13]. More precisely, for $r \geq \mathrm{e}$ the sets $E_{r,\beta}$ coincide with the rectangles $F_{r,\beta}$ defined in [13], while for $r < \mathrm{e}$ they are different. We choose different sets because for $r < \mathrm{e}$ the rectangles $F_{r,\beta}$ do not satisfy condition (3.2) which is necessary to involve the sets in the Calderón–Zygmund decomposition.

Let γ be a constant greater than $\frac{4B+2b+1}{2b-1}$. For each set $E_{r,\beta}$, we define its dilated set as

$$E^*_{r,\beta} = \begin{cases} B\big(e, \gamma \log r\big) & \text{if } 1 < r < \mathrm{e} \\ B_N\big(0_N, \gamma\, r^{\beta}\big) \times \big(\frac{1}{r^\gamma}, r^\gamma\big) & \text{if } r \geq \mathrm{e}\,. \end{cases}$$

Note that

$$\rho(E_{r,\beta}) \asymp \begin{cases} (\log r)^n & \text{if } 1 < r < \mathrm{e} \\ r^{2\beta Q} \log r & \text{if } r \geq \mathrm{e}\,. \end{cases} \tag{3.3}$$

The measures of $E_{r,\beta}$ and $E^*_{r,\beta}$ are comparable; more precisely, there exists a constant C^* such that $\rho(E^*_{r,\beta}) \leq C^*\,\rho(E_{r,\beta})$. For all $x_0 = (n_0, a_0)$ in S the left-translate of the set $E_{r,\beta}$ is

$$x_0 E_{r,\beta} = \begin{cases} B\big(x_0, \log r\big) & \text{if } 1 < r < \mathrm{e} \\ B_N\big(n_0, a_0^{1/2} r^{\beta}\big) \times \big(\frac{a_0}{r}, a_0 r\big) & \text{if } r \geq \mathrm{e}\,, \end{cases}$$

and put $(x_0 E_{r,\beta})^* = x_0 E^*_{r,\beta}$. We say that the set $x_0 E_{r,\beta}$ is centred at x_0.

Now we consider the family of all left-translates of the sets $E_{r,\beta}$ which contain the identity, i.e.

$$\mathcal{R} = \{x_0 E_{r,\beta} :\ x_0 \in S,\ r > 1,\ b < \beta < B,\ e \in x_0 E_{r,\beta}\}\,.$$

The associated maximal operator $M_\rho^{\mathcal{R}}$ is a left invariant noncentred maximal operator. Our result is the following.

Theorem 3.1. *The maximal operator $M_\rho^{\mathcal{R}}$ is bounded from $L^1(\rho)$ to $L^{1,\infty}(\rho)$.*

In order to prove Theorem 3.1, it suffices to study separately the maximal operators associated to "small sets" and "big sets". More precisely, define

$$\mathcal{R}^0 = \{x_0 E_{r,\beta} :\ x_0 \in S\,,\ 1 < r < \mathrm{e}\,,\ e \in x_0 E_{r,\beta}\}$$

and

$$\mathcal{R}^\infty = \{x_0 E_{r,\beta} :\ x_0 \in S\,,\ r \geq \mathrm{e},\ b < \beta < B,\ e \in x_0 E_{r,\beta}\}\,.$$

If we can show that $M_\rho^{\mathcal{R}^0}$ and $M_\rho^{\mathcal{R}^\infty}$ are of weak type $(1,1)$, then Theorem 3.1 follows.

3.1. The maximal operator $M_\rho^{\mathcal{R}^\infty}$

Given two sets $R_i = x_i E_{r_i,\beta_i}$, $i=1,2$, with $r_i \geq \mathrm{e}$, we say that $R_2 \leq R_1$ if $\rho(R_2) \leq \rho(R_1)$.

Lemma 3.2. *Let $R_i = x_i E_{r_i,\beta_i}$, $i=1,2$, with $r_i \geq \mathrm{e}$. If $R_2 \leq R_1$ and $R_1 \cap R_2 \neq \emptyset$, then $R_2 \subseteq R_1^*$.*

Proof. Let $R_i = x_i E_{r_i,\beta_i}$, where $1/2 < b < \beta_i < B$, $r_i \geq \mathrm{e}$, for $i=1,2$. Without loss of generalization we may suppose that R_2 is centred at the identity. Indeed, if this does not hold, then sets $x_2^{-1}R_i$ satisfy the hypothesis of the lemma and $x_2^{-1}R_2$ is centred at the identity. If the conclusion is true for sets $x_2^{-1}R_i$, then it obviously follows for sets R_i.

Then we suppose that $x_2 = e$ and $x_1 = (n_1, a_1)$. It is straightforward to check that the condition $R_2 \leq R_1$ implies that

$$\frac{(a_1^{1/2} r_1^{\beta_1})^{2Q}}{r_2^{2\beta_2 Q}} \frac{\log r_1}{\log r_2} \geq 1\,. \tag{3.4}$$

The fact that R_1 and R_2 intersect implies that

$$\frac{1}{r_1 r_2} < a_1 < r_1 r_2 \qquad \text{and} \qquad d_N(n_1, 0_N) < a_1^{1/2} r_1^{\beta_1} + r_2^{\beta_2}\,. \tag{3.5}$$

Let (n,a) be a point of R_2; we shall prove that it belongs to R_1^*.

From (3.5) we deduce that

$$\frac{1}{r_2^2 r_1} < \frac{a}{a_1} < r_2^2 r_1 \tag{3.6}$$

and

$$\begin{aligned} d_N(n, n_1) &\leq d_N(n, 0_N) + d_N(0_N, n_1) \\ &< 2\, r_2^{\beta_2} + a_1^{1/2}\, r_1^{\beta_1}\,. \end{aligned} \tag{3.7}$$

Now we examine separately two cases.

Case $r_2 \geq r_1$. In this case, from the inequality (3.4) we deduce that

$$\begin{aligned} r_2^{\beta_2} &\leq a_1^{1/2} r_1^{\beta_1} \Big(\frac{\log r_1}{\log r_2}\Big)^{1/2Q} \\ &\leq a_1^{1/2} r_1^{\beta_1}\,, \end{aligned}$$

and then from (3.7) we obtain that

$$\begin{aligned} d_N(n,n_1) &< 2\,r_2^{\beta_2} + a_1^{1/2} r_1^{\beta_1} \\ &\le 3\,a_1^{1/2} r_1^{\beta_1}\,. \end{aligned}$$

Again from (3.4) and (3.5) it follows that

$$\begin{aligned} r_2^{\beta_2} &\le {a_1}^{1/2} r_1^{\beta_1} \Big(\frac{\log r_1}{\log r_2}\Big)^{1/2Q} \\ &\le r_2^{1/2} r_1^{\beta_1+1/2}\,, \end{aligned}$$

and then $r_2 \le r_1^{\frac{2\beta_1+1}{2\beta_2-1}} \le r_1^{\frac{2B+1}{2b-1}}$. Thus, from (3.6)

$$\Big(\frac{1}{r_1}\Big)^{\frac{4B+2b+1}{2b-1}} \le \frac{1}{r_2^2 r_1} < \frac{a}{a_1} < r_2^2 r_1 \le r_1^{\frac{4B+2b+1}{2b-1}}\,.$$

Since $\gamma > \frac{4B+2b+1}{2b-1} > 3$ by assumption, we have that

$$\frac{1}{r_1^{\gamma}} < \frac{a}{a_1} < r_1^{\gamma} \qquad \text{and} \qquad d_N(n_1, 0_N) < \gamma\, a_1^{1/2} r_1^{\beta_1}\,.$$

Thus the point (n,a) is in R_1^* as required.

Case $r_2 < r_1$. In this case, using (3.6) we have that

$$\frac{1}{r_1^3} < \frac{1}{r_2^2 r_1} < \frac{a}{a_1} < r_2^2 r_1 < r_1^3\,.$$

It remains to verify that $d_N(n,n_1) < \gamma\, a_1^{1/2} r_1^{\beta_1}$. We examine two situations separately.

(i) If $r_2 < r_1^{\frac{2\beta_1-1}{2\beta_2+1}}$, then from (3.7) we obtain that

$$\begin{aligned} d_N(n,n_1) &< a_1^{1/2} r_1^{\beta_1} \Big(2\,\frac{r_2^{\beta_2}}{a_1^{1/2} r_1^{\beta_1}} + 1\Big) \\ &< a_1^{1/2} r_1^{\beta_1} \Big(2\,\frac{r_2^{\beta_2}}{r_1^{\beta_1}}\, r_1^{1/2} r_2^{1/2} + 1\Big) \\ &< 3\,a_1^{1/2} r_1^{\beta_1}\,. \end{aligned}$$

(ii) If $r_1^{\frac{2\beta_1-1}{2\beta_2+1}} \leq r_2 < r_1$, then from (3.4) we deduce that

$$\begin{aligned} r_2^{\beta_2} &\leq a_1^{1/2} r_1^{\beta_1} \Big(\frac{\log r_1}{\log r_2}\Big)^{1/2Q} \\ &\leq a_1^{1/2} r_1^{\beta_1} \Big(\frac{2\beta_2+1}{2\beta_1-1}\Big)^{1/2Q} \\ &\leq a_1^{1/2} r_1^{\beta_1} \Big(\frac{2B+1}{2b-1}\Big)^{1/2Q} . \end{aligned}$$

This implies that

$$\begin{aligned} d_N(n,n_1) &< 2\, r_2^{\beta_2} + a_1^{1/2} r_1^{\beta_1} \\ &\leq \left(2\Big(\frac{2B+1}{2b-1}\Big)^{1/2Q} + 1\right) a_1^{1/2} r_1^{\beta_1} . \end{aligned}$$

Since $\gamma > \frac{4B+2b+1}{2b-1} > 2\left(\frac{2B+1}{2b-1}\right)^{1/2Q} + 1 > 3$ by assumption, we have proved that

$$\frac{1}{r_1^{\gamma}} < \frac{a}{a_1} < r_1^{\gamma} \qquad \text{and} \qquad d_N(n_1, 0_N) < \gamma\, a_1^{1/2} r_1^{\beta_1} .$$

Thus the point (n,a) is in R_1^* as required. □

From Lemma 3.2 above, we deduce a standard covering lemma.

Lemma 3.3. *Given a finite collection of sets $\{R_i\}_i$ which are left-translates of sets E_{r_i,β_i}, with $r_i \geq \mathrm{e}$, there exists a subcollection of mutually disjoint sets $R_1, ..., R_k$ such that*

$$\bigcup_i R_i \subseteq \bigcup_{j=1}^{k} R_j^* .$$

Proof. This follows from a standard argument: at each step one selects the $\leq$-greatest set which does not intersect the sets already selected. □

As a straightforward consequence, we obtain the weak type property for the maximal operator $M_\rho^{\mathcal{R}^\infty}$.

Proposition 3.4. *The maximal operator $M_\rho^{\mathcal{R}^\infty}$ is bounded from $L^1(\rho)$ to $L^{1,\infty}(\rho)$.*

Proof. Let f be in $L^1(\rho)$ and $t > 0$. Let $\Omega_t = \{x \in S:\ M_\rho^{\mathcal{R}^\infty} f(x) > t\}$ and let F be any compact subset of Ω_t . By the compactness of F, we can select a finite collection of sets $\{R_i\}_i$ which cover F such that $R_i = x_i E_{r_i,\beta_i}$ and

$$\frac{1}{\rho(R_i)} \int_{R_i} |f| \, \mathrm{d}\rho > t\,.$$

By Lemma 3.3 we can select a disjoint subcollection $R_1, ..., R_k$ such that $F \subseteq \bigcup_{j=1}^k R_j^*$. Thus,

$$\rho(F) \le \sum_{j=1}^k \rho(R_j^*) \le C^* \sum_{j=1}^k \rho(R_j) \le \frac{C^*}{t} \sum_{j=1}^k \int_{R_j} |f| \, \mathrm{d}\rho \le \frac{C^*}{t} \|f\|_{L^1(\rho)}\,.$$

If we take the supremum over all such $F \subseteq \Omega_t$, then the conclusion is proved. □

3.2. **The maximal operator $M_\rho^{\mathcal{R}^0}$**

Given $R_i = x_i E_{r_i,\beta_i}$, with $1 < r_i < \mathrm{e}$, for $i = 1, 2$, we say that $R_2 \le R_1$ if $r_2 \le r_1$.

In this case an easy covering lemma holds. Indeed, if $R_2 \le R_1$ and $R_1 \cap R_2 \ne \emptyset$, then for each point x in R_2, we have that

$$d(x, x_1) \le d(x, x_2) + d(x_2, x_1) < 2\, \log r_2 + \log r_1 \le 3\, \log r_1\,.$$

Since $\gamma > 3$, the point x is in R_1^*.

In the same way as above one deduces a covering lemma and the weak type $(1, 1)$ property for the maximal operator $M_\rho^{\mathcal{R}^0}$.

As we have already remarked, since $M_\rho^{\mathcal{R}^\infty}$ and $M_\rho^{\mathcal{R}^0}$ are of weak type $(1, 1)$, Theorem 3.1 follows.

Remark 3.5. The hypothesis of Theorem 3.1 are that β is bounded away from both $1/2$ and ∞. If this does not hold, then the operator $M_\rho^{\mathcal{R}}$ is not of weak type $(1, 1)$, as the following argument shows.

Let $\tilde{\mathcal{R}}$ be the family

$$\tilde{\mathcal{R}} = \{x_0 E_{r,\beta}:\ x_0 \in S,\ r \ge \mathrm{e},\ \beta > 1/2,\ e \in x_0 E_{r,\beta}\}\,.$$

The maximal operator $M_\rho^{\tilde{\mathcal{R}}}$ is not of weak type $(1, 1)$. Indeed, if the weak type $(1, 1)$ inequality holds, then it can automatically be extended from

$L^1(\rho)$ functions to finite measures. Let δ_e be the unit point mass at the identity e. At a point $x=(n,a)$ we have that

$$M_\rho^{\tilde{\mathcal{R}}}\delta_e(x)\geq \sup_{r\geq \mathrm{e},\,\beta>1/2}\frac{1}{\rho\big(xE_{r,\beta}\big)}\,\delta_e(xE_{r,\beta})\,.$$

We have that $\delta_e(xE_{r,\beta})\neq 0$ if and only if

$$\frac{a}{r}<1<a\,r\qquad\text{and}\qquad d_N(n,0_N)<a^{1/2}r^{\beta}\,.$$

Since $\rho\big(xE_{r,\beta}\big)=a^Q r^{2\beta Q}\log r$,

$$M_\rho^{\tilde{\mathcal{R}}}\delta_e(x)\geq \sup\Big\{\frac{1}{a^Q r^{2\beta Q}\log r}:\ \beta>1/2,\ r\geq \mathrm{e},\ \frac{a}{r}<1<a\,r,\ a^{1/2}r^{\beta}>d_N(n,0_N)\Big\}\,.$$

Now let $a>\mathrm{e}$ and $d_N(n,0_N)>a$. We may choose $\beta=\log_a d-1/2>1/2$ and $r=a\Big(1+\frac{1}{\beta}\Big)$. Obviously

$$r>a\qquad\text{and}\qquad a^{1/2}r^{\beta}>a^{\beta+1/2}=d_N(n,0_N)\,.$$

Moreover,

$$\begin{aligned}a^Q r^{2\beta Q}\log r &= a^{2Q(\beta+1/2)}\Big(1+\frac{1}{\beta}\Big)^{2Q}\log\Big(a+\frac{a}{\beta}\Big)\\ &\leq C\big[d_N(n,0_N)\big]^{2Q}\log a\,.\end{aligned}$$

It follows that

$$M_\rho^{\tilde{\mathcal{R}}}\delta_e(x)\geq \frac{1}{C\big[d_N(n,0_N)\big]^{2Q}\log a}\,.$$

Estimating the level sets of the function $\frac{1}{\big[d_N(n,0_N)\big]^{2Q}\ \log a}$ in the region $\{x=(n,a)\in S:\ \ a>\mathrm{e},\, d_N(n,0_N)>a\}$, we disprove the weak type inequality.

Indeed, let $0<t<\mathrm{e}^{-2Q}$ and consider the set

$$\begin{aligned}\Omega_t &= \Bigg\{(n,a)\in S:\ a>\mathrm{e},\ d_N(n,0_N)>a,\ \frac{1}{\big[d_N(n,0_N)\big]^{2Q}\ \log a}>t\Bigg\}\\ &= \Big\{(n,a)\in S:\ \mathrm{e}<a<\alpha,\ a<d_N(n,0_N)<\frac{1}{(t\log a)^{1/2Q}}\Big\}\,,\end{aligned}$$

where $\alpha^{2Q}\log\alpha = \frac{1}{t}$. The right Haar measure of the set Ω_t is equal to

$$\begin{aligned}\rho(\Omega_t) &= \int_{\mathrm{e}}^{\alpha}\frac{\mathrm{d}a}{a}\int_{a^2}^{\frac{1}{(t\log a)^{1/Q}}}\sigma^{Q-1}\,\mathrm{d}\sigma\\ &= \frac{1}{Q}\int_{\mathrm{e}}^{\alpha}\Big(\frac{1}{t\log a}-a^{2Q}\Big)\frac{\mathrm{d}a}{a}\\ &= \frac{1}{Qt}\log\log\alpha-\frac{1}{2Q^2}\alpha^{2Q}+\frac{1}{2Q^2}\mathrm{e}^{2Q}\\ &\geq \frac{1}{Qt}\log\log\alpha-\frac{1}{2Q^2}\frac{1}{t\log\alpha}\\ &\geq \frac{1}{Qt}\Big(\log\log\alpha-\frac{1}{2Q}\Big),\end{aligned}$$

where we have used the integration formula for radial functions on N ([9, Proposition 1.15]).

It is easy to check that $\alpha > \frac{1}{t^{1/4Q}}$, and then

$$\begin{aligned}\rho(\Omega_t) &\geq \frac{1}{Qt}\Big(\log\log\Big(\frac{1}{t^{1/4Q}}\Big)-\frac{1}{2Q}\Big)\\ &\geq \frac{1}{Qt}\Big(\log\log\Big(\frac{1}{t}\Big)-\frac{1}{2Q}\Big),\end{aligned}$$

which is not bounded above by $\frac{C}{t}$. Thus, the weak type inequality for the maximal operator $M_\rho^{\tilde{\mathcal{R}}}$ does not hold.

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A MAXIMAL FUNCTION ON HARMONIC EXTENSIONS

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