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On representation theory of partition algebras for complex reflection groups

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Abstract This paper defines the partition algebra, denoted by $\mathcal{T}_k(r,p,n)$, for complex reflection group $G(r,p,n)$ acting on k-fold tensor product $(\mathbb{C}^n)^{\otimes k}$, where $\mathbb{C}^n$ is the reflection representation of $G(r,p,n)$. A basis of the centralizer algebra of this action of $G(r,p,n)$ was given by Tanabe and for $p=1$, the corresponding partition algebra was studied by Orellana. We also define a subalgebra $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ such that $\mathcal{T}_k(r,p,n) \subseteq \mathcal{T}_{k+\frac{1}{2}}(r,p,n) \subseteq \mathcal{T}_{k+1}(r,p,n)$ and establish this subalgebra as partition algebra of a subgroup of $G(r,p,n)$ acting on $(\mathbb{C}^n)^{\otimes k}$. We call the algebras $\mathcal{T}_k(r,p,n)$ and $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ Tanabe algebras. The aim of this paper is to study representation theory of Tanabe algebras: parametrization of their irreducible modules, and construction of Bratteli diagram for the tower of Tanabe algebras

$$\mathcal{T}_0(r,p,n) \subseteq \mathcal{T}_{\frac{1}{2}}(r,p,n) \subseteq \mathcal{T}_1(r,p,n) \subseteq \mathcal{T}_{\frac{3}{2}}(r,p,n) \subseteq \cdots \subseteq \mathcal{T}_{\lfloor \frac{n}{2} \rfloor}(r,p,n).$$

We conclude the paper by giving Jucys–Murphy elements of Tanabe algebras and their actions on the Gelfand–Tsetlin basis, determined by this multiplicity free tower, of irreducible modules.

Contents

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Keywords. Complex reflection groups, Tanabe algebras, Schur–Weyl duality, Jucys–Murphy elements.

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1. Introduction

The symmetric group S_k acts on the k-fold tensor product $V^{\otimes k}$ of the n-dimensional vector space $V = \mathbb{C}^n$ over the field of complex numbers $\mathbb{C}$. The general linear group $GL_n(\mathbb{C})$ acts on $V^{\otimes k}$ diagonally where V is the defining representation of $GL_n(\mathbb{C})$. These two actions commute; moreover, they generate the centralizers of each other. This is known as the classical Schur–Weyl duality [6].

Jones [9] and Martin [10], independently, defined partition algebra $\mathbb{C}A_k(q)$, where $q \in \mathbb{C}$, as a generalization of Temperley–Lieb algebras and Potts model in higher dimensional statistical mechanics. The symmetric group S_n, being the subgroup of permutation matrices in $GL_n(\mathbb{C})$, acts on $V^{\otimes k}$. Jones [9] proved Schur–Weyl duality between the partition algebra $\mathbb{C}A_k(n)$ and the symmetric group S_n acting on $V^{\otimes k}$. Furthermore, Martin and Saleur studied the structure of partition algebras in [13, 14] and proved that the partition algebra $\mathbb{C}A_k(n)$ is semisimple unless n is an integer such that $0 \leqslant n < 2k-1$.

The subalgebra $\mathbb{C}A_{k+\frac{1}{2}}(n)$ of partition algebra $\mathbb{C}A_{k+1}(n)$ was introduced by Martin and Rollet [12] (See also [11]). Halverson and Ram [8] showed Schur–Weyl duality between $\mathbb{C}A_{k+\frac{1}{2}}(n)$ and the subgroup S_{n-1} of S_n; and thus established it to be equally important as partition algebra $\mathbb{C}A_k(n)$. It was also shown in [8] that the branching rule is multiplicity free for $\mathbb{C}A_{l-\frac{1}{2}}(n) \subseteq \mathbb{C}A_l(n)$ for $l \in \frac{1}{2}\mathbb{Z}_{>0}$ whenever both the algebras are semisimple. Recursively building the Bratteli diagram for the tower of partition algebras

$$\mathbb{C}A_0(n) \subseteq \mathbb{C}A_{\frac{1}{2}}(n) \subseteq \mathbb{C}A_1(n) \subseteq \mathbb{C}A_{\frac{3}{2}}(n) \subseteq \cdots,$$

the Jucys–Murphy elements of partition algebras were given in [8, Theorem 3.37]. Later, the seminormal representations of parition algebra were derived by Enyang [5].

The complex reflection group $G(r,p,n)$, where r, p and n are positive integers such that p divides r, is a subgroup of $GL_n(\mathbb{C})$. The group $G(r,1,n)$ is the wreath product of the cyclic group $\mathbb{Z}/r\mathbb{Z}$ by the symmetric group S_n and $G(r,p,n)$ is a normal subgroup of index p of $G(r,1,n)$. Shephard and Todd [25] gave a classification of finite irreducible complex reflection groups. It was shown there that the families of groups S_n for $n > 1$, $\mathbb{Z}/r\mathbb{Z}$ for $r > 1$, and $G(r,p,n)$ (except when $(r,p,n) = (2,2,2)$ or $(1,1,1)$) are the only infinite families of finite irreducible complex reflection groups and there are exactly 34 more finite irreducible complex reflection groups. Also, they characterized the group $G(r,p,n)$, for $n > 1$, by showing that these (except the group $G(2,2,2)$) are the only finite irreducible imprimitive complex reflection groups up to isomorphism.

The restriction of the action of $GL_n(\mathbb{C})$ on V to $G(r,p,n)$ is the reflection representation of $G(r,p,n)$. Tanabe [27, Lemma 2.1] described a basis of the centralizer algebra of the action of $G(r,p,n)$ on the tensor space $V^{\otimes k}$. Orellana [19] defined a subalgebra $\mathcal{T}_k(n,r)$ of partition algebra $\mathbb{C}A_k(n)$, and proved Schur–Weyl duality between $\mathcal{T}_k(n,r)$ and $G(r,1,n)$ [19, Theorem 5.4]. Also, she recursively constructed the Bratteli diagram for the tower of algebras

$$\mathcal{T}_0(n,r) \subseteq \mathcal{T}_1(n,r) \subseteq \mathcal{T}_2(n,r) \subseteq \cdots$$

in [19, Proposition 5.6].

In this paper, we define a subalgebra, denoted by $\mathcal{T}_k(r,p,n)$, of partition algebra $\mathbb{C}A_k(n)$ such that there is Schur–Weyl duality between $\mathcal{T}_k(r,p,n)$ and the complex reflection group $G(r,p,n)$. In particular, for $p = 1$, the algebra $\mathcal{T}_k(r,1,n)$ is equal to the algebra $\mathcal{T}_k(n,r)$ defined by Orellana. Along the lines of [8], we introduce a subgroup $L(r,p,n)$ of $G(r,p,n)$ which plays a role analogous to the subgroup S_{n-1} of S_n in the study of partition algebra. We define a subalgebra, denoted by $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$, of partition algebra $\mathbb{C}A_{k+\frac{1}{2}}(n)$ and exhibit Schur–Weyl duality between $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$

and $L(r,p,n)$. Thus, the algebras $\mathcal{T}_k(r,p,n)$ and $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ are partition algebras for the complex reflection group $G(r,p,n)$ and its subgroup $L(r,p,n)$, respectively. We call the algebras $\mathcal{T}_k(r,p,n)$ and $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ *Tanabe algebras.*

The main results in this paper are as follows.

(a) For Tanabe algebras:
  (i) Decomposition of the centralizer algebras $\mathrm{End}_{G(r,p,n)}(V^{\otimes k})$ and $\mathrm{End}_{L(r,p,n)}(V^{\otimes k})$ into their irreducible modules which, in particular, gives parametrization of the irreducible modules of Tanabe algebras $\mathcal{T}_k(r,p,n)$ and $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ for $n \geqslant 2k$ and $n \geqslant 2k+1$, respectively (Theorem 6.4).
  (ii) Construction of Bratteli diagram recursively for the tower

$$\mathcal{T}_0(r,p,n) \subseteq \mathcal{T}_{\frac{1}{2}}(r,p,n) \subseteq \mathcal{T}_1(r,p,n) \subseteq \mathcal{T}_{\frac{3}{2}}(r,p,n) \subseteq \cdots \subseteq \mathcal{T}_{\lfloor \frac{n}{2} \rfloor}(r,p,n).$$

  In this case, the Bratteli diagram is a simple graph (Section 6).
  (iii) Description of a specific set of commuting elements, called Jucys–Murphy elements, which act as scalars on the canonical basis, called Gelfand–Tsetlin basis, of each irreducible module of Tanabe algebras (Theorem 7.6).

(b) For complex reflection groups:
  (i) Construction of a basis of irreducible $G(r,p,n)$-modules (Theorem 4.14) using a combination of ideas from Okounkov–Vershik approach to the representation theory of $G(r,1,n)$ in [16], Clifford theory and higher Specht polynomials in [17].
  (ii) Branching rule from $G(r,p,n)$ to $L(r,p,n)$ (Theorem 4.16).
  (iii) Decomposition of $V^{\otimes k}$ in terms of irreducible $G(r,p,n)$-modules and $L(r,p,n)$-modules (Theorem 6.3).

Using theory of the basic construction, [8, Theorem 3.27] shows that the necessary and sufficient condition for the semisimplicity of partition algebra $\mathbb{C}A_l(n)$, for $n \in \mathbb{Z}_{\geqslant 2}$ and $l \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$, is $l \leqslant \frac{n+1}{2}$. The corresponding question for Tanabe algebras is open. Also, another important question is to find a presentation by generators and relations for Tanabe algebras. A generating set for $\mathcal{T}_k(r,1,n)$ is given in [19, Proposition 5.1]. A set of generators for $\mathrm{End}_{G(r,p,n)}(V^{\otimes k})$ is described in [27, Section 3].

The inductive approach to the representation theory of symmetric groups was done by Okounkov and Vershik in [18, 28]. This approach considers the chain of symmetric groups

$$\{1\} = S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$$

to study their representation theory recursively. The advantage over the traditional approach is that the appearance of Young diagrams and standard Young tableaux is given a spectral explanation, and the branching rule is determined simultaneously. The Gelfand–Tsetlin decomposition, the Gelfand–Tsetlin algebra, the canonical Gelfand–Tsetlin basis of the irreducible representations, and the Jucys–Murphy elements, a set of generators of Gelfand–Tsetlin algebra, are fundamental to this approach. The corresponding approach in the case of $G(r,1,n)$ proves fruitful in giving new proofs of some known results and also in establishing new results in this paper.

This paper is organized in the following sections. Section 2 gives a brief introduction to partition algebra, Okounkov–Vershik approach to the representation theory of $G(r,1,n)$, and Clifford theory. In Section 3, we define Tanabe algebras, $\mathcal{T}_k(r,p,n)$ and $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$, as subspaces of partition algebras and prove that these subspaces are algebras.

Section 4 contains a description of representation theory of complex reflection group $G(r,p,n)$ and its subgroup $L(r,p,n)$ (Theorems 4.10 and 4.12). We review the representation theory of $G(r,p,n)$ using Clifford theory. We parametrize the irreducible $L(r,1,n)$-modules and, then by Clifford theory, determine the representation theory of $L(r,p,n)$. This section concludes with the branching rule from $G(r,p,n)$ to $L(r,p,n)$.

In Section 5, we demonstrate that $\mathcal{T}_k(r,p,n)$ and $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ are in Schur–Weyl duality with $G(r,p,n)$ and $L(r,p,n)$, respectively. Using results from Section 4, Section 6 starts with the decomposition of $V^{\otimes k}$ as $G(r,p,n)$-module and $L(r,p,n)$-module. Then, we give decomposition of $V^{\otimes k}$ as $(G(r,p,n),\mathcal{T}_k(r,p,n))$-bimodule and $(L(r,p,n),\mathcal{T}_{k+\frac{1}{2}}(r,p,n))$-bimodule (Theorem 6.5) and use it to construct Bratteli diagram of Tanabe algebras. In Section 7, we give Jucys–Murphy elements and their actions on the canonical Gelfand–Tsetlin basis of irreducible modules of Tanabe algebras.

Conventions. Throughout this paper, we assume that

(i) r,p,m and n are positive integers such that p divides r and $m=\frac{r}{p}$, and
(ii) we index the components in a w-tuple from $1,\ldots,w$, therefore, for a multiple t of w, the $t(\operatorname{mod} w)$-th component means the w-th component.

2. Preliminaries

In this section, we give an overview of partition algebra, Okounkov–Vershik approach and Clifford theory to set up notations and to state basic definitions and results used in the rest of the paper.

2.1. Partition algebra. For $k\in\mathbb{Z}_{\geqslant 0}$, let A_k be the set of all set partitions of $\{1,2,\ldots,k,1',2',\ldots,k'\}$. Given an element $d\in A_k$, we say that i and j are in the same block in d if i and j belong to the same set partition in d. The elements of A_k can be depicted as graphs, called partition diagrams, with the vertices $\{1,2,\ldots,k\}$ and $\{1',2',\ldots,k'\}$ in the top and bottom rows, respectively and two vertices in the same block are connected by an edge. By $d=\{B_1,B_2,\ldots,B_s\}$, we denote that there are exactly s blocks $B_1,B_2,\ldots,B_s$ in d. Also, $|d|$ denotes the number of blocks in d.

The multiplication of two elements $d_1,d_2\in A_k$, denoted by $d_1\circ d_2$, is obtained by concatenating the diagrams d_1 and d_2 in the following way: place d_1 above d_2, identify the vertices in the bottom row of d_1 with the vertices in the top row of d_2, then remove all the connected blocks which are entirely in the middle row. The multiplication $\circ$ makes $(A_k,\circ)$ a monoid with the identity element given in Figure 1.

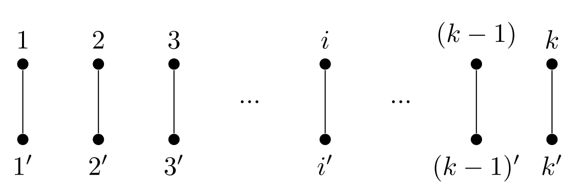

Figure 1. Identity element in partition monoid.

Define a subset $A_{k+\frac{1}{2}}$ of A_{k+1} consisting of those elements which have $(k+1)$ and $(k+1)'$ in the same block. It can be easily seen that $A_{k+\frac{1}{2}}$ is a submonoid of A_{k+1}. The monoids A_k and $A_{k+\frac{1}{2}}$ are called partition monoids.

Example 2.1. Taking $k=6$, the elements d_1 and d_2 in A_6 with

$$d_1=\{\{1,2,1'\},\{3,5,3'\},\{4\},\{6,5'\},\{2',4'\},\{6'\}\}\in A_6$$

and

$$d_2 = \{\{1,5,2',3'\},\{2,4\},\{3\},\{6,6'\},\{1',4'\},\{5'\}\} \in A_6,$$

can be written in terms of partition diagrams and the multiplication $d_1 \circ d_2$ is illustrated in Figure 2.

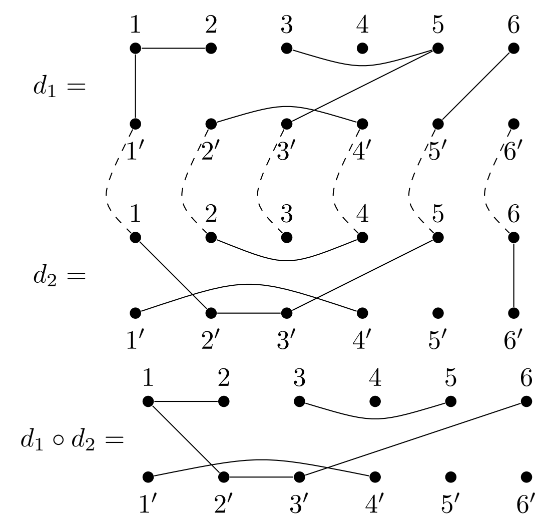

FIGURE 2. Example of multiplication in partition monoid.

For a complex number q, define

$$\mathbb{C}A_k(q) := \mathbb{C}\text{-}\operatorname{span}\{d \in A_k\}.$$

The multiplication of basis elements, which when extended linearly makes $\mathbb{C}A_k(q)$ an associative algebra, is defined as: for $d_1, d_2 \in A_k$, define

$$d_1 d_2 := q^l d_1 \circ d_2$$

where l is the number of blocks removed from the middle row while computing $d_1 \circ d_2$. Also, $\mathbb{C}A_{k+\frac{1}{2}}(q)$ is a subalgebra of $\mathbb{C}A_{k+1}(q)$. The algebras $\mathbb{C}A_k(q)$ and $\mathbb{C}A_{k+\frac{1}{2}}(q)$ are called *partition algebras*.

EXAMPLE 2.2. In Example 2.1, the product $d_1 d_2$ in $\mathbb{C}A_6(q)$ is illustrated in Figure 3 since one block has been removed from the middle row while computing $d_1 \circ d_2$ in Figure 2.

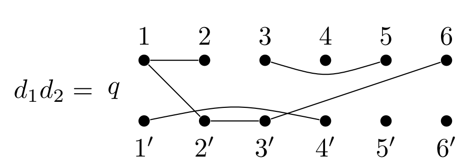

FIGURE 3. Example of multiplication in partition algebra.

2.2. The Okounkov–Vershik approach. Let G^n denote the direct product of n-copies of a finite group G and let $\mathbb{C}[G]$ be the group algebra of G. The action of the symmetric group S_n on G^n by permuting the coordinates defines the semidirect product of G^n by S_n. The group $G^n \rtimes S_n$ is also known as wreath product of G by S_n. The Okounkov–Vershik approach to the representation theory of $G^n \rtimes S_n$, where G is any finite group, was done in [16]. In this paper, we are interested in the particular case $G = \mathbb{Z}/r\mathbb{Z}$, the cyclic group of order r. We use the notation $G(r,1,n) := (\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$ where $\mathbb{Z}/r\mathbb{Z} = \langle \zeta \rangle$ with ζ being a primitive r-th root of unity. Thus,

$$G(r,1,n) = \{(g_1, g_2, \dots, g_n, \pi) \mid g_i \in \mathbb{Z}/r\mathbb{Z} \text{ for } i = 1, \dots, n \text{ and } \pi \in S_n\}.$$

In this section, we follow [16] and present here a brief summary of Okounkov–Vershik approach to the representation theory of $G(r,1,n)$.

For $1 \leqslant i \leqslant n$, let $H_{i,n} := \{(g_1, \dots, g_n, \pi) \in G(r,1,n) \mid \pi(j) = j \text{ for } i+1 \leqslant j \leqslant n\}$. We have the following chain of subgroups of $G(r,1,n)$

$$H_{1,n} \subseteq H_{2,n} \subseteq \cdots \subseteq H_{n,n} := G(r,1,n). \tag{1}$$

The irreducible representations of $H_{1,n} \cong G^n = (\mathbb{Z}/r\mathbb{Z})^n$ are one-dimensional.

The following result of Wigner [29, p. 677] gives a necessary and sufficient condition for the restriction to be multiplicity free. For a proof using double centralizer theorem, see [16, Theorem 4.1]. We use this result to prove the multiplicity freeness of chain (1).

Theorem 2.3. *Let A be a complex finite dimensional semisimple algebra and let B be a semisimple subalgebra. Then the relative commutant of the pair A and B, denoted by $Z(A,B)$, consisting of all the elements of A that commute with the elements of B is semisimple and the following conditions are equivalent:*

(a) *The restriction of any finite dimensional complex irreducible representation of A to B is multiplicity free.*
(b) *The relative commutant $Z(A,B)$ is commutative.*

Using [16, Theorem 4.13], we can conclude that the relative commutant of the pair of group algebras $\mathbb{C}[H_{m,n}]$ and $\mathbb{C}[H_{m-1,n}]$ is commutative for all $2 \leqslant m \leqslant n$. Thus, Theorem 2.3 implies that the chain (1) is multiplicity free.

For each $i = 1, 2, \dots, n$, suppose that $H_{i,n}^{\wedge}$ denotes the indexing set of irreducible $H_{i,n}$-modules and given $\lambda \in H_{i,n}^{\wedge}$, assume that V^{λ} denotes the corresponding $H_{i,n}$-module. Bratteli diagram of the chain (1) is a simple graph in which the vertices at i-th level are elements of $H_{i,n}^{\wedge}$ and a vertex $\mu \in H_{i-1,n}^{\wedge}$ is joined by an edge with a vertex $\lambda \in H_{i,n}^{\wedge}$ if V^{μ} appears in the restriction of V^{λ} to $H_{i-1,n}$.

For a fixed $1 \leqslant m \leqslant n$, consider the $H_{m,n}$-module V^{λ_m}, where $\lambda_m \in H_{m,n}^{\wedge}$. The branching rule being multiplicity free implies that the decomposition of V^{λ_m} into irreducible $H_{m-1,n}$-modules is canonical, and the decomposition is

$$V^{\lambda_m} = \bigoplus_{\lambda_{m-1}} V^{\lambda_{m-1}},$$

where the sum is over all $\lambda_{m-1} \in H_{m-1,n}^{\wedge}$ with an edge from λ_{m-1} to λ_m such that $V^{\lambda_{m-1}}$ is identified with the corresponding submodule of V^{λ_m}. We iterate the above decomposition for the chain of subgroups of $H_{m,n}$ in chain (1). This implies that the following decomposition of V^{λ_m} into irreducible $H_{1,n}^{\wedge}$-submodules is canonical:

$$V^{\lambda_m} = \bigoplus_{T} V_T, \tag{2}$$

where the sum is over all possible paths $T = (\lambda_1, \lambda_2, \dots, \lambda_m)$ from a vertex in $H_{1,n}$ to λ_m in Bratteli diagram with $\lambda_i \in H_{i,n}^{\wedge}$ for $1 \leqslant i \leqslant m$.

The decomposition given in (2) is called the *Gelfand–Tsetlin decomposition* (*GZ-decomposition*) of V^{λ_m} and each V_T in (2) is called a *Gelfand–Tsetlin subspace* (*GZ-subspace*) of V^{λ_m}. In our case, each GZ-subspace V_T is one-dimensional. Choose a non-zero vector $v_T \in V_T$. The basis

$$\{v_T \mid T \text{ is a path in the GZ-decomposition of } V^{\lambda_m}\}$$

of V^{λ_m} is called the *Gelfand–Tsetlin basis* (*GZ-basis*) of V^{λ_m} and it is unique up to scalars and

$$\mathbb{C}[H_{i,n}] \cdot v_T = V^{\lambda_i}, \quad i = 1, 2, \ldots, m.$$

The GZ-basis being canonical is significant in the case of $G(r, 1, n)$, since, in general, for the wreath product of a finite group by symmetric group, GZ-subspaces are canonical, GZ-basis may not necessarily be canonical.

Using the algebra isomorphism

$$\mathbb{C}[H_{m,n}] \cong \bigoplus_{\lambda_m \in H^{\wedge}_{m,n}} \mathrm{End}(V^{\lambda_m}), \tag{3}$$

given by

$$g \mapsto (V^{\lambda_m} \xrightarrow{g} V^{\lambda_m} \ : \ \lambda_m \in H^{\wedge}_{m,n}), \ \ g \in H_{m,n}, 1 \leqslant m \leqslant n,$$

we can define the *Gelfand–Tsetlin algebra* (*GZ-algebra*), denoted by $GZ_{m,n}$, a maximal commutative subalgebra of $\mathbb{C}[H_{m,n}]$ based on the GZ-decomposition (2):

$$GZ_{m,n} = \{a \in \mathbb{C}[H_{m,n}] \mid a \text{ acts diagonally in the GZ-basis of } V^{\lambda_m}, \text{ for all } \lambda_m \in H^{\wedge}_{m,n}\}.$$

Theorem 2.4 ([16, Theorem 3.1(i)]). *Let $Z_{i,n}$ denote the center of $\mathbb{C}[H_{i,n}]$ for each $i = 1, 2, \ldots, m$. Then,*

$$GZ_{m,n} = \langle Z_{1,n}, Z_{2,n}, \ldots, Z_{m,n}\rangle.$$

The theorem above implies the following result which is [16, Lemma 3.2].

Lemma 2.5.

(a) *Let $v \in V^{\lambda_m}$ for some $\lambda_m \in H^{\wedge}_{m,n}$ such that v is an eigenvector of every element of $GZ_{m,n}$, then (a scalar multiple of) v belongs to the GZ-basis of V^{λ_m}.*

(b) *Let v and u be elements in V^{λ_m} and V^{μ_m}, respectively for some $\lambda_m, \mu_m \in H^{\wedge}_{m,n}$ such that v and u have the same eigenvalues for every element of $GZ_{m,n}$, then $v = u$ and $\lambda_m = \mu_m$.*

A GZ-subspace of $H_{m,n}$ is a GZ-subspace of some irreducible $H_{m,n}$-module V^{λ_m}, $\lambda_m \in H^{\wedge}_{m,n}$. Thus, using Lemma 2.5, a GZ-subspace of $H_{m,n}$ is GZ-subspace of a unique irreducible representation of $H_{m,n}$.

The Jucys–Murphy elements for the wreath product of a finite group by a symmetric group were given in [20]. For our particular case $G(r, 1, n)$, the Jucys–Murphy elements can be written as:

$$X_1 = 0,$$

and

$$X_j = \sum_{i=1}^{j-1} \sum_{l=0}^{r-1} \zeta_i^l \zeta_j^{-l} s_{ij}, \ \ 2 \leqslant j \leqslant n, \tag{4}$$

where s_{ij} is the transposition (i, j) and

$$\zeta_i^l \zeta_j^{-l} s_{ij} = (1, \ldots, 1, \zeta^l, 1 \ldots, 1, \zeta^{-l}, 1, \ldots, 1, (i, j)) \in G(r, 1, n),$$

with ζ^l and ζ^{-l} as i-th and j-th coordinates, respectively. It is clear that the element X_j can be identified naturally with an element of $H_{j,n}$.

Theorem 2.6 ([16, Theorem 4.4]). *We have*

$$GZ_{m,n} = \langle Z[\mathbb{C}[G^n]], X_1, X_2, \dots, X_m \rangle.$$

A GZ-subspace V of $H_{m,n}$ is isomorphic to $\rho_1 \otimes \cdots \otimes \rho_n,\ \rho_i \in G^\wedge$ for all i. We call $\rho = (\rho_1, \dots, \rho_n)$ the *label* of V. And define the *weight* $\alpha(V)$ of V by

$$\alpha(V) = (\rho, \alpha_1, \dots, \alpha_m), \tag{5}$$

where α_i = eigenvalue of X_i on V. Using Lemma 2.5 and Theorem 2.6, it can be easily shown that a GZ-subspace is uniquely determined by its weight.

Let $\mathcal{Y}$ be the set of all Young diagrams. The unique Young diagram with zero boxes is empty Young diagram, denoted by $\varnothing$. For $\lambda \in \mathcal{Y}$, let $|\lambda|$ denote the number of boxes in λ. Define

$$\mathcal{Y}(r,n) := \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \mid \lambda_i \in \mathcal{Y} \text{ for all } i = 1, 2, \dots, r \text{ and } \sum_{i=1}^{r} |\lambda_i| = n\},$$

i.e. $\mathcal{Y}(r,n)$ is the set of r-tuples of Young diagrams such that the total number of boxes is n.

Let $\mu \in \mathcal{Y}$. A standard Young tableau of shape μ is obtained by filling the boxes in the Young diagram μ with the distinct numbers $1, 2, \dots, |\mu|$ such that the numbers in the boxes strictly increase along each row and each column of μ. For $\lambda \in \mathcal{Y}(r,n)$, a standard r-tuple of Young tableaux of shape λ is obtained by filling the n-boxes in the r-tuple λ with the distinct numbers $1, 2, \dots, n$ such that the numbers in the boxes strictly increase along each row and each column of all Young diagrams occurring in λ. Define $\mathrm{Tab}(r, \lambda)$ as the set of all standard r-tuple of Young tableaux and set $\mathrm{Tab}(r,n) := \cup_{\lambda \in \mathcal{Y}(r,n)} \mathrm{Tab}(r, \lambda)$.

For each $i = 1, 2, \dots, r$, define the irreducible representation σ_i of G:

$$\begin{aligned} \sigma_i :\ G &\to \mathbb{C}^* \\ \zeta &\mapsto \zeta^{i-1}. \end{aligned}$$

The irreducible representations of G are $\sigma_1, \sigma_2, \dots, \sigma_r$.

We draw Young diagrams by following the convention of writing down matrices with x-axis running downwards and y-axis running to the right. For a box b of a Young diagram, its content $c(b)$ is its y-coordinate $-$ its x-coordinate. Given $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{Y}(r,n)$, $T \in \mathrm{Tab}(r, \lambda)$ and $1 \leqslant i \leqslant n$, the number i resides in exactly one box of one of $\lambda_1, \dots, \lambda_r$, say λ_{j_i}, let $b_T(i)$ be this box in λ_{j_i} and let $r_T(i) := \sigma_{j_i}$.

The following result for $G(r,1,n)$ can be easily seen by [16, Theorem 6.5].

Theorem 2.7. *Let $\lambda \in \mathcal{Y}(r,n)$. Then the GZ-subspaces of V^λ can be parametrized by $T \in \mathrm{Tab}(r, \lambda)$ and the GZ-decomposition of V^λ can be written as*

$$V^\lambda = \bigoplus_{T \in \mathrm{Tab}(r,\lambda)} V_T, \tag{6}$$

where each V_T is closed under the action of G^n and, as a G^n-module, is isomorphic to the irreducible G^n-module

$$r_T(1) \otimes r_T(2) \otimes \cdots \otimes r_T(n)$$

For $i = 1, \dots, n$, the eigenvalue of X_i on V_T is given by $rc(b_T(i))$.

Let R denote the element of $\mathrm{Tab}(r, \lambda)$ defined as follows: for $\lambda = (\lambda_1, \dots, \lambda_r)$, we start with λ_1 by filling the Young diagram λ_1 with the numbers $1, \dots, |\lambda_1|$ in *row major order*, i.e. the first row is filled with $1, 2, \dots, l_1$ in increasing order where l_1 is the length of the first row, the second row is filled with $l_1+1,\dots,l_1+l_2$ in increasing order where l_2 is the length of the second row and so on till the last row of λ_1 has been filled. Then we fill the Young diagram λ_2 with $|\lambda_1|+1,\dots,|\lambda_1|+|\lambda_2|$ in row major order and so on till the last Young diagram λ_r.

The irreducible representations of $G(r,1,n)$ are parametrized by the elements of $\mathcal{Y}(r,n)$ and given $\lambda\in\mathcal{Y}(r,n)$, the GZ-basis elements (and hence, GZ-subspaces) of V^{λ} are parametrized by $T\in\mathrm{Tab}(r,\lambda)$.

2.3. Clifford Theory. We give an outline of Clifford theory for a finite group H and its normal subgroup N such that H/N is a cyclic group of order p as done in [1, 17, 26]. The pair H and N on which they have applied Clifford theory is the pair $G(r,1,n)$ and $G(r,p,n)$. The group $G(r,p,n)$ can be considered as the subgroup of $GL_n(\mathbb{C})$ consisting of generalized permutation matrices such that the m-th power of the product of nonzero entries is one. We discuss the complex reflection group $G(r,p,n)$ and its representation theory in detail in Section 4 and review Clifford theory for the rest of this section.

Let $H^{\wedge}$ denote the indexing set of irreducible representations of H. Identifying H/N with the group C consisting of one-dimensional representations of H which contain N in their kernel, we can define an action of C on the set of irreducible representations of H by

$$V^{\rho}\mapsto\delta\otimes V^{\rho}$$

where $\delta\in C$ and V^{ρ} is the irreducible representation of H indexed by $\rho\in H^{\wedge}$. Denote the orbit of V^{ρ} by $[\rho]$ with respect to the action of C. The irreducible representations of H which are in the same orbit are called associates of each other. Assume that V^{ρ} has $b(\rho)$ associates. Then the stabilizer subgroup of C with respect to V^{ρ}, denoted by C_{ρ}, has the order $u(\rho)=\frac{p}{b(\rho)}$. Suppose that δ_0 is a generator of C_{ρ}. It is easy to see that there exists a N-linear map $A:V^{\rho}\longrightarrow V^{\rho}$ such that $A(hv)=\delta_0(h)hA(v)$ for all $h\in H$ and $v\in V^{\rho}$. Then by Schur's lemma, the H-linear map $A^{u(\rho)}$ acts by a nonzero scalar. Normalizing the scalar, we obtain that $A^{u(\rho)}$ is the identity map on V^{ρ}. Such an A is called the *associator* of V^{ρ}. Also, if $\mu\in[\rho]$, then the stabilizer subgroup $C_{\mu}=C_{\rho}$. The following theorem gives parametrization of irreducible N-modules.

Theorem 2.8.

(a) *The eigenspace decomposition of V^{ρ} with respect to A is given by*

$$V^{\rho}\cong\bigoplus_{l=0}^{u(\rho)-1}E^{(l)}, \tag{7}$$

where $E^{(l)}$ is the eigenspace with eigenvalue $e^{\frac{2\pi il}{u(\rho)}}$. The group C_{ρ} can be identified with $\{e^{\frac{2\pi il}{u(\rho)}}\mid l=0,1,\dots,u(\rho)-1\}$.

(b) *The eigenspaces $E^{(l)}$, for $0\leqslant l\leqslant u(\rho)-1$, occuring in (7) are inequivalent irreducible N-modules, each of which is of dimension $\dim(V^{\rho})/u(\rho)$.*

(c) *For any $0\leqslant l\leqslant u(\rho)-1$, we have*

$$\mathrm{Ind}_N^H(E^{(l)})=\bigoplus_{\mu\in[\rho]}V^{\mu}.$$

(d) *Let $\mathcal{O}$ denote the set of all orbits in $H^{\wedge}$. The irreducible N-modules are parametrized by the pairs $([\rho],\epsilon)$ where $[\rho]\in\mathcal{O}$ and $\epsilon\in C_{\rho}$.*

3. Tanabe algebra

The partition monoid is a poset with the partial order given as: for $d, d' \in A_k$, $d' \leqslant d$ if d' is coarser than d, i.e. if i and j are in the same block of d, then i and j are in the same block of d'. For $d \in A_k$, define the unique element $x_d \in \mathbb{C}A_k(n)$ satisfying

$$d = \sum_{d' \leqslant d} x_{d'}. \tag{8}$$

This partial order on A_k can be extended to a total order on A_k. It can be easily seen that the transition matrix between $\{d \mid d \in A_k\}$ and $\{x_d \mid d \in A_k\}$ is an upper triangular matrix with $1'$s on the diagonal and thus, $\{x_d \mid d \in A_k\}$ is also a basis of the partition algebra $\mathbb{C}A_k(n)$, see also [8, p. 879].

An *internal block* in $d_1 \circ d_2$, for $d_1, d_2 \in A_k$, is a block that is entirely in the middle while computing $d_1 \circ d_2$. We say that *the bottom row of d_1 matches with the top row of d_2* if the following condition is satisfied: i' and j' are in the same block in d_1 if and only if i and j are in the same block in d_2 for $1 \leqslant i, j \leqslant k$. For every s in a block B of $d \in A_k$, if we put $i_s = t$ for some $1 \leqslant t \leqslant n$, then t is said to be a *mark of the block B*. The next lemma and the idea of its proof are from the online notes [23] and it also appears in [2, Section 4]. It gives the structure constants with respect to the basis $\{x_d \mid d \in A_k\}$ of $\mathbb{C}A_k(n)$.

Lemma 3.1. *For $d_1, d_2 \in A_k$, the multiplication of x_{d_1} and x_{d_2} in $\mathbb{C}A_k(n)$ is given by*

$$x_{d_1} x_{d_2} = \begin{cases} \sum\limits_{d \in A_k} c_d x_d, & \text{if the bottom row of } d_1 \text{ matches with the top row of } d_2, \\ 0, & \text{otherwise,} \end{cases}$$

where the sum is taken over all those d in A_k such that d is coarser than $d_1 \circ d_2$ and the coarsening is done by connecting a block of d_1 which is contained entirely in the top row of d_1 with a block of d_2 which is contained entirely in the bottom row of d_2 and

$$c_d = (n - |d|)_{[d_1 \circ d_2]},$$

where $|d|$ is the number of blocks in d, $[d_1 \circ d_2]$ is the number of internal blocks in $d_1 \circ d_2$, and for $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{\geqslant 0}$,

$$(a)_b := \begin{cases} a(a-1)\cdots(a-b+1), & \text{if } b > 0, \\ 1, & \text{if } b = 0, \end{cases}$$

such that when $a, b \in \mathbb{Z}_{\geqslant 0}$ and $a \geqslant b$, we have $(a)_b = {}^aP_b$, the number of permutations of a objects taken b at a time.

Proof. Let $n \geqslant 2k$. Then $\phi_k : \mathbb{C}A_k(n) \cong \mathrm{End}_{S_n}(V^{\otimes k})$ (by Schur–Weyl duality for partition algebras, Theorem 5.1). Identifying x_d with $\phi_k(x_d)$, we have

$$\begin{aligned} &(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})(x_{d_1} x_{d_2}) \\ &= \sum_{i_{1'}, i_{2'}, \dots, i_{k'}, i_{1''}, i_{2''}, \dots, i_{k''}} (v_{i_{1''}} \otimes v_{i_{2''}} \otimes \cdots \otimes v_{i_{k''}})(x_{d_1})^{i_1, i_2, \dots, i_k}_{i_{1'}, i_{2'}, \dots, i_{k'}} (x_{d_2})^{i_{1'}, i_{2'}, \dots, i_{k'}}_{i_{1''}, i_{2''}, \dots, i_{k''}}. \end{aligned}$$

If the bottom row of d_1 does not match with the top row of d_2, then using (20) it can be seen that $x_{d_1} x_{d_2} = 0$.

If the bottom row of d_1 matches with the top row of d_2, then again using (20) we have

$$\sum_{i_{1'},i_{2'},\ldots,i_{k'},i_{1''},i_{2''},\ldots,i_{k''}} (v_{i_{1''}} \otimes v_{i_{2''}} \otimes \cdots \otimes v_{i_{k''}})(x_{d_1})^{i_1,i_2,\ldots,i_k}_{i_{1'},i_{2'},\ldots,i_{k'}} (x_{d_2})^{i_{1'},i_{2'},\ldots,i_{k'}}_{i_{1''},i_{2''},\ldots,i_{k''}} = \sum_d \alpha_d x_d,$$

where α_d is some positive integer and the sum is over all d obtained by coarsening $d_1 \circ d_2$ which is done by connecting a block of d_1 contained entirely in the top row of d_1 and a block of d_2 contained entirely in the bottom row of d_2. So, $\alpha_d =$ number of ways the internal blocks of $d_1 \circ d_2$ can be marked distinctly after putting distinct marks on the blocks of $d = (n - |d|)_{[d_1 \circ d_2]} = c_d$.

Fix k and vary n. For a given n, fix $d_1, d_2, d \in A_k(n)$. Then the coefficient of x_d in the product $x_{d_1} x_{d_2}$ is a polynomial $f_d(n)$ in n. Then by above arguments, for $n \geqslant 2k$, we have $f_d(n) = (n - |d|)_{[d_1 \circ d_2]}$. The fundamental theorem of algebra implies that $f_d(n) = (n - |d|)_{[d_1 \circ d_2]}$ for all n. □

Let B be a block of $d \in A_k$. Suppose that $N(B)$ is the number of elements in $B \cap \{1, 2, \ldots, k\}$ and $M(B)$ is the number of elements in $B \cap \{1', 2', \ldots, k'\}$. Thus, $N(B)$ and $M(B)$ are the number of elements in the block B in top row and bottom row of d, respectively.

Define the following mutually disjoint subsets of A_k:

$$\begin{aligned}
\Pi_k(r) &:= \{d = \{B_1, B_2, \ldots, B_s\} \in A_k \mid s \geqslant 1 \text{ and} \\
&\qquad N(B_i) \equiv M(B_i) \pmod r (1 \leqslant i \leqslant s)\}, \\
\Lambda_k(r,p,n) &:= \{d = \{B_1, B_2, \ldots, B_n\} \in A_k \mid N(B_i) \equiv M(B_i) \pmod m, \\
&\qquad N(B_i) \not\equiv M(B_i) \pmod r, (1 \leqslant i \leqslant n), \text{ and} \\
&\qquad N(B_i) - M(B_i) \equiv N(B_j) - M(B_j) \pmod r, (1 \leqslant i, j \leqslant n)\}, \\
\Theta_k(r,p,n) &:= \{d = \{B_1, B_2, \ldots, B_y\} \in A_k \mid y > n, \\
&\qquad N(B_i) \equiv M(B_i) \pmod m, (1 \leqslant i \leqslant y), \\
&\qquad \text{and for some } j \in \{1, \ldots, y\}, N(B_j) \not\equiv M(B_j) \pmod r\}.
\end{aligned}$$

See Example 3.10 for the above defined sets in the case $k = 2$. Define $A_k(r,p,n)$, a subset of A_k, by setting

$$A_k(r,p,n) := \Pi_k(r) \cup \Lambda_k(r,p,n) \cup \Theta_k(r,p,n).$$

Definition 3.2. *Define $\mathcal{T}_k(r,p,n) := \mathbb{C}$-span$\{x_d \mid d \in A_k(r,p,n)\}$, a subspace of the partition algebra $\mathbb{C}A_k(n)$.*

Definition 3.3. *Define $\mathbb{C}A_k(r,p,n) := \mathbb{C}$-span$\{d \mid d \in A_k(r,p,n)\}$, a subspace of the partition algebra $\mathbb{C}A_k(n)$.*

The subspace $\mathbb{C}A_k(r,1,n)$ was defined by Orellana [19, p. 610] (she used the notation $\mathcal{T}_k(n,r)$). Also, she proved Schur–Weyl duality between $\mathbb{C}A_k(r,1,n)$ and $G(r,1,n)$ [19, Theorem 5.4]. In the next lemma, we show that the subspaces $\mathcal{T}_k(r,p,n)$ and $\mathbb{C}A_k(r,p,n)$ are equal when $p = 1$. For $p \neq 1$, the subspace $\mathbb{C}A_k(r,p,n)$ may not be a subalgebra of $\mathbb{C}A_k(n)$, and $\mathbb{C}A_k(r,p,n)$ may neither contain nor be contained in $\mathcal{T}_k(r,p,n)$. See Remark 3.11.

Lemma 3.4. *The subspaces $\mathcal{T}_k(r,1,n)$ and $\mathbb{C}A_k(r,1,n)$ are equal. Also, $\mathcal{T}_k(r,1,n)$ is a subalgebra of $\mathbb{C}A_k(n)$.*

Proof. The sets $\Lambda_k(r,1,n)$ and $\Theta_k(r,1,n)$ are empty. So,

$$\mathcal{T}_k(r,1,n) = \mathbb{C}\text{-}\operatorname{span}\{x_d \mid d \in \Pi_k(r)\} \text{ and } \mathbb{C}A_k(r,1,n) = \mathbb{C}\text{-}\operatorname{span}\{d \mid d \in \Pi_k(r)\}.$$

Also, for $d \in \Pi_k(r)$, the elements $d' \leqslant d$ belong to $\Pi_k(r)$ because the difference between the number of elements in top row and bottom row in each block remains $0 \pmod r$ even after coarsening. Using the definition of x_d in (8), we get that $d \in \mathcal{T}_k(r,1,n)$ for all $d \in \Pi_k(r)$. Thus, $\mathbb{C}A_k(r,1,n) \subseteq \mathcal{T}_k(r,1,n)$ and we get the equality because of their dimensions being equal.

The subset $\Pi_k(r)$ is a submonoid of A_k. Therefore, $\mathbb{C}A_k(r,1,n)$ is a subalgebra of $\mathbb{C}A_k(n)$. ☐

The explicit formula for the dimension of $\mathcal{T}_k(r,p,n)$, or even in the case of $\mathcal{T}_k(r,1,n)$, is not known. However, when $m \to \infty$ (recall $m = \frac{r}{p}$), then $\Lambda_k(r,p,n)$ and $\Theta_k(r,p,n)$ are empty sets and thus,

$$\lim_{m\to\infty} \dim(\mathcal{T}_k(r,p,n)) = \lim_{r\to\infty} \dim(\mathcal{T}_k(r,1,n)).$$

Also, $\lim\limits_{r\to\infty} \dim(\mathcal{T}_k(r,1,n))$ is described in [19, p. 611].

Let $V = \mathbb{C}^n$ be the n-dimensional vector space on which $GL_n(\mathbb{C})$ acts naturally. The action of $G(r,p,n)$ on V is given by the restriction of the action of $GL_n(\mathbb{C})$ on V. Also, $G(r,p,n)$ acts on the k-fold tensor product $V^{\otimes k}$ by the diagonal action.

Remark 3.5. The proof of Theorem 3.6 uses the following two known results (both of which are independent of the algebra structure of $\mathcal{T}_k(r,p,n)$):

(a) Schur–Weyl duality between the partition algebra $\mathbb{C}A_k(n)$ and the symmetric group S_n as stated in Theorem 5.1(a) (also see [8, Theorem 3.6] and [9]); and
(b) The basis of the centralizer algebra of the action of $G(r,p,n)$ on $V^{\otimes k}$ as given in Lemma 5.2(a) (also see [27, Lemma 2.1]).

Theorem 3.6. *The vector space $\mathcal{T}_k(r,p,n)$ is a subalgebra of $\mathbb{C}A_k(n)$.*

Proof. Let $d_1, d_2 \in A_k(r,p,n)$. It is sufficient to assume that the bottom row of d_1 matches with the top row of d_2. The multiplication $x_{d_1}x_{d_2}$ is given by

$$x_{d_1}x_{d_2} = \sum_{d\in A_k} c_d x_d \tag{9}$$

Case 1. If $d_1, d_2 \in \Pi_k(r)$, then by Lemma 3.4, we have $x_{d_1}x_{d_2} \in \mathcal{T}_k(r,1,n) \subseteq \mathcal{T}_k(r,p,n)$.

Case 2. One of d_1 or d_2 is in $\Theta_k(r,p,n)$. Without loss of generality, assume that $d_1 \in \Theta_k(r,p,n)$ and $d_2 \in A_k(r,p,n)$.

Claim. $c_d = 0$ for $d \notin A_k(r,p,n)$ in (9). Since d_1 has more than n blocks, therefore using Theorem 5.1(a) (see Remark 3.5), we get, in (9)

$$\sum_{\substack{d\in A_k,\\ |d|\leqslant n}} c_d \phi_k(x_d) = 0.$$

The linear independence of $\{\phi_k(x_d) \mid |d| \leqslant n\}$ implies that $c_d = 0$ for $d \in A_k, |d| \leqslant n$. Thus, c_d can be nonzero only when $|d| > n$. For such d, we show that either $d \in \Pi_k(r)$ or $d \in \Theta_k(r,p,n)$. Suppose $d \notin \Pi_k(r)$, then there exists $1 \leqslant j \leqslant |d|$ such that $N(B_j) \not\equiv M(B_j) \pmod r$.

Subcase 2.1. Suppose $d = d_1 \circ d_2$. If a block B in d_1 is connected with a block B' in d_2 then $N(B) \equiv M(B) \pmod m$, $N(B') \equiv M(B') \pmod m$ and $M(B) = N(B')$. Thus, $N(B) \equiv M(B') \pmod m$ and $d \in \Theta_k(r,p,n)$. This also includes the cases when either of B and B' are entirely in the top or bottom row of d_1 and d_2, respectively.

SUBCASE 2.2. Suppose that d is obtained by coarsening of $d_1 \circ d_2$ as in Lemma 3.1. Let the coarsening be done by connecting a block B entirely in the top row of d_1 with a block B' entirely in the bottom row of d_2. Then

$$N(B) \equiv 0(\operatorname{mod} m) \text{ and } 0 \equiv M(B')(\operatorname{mod} m).$$

Thus, $N(B) \equiv M(B')(\operatorname{mod} m)$ and $d \in \Theta_k(r,p,n)$.

CASE 3. One of d_1 and d_2 is in $\Pi_k(r)$ and the other is in $\Lambda_k(r,p,n)$. Without loss of generality, assume that $d_1 \in \Pi_k(r)$ and $d_2 \in \Lambda_k(r,p,n)$. If $|d_1| > n$, then we can argue similar to the case (ii) above. So, assume that $|d_1| \leqslant n$. From (9), we have

$$0 \neq \sum_{d \in A_k} c_d \phi_k(x_d) \in \operatorname{End}_{G(r,p,n)}(V^{\otimes k}).$$

Using the basis of $\operatorname{End}_{G(r,p,n)}(V^{\otimes k})$ as given in Lemma 5.2(a) (see Remark 3.5), it follows that, for d such that $|d| \leqslant n$, c_d can be nonzero only when $d \in \Pi_k(r) \cup \Lambda_k(r,p,n)$.

If there exists d in (9) with more than n blocks such that $c_d \neq 0$, then by the arguments similar to the case (ii), we get either $d \in \Pi_k(r)$ or $d \in \Lambda_k(r,p,n)$. ☐

Define the following mutually disjoint subsets of $A_{k+\frac{1}{2}}$:

$$\Pi_{k+\frac{1}{2}}(r) := \Pi_{k+1}(r) \cap A_{k+\frac{1}{2}},$$
$$\Lambda_{k+\frac{1}{2}}(r,p,n) := \Lambda_{k+1}(r,p,n) \cap A_{k+\frac{1}{2}},$$
$$\Theta_{k+\frac{1}{2}}(r,p,n) := \Theta_{k+1}(r,p,n) \cap A_{k+\frac{1}{2}}.$$

Also, define $A_{k+\frac{1}{2}}(r,p,n)$, a subset of $A_{k+\frac{1}{2}}$, by setting

$$A_{k+\frac{1}{2}}(r,p,n) := \Pi_{k+\frac{1}{2}}(r) \cup \Lambda_{k+\frac{1}{2}}(r,p,n) \cup \Theta_{k+\frac{1}{2}}(r,p,n).$$

DEFINITION 3.7. *Define* $\mathcal{T}_{k+\frac{1}{2}}(r,p,n) := \mathbb{C}\text{-}\operatorname{span}\{x_d \mid d \in A_{k+\frac{1}{2}}(r,p,n)\}$*, a subspace of partition algebra* $\mathbb{C}A_{k+\frac{1}{2}}(n)$.

THEOREM 3.8. *The vector space* $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ *is a subalgebra of* $\mathbb{C}A_{k+\frac{1}{2}}(n)$.

Proof. Note that $\mathcal{T}_{k+\frac{1}{2}}(r,p,n) = \mathcal{T}_{k+1}(r,p,n) \cap \mathbb{C}A_{k+\frac{1}{2}}(n)$, hence $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ is an algebra. ☐

DEFINITION 3.9 (Tanabe algebra). *We call the algebras* $\mathcal{T}_k(r,p,n)$ *and* $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ *Tanabe algebras.*

There is an injective algebra homomorphism (see [8, p. 879])

$$\mathbb{C}A_k(n) \hookrightarrow \mathbb{C}A_{k+\frac{1}{2}}(n)$$
$$d \mapsto d',$$

where $d \in A_k$ and $d' \in A_{k+\frac{1}{2}}$ has same blocks as d with an additional block $\{(k+1),(k+1)'\}$. It is easy to see that the corresponding element x_d is mapped to $(x_{d'} + \sum x_{d''})$, where the sum is over all $d'' \in A_{k+\frac{1}{2}} \smallsetminus \{d'\}$ obtained by connecting a block in d' with the block $\{(k+1),(k+1)'\}$. Using the description of the above map in terms of the elements x_d, we see that the algebra $\mathcal{T}_k(r,p,n)$ can be embedded inside the algebra $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$.

EXAMPLE 3.10. In this example, we describe the elements of the sets $\Pi_k(r), \Lambda_k(r,p,n)$ and $\Theta_k(r,p,n)$ for various specific values of r,p and n when $k=2$. The partition monoid $A_2 = \{d_1, d_2, \ldots, d_{15}\}$ with the elements in terms of partition diagrams is given in Figure 4.

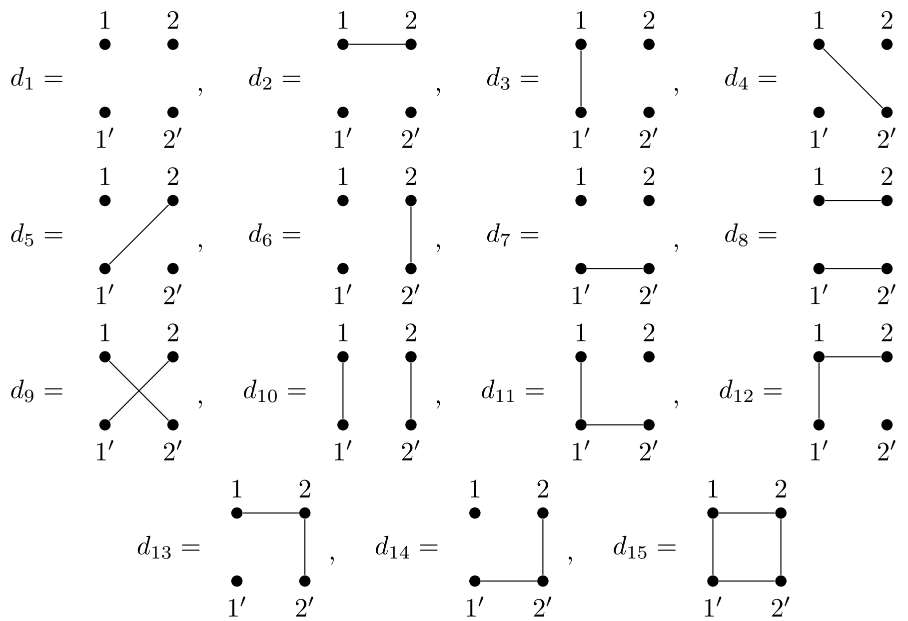

Figure 4. Elements of A_2.

(a) For $r=2$, we have $\Pi_2(2)=\{d_8,d_9,d_{10},d_{15}\}$.
   (i) For $p=2, n=2$, the sets
$$\Lambda_2(2,2,2)=\{d_{11},d_{12},d_{13},d_{14}\}$$
$$\text{and } \Theta_2(2,2,2)=\{d_1,d_2,d_3,d_4,d_5,d_6,d_7\}.$$
   Thus, $\mathcal{T}_2(2,2,2)$ is the partition algebra $\mathbb{C}A_2(2)$.
   (ii) For $p=2, n=3$, $\Lambda_2(2,2,3)$ is an empty set and $\Theta_2(2,2,3)=\{d_1\}$.
   (iii) For $p=2, n=4$, we have $\Lambda_2(2,2,4)=\{d_1\}$ and $\Theta_2(2,2,4)$ is an empty set.

(b) For $r\neq 1,2$, we have $\Pi_2(r)=\{d_9,d_{10},d_{15}\}$. For $r=3$, $\Lambda_2(r,p,n)$ is nonempty if and only if $(r,p,n)=(3,3,3)$; and $\Lambda_2(3,3,3)=\{d_2,d_7\}$. For $r=4$, $\Lambda_2(r,p,n)$ is nonempty if and only if $(r,p,n)=(4,2,2)$ or $(4,4,2)$; and $\Lambda_2(4,2,2)=\Lambda_2(4,4,2)=\{d_8\}$. For $r>4$, $\Lambda_2(r,p,n)$ is empty for all values of p and n. In general, for $r>2k$, $\Lambda_k(r,p,n)$ is empty for all values of p and n.

Remark 3.11. In Example 3.10(a), we see that $A_2(2,2,3)=\{d_1,d_8,d_9,d_{10},d_{15}\}$. The product $d_8d_1=d_2$ is not an element of $A_2(2,2,3)$ which implies that $\mathbb{C}A_2(2,2,3)$ is not an algebra. Using the definition of x_d in (8), we have (a) $d_1=\sum_{i=1}^{15} x_{d_i}$ and (b) the coefficient of d_2 in the expression of x_{d_1} as a linear combination of partition diagrams is -1. So, we can conclude that $\mathbb{C}A_2(2,2,3)\nsubseteq \mathcal{T}_2(2,2,3)$ and $\mathcal{T}_2(2,2,3)\nsubseteq \mathbb{C}A_2(2,2,3)$.

Remark 3.12. For $n\geqslant 2k$, $\Theta_k(r,p,n)$ is an empty set. Moreover, for $n\geqslant 2k$, the set $\Lambda_k(r,p,n)$ is nonempty if and only if $(r,p,n)=(2,2,2k)$; $\Lambda_k(2,2,2k)=\{d\}$, where d is a partition diagram with $2k$ blocks, i.e. each block consists of a single vertex. Using the multiplication rule in Lemma 3.1, it is easy to check that the corresponding x_d is a central element of Tanabe algebra $\mathcal{T}_k(2,2,2k)$. This remark, along with Remark 6.7, is useful in Section 7 for the special case $(r,p,n)=(2,2,2k)$.

4. Complex reflection groups

For an n-dimensional complex vector space W, a linear isomorphism of W of finite order is said to be a *reflection* in W if it has exactly $(n-1)$ eigenvalues equal to 1. A *complex reflection group* R in W is a group generated by reflections in W. The space W is called the *reflection representation* of R. We say R is irreducible if the R-invariant complement of the subspace W^R, which is fixed pointwise by R, in W is irreducible (see [4, Definitions (1.1)]). If there exists a direct sum $W = W_1 \oplus W_2 \oplus \cdots \oplus W_t$, where W_i is non-trivial proper subspace of W for each $1 \leqslant i \leqslant n$, such that $W_1, W_2, \ldots, W_t$ are permuted among themselves under the action of R, then we say that R is *imprimitive*. By Shephard–Todd classification, the groups $G(r,p,n)$, for $n > 1$, (except the group $G(2,2,2)$) are the only finite irreducible imprimitive complex reflection groups [25, Section 2]. The group $G(2,2,2)$ is imprimitive, but it is not irreducible [4, Theorem (2.4)].

Suppose that $G := \mathbb{Z}/r\mathbb{Z}$ is the cyclic group of order r with ζ, a primitive r-th root of unity. Define $\mathrm{D}(r,p,n)$ to be the subgroup of $GL_n(\mathbb{C})$ consisting of diagonal matrices as:

$$\mathrm{D}(r,p,n) := \left\{ \begin{bmatrix} \zeta^{i_1} & 0 & \ldots & 0 \\ 0 & \zeta^{i_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \zeta^{i_n} \end{bmatrix} \;\middle|\; i_1 + i_2 + \cdots + i_n \equiv 0 (\operatorname{mod} p) \right\}.$$

Let S_n be the group of $n \times n$ permutation matrices. Define $G(r,p,n)$ to be the subgroup of $GL(n,\mathbb{C})$ generated by $\mathrm{D}(r,p,n)$ and S_n. Since S_n normalizes $\mathrm{D}(r,p,n)$ and $\mathrm{D}(r,p,n) \cap S_n = \{I_n\}$, where I_n is the identity matrix, the group $G(r,p,n)$ is a semidirect product:

$$G(r,p,n) = \mathrm{D}(r,p,n) \rtimes S_n.$$

Thus, as a subgroup of $GL_n(\mathbb{C})$, the group $G(r,p,n)$ consists of generalized permutation matrices with nonzero entries being r-th roots of unity and the m-th power of the product of nonzero entries is one. Also, the elements of $G(r,p,n)$ can be written as $(n+1)$-tuple:

$$G(r,p,n) = \{(\zeta^{i_1}, \zeta^{i_2}, \ldots, \zeta^{i_n}, \pi) \mid i_1 + i_2 + \cdots + i_n \equiv 0 (\operatorname{mod} p), \pi \in S_n\}.$$

The particular case when $p = 1$ is the group $G(r,1,n)$, the wreath product of the group G by the symmetric group S_n, of order $r^n n!$. Taking the exact sequence

$$\begin{gathered} 1 \longrightarrow G(r,p,n) \longrightarrow G(r,1,n) \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 1 \\ (\zeta^{i_1}, \zeta^{i_2}, \ldots, \zeta^{i_n}, \pi) \mapsto \zeta^{i_1 + i_2 + \cdots + i_n}, \end{gathered}$$

we see that $G(r,p,n)$ is a normal subgroup of the group $G(r,1,n)$ of index p. So, the order of the group $G(r,p,n)$ is $(r^n n!)/p$.

Some families of groups which are special cases of $G(r,p,n)$ are:

(a) cyclic group of order r, i.e. $\mathbb{Z}/r\mathbb{Z} = G(r,1,1)$,
(b) dihedral group of order $2r$, $D_{2r} = G(r,r,2)$,
(c) symmetric group on n symbols, $S_n = G(1,1,n)$,
(d) Weyl group of type B_n (also called hyperoctahedral group) is $G(2,1,n)$,
(e) Weyl group of type D_n is $G(2,2,n)$.

Let $G(n)$ be an isomorphic copy of G $(= \mathbb{Z}/r\mathbb{Z})$ in $G(r,1,n)$ defined as

$$G(n) := \{(1, \ldots, 1, g_n, (1)) \mid g_n \in G\}.$$

Assume that S_{n-1} is the subgroup of S_n consisting of elements fixing n. The groups $G(r,1,n-1) \times G(n)$ and $G(r,p,n)$ are subgroups of $G(r,1,n)$. Let $L(r,p,n)$ be the subgroup of $G(r,p,n)$ defined as:

$$\begin{aligned}L(r,p,n) :=&(G(r,1,n-1)\times G(n))\cap G(r,p,n)\\ =&((G^{n-1}\rtimes S_{n-1})\times G(n))\cap(\mathrm{D}(r,p,n)\rtimes S_n)\\ =&(G^n\rtimes S_{n-1})\cap(\mathrm{D}(r,p,n)\rtimes S_n)\\ =&\mathrm{D}(r,p,n)\rtimes S_{n-1}.\end{aligned}$$

As a subgroup of $GL_n(\mathbb{C})$, the group $L(r,p,n)$ consists of those elements in $G(r,p,n)$ such that the (n,n)-th entry is nonzero. For $p=1$, we have

$$\begin{aligned}L(r,1,n) &= G^n\rtimes S_{n-1}\\ &= \big(G^{n-1}\rtimes S_{n-1}\big)\times G(n)\\ &= G(r,1,n-1)\times G(n).\end{aligned}$$

The order of $L(r,1,n)$ is $r^n(n-1)!$. Taking the exact sequence

$$\begin{aligned}1\longrightarrow L(r,p,n)\longrightarrow L(r,1,n)&\longrightarrow \mathbb{Z}/p\mathbb{Z}\longrightarrow 1\\ (g_1,g_2,\dots,g_n,\pi)&\mapsto g_1g_2\cdots g_n,\end{aligned}$$

we see that $L(r,p,n)$ is a normal subgroup of the group $L(r,1,n)$ of index p. Thus, the order of $L(r,p,n)$ is $(r^n(n-1)!)/p$.

Recall from Section 2.2 that $\mathcal{Y}(r,n)$ is the set of r-tuples of Young diagram such that the total number of boxes is n. Given an r-tuple of Young diagrams $(\lambda_1,\lambda_2,\dots,\lambda_r)\in\mathcal{Y}(r,n-1)$, choose one $i\in\{1,2,\dots,r\}$, take λ_i (λ_i may be empty also), color it by n and denote by λ_i^n. We note that λ_i^n denotes the same Young diagram λ_i, but it has the color n. The *(n,i)-colored r-tuple of Young diagrams*, denoted by $\lambda^{(n,i)} := (\lambda_1,\lambda_2,\dots,\lambda_{i-1},\lambda_i^n,\lambda_{i+1},\dots,\lambda_r)$, consists of the r-tuple $(\lambda_1,\lambda_2,\dots,\lambda_r)\in\mathcal{Y}(r,n-1)$ with i-th component λ_i colored by n. Let $\mathcal{Y}^n(r,n-1)$ denote the set of all (n,i)-colored r-tuples of Young diagrams with total $n-1$ boxes for $i=1,2,\dots,r$.

Lemma 4.1. *The irreducible $L(r,1,n)$-modules are indexed by the elements of $\mathcal{Y}^n(r,n-1)$.*

Proof. The irreducible representations of G are $\sigma_1,\sigma_2,\dots,\sigma_r$ (defined in Section 2.2). Suppose that V^λ is the irreducible representation of $G(r,1,n-1)$ corresponding to $\lambda\in\mathcal{Y}(r,n-1)$. Then,

$$\{V^\lambda\otimes\sigma_i \mid \lambda\in\mathcal{Y}(r,n-1), i=1,\dots,r\}$$

is the set of irreducible representations of $L(r,1,n)$ which is indexed by the elements of the set $\mathcal{Y}^n(r,n-1)$. □

Definition 4.2 (Inner corner and outer corner). *Let λ be a Young diagram with $|\lambda|$ boxes. A box, which when removed from λ, leaves a Young diagram with $|\lambda|-1$ boxes is called an inner corner of λ. A box, which when added to λ, gives a Young diagram with $|\lambda|+1$ boxes is called an outer corner of λ.*

We describe the branching rule from $G(r,1,n)$ to $L(r,1,n)$. For $\mu=(\mu_1,\mu_2,\dots,\mu_r)$ in $\mathcal{Y}(r,n)$ with $\mu_i\neq\varnothing$, let $\mu\downarrow i$ denote the set of elements $\nu^{(n,i)}\in\mathcal{Y}^n(r,n-1)$ such that ν is obtained from μ by removing the box at an inner corner of μ_i and then coloring the i-th component of ν by n to obtain (n,i)-colored r-tuple $\nu^{(n,i)}$. Assume that V^μ and $V^{\nu^{(n,i)}}$ are the irreducible $G(r,1,n)$-module and $L(r,1,n)$-module corresponding to $\mu\in\mathcal{Y}(r,n)$ and $\nu^{(n,i)}\in\mathcal{Y}^n(r,n-1)$, respectively.

Theorem 4.3 (Branching rule from $G(r,1,n)$ to $L(r,1,n)$). *We have*

$$\mathrm{Res}^{G(r,1,n)}_{L(r,1,n)}(V^{\mu}) = \bigoplus_{i=1}^{r}\left(\bigoplus_{\nu\in\mu\downarrow i} V^{\nu^{(n,i)}}\right). \tag{10}$$

Remark 4.4. We take equality in place of isomorphism because the restriction rule is multiplicity free which makes the decomposition canonical and we identify $V^{\nu^{(n,i)}}$ with the corresponding $L(r,1,n)$-submodule of V^{μ}.

Proof. Since $\nu_j = \mu_j$ for $j \neq i$ and $|\nu_i| = |\mu_i| - 1$, therefore given a GZ-subspace of V^{μ}, there exists a GZ-subspace of $V^{\nu^{(n,i)}} = V^{\nu} \otimes \sigma_i$ with the same label. Also, for $1 \leqslant i \leqslant n-1$, the action of $X_i \in GZ_{n-1,n} \subseteq GZ_{n,n}$ on GZ-subspace of $V^{\nu^{(n,i)}}$ is same as its action on GZ-subspace of V^{μ}. A GZ-subspace is uniquely determined by its weight and a GZ-subspace uniquely determines the parametrization of irreducible representation. Thus, $V^{\nu^{(n,i)}}$ appears in the restriction of V^{μ} as a $L(r,1,n)$-module with multiplicity one since the restriction from $G(r,1,n)$ to $L(r,1,n)$ is multiplicity free (follows from chain (1) since $H_{m-1,n} = L(r,1,n)$). ☐

The next step is the parametrization of the irreducible representations of $G(r,p,n)$ and $L(r,p,n)$ using Clifford theory. Consider the one-dimensional representation

$$\delta_0 : G(r,1,n) \longrightarrow \mathbb{C}^*$$
$$\delta_0(g_1, g_2, \ldots, g_n, \sigma) = g_1 g_2 \ldots g_n.$$

As a $G(r,1,n)$-module, δ_0 is parametrized by $(\varnothing, (n), \varnothing, \ldots, \varnothing)$ and $G(r,p,n) \subseteq \mathrm{Ker}(\delta_0^m)$. We use the same notation δ_0 to denote the restriction of δ_0 to $L(r,1,n)$. It will be clear from the context whether we consider δ_0 as a $G(r,1,n)$-module or as a $L(r,1,n)$-module. As a $L(r,1,n)$-module, δ_0 is parametrized by the $(n,2)$-colored r-tuple $(\varnothing, (n-1)^n, \varnothing, \ldots, \varnothing)$ and $L(r,p,n) \subseteq \mathrm{Ker}(\delta_0^m)$. The cyclic group C generated by δ_0^m is of order p. Thus

$$C \cong G(r,1,n)/G(r,p,n) \cong L(r,1,n)/L(r,p,n).$$

Definition 4.5. *Define the shift map on* $\mathcal{Y}(r,n)$ *as* $\mathrm{sh} : \mathcal{Y}(r,n) \longrightarrow \mathcal{Y}(r,n)$ *by*

$$(\lambda_1, \lambda_2, \ldots, \lambda_r) \mapsto (\lambda_r, \lambda_1, \lambda_2, \ldots, \lambda_{r-1}).$$

Using the same notation, the shift map on $\mathcal{Y}^n(r,n-1)$ *is defined as*

$$\mathrm{sh} : \mathcal{Y}^n(r,n-1) \longrightarrow \mathcal{Y}^n(r,n-1) \ \text{by}$$
$$(\mu_1, \mu_2, \ldots, \mu_i^n, \ldots, \mu_r) \mapsto (\mu_r, \mu_1, \mu_2, \ldots, \mu_{i-1}, \mu_i^n, \ldots, \mu_{r-1}),$$

where the r-tuples on the left hand side and the right hand side are (n,i)-colored and $(n,i+1)$-colored, respectively. For the definition of shift map on $\mathrm{Tab}(r,\lambda)$*, where* $\lambda \in \mathcal{Y}(r,n)$*, see Definition* 4.13.

Suppose that V^{λ} and $V^{\mu^{(n,i)}}$ denote the irreducible representations of $G(r,1,n)$ and $L(r,1,n)$ parametrized by the r-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{Y}(r,n)$ and (n,i)-colored r-tuple $\mu^{(n,i)} = (\mu_1, \mu_2, \ldots, \mu_i^n, \ldots, \mu_r) \in \mathcal{Y}^n(r,n-1)$ for some $i \in \{1,2,\ldots,r\}$, respectively.

The following lemma is proved using Okounkov–Vershik approach. Part (a) is [15, Theorem 24] and it was proved there using $*$-rim hook tableaux.

Lemma 4.6. *For* $\lambda \in \mathcal{Y}(r,n)$ *and* $\mu^{(n,i)} \in \mathcal{Y}^n(r,n-1)$*, the following are true:*

(a) *As* $G(r,1,n)$*-modules,*

$$\delta_0 \otimes V^{\lambda} \ \cong \ V^{\mathrm{sh}(\lambda)}.$$

(b) *As $L(r,1,n)$-modules,*

$$\delta_0 \otimes V^{\mu^{(n,i)}} \;\cong\; V^{\mathrm{sh}(\mu^{(n,i)})}.$$

Proof. (a) A GZ-subspace of an irreducible representation of $G(r,1,n)$ is uniquely determined by its weight. Also, a GZ-subspace uniquely determines the r-tuple of Young diagrams in $\mathcal{Y}(r,n)$ which parametrize the irreducible representation of which it is a GZ-subspace.

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathcal{Y}(r,n)$ with $y_i := |\lambda_i|$, let R be the standard r-tuple of Young tableaux written in row major order. The GZ-subspace of type V_R is isomorphic to

$$\underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{y_1\text{- fold}} \otimes \underbrace{\sigma_2 \otimes \cdots \otimes \sigma_2}_{y_2\text{- fold}} \otimes \cdots \otimes \underbrace{\sigma_r \otimes \cdots \otimes \sigma_r}_{y_r\text{- fold}}$$

as a G^n-module. For $i = 1, 2, \dots, n$ and GZ-basis element

$$v_R = \underbrace{v_1 \otimes \cdots \otimes v_1}_{y_1\text{- fold}} \otimes \underbrace{v_2 \otimes \cdots \otimes v_2}_{y_2\text{- fold}} \otimes \cdots \otimes \underbrace{v_r \otimes \cdots \otimes v_r}_{y_r\text{- fold}}$$

we have $X_i(v_R) = rc(b_R(i))(v_R)$ (for the definition of Jucys–Murphy elements X_i, see (4) in Section 2.2).

The GZ-subspace of δ_0 is given by n- fold $\sigma_2 \otimes \cdots \otimes \sigma_2$ with GZ-basis element given by n- fold $v_2 \otimes \cdots \otimes v_2$. Thus, the GZ-subspace of $\delta_0 \otimes V^\lambda$ is

$$\begin{aligned}
&\big(\underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{y_1\text{- fold}} \otimes \underbrace{\sigma_2 \otimes \cdots \otimes \sigma_2}_{y_2\text{- fold}} \otimes \cdots \otimes \underbrace{\sigma_r \otimes \cdots \otimes \sigma_r}_{y_r\text{- fold}}\big) \otimes \big(\underbrace{\sigma_2 \otimes \cdots \otimes \sigma_2}_{n\text{- fold}}\big)\\
&\cong\; \underbrace{\sigma_2 \otimes \cdots \otimes \sigma_2}_{y_1\text{- fold}} \otimes \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{y_2\text{- fold}} \otimes \cdots \otimes \underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{y_r\text{- fold}}\\
&\cong\; V_{\mathrm{sh}(R)},
\end{aligned}$$

isomorphic as G^n-module, with basis element v' being

$$\underbrace{(v_1 \otimes v_2) \cdots \otimes (v_1 \otimes v_2)}_{y_1\text{- fold}} \otimes \underbrace{(v_2 \otimes v_2) \cdots \otimes (v_2 \otimes v_2)}_{y_2\text{- fold}} \otimes \cdots \otimes \underbrace{(v_r \otimes v_2) \cdots \otimes (v_r \otimes v_2)}_{y_r\text{- fold}}).$$

Also, for $1 \leqslant i \neq j \leqslant n$, we have

$$\zeta_i^l \zeta_j^{-l} s_{ij}(v_2 \otimes \cdots \otimes v_2) = v_2 \otimes \cdots \otimes v_2$$

So, for $1 \leqslant i \leqslant n$,

$$\begin{aligned}
X_i(v') &= (v_2 \otimes \cdots \otimes v_2) \otimes X_i(v_1 \otimes \cdots \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_2 \otimes \cdots \otimes v_r \otimes \cdots \otimes v_r)\\
&= rc(b_R(i))(v')\\
&= rc(b_{\mathrm{sh}(R)}(i))(v_{\mathrm{sh}(R)}) = X_i(v_{\mathrm{sh}(R)})
\end{aligned}$$

which implies that $v' = v_{\mathrm{sh}(R)}$.

We have shown that $V_{\mathrm{sh}(R)}$ is a GZ-subspace of $\delta_0 \otimes V^\lambda$. Thus, $V^{\mathrm{sh}(\lambda)}$, corresponding to r-tuple $\mathrm{sh}(\lambda)$, is a $G(r,1,n)$-submodule of $\delta_0 \otimes V^\lambda$. The irreducibility of $\delta_0 \otimes V^\lambda$ implies the result.

(b) This part can be proved by arguments similar to those in part (a). To be able to do so, we note that a GZ-subspace of an irreducible representation of $L(r,1,n)$ is uniquely determined by its weight, i.e. its label and the action of Jucys–Murphy elements $X_1, X_2, \dots, X_{n-1}$ on it. ☐

Lemma 4.6 implies Corollaries 4.7 and 4.8. Part (a) of Corollary 4.7 is [15, Corollary 25] and is also stated as Theorem 2.1 in [19].

Corollary 4.7. *For $\lambda \in \mathcal{Y}(r,n)$ and $\mu^{(n,i)} \in \mathcal{Y}^n(r,n-1)$, the following are true for $t \in \mathbb{Z}$:*

(a) *As a $G(r,1,n)$-module,*

$$\delta_0^t \otimes V^{\lambda} \cong V^{\mathrm{sh}^t(\lambda)}.$$

(b) *As a $L(r,1,n)$-module,*

$$\delta_0^t \otimes V^{\mu^{(n,i)}} \cong V^{\mathrm{sh}^t(\mu^{(n,i)})}.$$

Corollary 4.8. *For $t \in \mathbb{Z}$:*

(a) *As a $G(r,1,n)$-module, δ_0^t is parametrized by $(\varnothing,\dots,\varnothing,(n),\varnothing,\dots,\varnothing) \in \mathcal{Y}(r,n)$, where (n) occurs at $(t+1)(\mathrm{mod}\, r)$-th component.*

(b) *As a $L(r,1,n)$-module, δ_0^t is parametrized by the $(n,(t+1)(\mathrm{mod}\, r))$-colored r-tuple $(\varnothing,\dots,\varnothing,(n-1)^n,\varnothing,\dots,\varnothing) \in \mathcal{Y}^n(r,n-1)$, where $(n-1)^n$ occurs at $(t+1)(\mathrm{mod}\, r)$-th component.*

We define a combinatorial object (m,p)-necklace as in [7, p. 174] which will be useful in parametrization of irreducible $G(r,p,n)$-modules and $L(r,p,n)$-modules.

Let $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_r) \in \mathcal{Y}(r,n)$. For each i such that $1 \leqslant i \leqslant m$, consider the p-tuple

$$\tilde{\lambda}_{(i)} := (\lambda_i, \lambda_{m+i}, \lambda_{2m+i}, \dots, \lambda_{(p-1)m+i}).$$

Depict $\tilde{\lambda}_{(i)}$ as a p-necklace in the following way: the circular necklace, with centre on the x-axis, has p nodes and lies in a vertical xy-plane with the first node λ_i placed at the point, where tangent to the necklace in $(y>0)$-half plane is parallel to the x-axis. The placement of nodes is done in clockwise direction with the j-th node being $\lambda_{(j-1)m+i}$ and placed at a clockwise angle of $2\pi/(j-1)$ with y-axis for $j=2,\dots,p$. A (m,p)-*necklace* of total n boxes obtained from $\lambda \in \mathcal{Y}(r,n)$, denoted by $\tilde{\lambda}$, is a m-tuple

$$\tilde{\lambda} = (\tilde{\lambda}_{(1)}, \tilde{\lambda}_{(2)}, \dots, \tilde{\lambda}_{(m)}),$$

where $\tilde{\lambda}_{(i)}$ is a p-necklace for each $1 \leqslant i \leqslant m$. For $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant m$, let $\tilde{\lambda}_{(i,j)}$ denote the j-th node in $\tilde{\lambda}_{(i)}$, i.e. $\tilde{\lambda}_{(i,j)} = \lambda_{(j-1)m+i}$. Thus, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{p} \tilde{\lambda}_{(i,j)} = n.$$

Two (m,p)-necklaces, $\tilde{\lambda}$ and $\tilde{\mu}$, both of total boxes n, are said to be *equivalent* if for some integer t, $\tilde{\lambda}_{(i,j)} = \tilde{\mu}_{(i,(j+t)(\mathrm{mod}\, p))}$ for all $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant m$. Let $\mathcal{Y}(m,p,n)$ denote the set of inequivalent (m,p)-necklaces of total n boxes.

Note that for any element $\mu \in [\lambda]$, the stabilizer subgroup $C_\mu = C_\lambda$. So, the stabilizer subgroup of a representative of $[\lambda]$ can be written as C_λ while considering (m,p)-necklace $\tilde{\lambda}$.

Example 4.9. The $(3,4)$-necklace of total 30 boxes obtained from

$$\lambda = ((2,1),(3,2),(2,1,1),(1),(1,1),(1,1),(1,1,1),(2,1),(1),(2),(2),(2))$$

is given in Figure 5.

Theorem 4.10 and its proof follow the expositions in [1, 17, 26].

Theorem 4.10. *The irreducible $G(r,p,n)$-modules are parametrized by the ordered pairs $(\tilde{\lambda},\delta)$, where $\tilde{\lambda} \in \mathcal{Y}(m,p,n)$ and $\delta \in C_\lambda$. Given $\lambda \in \mathcal{Y}(r,n)$, the restriction of the corresponding $G(r,1,n)$-module V^λ to $G(r,p,n)$ has multiplicity free decomposition given as:*

$$\mathrm{Res}^{G(r,1,n)}_{G(r,p,n)}(V^\lambda) \;=\; \bigoplus_{\delta\in C_\lambda} V^{(\tilde{\lambda},\delta)}.$$

Also, for $\mu \in [\lambda]$,

$$\mathrm{Res}^{G(r,1,n)}_{G(r,p,n)}(V^\lambda) \;\cong\; \mathrm{Res}^{G(r,1,n)}_{G(r,p,n)}(V^\mu).$$

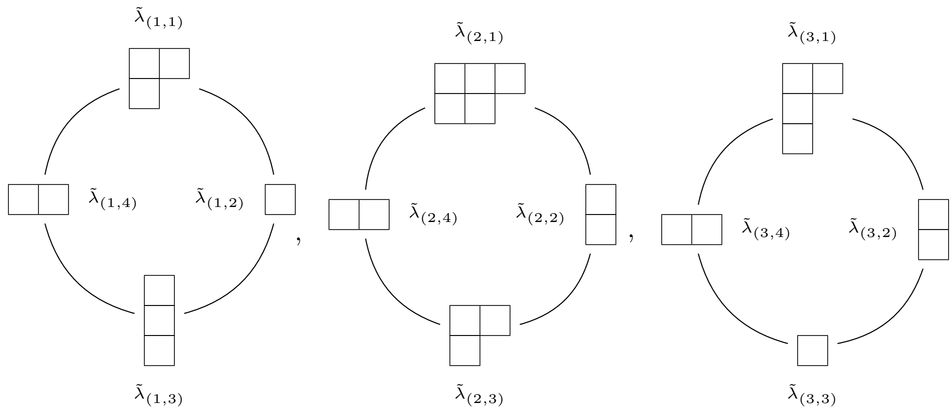

Figure 5. The $(3,4)$-necklace obtained from λ in Example 4.9.

Proof. The group $C = \langle \delta_0^m \rangle$ acts on the set of irreducible $G(r,1,n)$-modules. For $\lambda \in \mathcal{Y}(r,n)$, suppose that $[\lambda]$ denotes the set of elements in $\mathcal{Y}(r,n)$ which parametrize the irreducible $G(r,1,n)$-modules in the orbit of V^λ. Using Corollary 4.7(a), we have

$$[\lambda] = \{\nu \mid \nu = \mathrm{sh}^{im}(\lambda) \text{ for some } i = 0,1,\ldots,p-1\}.$$

Let the order of the orbit $[\lambda]$ be $b(\lambda)$. Then, the order of the stabilizer subgroup C_λ is $u(\lambda) := \frac{p}{b(\lambda)}$. Also, C_λ is generated by $\delta_0^{b(\lambda)m}$. The result follows from Theorem 2.8. ☐

Given $\tilde\mu \in \mathcal{Y}(m,p,n-1)$, the (n,i,j)-*colored* (m,p)-*necklace*, denoted by $\tilde\mu^{(n,i,j)}$, is obtained by coloring $\tilde\mu_{(i,j)}$ by n, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$. The colored (m,p)-necklaces, $\tilde\mu^{(n,i,j)}$ and $\tilde\nu^{(n,s,t)}$, are *equivalent* if and only if

(i) $i = s$, and $j = (t+l)(\mathrm{mod}\, p)$ for some $l \in \mathbb{Z}$, and
(ii) the corresponding $\tilde\mu$ and $\tilde\nu$ are equivalent as (m,p)-necklaces using the same l as in (i), i.e. $\tilde\mu_{(a,b)} = \tilde\nu_{(a,(b+l)(\mathrm{mod}\, p))}$ for all $1 \leqslant a \leqslant m$ and $1 \leqslant b \leqslant p$.

Let $\mathcal{Y}^n(m,p,n-1)$ be the set of inequivalent (n,i,j)-colored (m,p)-necklaces of total $n-1$ boxes for all $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant p$.

Example 4.11. Consider the $(3,4)$-necklace $\tilde\lambda$ with total 30 boxes in Example 4.9. We obtain the $(31,2,3)$-colored $(3,4)$-necklace in Figure 6, denoted by $\tilde\lambda^{(31,2,3)}$, by coloring $\tilde\lambda_{(2,3)}$ by 31.

By depicting $\mu = (\mu_1, \mu_2, \ldots, \mu_t^n, \ldots, \mu_r) \in \mathcal{Y}^n(r,n-1)$ as a (m,p)-necklace, we get $\tilde\mu^{(n,i,j)} \in \mathcal{Y}^n(m,p,n-1)$, where $t = (j-1)m + i$ for a unique pair (i,j) such that $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$.

Theorem 4.12. *The irreducible $L(r,p,n)$-modules are parametrized by the elements of $\mathcal{Y}^n(m,p,n-1)$. For $\mu^{(n,t)} \in \mathcal{Y}^n(r,n-1)$, the restriction of the corresponding irreducible $L(r,1,n)$-module $V^{\mu^{(n,t)}}$ to $L(r,p,n)$ has multiplicity free decomposition given as:*

$$\mathrm{Res}^{L(r,1,n)}_{L(r,p,n)}(V^{\mu^{(n,t)}}) \;=\; V^{\tilde\mu^{(n,i,j)}},$$

where $t = (j-1)m + i$ for a unique pair (i,j) such that $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$. Also, for any $\nu^{(n,s)} \in [\mu^{(n,t)}]$,

$$\mathrm{Res}^{L(r,1,n)}_{L(r,p,n)}(V^{\mu^{(n,t)}}) \;\cong\; \mathrm{Res}^{L(r,1,n)}_{L(r,p,n)}(V^{\nu^{(n,s)}}).$$

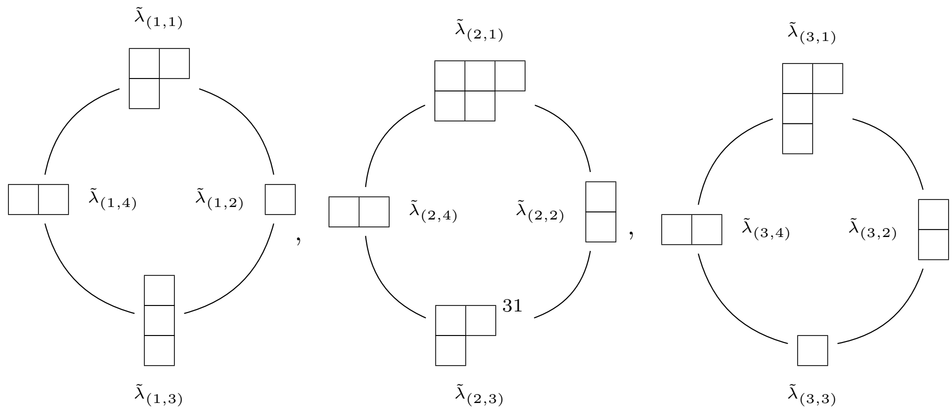

Figure 6. The $(31, 2, 3)$-colored $(3, 4)$-necklace in Example 4.11.

Proof. The group $C = \langle \delta_0^m \rangle$ acts on the set of irreducible $L(r, 1, n)$-modules. For $\mu^{(n,t)} \in \mathcal{Y}^n(r, n-1)$, suppose that $[\mu^{(n,t)}]$ denotes the set of elements in $\mathcal{Y}^n(r, n-1)$ which parametrize the irreducible $L(r, 1, n)$-modules in the orbit of $V^{\mu^{(n,t)}}$. Using Corollary 4.7(b), we have

$$[\mu^{(n,t)}] = \{\omega^{(n,y)} \mid \omega^{(n,y)} = \mathrm{sh}^{zm}(\mu^{(n,t)}) \text{ for some } z = 0, 1, \ldots, p-1\}.$$

Since the color is also shifting, therefore, the number of elements in the orbit is p and thus the stabilizer subgroup consists of identity element only. The results follow from Theorem 2.8. $\square$

Branching rule from $G(r, p, n)$ to $L(r, p, n)$. The construction of higher Specht polynomials for $G(r, p, n)$ from the higher Specht polynomials for $G(r, 1, n)$ was described in [17] to decompose a module isomorphic to left regular $G(r, p, n)$-module into its irreducible submodules. Applying a similar (but not identical) construction on the canonical GZ-bases of irreducible $G(r, 1, n)$-modules obtained in Okounkov–Vershik approach in Section 2, we construct the bases of irreducible $G(r, p, n)$-modules in Theorem 4.14. We use such constructed basis to show in Theorem 4.15 that the irreducible $G(r, p, n)$-modules $V^{(\tilde\lambda,\delta_1)}$ and $V^{(\tilde\lambda,\delta_2)}$, for $\tilde\lambda \in \mathcal{Y}(m, p, n)$ and $\delta_1, \delta_2 \in C_\lambda$, are isomorphic as $L(r, p, n)$-modules. Theorem 4.15 is useful in the proof of Theorem 4.16 for description of branching rule from $G(r, p, n)$ to $L(r, p, n)$.

Definition 4.13. *Fix* $\lambda \in \mathcal{Y}(r, n)$. *Define the shift map* $\mathrm{sh} : \mathrm{Tab}(r, \lambda) \longrightarrow \mathrm{Tab}(r, \lambda)$ *by*

$$(T_1, T_2, \ldots, T_r) \mapsto (T_r, T_1, \ldots, T_{r-1}).$$

Since C_λ is generated by $\delta_0^{mb(\lambda)}$, the $G(r, 1, n)$-modules V^λ and $\delta_0^{mb(\lambda)} \otimes V^\lambda$ are isomorphic. Suppose that $T \in \mathrm{Tab}(r, \lambda)$ and $\mathbf{1}_{\delta_0^{mb(\lambda)}}$ is the basis element of one-dimensional $G(r, 1, n)$-module $\delta_0^{mb(\lambda)}$. Using Corollary 4.7(a), define the $G(r, 1, n)$-linear isomorphism $\mathcal{E} : V^\lambda \longrightarrow \delta_0^{mb(\lambda)} \otimes V^\lambda$ by

$$v_T \mapsto \mathbf{1}_{\delta_0^{mb(\lambda)}} \otimes v_{\mathrm{sh}^{-mb(\lambda)}(T)}.$$

Also, the map $\mathcal{F} : \delta_0^{mb(\lambda)} \otimes V^\lambda \longrightarrow V^\lambda$ given by $\mathbf{1}_{\delta_0^{mb(\lambda)}} \otimes v_T \mapsto v_T$ is a $G(r, p, n)$-linear isomorphism.

The associator of V^λ is given by

$$\begin{aligned} \mathcal{A}^\lambda = \mathcal{F}\mathcal{E} : V^\lambda &\longrightarrow V^\lambda \\ v_T &\mapsto v_{\mathrm{sh}^{-mb(\lambda)}(T)}. \end{aligned} \tag{11}$$

For $h = 1, 2, \ldots, r$, we define

$$\mathrm{Tab}(r,\lambda)_h = \{T = (T^1, T^2, \ldots, T^r) \in \mathrm{Tab}(r,\lambda) \mid n \in T^{r-\nu}, 0 \leqslant \nu < h\}.$$

For $T \in \mathrm{Tab}(r,\lambda)_{mb(\lambda)}$, we get the following $u(\lambda)$ distinct standard r-Young tableaux:

$$T, \mathrm{sh}^{mb(\lambda)}(T), \mathrm{sh}^{2mb(\lambda)}(T), \ldots, \mathrm{sh}^{(u(\lambda)-1)mb(\lambda)}(T).$$

An element $\delta \in C_\lambda = \langle \delta_0^{mb(\lambda)} \rangle$ can be identified with $\zeta^{lmb(\lambda)}$ for some $0 \leqslant l \leqslant u(\lambda)-1$. Fixing $\delta \in C_\lambda$, we define, for each $T \in \mathrm{Tab}(r,\lambda)_{mb(\lambda)}$,

$$v_T^{(\delta)} := \sum_{t=0}^{u(\lambda)-1} \zeta^{tlmb(\lambda)} v_{\mathrm{sh}^{tmb(\lambda)}(T)}.$$

The linear independence of $\{v_T \mid T \in \mathrm{Tab}(r,\lambda)\}$ implies that $\{v_T^{(\delta)} \mid T \in \mathrm{Tab}(r,\lambda)_{mb(\lambda)}\}$, for a fixed $\delta \in C_\lambda$, is linearly independent.

Theorem 4.14. *For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{Y}(r,n)$, consider $\tilde{\lambda} \in \mathcal{Y}(m,p,n)$. For each $\delta \in C_\lambda$, define*

$$V^{(\tilde{\lambda},\delta)} := \mathbb{C}\text{-}\mathrm{span}\{v_T^{(\delta)} \mid T \in \mathrm{Tab}(r,\lambda)_{mb(\lambda)}\}.$$

The following are true:

(a) *The eigenspace decomposition of V^λ with respect to the associator $\mathcal{A}^\lambda$ is:*

$$V^\lambda = \bigoplus_{\delta \in C_\lambda} V^{(\tilde{\lambda},\delta)}. \tag{12}$$

(b) *The eigenspace $V^{(\tilde{\lambda},\delta)}$, for $\delta \in C_\lambda$, is an irreducible $G(r,p,n)$-module.*

(c) *The set $\{V^{(\tilde{\lambda},\delta)} \mid \tilde{\lambda} \in \mathcal{Y}(m,p,n), \delta \in C_\lambda\}$ is the complete set of irreducible $G(r,p,n)$-modules.*

Proof. It can be seen from the definition of the associator $\mathcal{A}^\lambda$ in (11) that

$$\mathcal{A}^\lambda(v_T^{(\delta)}) = \zeta^{lmb(\lambda)}(v_T^{(\delta)}), \text{ for } \delta \in C_\lambda.$$

This implies that the subspaces $V^{(\tilde{\lambda},\delta)}$, for $\delta \in C_\lambda$, are contained in the distinct eigenspaces of $\mathcal{A}^\lambda$. Thus, we have

$$\bigoplus_{\delta \in C_\lambda} V^{(\tilde{\lambda},\delta)} \subset V^\lambda. \tag{13}$$

Also for each $\delta \in C_\lambda$, the dimension of $V^{(\tilde{\lambda},\delta)}$ is equal to the number of elements in $\mathrm{Tab}(r,\lambda)_{mb(\lambda)}$, denoted by $\#(\mathrm{Tab}(r,\lambda))$. This implies that we have

$$\dim V^{(\tilde{\lambda},\delta)} = \frac{1}{u(\lambda)} \#(\mathrm{Tab}(r,\lambda)) = \frac{1}{u(\lambda)} \dim V^\lambda.$$

Thus, the dimensions of both sides in (13) are equal which implies equality in (13). This proves part (a). The proofs of parts (b) and (c) follow from Clifford theory and part (a) of this theorem. □

Theorem 4.15. *For a fixed $\tilde{\lambda} \in \mathcal{Y}(m,p,n)$ and $\delta_1, \delta_2 \in C_\lambda$, we have*

$$\mathrm{Res}_{L(r,p,n)}^{G(r,p,n)}(V^{(\tilde{\lambda},\delta_1)}) \;\cong\; \mathrm{Res}_{L(r,p,n)}^{G(r,p,n)}(V^{(\tilde{\lambda},\delta_2)}).$$

Proof. The linear map $\theta : V^{(\tilde{\lambda},\delta_1)} \longrightarrow V^{(\tilde{\lambda},\delta_2)}$ defined by setting

$$v_T^{(\delta_1)} \mapsto v_T^{(\delta_2)}, \text{ for } T \in \mathrm{Tab}(r,\lambda)_{mb(\lambda)},$$

is an $L(r,p,n)$-module isomorphism. □

Given $\tilde{\lambda} \in \mathcal{Y}(m,p,n), 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant p$, let $\tilde{\lambda} \downarrow (i,j)$ denote the set of all elements in $\mathcal{Y}^n(m,p,n-1)$ obtained by deleting a box from an inner corner in $\tilde{\lambda}_{(i,j)}$ and then coloring the corresponding node by n. For a fixed $1 \leqslant i \leqslant m$, define $J(i) \subseteq \{1,2,\ldots,p\}$ such that for $s,t \in J(i), s \neq t$, we have $\tilde{\lambda} \downarrow (i,s) \cap \tilde{\lambda} \downarrow (i,t) = \varnothing$. If $\tilde{\lambda} \downarrow (i,s) \cap \tilde{\lambda} \downarrow (i,t) \neq \varnothing$, then $\tilde{\lambda} \downarrow (i,s) = \tilde{\lambda} \downarrow (i,t)$.

Theorem 4.16 (Branching rule from $G(r,p,n)$ to $L(r,p,n)$). *For $\tilde{\lambda} \in \mathcal{Y}(m,p,n)$ and $\delta \in C(\lambda)$, we have*

$$\mathrm{Res}^{G(r,p,n)}_{L(r,p,n)}(V^{(\tilde{\lambda},\delta)}) \cong \bigoplus_{i=1}^{m} \bigoplus_{j\in J(i)} \left(\bigoplus_{\tilde{\mu}^{(n,i,j)} \in \tilde{\lambda}\downarrow(i,j)} V^{\tilde{\mu}^{(n,i,j)}} \right),$$

and the branching rule from $G(r,p,n)$ to $L(r,p,n)$ is multiplicity free.

Proof. We use the transitivity of restriction from $G(r,1,n)$ to $L(r,p,n)$:

$$G(r,1,n) \supset L(r,1,n) \supset L(r,p,n) \text{ and } G(r,1,n) \supset G(r,p,n) \supset L(r,p,n).$$

Given $\tilde{\lambda}$, we have $\lambda \in \mathcal{Y}(r,n)$. Considering V^{λ} as $L(r,1,n)$-module, Theorem 4.3 implies that

$$\mathrm{Res}^{G(r,1,n)}_{L(r,1,n)}(V^{\lambda}) \cong \bigoplus_{t=1}^{r} \Big(\bigoplus_{\mu\in\lambda\downarrow t} V^{\mu^{(n,t)}} \Big). \tag{14}$$

Writing $t = (j-1)m + i$, where $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant p$, we note that $u(\lambda)$ distinct elements of $\mathcal{Y}^n(r,n-1)$

$$\mu^{(n,t)}, \mathrm{sh}^{mb(\lambda)}(\mu^{(n,t)}), \ldots, \mathrm{sh}^{(u(\lambda)-1)mb(\lambda)}(\mu^{(n,t)}) \tag{15}$$

give rise to the same $\tilde{\mu}^{(n,i,j)} \in \mathcal{Y}^n(m,p,n-1)$ and $j \in J(i)$. Also, from $\mu^{(n,s)}$ (not in (15)) such that $s = (y-1)m+i$, where $1 \leqslant y \leqslant p$, we get $\tilde{\mu}^{(n,i,y)} \in \mathcal{Y}^n(m,p,n-1)$, not equivalent to $\tilde{\mu}^{(n,i,j)}$, and thus $y \in J(i)$ and $y \neq j$. Restricting V^{λ} as $L(r,p,n)$-module in (14), Theorem 4.12 implies that

$$\mathrm{Res}^{G(r,1,n)}_{L(r,p,n)}(V^{\lambda}) \cong \left(\bigoplus_{i=1}^{m} \bigoplus_{j\in J(i)} \left(\bigoplus_{\tilde{\mu}^{(n,i,j)} \in \lambda\downarrow(i,j)} V^{\tilde{\mu}^{(n,i,j)}} \right) \right)^{\oplus u(\lambda)}. \tag{16}$$

Considering V^{λ} as $G(r,p,n)$-module, Theorem 4.10 implies that

$$\mathrm{Res}^{G(r,1,n)}_{G(r,p,n)}(V^{\lambda}) = \bigoplus_{\delta\in C_\lambda} V^{(\tilde{\lambda},\delta)}. \tag{17}$$

Further restricting V^{λ} as $L(r,p,n)$-module in (17), using Theorem 4.15 and the order of C_λ being $u(\lambda)$, we get

$$\mathrm{Res}^{G(r,1,n)}_{L(r,p,n)}(V^{\lambda}) \cong \left(\mathrm{Res}^{G(r,p,n)}_{L(r,p,n)}(V^{(\tilde{\lambda},\delta)}) \right)^{\oplus u(\lambda)}, \tag{18}$$

where $\delta \in C_\lambda$. The result follows from (16) and (18). □

Next, we illustrate Theorem 4.16 with an example.

Example 4.17. For the group $G(4,2,8)$, consider its irreducible representation $V^{((\tilde\lambda_1,\tilde\lambda_2),\delta)}$ of $G(4,2,8)$ parametrized by $((\tilde\lambda_1,\tilde\lambda_2),\delta)$, where $(\tilde\lambda_1,\tilde\lambda_2)$ is the $(2,2)$-necklace obtained from $\lambda = ((2,1),(1),(2,1),(1))$ and δ is a fixed element of the stabilizer subgroup C_λ. Thus, the $(2,2)$-necklace $(\tilde\lambda_1,\tilde\lambda_2)$ is as given in Figure 7.

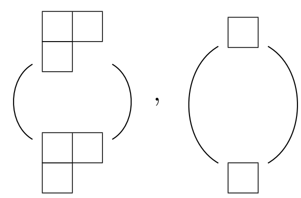

Figure 7. The $(2,2)$-necklace obtained from $\lambda = ((2,1),(1),(2,1),(1))$.

It can be easily seen that the sets $\tilde\lambda \downarrow (1,1)$ and $\tilde\lambda \downarrow (1,2)$ are equal and so, $J(1)$ contains only one element which can be chosen to be either 1 or 2. We choose $J(1) = \{2\}$; and $\tilde\lambda \downarrow (1,2)$ contains two elements written as $(8,1,2)$-colored $(2,2)$-necklaces as given in Figure 8.

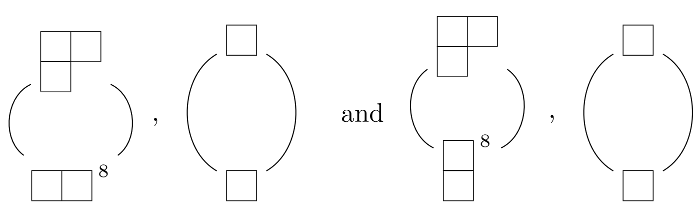

Figure 8. Elements of $\tilde\lambda \downarrow (1,2)$.

Also, the sets $\tilde\lambda \downarrow (2,1)$ and $\tilde\lambda \downarrow (2,2)$ are equal. We choose $J(2) = \{2\}$; and $\tilde\lambda \downarrow (2,2)$ contains a single element written as a $(8,2,2)$-colored $(2,2)$-necklace as given in Figure 9.

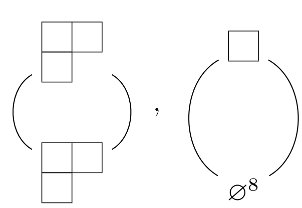

Figure 9. Element of $\tilde\lambda \downarrow (2,2)$.

Theorem 4.16 implies that as a $L(4,2,8)$-module, $V^{((\tilde\lambda_1,\tilde\lambda_2),\delta)}$ has a multiplicity free decomposition into a direct sum of irreducible representations parametrized by the elements of $\tilde\lambda \downarrow (1,2) \cup \tilde\lambda \downarrow (2,2)$.

5. Schur–Weyl duality for Tanabe algebras

Let $V = \mathbb{C}^n$ be the n-dimensional vector space with standard basis $\{v_1, v_2, \ldots, v_n\}$. There is a natural action of $GL_n(\mathbb{C})$ on V. For $k \in \mathbb{Z}_{\geqslant 0}$, consider the k-fold tensor product $V^{\otimes k} = V \otimes V \otimes \cdots \otimes V$ with the basis

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \mid 1 \leqslant i_1, i_2, \ldots, i_k \leqslant n\}.$$

With respect to this basis, $F \in \mathrm{End}(V^{\otimes k})$ can be written as a matrix $\left(F^{i_1,\cdots,i_k}_{i_{1'},\cdots,i_{k'}}\right)$ such that

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})F = \sum_{1 \leqslant i_{1'}, i_{2'}, \ldots, i_{k'} \leqslant n} F^{i_1,\cdots,i_k}_{i_{1'},\cdots,i_{k'}} (v_{i_{1'}} \otimes v_{i_{2'}} \otimes \cdots \otimes v_{i_{k'}}).$$

The action of $GL_n(\mathbb{C})$ on $V^{\otimes k}$ is given by

$$g(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = gv_{i_1} \otimes gv_{i_2} \otimes \cdots \otimes gv_{i_k}$$

for $g \in GL_n(\mathbb{C})$ and $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \in V^{\otimes k}$. The symmetric group S_n can be identified with the subgroup of permutation matrices of $GL_n(\mathbb{C})$. Also, we can identify the subgroup S_{n-1} of S_n fixing n with the subgroup of of permutation matrices having (n,n)-th entry as 1 of $GL_n(\mathbb{C})$. The action of S_n on $V^{\otimes k}$ is given by the restriction of the action of $GL_n(\mathbb{C})$ to S_n. Define $V^{\otimes(k+\frac{1}{2})} := V^{\otimes k} \otimes v_n$, a subspace of $V^{\otimes(k+1)}$, which is isomorphic to $V^{\otimes k}$ as a S_{n-1}-module.

Define a map

$$\begin{aligned} \phi_k :\ \mathbb{C}A_k(n) &\to \mathrm{End}(V^{\otimes k}) \\ d &\mapsto \phi_k(d) \end{aligned}$$

such that for $d \in A_k$ and for $1 \leqslant i_1, i_2, \ldots, i_k, i_{1'}, i_{2'}, \ldots, i_{k'} \leqslant n$,

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})(\phi_k(d)) = \sum_{1 \leqslant i_{1'}, i_{2'}, \ldots, i_{k'} \leqslant n} (\phi_k(d))^{i_1,i_2,\ldots,i_k}_{i_{1'},i_{2'},\ldots,i_{k'}} (v_{i_{1'}} \otimes v_{i_{2'}} \otimes \cdots \otimes v_{i_{k'}})$$

where

$$(\phi_k(d))^{i_1,i_2,\ldots,i_k}_{i_{1'},i_{2'},\ldots,i_{k'}} = \begin{cases} 1, & \text{if } i_r = i_s \text{ when } r \text{ and } s \text{ are} \\ & \qquad\qquad \text{in the same block of } d, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

This defines a right action of $\mathbb{C}A_k(n)$ on $V^{\otimes k}$ as:

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})d \ := \ (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})(\phi_k(d)).$$

It follows from (8) and (19) that for all $d \in A_k$,

$$(\phi_k(x_d))^{i_1,i_2,\ldots,i_k}_{i_{1'},i_{2'},\ldots,i_{k'}} = \begin{cases} 1, & \text{if } i_r = i_s \text{ if and only if } r \text{ and } s \\ & \qquad\qquad \text{are in the same block of } d, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

The action of the partition algebra $\mathbb{C}A_{k+\frac{1}{2}}(n)$ on $V^{\otimes(k+\frac{1}{2})}$ is

$$\phi_{k+\frac{1}{2}} : \mathbb{C}A_{k+\frac{1}{2}}(n) \longrightarrow \mathrm{End}(V^{\otimes(k+\frac{1}{2})})$$

given by $\phi_{k+\frac{1}{2}} = \phi_{k+1}|_{\mathbb{C}A_{k+\frac{1}{2}}(n)}$.

The following theorem is [8, Theorem 3.6] which shows that $\mathbb{C}A_k(n)$ and $\mathbb{C}A_{k+\frac{1}{2}}(n)$ are in Schur–Weyl duality with S_n and S_{n-1} acting on $V^{\otimes k}$ and $V^{\otimes(k+\frac{1}{2})}$, respectively.

Theorem 5.1.

(a) *The image of the map* $\phi_k :\ \mathbb{C}A_k(n) \to \mathrm{End}(V^{\otimes k})$ *is given by* $\mathrm{End}_{S_n}(V^{\otimes k})$ *and the kernel is given by*

$$\mathbb{C}\text{-}\operatorname{span}\{x_d \mid d \text{ has more than } n \text{ blocks}\}.$$

Thus, the partition algebra $\mathbb{C}A_k(n)$ *is isomorphic to* $\mathrm{End}_{S_n}(V^{\otimes k})$ *if and only if* $n \geqslant 2k$.

(b) *The image of the map* $\phi_{k+\frac{1}{2}} : \mathbb{C}A_{k+\frac{1}{2}}(n) \to \mathrm{End}(V^{\otimes(k+\frac{1}{2})})$ *is given by* $\mathrm{End}_{S_{n-1}}(V^{\otimes(k+\frac{1}{2})})$ *and the kernel is given by*

$$\mathbb{C}\text{-}\operatorname{span}\{x_d \mid d \text{ has more than } n \text{ blocks}\}.$$

Thus, the partition algebra $\mathbb{C}A_{k+\frac{1}{2}}(n)$ *is isomorphic to* $\mathrm{End}_{S_{n-1}}(V^{\otimes(k+\frac{1}{2})})$ *if and only if* $n \geqslant 2k+1$.

Let $\Pi_k(r,n)$ and $\Pi_{k+\frac{1}{2}}(r,n)$ be subsets of $\Pi_k(r)$ and $\Pi_{k+\frac{1}{2}}(r)$ (defined in Section 3), respectively, consisting of those elements which have at most n blocks. Define

$$\Pi_k(r,p,n) := \Pi_k(r,n) \cup \Lambda_k(r,p,n), \text{ and}$$
$$\Pi_{k+\frac{1}{2}}(r,p,n) := \Pi_{k+\frac{1}{2}}(r,n) \cup \Lambda_{k+\frac{1}{2}}(r,p,n),$$

subsets of $A_k(r,p,n)$ and $A_{k+\frac{1}{2}}(r,p,n)$, respectively.

The actions of $G(r,p,n)$ and $L(r,p,n)$ on V are given by restrictions of the action of $GL_n(\mathbb{C})$ on V. Also, V is the reflection representation of $G(r,p,n)$. We note that $\mathbb{C}$-span$\{v_n\}$ is a $L(r,p,n)$-invariant subspace of V. The following lemma gives bases of the centralizer algebras of the diagonal actions of $G(r,p,n)$ and $L(r,p,n)$ on $V^{\otimes k}$ and $V^{\otimes(k+\frac{1}{2})}$, respectively. Part (a) is [27, Lemma 2.1] and we follow the proof there to prove part (b) here.

Lemma 5.2.

(a) $\{\phi_k(x_d)) \mid d \in \Pi_k(r,p,n)\}$ *is a basis of* $\mathrm{End}_{G(r,p,n)}(V^{\otimes k})$.

(b) $\{\phi_{k+\frac{1}{2}}(x_d) \mid d \in \Pi_{k+\frac{1}{2}}(r,p,n)\}$ *is a basis of* $\mathrm{End}_{L(r,p,n)}(V^{\otimes(k+\frac{1}{2})})$.

Proof. (b) An element $d \in \Pi_{k+\frac{1}{2}}(r,p,n)$ has at most n blocks. By part (b) of Theorem 5.1, $\phi_{k+\frac{1}{2}}(x_d) \neq 0$. Also,

$$\{\phi_{k+\frac{1}{2}}(x_d) \mid d \in \Pi_{k+\frac{1}{2}}(r,p,n)\} \subset \{\phi_{k+\frac{1}{2}}(x_d) \mid d \in A_{k+\frac{1}{2}}\}$$

is a linearly independent set.

Since S_{n-1} is a subgroup of $L(r,p,n)$, thus we have

$$\mathrm{End}_{L(r,p,n)}(V^{\otimes(k+\frac{1}{2})}) \subset \mathrm{End}_{S_{n-1}}(V^{\otimes(k+\frac{1}{2})}).$$

Choose $0 \neq F \in \mathrm{End}_{L(r,p,n)}(V^{\otimes(k+\frac{1}{2})})$. Then, F can be written as

$$\begin{aligned} F &= \sum_{d\in A_{k+\frac{1}{2}}} a_d \phi_{k+\frac{1}{2}}(x_d) \\ &= \sum_{d\in \Pi_{k+\frac{1}{2}}(r,p,n)} a_d \phi_{k+\frac{1}{2}}(x_d) + \sum_{\substack{d\in A_{k+\frac{1}{2}},\\ d\notin \Pi_{k+\frac{1}{2}}(r,p,n)}} a_d \phi_{k+\frac{1}{2}}(x_d). \end{aligned}$$

with $\phi_{k+\frac{1}{2}}(x_d) \neq 0$ and $a_d \neq 0$ for some $d \in A_{k+\frac{1}{2}}$. Fix such a $d \in A_{k+\frac{1}{2}}$ and let $1 \leqslant i_1, \dots, i_k, i_{1'}, \dots, i_{k'} \leqslant n$ with $i_{k+1} = i_{(k+1)'} = n$ such that

$$(\phi_k(x_d))^{i_1,i_2,\dots,i_k,n}_{i_{1'},i_{2'},\dots,i_{k'},n} = 1.$$

For $1 \leqslant u \leqslant n$, define

$$B_u := \{j \in \{1,\dots,k+1,1',\dots,(k+1)'\} \mid i_j = u\}.$$

Note that $d = \{B_1, B_2, \dots, B_n\}$, where some of the blocks $B_1, B_2, \dots, B_{n-1}$ may be empty and $\{k+1,(k+1)'\} \subseteq B_n$.

For $1 \leqslant i \leqslant n$, define

$$t_i := (1,\dots,1,\zeta^p,1,\dots,1),$$

where ζ^p is i-th component, and for $1 \leqslant i \neq j \leqslant n$, define

$$h_{ij} := (1, \dots, 1, \zeta, 1, \dots, 1, \zeta^{-1}, 1, \dots, 1),$$

where ζ and ζ^{-1} are i-th and j-th components, respectively. The elements t_i, for $1 \leqslant i \leqslant n$, and the elements h_{ij}, for $1 \leqslant i \neq j \leqslant n$, together generate $\mathrm{D}(r,p,n)$.

For $1 \leqslant i \leqslant n$,

$$t_i^{-1} F t_i (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \otimes v_{i_{k+1}}) = (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \otimes v_{i_{k+1}}) F$$

implies that

$$\begin{aligned} &\sum_{i_{1'}, i_{2'}, \dots, i_{k'}} \zeta^{p(N(B_i)-M(B_i))} F^{i_1, \cdots, i_k, i_{k+1}}_{i_{1'}, \cdots, i_{k'}, i_{(k+1)'}} (v_{i_{1'}} \otimes v_{i_{2'}} \otimes \cdots \otimes v_{i_{k'}} \otimes v_{i_{(k+1)'}}) \\ (21) \qquad &= \sum_{i_{1'}, i_{2'}, \dots, i_{k'}} F^{i_1, \cdots, i_k, i_{k+1}}_{i_{1'}, \cdots, i_{k'}, i_{(k+1)'}} (v_{i_{1'}} \otimes v_{i_{2'}} \otimes \cdots \otimes v_{i_{k'}} \otimes v_{i_{(k+1)'}}). \end{aligned}$$

For $1 \leqslant i \neq j \leqslant n$,

$$h_{ij}^{-1} F h_{ij} (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \otimes v_{i_{k+1}}) = (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \otimes v_{i_{k+1}}) F$$

implies that

$$\begin{aligned} &\sum_{i_{1'}, i_{2'}, \dots, i_{k'}} \zeta^{(N(B_i)-M(B_i))-(N(B_j)-M(B_j))} F^{i_1, \cdots, i_k, i_{k+1}}_{i_{1'}, \cdots, i_{k'}, i_{(k+1)'}} (v_{i_{1'}} \otimes v_{i_{2'}} \otimes \cdots \otimes v_{i_{k'}} \otimes v_{i_{(k+1)'}}) \\ (22) \qquad &= \sum_{i_{1'}, i_{2'}, \dots, i_{k'}} F^{i_1, \cdots, i_k, i_{k+1}}_{i_{1'}, \cdots, i_{k'}, i_{(k+1)'}} (v_{i_{1'}} \otimes v_{i_{2'}} \otimes \cdots \otimes v_{i_{k'}} \otimes v_{i_{(k+1)'}}). \end{aligned}$$

From (21) and (22) we have

$$(23) \qquad N(B_i) \equiv M(B_i) (\operatorname{mod} m), \text{ for } 1 \leqslant i \leqslant n,$$

$$(24) \qquad N(B_i) - M(B_i) \equiv N(B_j) - M(B_j) (\operatorname{mod} r), \text{ for } 1 \leqslant i \neq j \leqslant n.$$

The following two cases arise.

(i) If $N(B_1) \equiv M(B_1) (\operatorname{mod} r)$, then (24) implies that $N(B_i) \equiv M(B_i) (\operatorname{mod} r)$ for all $1 \leqslant i \leqslant n$. So, we have

$$d \in \Pi_{k+\frac{1}{2}}(r,n).$$

(ii) If $N(B_1) \not\equiv M(B_1) (\operatorname{mod} r)$, then (24) implies that $N(B_i) \not\equiv M(B_i) (\operatorname{mod} r)$ for all $1 \leqslant i \leqslant n$. Thus, the number of elements, $N(B_i) + M(B_i)$, in the block B_i is nonzero for all $1 \leqslant i \leqslant n$. So, all the n blocks, $B_1, \dots, B_n$, in d are nonempty. Along with (23), we get $d \in \Lambda_{k+\frac{1}{2}}(r,p,n)$.

Combining both the cases we get that $d \in \Pi_{k+\frac{1}{2}}(r,p,n)$. ☐

Recall from Section 3 that Tanabe algebras $\mathcal{T}_k(r,p,n)$ and $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ are subalgebras of partition algebras $\mathbb{C}A_k(n)$ and $\mathbb{C}A_{k+\frac{1}{2}}(n)$, respectively. The actions of $\mathcal{T}_k(r,p,n)$ and $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ on $V^{\otimes k}$ and $V^{\otimes (k+\frac{1}{2})}$, respectively, are given by:

$$\begin{aligned} \psi_k :\ \mathcal{T}_k(r,p,n) &\longrightarrow \operatorname{End}(V^{\otimes k}) \text{ with } \psi_k := \phi_{k|_{\mathcal{T}_k(r,p,n)}}, \\ \text{and } \psi_{k+\frac{1}{2}} :\ \mathcal{T}_{k+\frac{1}{2}}(r,p,n) &\longrightarrow \operatorname{End}(V^{\otimes (k+\frac{1}{2})}) \text{ with } \psi_{k+\frac{1}{2}} := \phi_{k+\frac{1}{2}|_{\mathcal{T}_{k+\frac{1}{2}}(r,p,n)}}. \end{aligned}$$

Using Theorem 5.3 and Corollary 5.5, we get that $\mathcal{T}_k(r,p,n)$ and $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ are in Schur–Weyl duality with $G(r,p,n)$ and $L(r,p,n)$ acting on $V^{\otimes k}$ and $V^{\otimes (k+\frac{1}{2})}$, respectively.

Theorem 5.3.

(a) *The image of the map* $\psi_k : \mathcal{T}_k(r,p,n) \to \operatorname{End}(V^{\otimes k})$ *is given by* $\operatorname{End}_{G(r,p,n)}(V^{\otimes k})$ *and the kernel is given by*

$$\mathbb{C}\text{-}\operatorname{span}\{x_d \mid d \in A_k(r,p,n) \textit{ has more than } n \textit{ blocks}\}.$$

Thus, Tanabe algebra $\mathcal{T}_k(r,p,n)$ *is isomorphic to* $\operatorname{End}_{G(r,p,n)}(V^{\otimes k})$ *if and only if* $n \geqslant 2k$.

(b) *The image of the map* $\psi_{k+\frac{1}{2}} : \mathcal{T}_{k+\frac{1}{2}}(r,p,n) \to \operatorname{End}(V^{\otimes(k+\frac{1}{2})})$ *is given by* $\operatorname{End}_{L(r,p,n)}(V^{\otimes(k+\frac{1}{2})})$ *and the kernel is given by*

$$\mathbb{C}\text{-}\operatorname{span}\{x_d \mid d \in A_{k+\frac{1}{2}}(r,p,n) \textit{ has more than } n \textit{ blocks}\}.$$

Thus, Tanabe algebra $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ *is isomorphic to* $\operatorname{End}_{L(r,p,n)}(V^{\otimes(k+\frac{1}{2})})$ *if and only if* $n \geqslant 2k+1$.

Proof. (a) For $r = 1$, this is Theorem 5.1(a) which is Schur–Weyl duality between $\mathbb{C}A_k(n)$ and S_n acting on $V^{\otimes k}$. Now consider $r \geqslant 2$. Using Lemma 5.2(a), we have

$$\operatorname{End}_{G(r,p,n)}(V^{\otimes k}) = \mathbb{C}\text{-}\operatorname{span}\{\psi_k(x_d) \mid d \in \Pi_k(r,p,n)\} \subset \psi_k(\mathcal{T}_k(r,p,n)).$$

The element $d \in A_k(r,p,n) \smallsetminus \Pi_k(r,p,n)$ has more than n blocks. So, Theorem 5.1(a) implies that

$$(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})\psi_k(x_d) = 0,$$

for $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \in V^{\otimes k}$. Thus, we get the image and kernel as stated in the theorem. The kernel of ψ_k is zero if and only if $n \geqslant 2k$.

(b) The proof of this part is along the similar lines as that of part (a) using Lemma 5.2(b) and Theorem 5.1(b). □

Remark 5.4. Putting $p = 1$ in Theorem 5.3(a) we recover Schur–Weyl duality between $\mathcal{T}_k(r,1,n)$ and $G(r,1,n)$ as given in [19, Theorem 5.4].

Using the diagonal actions of $G(r,p,n)$ and $L(r,p,n)$ on $V^{\otimes k}$ and $V^{\otimes(k+\frac{1}{2})}$, respectively, we define the following maps

$$\vartheta_k : \mathbb{C}[G(r,p,n)] \to \operatorname{End}(V^{\otimes k}) \text{ and } \vartheta_{k+\frac{1}{2}} : \mathbb{C}[L(r,p,n)] \to \operatorname{End}(V^{\otimes(k+\frac{1}{2})}).$$

Corollary 5.5. *Assume that* $n \geqslant 2k$. *Then,*

(a) *The image of the map* $\vartheta_k : \mathbb{C}[G(r,p,n)] \to \operatorname{End}(V^{\otimes k})$ *is given by* $\operatorname{End}_{\mathcal{T}_k(r,p,n)}(V^{\otimes k})$.

(b) *The image of the map* $\vartheta_{k+\frac{1}{2}} : \mathbb{C}[L(r,p,n)] \to \operatorname{End}(V^{\otimes(k+\frac{1}{2})})$ *is given by* $\operatorname{End}_{\mathcal{T}_{k+\frac{1}{2}}(r,p,n)}(V^{\otimes(k+\frac{1}{2})})$.

Proof. (a) Theorem 5.3(a) implies that for $n \geqslant 2k$, the algebra $\mathcal{T}_k(r,p,n)$ is isomorphic to $\operatorname{End}_{G(r,p,n)}(V^{\otimes k})$ and the action of $\mathcal{T}_k(r,p,n)$ on $V^{\otimes k}$ commutes with the action of $G(r,p,n)$ on $V^{\otimes k}$. So, $\vartheta_k(\mathbb{C}[G(r,p,n)]) \subseteq \operatorname{End}_{\mathcal{T}_k(r,p,n)}(V^{\otimes k})$. Using double centralizer theorem [3, Theorem 1.3], we get that $\vartheta_k(\mathbb{C}[G(r,p,n)]) = \operatorname{End}_{\mathcal{T}_k(r,p,n)}(V^{\otimes k})$.

(b) Theorem 5.3(b) and double centralizer theorem imply part (b). □

6. Bratteli diagram of Tanabe algebras

Let us first study the decomposition of $V^{\otimes k}$ and $V^{\otimes(k+\frac{1}{2})}$ as $G(r,p,n)$-module and as $L(r,p,n)$-module, respectively. For the rest of the paper, we assume that $r \geqslant 2$.

It can be easily seen using Okounkov–Vershik approach that the $G(r,1,n)$-module V is an irreducible module parametrized by $((n-1),(1),\varnothing,\dots,\varnothing) \in \mathcal{Y}(r,n)$. Using the theory of Section 4, we see that for $(r,p,n) \neq (2,2,2)$, the $G(r,p,n)$-module V is an irreducible module parametrized by $(\tilde{\lambda}, \delta)$, where $\tilde{\lambda} \in \mathcal{Y}(m,p,n)$ and $\delta \in C_\lambda$ are as follows:

(i) If $p \neq r$, then $\tilde{\lambda} = (\tilde{\lambda}_{(1)}, \dots, \tilde{\lambda}_{(m)})$ with

$$\begin{aligned} \tilde{\lambda}_{(1)} &= ((n-1), \varnothing, \dots, \varnothing), \\ \tilde{\lambda}_{(2)} &= ((1), \varnothing, \dots, \varnothing), \\ \tilde{\lambda}_{(i)} &= (\varnothing, \dots, \varnothing), \quad \text{for } i = 3, \dots, m, \end{aligned}$$

and $\delta = 1$ since $C_\lambda = \{1\}$.

(ii) If $p = r$ and $(r,p,n) \neq (2,2,2)$, then $\tilde{\lambda} = (\tilde{\lambda}_{(1)}) = ((n-1),(1),\varnothing,\dots,\varnothing)$ and $\delta = 1$ since $C_\lambda = \{1\}$.

For $(r,p,n) = (2,2,2)$, V is the direct sum of irreducible $G(2,2,2)$-modules parametrized by $(((1),(1)),1)$ and $(((1),(1)),-1)$.

Suppose that $\mathbf{1}_n$ is the trivial representation of $G(r,1,n)$. Then, $\sigma = \mathbf{1}_{n-1} \otimes \sigma_2$ is a one-dimensional representation of $L(r,1,n)$ and thus, by restriction, a representation of $L(r,p,n)$. The parametrization of σ as a $L(r,1,n)$-module is $\mu = ((n-1), \varnothing^n, \varnothing, \dots, \varnothing) \in \mathcal{Y}^n(r, n-1)$. The parametrization of σ as a $L(r,p,n)$-module is $\tilde{\mu}^{(n,i,j)} \in \mathcal{Y}^n(m,p,n-1)$ given as follows:

(i) If $p \neq r$, then $i = 2, j = 1$ and $\tilde{\mu}^{(n,2,1)} = (\tilde{\mu}_{(1)}, \tilde{\mu}^{(n,1)}_{(2)}, \dots, \tilde{\mu}_{(m)})$ with

$$\begin{aligned} \tilde{\mu}_{(1)} &= ((n-1), \varnothing, \dots, \varnothing), \\ \tilde{\mu}^{(n,1)}_{(2)} &= (\varnothing^n, \varnothing, \dots, \varnothing), \\ \tilde{\mu}_{(i)} &= (\varnothing, \dots, \varnothing) \text{ for } i = 3, \dots, m. \end{aligned}$$

(ii) If $p = r$, then $i = 1, j = 2$ and $\tilde{\mu}^{(n,1,2)} = (\tilde{\mu}^{(n,2)}_{(1)}) = ((n-1), \varnothing^n, \varnothing, \dots, \varnothing)$.

Using the above parametrizations of V and σ, by Frobenius reciprocity and Theorem 4.16, we have (for $(r,p,n) = (2,2,2)$ also)

$$V \cong \mathrm{Ind}^{G(r,p,n)}_{L(r,p,n)}(\sigma).$$

Let M be a $G(r,p,n)$-module. Then using the tensor identity, we have

$$\begin{aligned} \mathrm{Ind}^{G(r,p,n)}_{L(r,p,n)}\left(\mathrm{Res}^{G(r,p,n)}_{L(r,p,n)}(M) \otimes \sigma\right) &\cong M \otimes \mathrm{Ind}^{G(r,p,n)}_{L(r,p,n)}(\sigma) \\ &\cong M \otimes V. \end{aligned}$$

Thus, taking $M = V^{\otimes(k-1)}$ for $k \geqslant 1$, we have

$$\mathrm{Ind}^{G(r,p,n)}_{L(r,p,n)}\left(\mathrm{Res}^{G(r,p,n)}_{L(r,p,n)}(V^{\otimes(k-1)}) \otimes \sigma\right) \cong V^{\otimes k} \tag{25}$$

$$\text{and } \mathrm{Res}^{G(r,p,n)}_{L(r,p,n)}\left(\mathrm{Ind}^{G(r,p,n)}_{L(r,p,n)}\left(\mathrm{Res}^{G(r,p,n)}_{L(r,p,n)}(V^{\otimes(k-1)}) \otimes \sigma\right)\right) \cong V^{\otimes k} \tag{26}$$

as $G(r,p,n)$-module and $L(r,p,n)$-module, respectively.

It will be clear from the context whether we consider σ as a $L(r,1,n)$-module or as a $L(r,p,n)$-module. Given $\lambda^{(n,t)} \in \mathcal{Y}^n(r, n-1)$, assume that $V^{\lambda^{(n,t)}}$ is the corresponding irreducible $L(r,1,n)$-module.

Lemma 6.1. *For $\lambda^{(n,t)} \in \mathcal{Y}^n(r, n-1)$*

$$V^{\lambda^{(n,t)}} \otimes \sigma = V^{\lambda^{(n,z)}} \tag{27}$$

where $\lambda^{(n,z)} \in \mathcal{Y}^n(r, n-1)$ and $z = (t+1)(\mathrm{mod}\, r)$.

Proof. Noting that the GZ-subspace of σ is given by $\underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{(n-1)\text{- fold}} \otimes \sigma_2$ with GZ-basis element given by $\underbrace{v_1 \otimes \cdots \otimes v_1}_{(n-1)\text{- fold}} \otimes v_2$, the proof is similar to that of Theorem 4.6. □

Given an (n,i,j)-colored (m,p)-necklace $\tilde{\lambda}^{(n,i,j)} \in \mathcal{Y}^n(m,p,n-1)$, suppose that $V^{\tilde{\lambda}^{(n,i,j)}}$ is the corresponding irreducible $L(r,p,n)$-module.

Lemma 6.2. *For* $\tilde{\lambda}^{(n,i,j)} \in \mathcal{Y}^n(m,p,n-1)$

$$V^{\tilde{\lambda}^{(n,i,j)}} \otimes \sigma = V^{\tilde{\lambda}^{(n,x,y)}} \tag{28}$$

where $\tilde{\lambda}^{(n,x,y)} \in \mathcal{Y}^n(m,p,n-1)$ *is obtained from* $\tilde{\lambda}^{(n,i,j)}$ *by the following rule:*

(a) *If* $i < m$, *then* $x = i+1$ *and* $y = j$;
(b) *If* $i = m$, *then* $x = 1$ *and* $y = (j+1)(\operatorname{mod} p)$.

Proof. The proof follows by using Lemma 6.1 and Theorem 4.12. □

Define the sets $\Omega_k(r,p,n)$ and $\Omega_{k+\frac{1}{2}}(r,p,n)$ as follows. Let

$$\Omega_0(r,p,n) = \{(\tilde{\lambda}, 1)\}$$

where $\tilde{\lambda} = (((n), \varnothing, \ldots, \varnothing), (\varnothing, \ldots, \varnothing), \ldots, (\varnothing, \ldots, \varnothing)) \in \mathcal{Y}(m,p,n)$. For $k \in \mathbb{Z}_{>0}$ the sets $\Omega_k(r,p,n) \subseteq \mathcal{Y}(m,p,n) \times C$ and $\Omega_{k+\frac{1}{2}}(r,p,n) \subseteq \mathcal{Y}^n(m,p,n-1)$ are obtained by the following recursive rule.

From $\Omega_k(r,p,n)$ *to* $\Omega_{k+\frac{1}{2}}(r,p,n)$:

For $(\tilde{\lambda}, \delta) \in \Omega_k(r,p,n)$, let $\tilde{\lambda}_{(i,j)-} \in \mathcal{Y}(m,p,n-1)$ be the set of (m,p)-necklaces obtained by deleting an inner corner from $\tilde{\lambda}_{(i,j)}$. For $\tilde{\mu} \in \tilde{\lambda}_{(i,j)-}$, color $\tilde{\mu}$ by (n,i,j) to obtain $\tilde{\mu}^{(n,i,j)} \in \Omega_{k+\frac{1}{2}}(r,p,n)$.

From $\Omega_{k+\frac{1}{2}}(r,p,n)$ *to* $\Omega_{k+1}(r,p,n)$:

For $\tilde{\mu}^{(n,i,j)} \in \Omega_{k+\frac{1}{2}}(r,p,n)$, remove the color (n,i,j) to get $\tilde{\mu} \in \mathcal{Y}(m,p,n-1)$ and then add a box to an outer corner, either in the j-th node of $(i+1)$-th component of $\tilde{\mu}$ if $1 \leqslant i \leqslant m-1$ or in the $(j+1)(\operatorname{mod} p)$-th node of the first component of $\tilde{\mu}$ if $i = m$, to obtain $\tilde{\nu} \in \mathcal{Y}(m,p,n)$. Let C_ν be the corresponding stabilizer subgroup. For $\delta \in C_\nu \subseteq C$, the element $(\tilde{\nu}, \delta) \in \Omega_{k+1}(r,p,n)$.

Theorem 6.3. *The indexing sets of the irreducible* $G(r,p,n)$*-modules occuring in* $V^{\otimes k}$ *and of the irreducible* $L(r,p,n)$*-modules occuring in* $V^{\otimes (k+\frac{1}{2})}$ *are* $\Omega_k(r,p,n)$ *and* $\Omega_{k+\frac{1}{2}}(r,p,n)$, *respectively.*

Proof. The proof follows from (25), (26), Lemma 6.2, branching rule from $G(r,p,n)$ to $L(r,p,n)$ in Theorem 4.16, Frobenius reciprocity and the observation that the spaces $V^{\otimes (k+\frac{1}{2})}$ and $V^{\otimes k}$ are isomorphic as $L(r,p,n)$-modules. □

Theorem 6.4. *The indexing sets of the irreducible* $\operatorname{End}_{G(r,p,n)}(V^{\otimes k})$*-modules and of the irreducible* $\operatorname{End}_{L(r,p,n)}(V^{\otimes (k+\frac{1}{2})})$*-modules are* $\Omega_k(r,p,n)$ *and* $\Omega_{k+\frac{1}{2}}(r,p,n)$, *respectively.*

Proof. The proof is a consequence of the centralizer theorem ([8, Theorem 5.4]) and Theorem 6.3. □

Theorem 6.5. *Let* n *and* k *be nonnegative integers.*

(a) *For* $n \geqslant 2k$, *as* $(\mathbb{C}[G(r,p,n)], \mathcal{T}_k(r,p,n))$*-bimodule,*

$$V^{\otimes k} \cong \bigoplus_{(\tilde{\lambda},\delta) \in \Omega_k(r,p,n)} \left(V^{(\tilde{\lambda},\delta)} \otimes \mathcal{T}_k^{(\tilde{\lambda},\delta)} \right),$$

where $V^{(\tilde{\lambda},\delta)}$ is the irreducible $G(r,p,n)$-module and $\mathcal{T}_k^{(\tilde{\lambda},\delta)}$ is the irreducible $\mathcal{T}_k(r,p,n)$-module parametrized by $(\tilde{\lambda},\delta) \in \Omega_k(r,p,n)$. Also

$$\begin{aligned}\dim(\mathcal{T}_k^{(\tilde{\lambda},\delta)}) = & \text{ the number of paths from } (((n),\varnothing,\dots,\varnothing),1) \in \Omega_0(r,p,n) \\ & \text{ to } (\tilde{\lambda},\delta) \in \Omega_k(r,p,n) \text{ in the Bratteli diagram } \widehat{\mathcal{T}}(r,p,n).\end{aligned}$$

(b) *For $n \geqslant 2k+1$, as $(\mathbb{C}[L(r,p,n)], \mathcal{T}_{k+\frac{1}{2}}(r,p,n)))$-bimodule,*

$$V^{\otimes(k+\frac{1}{2})} \cong \bigoplus_{\tilde{\mu}^{(n,i,j)} \in \Omega_{k+\frac{1}{2}}(r,p,n)} \left(V^{\tilde{\mu}^{(n,i,j)}} \otimes \mathcal{T}_{k+\frac{1}{2}}^{\tilde{\mu}^{(n,i,j)}}\right),$$

where $V^{\tilde{\mu}^{(n,i,j)}}$ is the irreducible $L(r,p,n)$-module and $\mathcal{T}_{k+\frac{1}{2}}^{\tilde{\mu}^{(n,i,j)}}$ is the irreducible $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$-module parametrized by $\tilde{\mu}^{(n,i,j)} \in \Omega_{k+\frac{1}{2}}(r,p,n)$ and

$$\begin{aligned}\dim(\mathcal{T}_k^{\tilde{\mu}^{(n,i,j)}}) = & \text{ the number of paths from } (((n),\varnothing,\dots,\varnothing),1) \in \Omega_0(r,p,n) \\ & \text{ to } \tilde{\mu}^{(n,i,j)} \in \Omega_{k+\frac{1}{2}}(r,p,n) \text{ in the Bratteli diagram } \widehat{\mathcal{T}}(r,p,n).\end{aligned}$$

Proof. The proofs of (a) and (b) follow from Theorem 5.3(a) and (b), respectively along with the centralizer theorem, Theorem 6.3 and Theorem 6.4. □

For $(\tilde{\lambda},\delta) \in \Omega_k(r,p,n)$, define the set $A_{k-\frac{1}{2}}^{(\tilde{\lambda},\delta)}$ as consisting of the elements $\tilde{\mu}^{(n,i,j)} \in \Omega_{k-\frac{1}{2}}(r,p,n)$ for some $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$ such that $(\tilde{\lambda},\delta)$ is obtained from $\tilde{\mu}^{(n,i,j)}$ while constructing $\Omega_k(r,p,n)$ from $\Omega_{k-\frac{1}{2}}(r,p,n)$. For $\tilde{\mu}^{(n,i,j)} \in \Omega_{k+\frac{1}{2}}(r,p,n)$, define the set $A_k^{\tilde{\mu}^{(n,i,j)}}$ as consisting of the elements $(\tilde{\lambda},\delta) \in \Omega_k(r,p,n)$ such that $\tilde{\mu}^{(n,i,j)}$ is obtained from $(\tilde{\lambda},\delta)$ while constructing $\Omega_{k+\frac{1}{2}}(r,p,n)$ from $\Omega_k(r,p,n)$.

Corollary 6.6.

(a) *For $n \geqslant 2k$ and for $(\tilde{\lambda},\delta) \in \Omega_k(r,p,n)$, we have*

$$\mathrm{Res}^{\mathcal{T}_k(r,p,n)}_{\mathcal{T}_{k-\frac{1}{2}}(r,p,n)}(\mathcal{T}_k^{(\tilde{\lambda},\delta)}) = \bigoplus_{\tilde{\mu}^{(n,i,j)} \in A_{k-\frac{1}{2}}^{(\tilde{\lambda},\delta)}} \mathcal{T}_{k-\frac{1}{2}}^{\tilde{\mu}^{(n,i,j)}}.$$

(b) *For $n \geqslant 2k+1$ and for $\tilde{\mu}^{(n,i,j)} \in \Omega_{k+\frac{1}{2}}(r,p,n)$, we have*

$$\mathrm{Res}^{\mathcal{T}_{k+\frac{1}{2}}(r,p,n)}_{\mathcal{T}_k(r,p,n)}(\mathcal{T}_{k+\frac{1}{2}}^{\tilde{\mu}^{(n,i,j)}}) = \bigoplus_{(\tilde{\lambda},\delta) \in A_k^{\tilde{\mu}^{(n,i,j)}}} \mathcal{T}_k^{(\tilde{\lambda},\delta)}.$$

Proof. (a) Using Theorem 6.5(a) and (25),

$$\begin{aligned}\mathcal{T}_k^{(\tilde{\lambda},\delta)} &\cong \mathrm{Hom}_{G(r,p,n)}(V^{\otimes k}, V^{(\tilde{\lambda},\delta)}) \\ &\cong \mathrm{Hom}_{G(r,p,n)}(\mathrm{Ind}^{G(r,p,n)}_{L(r,p,n)}((\mathrm{Res}^{G(r,p,n)}_{L(r,p,n)} V^{\otimes(k-1)}) \otimes \sigma), V^{(\tilde{\lambda},\delta)}).\end{aligned}$$

From the above isomorphism and Frobenius reciprocity, we get

$$\begin{aligned}&\mathrm{Res}^{\mathcal{T}_k(r,p,n)}_{\mathcal{T}_{k-\frac{1}{2}}(r,p,n)} \mathcal{T}_k^{(\tilde{\lambda},\delta)} \\ &\cong \mathrm{Res}^{\mathcal{T}_k(r,p,n)}_{\mathcal{T}_{k-\frac{1}{2}}(r,p,n)} \mathrm{Hom}_{L(r,p,n)}((\mathrm{Res}^{G(r,p,n)}_{L(r,p,n)} V^{\otimes(k-1)}) \otimes \sigma, \mathrm{Res}^{G(r,p,n)}_{L(r,p,n)} V^{(\tilde{\lambda},\delta)}) \\ &\cong \mathrm{Hom}_{L(r,p,n)}(V^{\otimes(k-\frac{1}{2})}, \sigma' \otimes \mathrm{Res}^{G(r,p,n)}_{L(r,p,n)} V^{(\tilde{\lambda},\delta)}),\end{aligned} \tag{29}$$

where σ' is the contragredient representation of σ. As $L(r,1,n)$-representation, $\sigma' = \sigma^{-1} = \underbrace{\sigma \otimes \cdots \otimes \sigma}_{(r-1)\text{- fold}}$ is parametrized by $((n-1),\varnothing,\dots,\varnothing,\varnothing^n) \in \mathcal{Y}^n(r,n)$. First using

Theorem 4.16 and then by the repeated application of Lemma 6.2, we compute $\sigma' \otimes \mathrm{Res}^{G(r,p,n)}_{L(r,p,n)} V^{(\tilde{\lambda},\delta)}$. Then, from (29) and Theorem 6.5(b), we have the restriction rule.

(b) The proof is along the similar lines as that of part (a) using Theorem 6.5(b), (26), Theorem 4.16, Frobenius reciprocity and Theorem 6.5(a). $\square$

Orellana [19, p. 614] describes the rule for recursively constructing Bratteli diagram for the tower of algebras

$$\mathcal{T}_0(r,1,n) \subseteq \mathcal{T}_1(r,1,n) \subseteq \mathcal{T}_2(r,1,n) \subseteq \cdots.$$

We consider the tower of Tanabe algebras

$$\mathcal{T}_0(r,p,n) \subseteq \mathcal{T}_{\frac{1}{2}}(r,p,n) \subseteq \mathcal{T}_1(r,p,n) \subseteq \mathcal{T}_{\frac{3}{2}}(r,p,n) \subseteq \cdots \subseteq \mathcal{T}_{\lfloor \frac{n}{2} \rfloor}(r,p,n) \tag{30}$$

and using Theorems 6.3, 6.4 and Corollary 6.6, construct the corresponding Bratteli diagram $\widehat{\mathcal{T}}(r,p,n)$ recursively by the following rule: For $l \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$, the vertices at l-th level of Bratteli diagram are elements of the set $\Omega_l(r,p,n)$. A vertex v_l at l-th level is connected by an edge with a vertex $v_{l+\frac{1}{2}}$ at $(l+\frac{1}{2})$-th level if and only if $v_{l+\frac{1}{2}}$ is obtained from v_l while constructing $\Omega_{l+\frac{1}{2}}(r,p,n)$ from $\Omega_l(r,p,n)$. The Bratteli diagram of Tanabe algebras is a simple graph.

Remark 6.7. For $t \in \mathbb{Z}_{\geqslant 0}$, $t \leqslant \lfloor \frac{n}{2} \rfloor$ and $(\tilde{\lambda},\delta) \in \Omega_t(r,p,n)$, the stabilizer subgroup C_λ is non-trivial if and only if $(r,p,n) = (2,2,2k)$ and $t = k$; in this case $C_\lambda = \{1,-1\}$. Thus, for $n \geqslant 2k$, there is a one-to-one correspondence between the irreducible representations of the same degree occuring at t-th level in Bratteli diagrams for $\mathcal{T}_k(r,1,n)$ and $\mathcal{T}_k(r,p,n)$ if and only if $(r,p,n,t) \neq (2,2,2k,k)$; the correspondence in terms of parametrization is $\lambda \mapsto (\tilde{\lambda},1)$.

We draw Bratteli diagram, up to level $k=2$, of Tanabe algebras for $(r,p,n) = (2,2,4)$ in Figure 10. Note that at level $k=2$, a node parametrized by $(\tilde{\lambda},-1)$ also appears when $\lambda = ((2),(2))$ because C_λ is nontrivial.

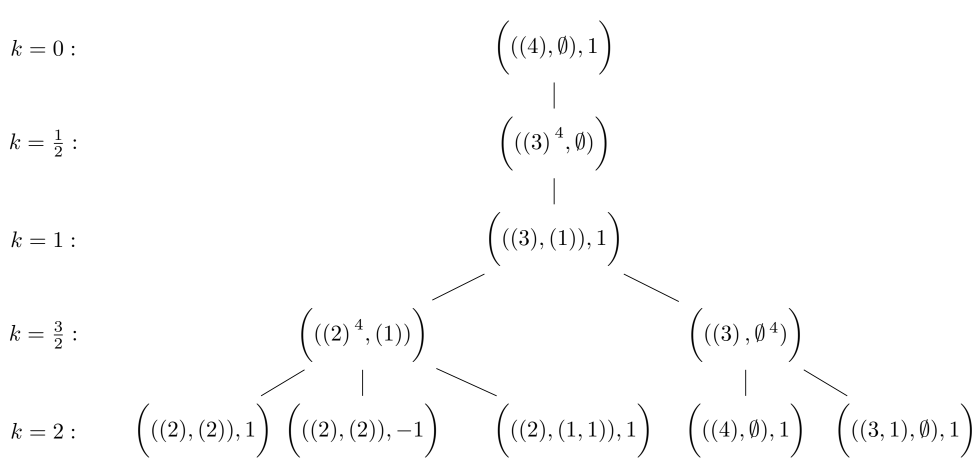

Figure 10. Bratteli diagram, up to level $k=2$, of Tanabe algebras for $(r,p,n) = (2,2,4)$.

In Figure 11, we draw Bratteli diagram, up to level $k = \frac{5}{2}$, of Tanabe algebras for $(r,p,n) = (6,2,6)$. Note that the representation $\mathcal{T}^v_{\frac{5}{2}}$ corresponding to $v = \big(((4),\varnothing),((1)^6,\varnothing),(\varnothing,\varnothing)\big)$ is of dimension two. To accommodate the figure, the last two vertices in the level $k=2$ in Figure 11 have been denoted by v_1 and v_2, where $v_1 = (((4)^6,\varnothing),(\varnothing,\varnothing),((1),\varnothing))$ and $v_2 = (((5),\varnothing),(\varnothing,\varnothing),(\varnothing^6,\varnothing))$.

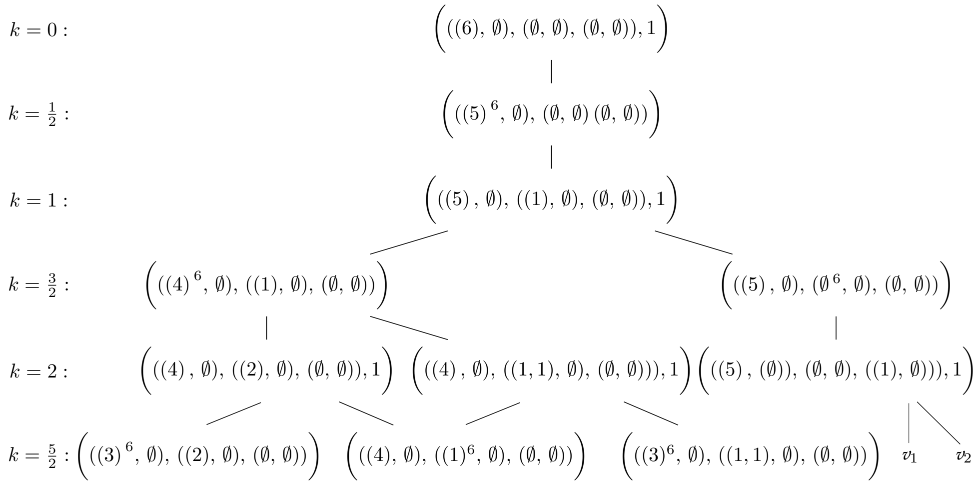

FIGURE 11. Bratteli diagram, up to level $k=\frac{5}{2}$, of Tanabe algebras for $(r,p,n)=(6,2,6)$.

7. Jucys–Murphy elements for Tanabe algebras

Recall from Section 2 that the Jucys–Murphy elements for $G(r,1,n)$ are:

$$X_1 = 0,$$
$$\text{and } X_j = \sum_{i=1}^{j-1}\sum_{l=0}^{r-1} \zeta_i^l \zeta_j^{-l} s_{ij}, \quad 2 \leqslant j \leqslant n.$$

For $T \in \mathrm{Tab}(r,\lambda)$, we have

$$\sum_{b\in\lambda} c(b) := \sum_{j=1}^{n} c(b_T(j))$$

and it is easily seen that $\sum_{b\in\lambda} c(b)$ is independent of the choice of $T \in \mathrm{Tab}(r,\lambda)$.

LEMMA 7.1.

(a) *For $r,n \in \mathbb{Z}_{\geqslant 0}$,*

$$\kappa_{r,n} := \frac{1}{r}\sum_{l=0}^{r-1}\sum_{1\leqslant i<j\leqslant n} \zeta_i^l\zeta_j^{-l} s_{ij}$$

is a central element of $\mathbb{C}[G(r,p,n)]$ and $\kappa_{r,n} = \sum_{b\in\lambda} c(b)$ as operators on $V^{(\tilde{\lambda},\delta)}$, the irreducible $G(r,p,n)$-module parametrized by $(\tilde{\lambda},\delta) \in \mathcal{Y}(m,p,n)\times C_\lambda$.

(b) *For $r,n \in \mathbb{Z}_{\geqslant 0}$,*

$$\kappa_{r,n-1} := \frac{1}{r}\sum_{l=0}^{r-1}\sum_{1\leqslant i<j\leqslant n-1} \zeta_i^l\zeta_j^{-l} s_{ij}$$

is a central element of $\mathbb{C}[L(r,p,n)]$ and $\kappa_{r,n-1} = \sum_{b\in\mu} c(b)$ as operators on $V^{\mu^{(n,i)}}$, the irreducible $L(r,1,n)$-module parametrized by $\mu^{(n,i)} \in \mathcal{Y}^n(r,n-1)$.

Proof. (a) First, we consider the case $p = 1$. Being the sum of elements in the conjugacy class of $(1, 1, \dots, 1, s_{12})$, $\kappa_{r,n}$ is a central element of $\mathbb{C}[G(r, 1, n)]$ and

$$\kappa_{r,n} = \frac{1}{r}\sum_{j=1}^{n} X_j.$$

For the irreducible $G(r, 1, n)$-module V^λ parametrized by $\lambda \in \mathcal{Y}(r, n)$, the canonical decomposition of V^λ into GZ-subspaces is

$$V^\lambda = \bigoplus_{T \in Tab(r,\lambda)} V_T.$$

Using Theorem 2.7 for the action of Jucys–Murphy elements of $G(r, 1, n)$, we have

$$\kappa_{r,n}(v_T) = \sum_{j=1}^{n} c(b_T(j))(v_T) = \bigg(\sum_{b\in\lambda} c(b)\bigg) v_T$$

where v_T is GZ-basis element corresponding to $T \in \mathrm{Tab}(r, \lambda)$. Thus, $\kappa_{r,n} = \sum_{b\in\lambda} c(b)$ as operators on V^λ. For a divisor p of r, note that $\kappa_{r,n} \in \mathbb{C}[G(r, p, n)] \subseteq \mathbb{C}[G(r, 1, n)]$. Thus, $\kappa_{r,n}$ is a central element of $\mathbb{C}[G(r, p, n)]$ also, and its action on the irreducible $G(r, p, n)$-module $V^{(\tilde{\lambda},\delta)}$ follows by restricting the action of $G(r, 1, n)$ on V^λ.

(b) The proof is along the similar lines as that of part (a). □

Now, we describe a particular central element in $\mathbb{C}[G(2, 2, 2k)]$. The conjugacy class C of the element $(1, 1, \dots, 1,(1, 2)(3, 4)\cdots(2k - 1, 2k))$ in $G(2, 1, 2k)$ consists of the elements of the form $(a_1, a_2, \dots, a_{2k},(i_1, i_2)(i_3, i_4)\cdots(i_{2k-1}, i_{2k}))$ such that $(i_1, i_2),(i_3, i_4), \dots,(i_{2k-1}, i_{2k})$ are mutually disjoint transpositions in S_{2k}, and $a_{i_j} a_{i_{j+1}} = 1$ for all $j = 1, 3, \dots, 2k - 1$ with $a_i \in G = \mathbb{Z}/2\mathbb{Z} = \{1, -1\}$ for all $i = 1, \dots, 2k$. Using [24, Theorem 11], the conjugacy class of $(1, 1, \dots, 1,(1, 2)(3, 4)\cdots(2k - 1, 2k))$ in $G(2, 1, 2k)$ decomposes into two conjugacy classes, denoted by C_1 and C_2, in $G(2, 2, 2k)$ with representatives

$$\begin{aligned} c_1 &= (-1, -1, 1, \dots, 1,(1, 2)(3, 4)\cdots(2k - 1, 2k)) \\ \text{and } c_2 &= (1, 1, \dots, 1,(1, 2)(3, 4)\cdots(2k - 1, 2k)), \end{aligned}$$

respectively. The classes C_1 and C_2 consist of those elements in C such that the number of pairs $(a_{i_j}, a_{i_{j+1}}) = (-1, -1)$, where $j = 1, 3, \dots, 2k - 1$, is odd and even, respectively. Let z_1 and z_2 be the conjugacy class sums of C_1 and C_2, respectively. Define $z = z_2 - z_1$ which is a central element in $\mathbb{C}[G(2, 2, 2k)]$.

Lemma 7.2. *Let $\lambda \in \mathcal{Y}(2, 2k)$. Then,*

(a) *For $\lambda \neq ((k),(k))$, $z = 0$ as operators on the irreducible $G(2, 2, 2k)$-module $V^{(\tilde{\lambda},1)}$.*

(b) *For $\lambda = ((k),(k))$, $z = 2^k k!$ as operators on the irreducible $G(2, 2, 2k)$-module $V^{(\tilde{\lambda},1)}$ and $z = -2^k k!$ as operators on the irreducible $G(2, 2, 2k)$-module $V^{(\tilde{\lambda},-1)}$.*

Proof. In the following, we use [16, Theorem 6.10] to describe the action of z on irreducible $G(2, 2, 2k)$-modules. The irreducible $G(2, 1, 2k)$-module V^λ parametrized by $\lambda = (\lambda_1, \lambda_2) \in \mathcal{Y}(2, 2k)$ has a GZ-basis element v_R where $R = (R_1, R_2)$ is the element of $\mathrm{Tab}(2, \lambda)$ written in row major order, i.e. the entries in R_1 are in from $1, \dots, |\lambda_1|$ and entries in R_2 are from $|\lambda_1| + 1, \dots, |\lambda_1| + |\lambda_2|$, both filled in row major order. We have the following cases:

(a) $\lambda \neq ((k),(k))$: V^λ remains irreducible as $G(2, 2, 2k)$-module and $V^{(\tilde{\lambda},1)} = V^\lambda$ with $v_R^{(1)} = v_R$ as one of the basis elements using the parametrization of irreducible

$G(2,2,2k)$-module in Theorem 4.10 and construction of basis of irreducible $G(2,2,2k)$-modules.

Let Y be the set of those $\pi \in S_{2k}$ which can be written as a product of disjoint transpositions such that the elements of each transposition are either in R_1 or in R_2. For a fixed $\pi \in Y$, the action of $\sum(a_1,\dots,a_{2k},\pi)$ on $v_R^{(1)}$, where the sum is over all such elements in C_1, is equal to the action of $\sum(b_1,\dots,b_{2k},\pi)$ on $v_R^{(1)}$, where the sum is over all such elements in C_2.

The coefficient of v_R in $tv_R^{(1)}$ is zero for any $t \in C_1 \cup C_2$ which is of the form $t = (a_1,a_2,\dots,a_{2k},(i_1,i_2)(i_3,i_4)\cdots(i_{2k-1},i_{2k}))$ such that there is at least one transposition (i_y,i_{y+1}) with one of i_y,i_{y+1} being from the entries in R_1 and the other being from the entries in R_2.

Thus, we have $zv_R^{(1)} = 0$.

(b) $\lambda = ((k),(k))$: V^{λ} decomposes into two irreducible as $G(2,2,2k)$-modules $V^{(\tilde{\lambda},1)}$ and $V^{(\tilde{\lambda},-1)}$ with $v_R^{(1)} = v_R + v_{\mathrm{sh}(R)}$ and $v_R^{(-1)} = v_R - v_{\mathrm{sh}(R)}$ as one of their basis elements, respectively. Analogous to part (a), for a fixed $\pi \in Y$, the action of $\sum(a_1,\dots,a_{2k},\pi)$ on v_R and $v_{\mathrm{sh}(R)}$, where the sum is over all such elements in C_1, is equal to the action of $\sum(b_1,\dots,b_{2k},\pi)$ on v_R and $v_{\mathrm{sh}(R)}$, where the sum is over all such elements in C_2, respectively.

Let P be the set of those $\beta \in S_{2k}$ which can be written as a product of disjoint transpositions such that one element of each transposition is from $1,\dots,k$ and the other one is from $k+1,\dots,2k$. The order of P is $k!$. For $(a_1,\dots,a_{2k},\beta) \in C_1$ and $\beta \in P$, $(a_1,\dots,a_{2k},\beta)v_R = -v_{\mathrm{sh}(R)}$ and $(a_1,\dots,a_{2k},\beta)v_{\mathrm{sh}(R)} = -v_R$. For $(a_1,\dots,a_{2k},\beta) \in C_2$ and $\beta \in P$, $(a_1,\dots,a_{2k},\beta)v_R = v_{\mathrm{sh}(R)}$ and $(a_1,\dots,a_{2k},\beta)v_{\mathrm{sh}(R)} = v_R$.

For those elements $(a_1,\dots,a_{2k},\gamma) \in C_1 \cup C_2$ such that $\gamma \notin Y \cup P$, the coefficients of both the elements v_R and $v_{\mathrm{sh}(R)}$ in both $(a_1,\dots,a_{2k},\gamma)v_R$ and $(a_1,\dots,a_{2k},\gamma)v_{\mathrm{sh}(R)}$ are zero. Thus,

$$zv_R^{(1)} = (2^k k!)v_R^{(1)} \text{ and } zv_R^{(-1)} = -(2^k k!)v_R^{(-1)}.$$

Thus, we get the scalars as stated in the theorem. □

Assume that S is a subset of $\{1,2,\dots,k\}$, I is a subset of $S \cup S'$ and I^c denotes the complement of I in $S \cup S'$, where S' is the set of all j' such that $j \in S$. Define the elements b_S and d_I of the partition monoid A_k:

$$b_S = \{S \cup S', \{l,l'\}_{l \notin S}\} \text{ and } d_I = \{I, I^c, \{l,l'\}_{l \notin S}\}.$$

Thus, $b_S \in \Pi_k(r)$. Also, it is easy to see that

$$d_I = d_{I^c},\ d_{S\cup S'} = d_{\varnothing} = b_S, \text{ and } d_{\{l,l'\}} = d_{\{l,l'\}^c} = b_{S \smallsetminus \{l\}}.$$

Example 7.3. For $k=6$, $S = \{1,2,4\}$, and $I = \{1,4'\} \subset S \cup S'$, b_S and d_I are given in Figure 12.

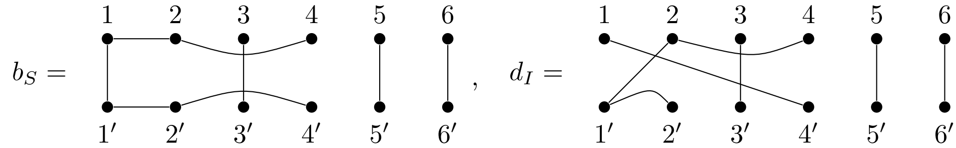

Figure 12. Example of b_S and d_I.

Following the notation of Section 3, let $N(I)$ and $M(I)$ denote the number of elements in top row and bottom row of the block I, respectively. For $k \in \mathbb{Z}_{\geqslant 0}$, we define an element $Z_{k,r} \in \mathcal{T}_k(r,1,n) \subseteq \mathcal{T}_k(r,p,n)$:

$$Z_{k,r} = \binom{n}{2} + \sum_{|S|\geqslant 1} (-1)^{|S|}\left((n-1)b_S + \sum_{N(I)\equiv M(I) (\operatorname{mod} r)} (-1)^{N(I)-M(I)}(d_I - b_S)\right),$$

where the outer sum is over all the nonempty subsets S of $\{1,2,\ldots,k\}$ and the inner sum is over $I \subseteq S \cup S'$ such that $d_I \in \Pi_k(r)$ and $d_I \neq b_S$.

Define an element $Z_{k+\frac{1}{2},r} \in \mathcal{T}_{k+\frac{1}{2}}(r,1,n) \subseteq \mathcal{T}_{k+\frac{1}{2}}(r,p,n)$ as follows:

$$\begin{aligned} Z_{k+\frac{1}{2},r} = \binom{n}{2} &+ \sum_{\substack{|S|\geqslant 1\\ k+1\notin S}} (-1)^{|S|}\left((n-1)b_S + \sum_{N(I)\equiv M(I) (\operatorname{mod} r)} (-1)^{N(I)-M(I)}(d_I - b_S)\right) \\ &+ \sum_{\substack{k+1\in S\\ |S|\geqslant 1}} (-1)^{|S|}\left((n-1)b_S + \sum_{\substack{\{k+1,(k+1)'\}\subset I \text{ or } I^c\\ N(I)\equiv M(I) (\operatorname{mod} r)}} (-1)^{N(I)-M(I)}(d_I - b_S)\right), \end{aligned}$$

where the first outer sum is over all the nonempty subsets S of $\{1,2,\ldots,k+1\}$ such that $k+1 \notin S$ and the inner sum in that is over $I \subseteq S \cup S'$ such that $d_I \in \Pi_{k+\frac{1}{2}}(r)$ and $d_I \neq b_S$; and the second outer sum is over all the nonempty subsets S of $\{1,2,\ldots,k,k+1\}$ such that $k+1 \in S$ and the inner sum in that is over $I \subseteq S \cup S'$ such that

$$\{k+1,(k+1)'\} \subseteq I \text{ or } I^c,\ d_I \in \Pi_{k+\frac{1}{2}}(r) \text{ and } d_I \neq b_S.$$

The elements $Z_{k,r}$ and $Z_{k+\frac{1}{2},r}$ and the idea of the proof of the next theorem are from the online notes [22].

Theorem 7.4.

(a) *Let $k \in \mathbb{Z}_{\geqslant 0}$. Then,*

$$\kappa_{r,n} = Z_{k,r} \text{ and } \kappa_{r,n-1} = Z_{k+\frac{1}{2},r}$$

as operators on $V^{\otimes k}$ and $V^{\otimes(k+\frac{1}{2})}$, respectively.

(b) *Let $k \in \mathbb{Z}_{\geqslant 0}$. Then $Z_{k,r}$ is a central element of $\mathcal{T}_k(r,p,n)$. For $n \geqslant 2k$*

$$Z_{k,r} = \sum_{b\in\lambda} c(b)$$

as operators on $\mathcal{T}_k^{(\tilde{\lambda},\delta)}$, the irreducible $\mathcal{T}_k(r,p,n)$-module parametrized by $(\tilde{\lambda},\delta) \in \Omega_k(r,p,n)$.

Also, $Z_{k+\frac{1}{2},r}$ is a central element of $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$. For $n \geqslant 2k+1$,

$$Z_{k+\frac{1}{2},r} = \sum_{b\in\mu} c(b)$$

as operators on $\mathcal{T}_{k+\frac{1}{2}}^{\tilde{\mu}^{(n,i,j)}}$, the irreducible $\mathcal{T}_{k+\frac{1}{2}}(r,p,n)$-module parametrized by $\tilde{\mu}^{(n,i,j)} \in \Omega_{k+\frac{1}{2}}(r,p,n)$.

Proof. (a) We express the action of $\kappa_{r,n}$ in terms of matrices $E_{i,j}$.

$$
\begin{aligned}
(31)\quad & 2\kappa_{r,n}(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k})\\
&= \frac{1}{r}\sum_{l=0}^{r-1}\sum_{1\leqslant i\neq j\leqslant n}\zeta_i^l\zeta_j^{-l}s_{ij}v_{i_1}\otimes\zeta_i^l\zeta_j^{-l}s_{ij}v_{i_2}\otimes\cdots\otimes\zeta_i^l\zeta_j^{-l}s_{ij}v_{i_k}\\
&= \frac{1}{r}\sum_{l=0}^{r-1}\sum_{1\leqslant i\neq j\leqslant n}(1-E_{ii}-E_{jj}+\zeta^lE_{ij}+\zeta^{-l}E_{ji})v_{i_1}\otimes\cdots\\
&\qquad\qquad\otimes(1-E_{ii}-E_{jj}+\zeta^lE_{ij}+\zeta^{-l}E_{ji})v_{i_k}.
\end{aligned}
$$

Let S be a subset of $\{1,2,\ldots,k\}$ such that S^c corresponds to the tensor positions where 1 is acting, $I\subset S\cup S'$ corresponds to the tensor positions that must equal i and I^c corresponds to the tensor positions that must equal j. Let

$$
\begin{aligned}
c_{S,I} := &\prod_{t\in S^c}(\delta_{i_ti_{t'}})(-1)^{\#(\{t,t'\}\subset I)+\#(\{t,t'\}\subset I^c)}\\
&\times\zeta^{l(\#(\{t\in I,t'\in I^c\})-\#(\{t\in I^c,t'\in I\}))}\prod_{t\in I}(\delta_{i_ti})\prod_{t\in I^c}(\delta_{i_tj})
\end{aligned}
$$

Thus, expanding (31), we get that $2\kappa_{r,n}(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k})$ equals

$$
(32)\qquad \frac{1}{r}\sum_{S\subset\{1,\ldots,k\}}\sum_{i_{1'},i_{2'},\ldots,i_{k'}}\sum_{l=0}^{r-1}\sum_{1\leqslant i\neq j\leqslant n}\sum_{I\subset S\cup S'}c_{S,I}(v_{i_{1'}}\otimes v_{i_{2'}}\otimes\cdots\otimes v_{i_{k'}}).
$$

Now, we take various cases of S and I to compute the above expression (32). Let $|S|=0$, then I is empty set and

$$c_{S,I}=c_{\varnothing,\varnothing}=\prod_{t\in\{1,\ldots,k\}}\delta_{i_ti_{t'}}.$$

The corresponding summand in (32) is

$$(n^2-n)(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k}).$$

Assume that $|S|\geqslant 1$ and we consider various cases of $I\subset S\cup S'$. Since the whole sum is symmetric in i and j and in I and I^c, therefore, the sum obtained is same when I is interchanged with I^c. If $I=S\cup S'$, then

$$c_{S,I}=\prod_{t\in S^c}(\delta_{i_ti_{t'}})(-1)^{|S|}\prod_{t\in S\cup S'}(\delta_{i_ti}),$$

and thus the corresponding summand in expression (32) is

$$
\begin{aligned}
\frac{1}{r}\sum_{i_{1'},i_{2'},\ldots,i_{k'}}\sum_{l=0}^{r-1}(n-1)\sum_{1\leqslant i\leqslant n}&c_{S,I}(v_{i_{1'}}\otimes v_{i_{2'}}\otimes\cdots\otimes v_{i_{k'}})\\
&=(n-1)(-1)^{|S|}b_S(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k}).
\end{aligned}
$$

We get an identical summand for the case $I=\varnothing$.

Consider $I\subsetneq S\cup S'$ and $N(I)\not\equiv M(I)(\operatorname{mod} r)$. Let

$$
\begin{aligned}
T(I)&=\{t\in\{1,2,\ldots,k\}\mid t\in I\},\\
D(I)&=\{t'\in\{1',2',\ldots,k'\}\mid t'\in I\},
\end{aligned}
$$

and

$$B(I)=\{t\in I\mid t'\in I\}.$$

Thus, $N(I)=|T(I)|,\ M(I)=|D(I)|$. Also, we can see that

$$\#(\{t\in I,t'\in I^c\})=N(I)-|B(I)|$$

and

$$\#(\{t\in I^c,t'\in I\})=M(I)-|B(I)|.$$

Thus,

$$\#(\{t\in I,t'\in I^c\})-\#(\{t\in I^c,t'\in I\})=N(I)-M(I)\not\equiv 0(\operatorname{mod} r).$$

In this case, since the sum of all the r-th roots of unity is zero, the summand for all such I in expression (32) is zero.

Now, consider those subsets $I\subsetneq S\cup S'$ such that $N(I)\equiv M(I)(\operatorname{mod} r)$. Define $B(I)':=\{t'\mid t\in B(I)\}$, thus

$$\begin{gathered}\{t\in\{1,2,\ldots,k\}\mid\{t,t'\}\subset I\}=B(I),\\ \text{and } \{t\in\{1,2,\ldots,k\}\mid\{t,t'\}\subset I^c\}=(S\smallsetminus T(I))\smallsetminus(D(I)\smallsetminus B(I)').\end{gathered}$$

This implies that

$$\begin{aligned}(-1)^{\#(\{t,t'\}\subset I)+\#(\{t,t'\}\subset I^c)}&=(-1)^{|B(I)|+(|S|-N(I))-(M(I)-|B(I)|)}\\&=(-1)^{|B(I)|+(|S|-N(I))+(M(I)-|B(I)|)}\\&=(-1)^{|S|-(N(I)-M(I))}\\&=(-1)^{|S|+(N(I)-M(I))}.\end{aligned}$$

Thus, for the subsets I such that $N(I)\equiv M(I)(\operatorname{mod} r)$, we get the summand in expression (32) as:

$$\begin{aligned}&\frac{1}{r}(-1)^{\#(\{t,t'\}\subset I)+\#(\{t,t'\}\subset I^c)}\sum_{i_{1'},i_{2'},\ldots,i_{k'}}\sum_{l=0}^{r-1}\prod_{t\in S^c}(\delta_{i_ti_{t'}})\\&\qquad\qquad\times\sum_{1\leqslant i\neq j\leqslant n}\prod_{t\in I}(\delta_{i_ti})\prod_{t\in I^c}(\delta_{i_tj})(v_{i_{1'}}\otimes v_{i_{2'}}\otimes\cdots\otimes v_{i_{k'}})\\&=(-1)^{|S|+(N(I)-M(I))}\sum_{i_{1'},i_{2'},\ldots,i_{k'}}\prod_{t\in S^c}(\delta_{i_ti_{t'}})\left(\sum_{1\leqslant i,j\leqslant n}\prod_{t\in I}(\delta_{i_ti})\prod_{t\in I^c}(\delta_{i_tj})\right.\\&\qquad\qquad\left.-\sum_{1\leqslant i=j\leqslant n}\prod_{t\in I}(\delta_{i_ti})\prod_{t\in I^c}(\delta_{i_tj})\right)(v_{i_{1'}}\otimes v_{i_{2'}}\otimes\cdots\otimes v_{i_{k'}})\\&=(-1)^{|S|+(N(I)-M(I))}(d_I-b_S)(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k}).\end{aligned}$$

Also, for the subsets I such that $N(I)\equiv M(I)(\operatorname{mod} r)$, we also have $N(I^c)\equiv M(I^c)(\operatorname{mod} r)$ and thus we get an identical summand by interchanging I and I^c.

Combining all the above cases together, we get that, as operators on $V^{\otimes k}$,

$$\kappa_{r,n}=Z_{k,r}.$$

Now we prove the second part of (b). We have

$$(1-E_{ii}-E_{jj}+E_{ii}E_{jj})(v_n)=\begin{cases}0, & \text{if } i=n \text{ or } j=n,\\ v_n, & \text{otherwise.}\end{cases}$$

Thus,

$$
\begin{aligned}
(33)\quad & 2\kappa_{r,n-1}(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k}\otimes v_n)\\
&= \frac{1}{r}\sum_{l=0}^{r-1}\sum_{1\leqslant i\neq j\leqslant n-1}\zeta_i^l\zeta_j^{-l}s_{ij}(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k}\otimes v_n)\\
&= \frac{1}{r}\sum_{l=0}^{r-1}\sum_{1\leqslant i\neq j\leqslant n}\zeta_i^l\zeta_j^{-l}s_{ij}v_{i_1}\otimes\zeta_i^l\zeta_j^{-l}s_{ij}v_{i_2}\otimes\\
&\qquad\cdots\otimes\zeta_i^l\zeta_j^{-l}s_{ij}v_{i_k}\otimes(1-E_{ii}-E_{jj}+E_{ii}E_{jj})(v_n)\\
&= \frac{1}{r}\sum_{l=0}^{r-1}\sum_{1\leqslant i\neq j\leqslant n}\zeta_i^l\zeta_j^{-l}s_{ij}(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k})\otimes v_n\\
&\quad+\frac{1}{r}\sum_{l=0}^{r-1}\sum_{1\leqslant i\neq j\leqslant n}\Big((1-E_{ii}-E_{jj}+\zeta^lE_{ij}+\zeta^{-l}E_{ji})v_{i_1}\otimes\cdots\\
&\quad\otimes(1-E_{ii}-E_{jj}+\zeta^lE_{ij}+\zeta^{-l}E_{ji})v_{i_k}\Big)\otimes(-E_{ii}-E_{jj})v_n\\
&\quad+\frac{1}{r}\sum_{l=0}^{r-1}\sum_{1\leqslant i\neq j\leqslant n}\zeta_i^l\zeta_j^{-l}s_{ij}v_{i_1}\otimes\zeta_i^l\zeta_j^{-l}s_{ij}v_{i_2}\otimes\cdots\otimes\zeta_i^l\zeta_j^{-l}s_{ij}v_{i_k}\otimes E_{ii}E_{jj}v_n.
\end{aligned}
$$

In the expression (33), the first summand is equal to $2\kappa_{r,n}(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k})$ which has been calculated in the first part of (b). Since $i\neq j$, the last summand is zero. Expanding the middle summand gives

$$
\frac{1}{r}\sum_{\substack{S\subset\{1,\dots,k+1\}\\ k+1\in S}}\ \sum_{i_{1'},i_{2'},\dots,i_{k'}}\ \sum_{l=0}^{r-1}\ \sum_{1\leqslant i\neq j\leqslant n}\ \sum_{\substack{I\subset S\cup S'\\ \{k+1,(k+1)'\}\subset I\text{ or }I^c}} c_{S,I}(v_{i_{1'}}\otimes v_{i_{2'}}\otimes\cdots\otimes v_{i_{k'}}).
$$

The case $|S|=0$ does not arise because $k+1\in S$. For $|S|>1$, we consider various cases of $I\subset S\cup S'$ which are:

(i) $I=S\cup S'$,
(ii) $\{k+1,(k+1)'\}\subset I\subsetneq S\cup S'$, $N(I)\not\equiv M(I)(\operatorname{mod} r)$, and
(iii) $\{k+1,(k+1)'\}\subset I\subsetneq S\cup S'$, $N(I)\equiv M(I)(\operatorname{mod} r)$.

and identical summands arise when I is interchanged with I^c in the cases (i), (ii), and (iii). Thus, the middle summand gives us

$$
\sum_{\substack{k+1\in S\\ |S|\geqslant 1}}(-1)^{|S|}2\Bigg((n-1)b_S+\sum_{\substack{\{k+1,(k+1)'\}\subset I\text{ or }I^c\\ N(I)\equiv M(I)(\operatorname{mod} r)}}(-1)^{N(I)-M(I)}(d_I-b_S)\Bigg).
$$

So, as operators on $V^{\otimes(k+\frac12)}$, we have $\kappa_{r,n-1}=Z_{k+\frac12,r}$.

(b) Using Theorem 6.5(a) and using Lemma 7.1(a), we get that for $n\geqslant 2k$, $Z_{k,r}$ acts on $\mathcal{T}_k^{(\tilde\lambda,\delta)}$ as the constant stated in the theorem. Therefore, $Z_{k,r}$ is a central element of $\mathcal{T}_k(r,p,n)$ for $n\geqslant 2k$. Since the multiplication of elements of $\mathcal{T}_k(r,p,n)$ is a polynomial in n, therefore the equality

$$Z_{k,r}x_d=x_dZ_{k,r}\text{ for all }x_d\in\mathcal{T}_k(r,p,n)\text{ and }n\geqslant 2k$$

implies the equality

$$Z_{k,r}x_d=x_dZ_{k,r}\text{ for all }x_d\in\mathcal{T}_k(r,1,n)\text{ and for all }n.$$

Theorem 6.5(b) and Lemma 7.1(b) along with the arguments similar to the above imply the result for $Z_{k+\frac{1}{2},r}$. $\square$

In the light of Remarks 3.12 and 6.7, $(r,p,n)=(2,2,2k)$ is the special case. Define $M_{k,2,2}:=x_d\in\mathcal{T}_k(2,2,2k)$, where d is the only element in $\Lambda_k(2,2,2k)$ and d consists of $2k$ blocks, each vertex being a block. The element $M_{k,2,2}$ is a central element of $\mathcal{T}_k(2,2,2k)$.

Theorem 7.5.

(a) *Let $k\in\mathbb{Z}_{\geqslant 0}$. Then, $M_{k,2,2}=\frac{1}{2^k}z$ as operators on $V^{\otimes k}$.*

(b) *Let $k\in\mathbb{Z}_{\geqslant 0}$. Then, for $\lambda\neq((k),(k))$, $M_{k,2,2}=0$ as operators on the irreducible $\mathcal{T}_k(2,2,2k)$-module $\mathcal{T}_k^{(\tilde\lambda,1)}$. For $\lambda=((k),(k))$,*

$$M_{k,2,2}=k!$$

as operators on the irreducible $\mathcal{T}_k(2,2,2k)$-module $\mathcal{T}_k^{(\tilde\lambda,1)}$ and

$$M_{k,2,2}=-k!$$

as operators on the irreducible $\mathcal{T}_k(2,2,2k)$-module $\mathcal{T}_k^{(\tilde\lambda,-1)}$.

Proof. (a) The action of $M_{k,2,2}$ on $V^{\otimes k}$ is:

$$M_{k,2,2}(v_{i_1}\otimes\cdots\otimes v_{i_k})=\begin{cases}\sum_\pi v_{\pi(j_1)}\otimes\cdots\otimes v_{\pi(j_k)}, & \text{if } i_1,i_2,\dots,i_k \text{ are distinct}\\ & \text{elements of } \{1,\dots,2k\},\\ 0, & \text{otherwise,}\end{cases}$$

where π varies over all the permutations of the set

$$\{j_1,\dots,j_k\}=\{1,\dots,2k\}\smallsetminus\{i_1,\dots,i_k\}.$$

Now, we discuss the action of z on $V^{\otimes k}$. Consider $v_{i_1}\otimes\cdots\otimes v_{i_k}\in V^{\otimes k}$ such that $i_1,\dots,i_k$ are distinct elements of $\{1,\dots,2k\}$. Then,

$$(a_1,\dots,a_{2k},(i_1,j_1)\cdots(i_k,j_k))(v_{i_1}\otimes\cdots\otimes v_{i_k})=-(v_{j_1}\otimes\cdots\otimes v_{j_k})$$

if $(a_1,\dots,a_{2k},(i_1,j_1)\cdots(i_k,j_k))\in C_1$, and

$$(a_1,\dots,a_{2k},(i_1,j_1)\cdots(i_k,j_k))(v_{i_1}\otimes\cdots\otimes v_{i_k})=(v_{j_1}\otimes\cdots\otimes v_{j_k})$$

if $(a_1,\dots,a_{2k},(i_1,j_1)\cdots(i_k,j_k))\in C_2$, where in each case

$$\{j_1,\dots,j_k\}=\{1,\dots,2k\}\smallsetminus\{i_1,\dots,i_k\}.$$

For a fixed $(i_1,j_1)\cdots(i_k,j_k)$ element in S_{2k}, there are 2^{k-1} elements of the form $(a_1,\dots,a_{2k},(i_1,j_1)\cdots(i_k,j_k))$ in each of C_1 and C_2.

Consider an element of the form $(a_1,\dots,a_{2k},(x_1,y_1)\cdots(x_k,y_k))\in C$ such that at least one pair, say $\{x_1,y_1\}\subset\{i_1,\dots,i_k\}$. Then, one of $x_2,\dots,x_k$, say x_k, is different from $i_1,\dots,i_k$ and one can choose $y_k\in\{1,\dots,2k\}\smallsetminus\{i_1,\dots,i_k,x_k,y_2,\dots,y_{k-1}\}$. Now, $(a_{x_k},a_{y_k})=(1,1)$ or $(a_{x_k},a_{y_k})=(-1,-1)$ keeps the sign of the action of $(a_1,\dots,a_{2k},(x_1,y_1)\cdots(x_k,y_k))$ on $(v_{i_1}\otimes\cdots\otimes v_{i_k})$ same.

Given $(b_1,\dots,b_{2k},(x_1,y_1)\cdots(x_k,y_k))\in C_1$ such that $(b_{x_1},b_{y_1})=(1,1)$, we have the element $(f_1,\dots,f_{2k},(x_1,y_1)\cdots(x_k,y_k))\in C_2$, such that $(f_{x_i},f_{y_i})=(b_{x_i},b_{y_i})$ for $i\neq k$ and $(f_{x_k},f_{y_k})=-(b_{x_k},b_{y_k})$ and

$$\begin{aligned}(b_1,\dots,b_{2k},&(x_1,y_1)\cdots(x_k,y_k))(v_{i_1}\otimes\cdots\otimes v_{i_k})\\&=(f_1,\dots,f_{2k},(x_1,y_1)\cdots(x_k,y_k))(v_{i_1}\otimes\cdots\otimes v_{i_k}).\end{aligned}$$

A similar analysis can be done if $(b_{x_1},b_{y_1})=(-1,-1)$.

If at least two of $i_1,\dots,i_k$ are same, say $i_1=i_2$, then we can find a pair (a_{x_k},a_{y_k}) such that the action of $(a_1,\dots,a_{2k},(x_1,y_1)\cdots(x_k,y_k))$ on $(v_{i_1}\otimes\cdots\otimes v_{i_k})$ has the same sign whether $(a_{x_k}, a_{y_k}) = (1,1)$ or $(-1,-1)$. A similar analysis as above shows that corresponding to any element $(b_1, \dots, b_{2k}, (x_1, y_1) \cdots (x_k, y_k)) \in C_1$ such that we can find the element $(f_1, \dots, f_{2k}, (x_1, y_1) \cdots (x_k, y_k)) \in C_2$, such that

$$\begin{aligned}(b_1, \dots, b_{2k}, (x_1, y_1) \cdots (x_k, y_k))&(v_{i_1} \otimes \cdots \otimes v_{i_k}) \\ &= (f_1, \dots, f_{2k}, (x_1, y_1) \cdots (x_k, y_k))(v_{i_1} \otimes \cdots \otimes v_{i_k}).\end{aligned}$$

Collecting all the cases, we have

$$z(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \begin{cases} (2^k) \sum_{\pi} v_{\pi(j_1)} \otimes \cdots \otimes v_{\pi(j_k)}, & \text{if } i_1, \dots, i_k \text{ are distinct} \\ & \text{elements of } \{1, \dots, 2k\}, \\ 0, & \text{otherwise,} \end{cases}$$

where π varies over all the permutations of the set

$$\{j_1, \dots, j_k\} = \{1, \dots, 2k\} \setminus \{i_1, \dots, i_k\}.$$

(b) The proof is clear by using part (a) of this theorem, Theorem 6.5 (a) and Lemma 7.2. □

For $l \in \frac{1}{2}\mathbb{Z}_{>0}$, define the Jucys–Murphy elements of $\mathcal{T}_l(r,p,n)$ as follows:

$$\begin{aligned} M_{\frac{1}{2},r} &= 1, \\ \text{and } M_{y,r} &= Z_{y,r} - Z_{y-\frac{1}{2},r}, \text{ for } y \in \frac{1}{2}\mathbb{Z}_{>0} \text{ and } 1 \leqslant y \leqslant l. \end{aligned}$$

In addition to these elements, $\mathcal{T}_k(2,2,2k)$ has one more Jucys–Murphy element which is $M_{k,2,2}$.

Theorem 7.6. *Let $l \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and let n be a positive integer.*

(a) *The elements $M_{\frac{1}{2},r}, M_{1,r}, \dots, M_{l-\frac{1}{2},r}, M_{l,r}$ commute with each other in Tanabe algebra $\mathcal{T}_l(r,p,n)$.*

(b) *Assume that $n \geqslant 2l$. Let $v_l \in \Omega_l(r,p,n)$ and $\mathcal{T}_l^{v_l}$ be the irreducible $\mathcal{T}_l(r,p,n)$-module parametrized by v_l. Then there is a unique, up to scalars, basis*

$$\{u_{\mathcal{P}} \mid \mathcal{P} \text{ is a path in } \widehat{\mathcal{T}}(r,p,n) \text{ from } v_0 = ((n), \varnothing, \dots, \varnothing) \text{ to } v_l\}$$

of $\mathcal{T}_l^{v_l}$ such that, for all $\mathcal{P} = (v_0, v_{\frac{1}{2}}, v_1, \dots, v_l)$, and for all $k \in \mathbb{Z}_{\geqslant 0}$, $k \leqslant l$

$$M_{k,r}(u_{\mathcal{P}}) = c(v_k / v_{k-\frac{1}{2}}) u_{\mathcal{P}},$$

and

$$M_{k+\frac{1}{2},r}(u_{\mathcal{P}}) = -c(v_k / v_{k+\frac{1}{2}}) u_{\mathcal{P}},$$

where $v_k / v_{k-\frac{1}{2}}$ and $v_k / v_{k+\frac{1}{2}}$ denote the box by which v_k differs from $v_{k-\frac{1}{2}}$ and $v_{k+\frac{1}{2}}$ as r-tuple of Young diagrams, respectively.

(c) *For $(r,p,n) = (2,2,2k)$, the element $M_{k,2,2}$ commutes with the elements $M_{\frac{1}{2},2}$, $M_{1,2}, \dots, M_{2k-\frac{1}{2},2}, M_{2k,2}$. The scalars by which the Jucys–Murphy elements of $\mathcal{T}_k(2,2,2k)$ act on the basis (as given by part (b)) of $\mathcal{T}_k^{(((k),(k)),1)}$ and $\mathcal{T}_k^{(((k),(k)),-1)}$ are same except for $M_{k,2,2}$.*

Proof. (a) For $i, j \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and $i \leqslant j \leqslant l$, we have $Z_{i,r}, Z_{j,r} \in \mathcal{T}_j(r,p,n)$ and $Z_{j,r}$ is a central element of $\mathcal{T}_j(r,p,n) \subseteq \mathcal{T}_l(r,p,n)$, thus $Z_{i,r} Z_{j,r} = Z_{j,r} Z_{i,r}$. Since $M_{j,r} = Z_{j,r} - Z_{j-\frac{1}{2},r}$, thus Jucys–Murphy elements commute with each other in $\mathcal{T}_l(r,p,n)$.

(b) The branching rule from $\mathcal{T}_j(r,p,n)$ to $\mathcal{T}_{j-\frac{1}{2}}(r,p,n)$ is multiplicity free for all $j \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and $n \geqslant 2j$. Thus, $\mathcal{T}_l^{v_l}$ has canonical decomposition as irreducible $\mathcal{T}_{l-\frac{1}{2}}$-module:

$$\mathcal{T}_l^{v_l} = \bigoplus_{v_{l-\frac{1}{2}} \in \Omega_{l-\frac{1}{2}}(r,p,n)} \mathcal{T}_{l-\frac{1}{2}}^{v_{l-\frac{1}{2}}}$$

such that there is an edge from $v_{l-\frac{1}{2}}$ to v_l in $\widehat{\mathcal{T}}(r,p,n)$. Further, iterating this decomposition, a canonical decomposition of $\mathcal{T}_l^{v_l}$ into irreducible $\mathcal{T}_0(r,p,n)$-modules is obtained:

$$\mathcal{T}_l^{v_l} = \bigoplus_{\mathcal{P}} \mathcal{T}_{\mathcal{P}}, \tag{34}$$

where $\mathcal{T}_{\mathcal{P}}$ are one-dimensional $\mathcal{T}_0(r,p,n)$-modules and the sum is over all paths $\mathcal{P} = (v_0, v_{\frac{1}{2}}, v_1, \ldots, v_l)$ such that $v_j \in \Omega_j(r,p,n)$. The basis of $\mathcal{T}_l^{v_l}$ is obtained by choosing a nonzero vector $u_{\mathcal{P}}$ in each $\mathcal{T}_{\mathcal{P}}$ in the decomposition (34). Such a basis is called the *Gelfand–Tsetlin basis* of the corresponding irreducible representation and it is unique, up to scalars. Using the decomposition (34) and the definition of $u_{\mathcal{P}}$, we get

$$\mathcal{T}_j(r,p,n)u_{\mathcal{P}} = \mathcal{T}_j^{v_j},$$

for all $j \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and $j \leqslant l$,which implies that $u_{\mathcal{P}}$ is a basis element of $\mathcal{T}_j^{v_j}$. Thus, for all $j \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and $j \leqslant l$, the action of $Z_{j,r}$ on $u_{\mathcal{P}}$ is as a scalar given in Theorem 7.4(b). Now, by the definition of Jucys–Murphy elements, we get their actions on $u_{\mathcal{P}}$.

(c) The element $M_{k,2,2}$ is a central element of $\mathcal{T}_k(2,2,2k)$. For $(r,p,n) = (2,2,2k)$ and $\lambda = ((k),(k))$, let $v_k = (\tilde{\lambda}, 1) \in \Omega_k(2,2,2k)$, $v_k' = (\tilde{\lambda}, -1) \in \Omega_k(2,2,2k)$. Then

$$\mathcal{T}_k^{v_k} \cong \mathcal{T}_k^{v_k'}$$

as $\mathcal{T}_{k-\frac{1}{2}}(r,p,n)$-modules. Thus, the part of the paths from v_0 to v_k and v_k' are same for $l < k$, $l \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and so, we have

$$M_j(u_{\mathcal{P}}) = M_j(u_{\mathcal{P}'}), \quad j \leqslant k \text{ and } j \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$$

where $u_{\mathcal{P}}$ and $u_{\mathcal{P}'}$ are Gelfand–Tsetlin basis elements of $\mathcal{T}_k^{v_k}$ and $\mathcal{T}_k^{v_k'}$, respectively. However, by Theorem 7.5(b), we get that

$$M_{k,2,2}u_{\mathcal{P}} = (k!)u_{\mathcal{P}} \text{ and } M_{k,2,2}u_{\mathcal{P}'} = -(k!)u_{\mathcal{P}'},$$

which proves the result. □

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