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Independent Spaces of q-Polymatroids

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Abstract This paper is devoted to the study of independent spaces of q-polymatroids. With the aid of an auxiliary q-matroid it is shown that the collection of independent spaces satisfies the same properties as for q-matroids. However, in contrast to q-matroids, the rank value of an independent space does not agree with its dimension. Nonetheless, the rank values of the independent spaces fully determine the q-polymatroid, and this fact can be exploited to derive a cryptomorphism of q-polymatroids. Finally, the notions of minimal spanning spaces, maximally strongly independent spaces, and bases will be elaborated on.

1. Introduction

Thanks to their relation to rank-metric codes, q-matroids and q-polymatroids have recently garnered a lot of attention, [1, 2, 3, 4, 5, 6, 7, 10]. Indeed, $\mathbb{F}_{q^m}$-linear rank-metric codes in $\mathbb{F}_{q^m}^n$ give rise to q-matroids, whereas $\mathbb{F}_q$-linear rank-metric codes induce q-polymatroids. This leads to an abundance of examples of q-(poly)matroids. In either case, the q-(poly)matroid induced by a rank-metric code arises via a rank function which captures the dimension of certain characteristic subspaces of the code in question. As a consequence, the q-(poly)matroid reflects many of the algebraic and combinatorial properties of the code, such as the generalized weights [5, 6] and the rank-weight enumerator [1, 10].

For q-matroids a variety of cryptomorphic definitions are known [3, 7]. They are based on independent spaces, bases, circuits, spanning spaces, flats and many more; see the comprehensive account in [3]. For q-polymatroids, most of these notions have yet to be defined. As to our knowledge the only existing notion are flats, which have been introduced in [5].

In this paper we introduce the notion of independent spaces and bases for q-polymatroids. As it turns out, the 'standard notion' of independence, namely the equality of rank value and dimension, is too restrictive for q-polymatroids. For this reason we introduce a more general notion of independence, which is inspired by the analogue for classical polymatroids in [8, Sec. 11]. In order to derive properties of the independent spaces, we introduce an auxiliary q-matroid on the same ground space (akin to a construction in [8]). Since the independent spaces of the auxiliary q-matroid coincide with those of the q-polymatroid, the latter inherits all properties known for

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independent spaces of q-matroids — as long as these properties do not involve the rank function.

The independent spaces naturally give rise to a notion of basis, namely the inclusion-maximal independent subspaces. Despite the lack of rigidity of the rank function in a q-polymatroid, it turns out that all bases of a subspace have the same rank value and this value agrees with the rank value of the subspace. In other words, the rank function restricted to the independent spaces fully determines the q-polymatroid. This result allows us to provide a cryptomorphism for q-polymatroids based on independent spaces: we characterize the collections of spaces endowed with a rank function on these spaces that give rise to a q-polymatroid with exactly these spaces as independent spaces and whose rank function restricts to the given one. Examples show that no such cryptomorphism is possible using only bases, dependent spaces, or circuits.

We finally turn to spanning spaces. These are the spaces that share the same rank value as the ground space. It turns out that in a q-polymatroid every minimal spanning space is contained in a basis, but is, in general, not a basis itself. Thus the notions 'minimal spanning' and 'maximally independent' do not agree. On the plus side, 'minimal spanning' is the dual notion to 'maximally strongly independent'. This simple fact may be regarded as the generalization of the duality result for bases in q-matroids. The latter states that in a q-matroid a space is a basis if and only if its orthogonal is a basis of the dual q-matroid. It turns out that this duality characterizes q-matroids.

In order to streamline our discussion, examples are postponed to Section 6. They serve to illustrate the most crucial differences between q-polymatroids and q-matroids.

Notation: We fix a finite field $\mathbb{F} = \mathbb{F}_q$ with q elements and a finite-dimensional $\mathbb{F}$-vector space E. We write $V \leqslant E$ if V is a subspace of E and denote by $\mathcal{V}(E)$ the collection of all subspaces of E. The standard basis vectors in $\mathbb{F}^n$ are denoted by $e_1, \ldots, e_n$. We write $[n]$ for the set $\{1, \ldots, n\}$.

2. Basic Notions of q-Polymatroids

In this section we define q-polymatroids and present some basic properties. We also introduce the main class of examples, namely q-polymatroids induced by rank-metric codes. The section is based on the material in [5].

Definition 2.1. *Set $\mathcal{V} = \mathcal{V}(E)$. A* q-rank function *on E is a map $\rho : \mathcal{V} \longrightarrow \mathbb{Q}_{\geqslant 0}$ satisfying:*

- (R1) *Dimension-Boundedness: $0 \leqslant \rho(V) \leqslant \dim V$ for all $V \in \mathcal{V}$;*
- (R2) *Monotonicity: $V \leqslant W \Longrightarrow \rho(V) \leqslant \rho(W)$ for all $V, W \in \mathcal{V}$;*
- (R3) *Submodularity: $\rho(V + W) + \rho(V \cap W) \leqslant \rho(V) + \rho(W)$ for all $V, W \in \mathcal{V}$.*

A q-polymatroid (q-PM) on E *is a pair $\mathcal{M} = (E, \rho)$, where $\rho : \mathcal{V} \longrightarrow \mathbb{Q}_{\geqslant 0}$ is a q-rank function. The value $\rho(E)$ is called the* rank *of the q-PM. A number $\mu \in \mathbb{Q}_{>0}$ is called a* denominator *of ρ (and $\mathcal{M}$) if $\mu\rho(V) \in \mathbb{N}_0$ for all $V \in \mathcal{V}$. In that case we call the map $\tau_\mu := \mu\rho$ the* induced integer ρ-function *w.r.t. μ. The smallest denominator is called the* principal denominator. *A q-PM with principal denominator 1 (i.e., $\rho(V) \in \mathbb{N}_0$ for all V) is called a* q-matroid. *If (E, ρ) is the* trivial *q-PM, i.e., ρ is the zero map, we declare 1 to be its principal denominator.*

We will often make use of the induced integer ρ-function τ_μ. Clearly τ_μ is also monotonic and submodular, and instead of (R1) it satisfies $0 \leqslant \tau_\mu(V) \leqslant \mu \dim V$ for all $V \in \mathcal{V}$.

The above definition appears in various forms in the literature. In [6, Def. 4.1] of Gorla et al. the same definition occurs with the only difference that the rank function may assume arbitrary real numbers. Next, a q-matroid in the sense of Jurrius/Pellikaan [7] is exactly a q-matroid as defined above. Finally, for any $r \in \mathbb{N}$ a (q, r)-polymatroid as in [10, Def. 2] by Shiromoto, [4, Def. 1] by Ghorpade/Johnson and [1, Def. 1] by Byrne et al. can be turned into a q-PM with denominator r by dividing the rank function by r. Conversely, given a q-PM (E, ρ) with denominator μ, then $(E, \mu\rho)$ is a $(q, \lceil \mu \rceil)$-polymatroid in the sense of these papers.

For later reference we record the following fact.

Remark 2.2 ([5, Rem. 2.3]). Let (E, ρ) be a non-trivial q-PM with principal denominator μ. Then $\mu \geqslant 1$ and the set of all denominators of (E, ρ) is $\mu\mathbb{N}$.

In order to discuss duality and details on independent spaces we need the following notions of equivalence. They extend [6, Def. 4.4]. q-PMs are scaling-equivalent if they differ only by an isomorphism between the ground spaces and a non-zero factor of the rank functions, and they are equivalent if the factor is 1.

Definition 2.3. *Let E_i, $i = 1, 2$, be $\mathbb{F}$-vector spaces of the same finite dimension and let $\mathcal{M}_i = (E_i, \rho_i)$ be q-PMs.*

(a) *$\mathcal{M}_1$ and $\mathcal{M}_2$ are called* scaling-equivalent *if there exists an $\mathbb{F}$-isomorphism $\alpha \in \mathrm{Hom}_{\mathbb{F}}(E_1, E_2)$ and $a \in \mathbb{Q}_{>0}$ such that $\rho_2(\alpha(V)) = a\rho_1(V)$ for all $V \in \mathcal{V}(E_1)$.*

(b) *We call $\mathcal{M}_1$ and $\mathcal{M}_2$* equivalent*, denoted by $\mathcal{M}_1 \approx \mathcal{M}_2$, if there exists an $\mathbb{F}$-isomorphism $\alpha \in \mathrm{Hom}_{\mathbb{F}}(E_1, E_2)$ such that $\rho_2(\alpha(V)) = \rho_1(V)$ for all $V \in \mathcal{V}(E_1)$.*

Theorem 2.4 ([5, Thm. 2.8], [6, 4.5–4.7], and for q-matroids [7, Thm. 42]). *Let $\langle \cdot \,|\, \cdot \rangle$ be a non-degenerate symmetric bilinear form on E. For $V \in \mathcal{V}(E)$ define $V^\perp = \{w \in E \mid \langle v \,|\, w \rangle = 0 \text{ for all } v \in V\}$. Let $\mathcal{M} = (E, \rho)$ be a q-PM and set*

$$\rho^*(V) = \dim V + \rho(V^\perp) - \rho(E). \tag{1}$$

Then ρ^* is a q-rank function on E and $\mathcal{M}^* = (E, \rho^*)$ is a q-PM. It is called the dual *of $\mathcal{M}$ with respect to the form $\langle \cdot \,|\, \cdot \rangle$. Furthermore, the bidual $\mathcal{M}^{**} = (\mathcal{M}^*)^*$ satisfies $\mathcal{M}^{**} = \mathcal{M}$, and $\mathcal{M}$ and $\mathcal{M}^*$ have the same set of denominators. Finally, the equivalence class of $\mathcal{M}^*$ does not depend on the choice of the bilinear form. More precisely, if $\langle\!\langle \cdot \,|\, \cdot \rangle\!\rangle$ is another non-degenerate symmetric bilinear form on E and $\mathcal{M}^{\hat{*}} = (E, \rho^{\hat{*}})$ is the resulting dual q-PM, then $\mathcal{M}^{\hat{*}} \approx \mathcal{M}^*$.*

The next result has been proven in [6] for q-PMs on $\mathbb{F}^n$, endowed with the standard dot product. Thanks to the invariance of the dual, it generalizes without the need to specify bilinear forms.

Proposition 2.5 ([6, Prop. 4.7]). *Let $\mathcal{M} = (E, \rho)$ and $\hat{\mathcal{M}} = (\hat{E}, \hat{\rho})$ be q-PMs. Then $\mathcal{M} \approx \hat{\mathcal{M}}$ implies $\mathcal{M}^* \approx \hat{\mathcal{M}}^*$.*

Example 2.6 ([7, Ex. 4 and Ex. 47]). Let $\mathcal{U}_k(E) = (E, \rho)$ be the *uniform q-matroid of rank k*, that is, $\rho(V) = \min\{k, \dim V\}$ for all $V \in \mathcal{V}(E)$. Then $\mathcal{U}_k(E)^* = \mathcal{U}_{\dim E - k}(E)$.

The rest of this section is devoted to q-PMs induced by rank-metric codes. This will provide us with plenty of examples. As usual, we endow $\mathbb{F}^{n\times m}$ with the rank-metric given by $\mathrm{d}(A, B) = \mathrm{rk}(A - B)$. We only consider *linear rank-metric codes*, that is, subspaces of the metric space $(\mathbb{F}^{n\times m}, \mathrm{d})$. The following remark collects standard facts and terminology from the theory of rank-metric codes. For $V \leqslant \mathbb{F}^n$ denote by $V^\perp \leqslant \mathbb{F}^n$ the orthogonal space with respect to the standard dot product.

Definition 2.7. *Let $\mathcal{C} \leqslant \mathbb{F}^{n\times m}$ be a rank-metric code.*

(a) *The* rank distance *of* $\mathcal{C}$ *is defined as* $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = \min\{\mathrm{rk}(M) \mid M \in \mathcal{C} \smallsetminus 0\}$. *If* $d = \mathrm{d}_{\mathrm{rk}}(\mathcal{C})$, *then* $\dim(\mathcal{C}) \leqslant \max\{m,n\}(\min\{m,n\} - d + 1)$, *which is known as the* Singleton bound. *If* $\dim(\mathcal{C}) = \max\{m,n\}(\min\{m,n\} - d + 1)$, *then* $\mathcal{C}$ *is called an* MRD code.

(b) *The* dual code *of* $\mathcal{C}$ *is defined as* $\mathcal{C}^\perp = \{M \in \mathbb{F}^{n\times m} \mid \mathrm{tr}(MN^\mathsf{T}) = 0 \text{ for all } N \in \mathcal{C}\}$, *where* $\mathrm{tr}(\cdot)$ *denotes the trace of the given matrix.*

(c) *For* $V \in \mathcal{V}(\mathbb{F}^n)$ *we set* $\mathcal{C}(V,\mathrm{c}) = \{M \in \mathcal{C} \mid \mathrm{colsp}(M) \leqslant V\}$, *where* $\mathrm{colsp}(M)$ *denotes the column space of* M. *Then* $\mathbb{F}^{n\times m}(V,\mathrm{c})^\perp = \mathbb{F}^{n\times m}(V^\perp,\mathrm{c})$ *and by* [9, Lem. 28]

$$\dim\mathcal{C}(V^\perp,\mathrm{c}) = \dim\mathcal{C} - m\dim V + \dim\mathcal{C}^\perp(V,\mathrm{c}).$$

Rank-metric codes induce q-PMs. This has been shown first in [6]. The statement in (2) below is immediate with Definition 2.7(c).

Theorem 2.8 ([6, Thm. 5.3]). *For a nonzero rank-metric code* $\mathcal{C} \leqslant \mathbb{F}^{n\times m}$ *define*

$$\rho_\mathrm{c} : \mathcal{V}(\mathbb{F}^n) \longrightarrow \mathbb{Q}_{\geqslant 0}, \quad V \longmapsto \frac{\dim\mathcal{C} - \dim\mathcal{C}(V^\perp,\mathrm{c})}{m}.$$

Then ρ_c *is a* q*-rank function with denominator* m *(which in general is not principal). The* q*-PM* $\mathcal{M}_\mathrm{c}(\mathcal{C}) := (\mathbb{F}^n, \rho_\mathrm{c})$ *is called the* (column) q-polymatroid *of* $\mathcal{C}$. *Its rank is* $\dim\mathcal{C}/m$. *The rank function satisfies*

$$\rho_\mathrm{c}(V) = \dim V - \frac{1}{m}\dim\mathcal{C}^\perp(V,\mathrm{c}). \tag{2}$$

Similarly we can define the *row* q*-PM* of a rank-metric code, which then has denominator n. Since it is the same as the column q-PM of the transposed code, it suffices to consider column q-PMs.

It is worth noting that not every q-PM (and not even every q-matroid) is induced by a rank-metric code; an example can be found in [5, Thm. 4.9].

The rank distances of a code and its dual are closely related to the q-PM. Indeed, let $\mathcal{C} \leqslant \mathbb{F}^{n\times m}$ be a nonzero rank-metric code with rank-distance d, and let $d^\perp$ be the rank distance of $\mathcal{C}^\perp$. Then any $V \in \mathcal{V}(\mathbb{F}^n)$ satisfies (see [6, Prop. 6.2], [7, Lem. 30], and [5, Rem. 3.8])

$$\rho_\mathrm{c}(V) = \begin{cases} \dim V, & \text{if } \dim V < d^\perp, \\ \frac{\dim\mathcal{C}}{m}, & \text{if } \dim V > n - d. \end{cases}$$

For MRD codes we can give more detailed information. The following result shows that if $m \geqslant n-1$, then the column q-PM of an MRD code in $\mathbb{F}^{n\times m}$ depends only on the parameters $(n, d, |\mathbb{F}|)$. Example 6.1 in Section 6 shows that this is not the case for $m < n-1$. Therein, two MRD codes in $\mathbb{F}_2^{5\times 2}$ with the same rank-distance but non-equivalent q-PMs are presented.

Proposition 2.9 ([6, Cor. 6.6], [5, Thm. 3.10]). *Let* $\mathcal{C} \leqslant \mathbb{F}^{n\times m}$ *be an MRD code with* $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = d$.

(a) *Let* $n \leqslant m$. *Then* $\mathcal{M}_\mathrm{c}(\mathcal{C}) = \mathcal{U}_{n-d+1}(\mathbb{F}^n)$, *that is,* $\mathcal{M}_\mathrm{c}(\mathcal{C})$ *is the uniform* q*-matroid of rank* $n-d+1$. *Thus* $\mathcal{M}_\mathrm{c}(\mathcal{C})$ *depends only on* $(n, d, |\mathbb{F}|)$.

(b) *Let* $n \geqslant m$. *Then* $\mathcal{M}_\mathrm{c}(\mathcal{C})$ *satisfies*

$$\rho_\mathrm{c}(V) = \begin{cases} \dim V, & \text{if } \dim V \leqslant m - d + 1, \\ \frac{n(m-d+1)}{m}, & \text{if } \dim V \geqslant n - d + 1, \end{cases}$$

and $\rho_\mathrm{c}(V) \geqslant \max\{1, (\dim V)/m\}(m-d+1)$ *if* $\dim V \in [m-d+2,\, n-d]$. *Thus, if* $m = n-1$, *then* $\mathcal{M}_\mathrm{c}(\mathcal{C})$ *depends only on* $(n, d, |\mathbb{F}|)$.

Equivalence of codes, in the usual sense, translates into equivalence of the associated q-PMs.

Proposition 2.10 ([5, Prop. 3.5] and [6, Prop. 6.7]). *Let $\mathcal{C}, \mathcal{C}' \leqslant \mathbb{F}^{n\times m}$ be rank-metric codes.*

(a) *Suppose $\mathcal{C}, \mathcal{C}'$ are* equivalent*, i.e., $\mathcal{C}' = X\mathcal{C}Y := \{XMY \mid M \in \mathcal{C}\}$ for some $X \in \mathrm{GL}_n(\mathbb{F})$ and $Y \in \mathrm{GL}_m(\mathbb{F})$. Then $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$ and $\mathcal{M}_{\mathrm{c}}(\mathcal{C}')$ are equivalent via $\beta \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^n)$ given by $x \mapsto (X^{\mathsf{T}})^{-1}x$.*

(b) *Let $n = m$ and suppose $\mathcal{C}, \mathcal{C}'$ are* transposition-equivalent*, that is, $\mathcal{C}' = X\mathcal{C}^{\mathsf{T}}Y$ for some $X, Y \in \mathrm{GL}_n(\mathbb{F})$, and where $\mathcal{C}^{\mathsf{T}} = \{M^{\mathsf{T}} \mid M \in \mathcal{C}\}$. Then $\mathcal{M}_{\mathrm{c}}(\mathcal{C}^{\mathsf{T}})$ and $\mathcal{M}_{\mathrm{c}}(\mathcal{C}')$ are equivalent via β, where β is as in* (a)*.*

Duality of q-PMs (see Theorem 2.4) corresponds to duality of codes.

Theorem 2.11 ([6, Thm. 8.1]). *Let $\mathcal{C} \leqslant \mathbb{F}^{n\times m}$ be a rank-metric code and $\mathcal{C}^{\perp}$ be its dual. Then $\mathcal{M}_{\mathrm{c}}(\mathcal{C})^* = \mathcal{M}_{\mathrm{c}}(\mathcal{C}^{\perp})$, where $\mathcal{M}_{\mathrm{c}}(\mathcal{C})^*$ is the dual of $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$ w.r.t. the standard dot product on $\mathbb{F}^n$.*

Another instance of the interplay between duality of q-PMs and duality of rank-metric codes has been presented in [5, Thms. 5.3 and 5.5]. Therein, it is shown that contraction and deletion of q-PMs are mutually dual and correspond to shortening and puncturing of rank-metric codes.

Occasionally, we will consider $\mathbb{F}_{q^m}$-linear rank-metric codes, which we introduce as follows. Recall that $\mathbb{F} = \mathbb{F}_q$. Let $\mathcal{B} = (v_1, \ldots, v_m)$ be a basis of the $\mathbb{F}$-vector space $\mathbb{F}_{q^m}$ and let $\psi : \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_q^m$ be the associated coordinate map, i.e., $\psi(\sum_{i=1}^m \lambda_i v_i) = (\lambda_1, \ldots, \lambda_m)$. Extending ψ entry-wise, we obtain, for any n, an isomorphism $\Psi_{\mathcal{B}} : \mathbb{F}_{q^m}^n \longrightarrow \mathbb{F}_q^{n\times m}$ that maps $(c_1, \ldots, c_n)$ to the matrix with rows $\psi(c_1), \ldots, \psi(c_n)$. It follows from basic linear algebra that the following definition of $\mathbb{F}_{q^s}$-linearity does not depend on the choice of basis $\mathcal{B}$.

Definition 2.12. *Fix an $\mathbb{F}_q$-basis $\mathcal{B}$ of $\mathbb{F}_{q^m}$. Let $\mathbb{F}_{q^s}$ be a subfield of $\mathbb{F}_{q^m}$ and $\mathcal{C} \leqslant \mathbb{F}_q^{n\times m}$ be a rank-metric code (hence an $\mathbb{F}_q$-linear subspace). Then $\mathcal{C}$ is called* right $\mathbb{F}_{q^s}$-linear *if $\Psi_{\mathcal{B}}^{-1}(\mathcal{C})$ is an $\mathbb{F}_{q^s}$-subspace of $\mathbb{F}_{q^m}^n$.* Left linearity *over $\mathbb{F}_{q^n}$ and its subfields is defined analogously.*

Obviously, the qualifiers left/right are needed only in the case where $\mathbb{F}_{q^s}$ is a subfield of both $\mathbb{F}_{q^n}$ and $\mathbb{F}_{q^m}$. It is easy to verify (see [5, Sec. 3]) that if $\mathcal{C} \leqslant \mathbb{F}_q^{n\times m}$ is a right $\mathbb{F}_{q^s}$-linear rank-metric code, then $\mu = m/s$ is a denominator of $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$. In particular, for $s = m$ the q-PM $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$ is a q-matroid. These are exactly the q-matroids studied in [7]. However, one should keep in mind that $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$ may be a q-matroid even if $\mathcal{C}$ is not right $\mathbb{F}_{q^m}$-linear. Examples are given by MRD codes for $n \leqslant m$ (see Proposition 2.9(a)) as well as by [5, Thm. 5.5], where it is shown that if $\mathcal{C}$ induces a q-matroid, then so does every shortening and puncturing of $\mathcal{C}$.

3. Independent Spaces

We now introduce independent spaces for q-PMs. We show that the collection of independent spaces satisfies properties analogous to those for q-matroids. However, different from the latter, the independent spaces do not fully determine the q-PM. Only if we also take their rank values into account, can we fully recover the q-PM. This will be dealt with in the next section. As mentioned earlier, supporting examples are postponed to Section 6.

Considering the theory of classical matroids and q-matroids, one may be inclined to declare a space V in a q-PM (E, ρ) independent if $\rho(V) = \dim V$. While this is indeed the right notion for q-matroids, it turns out to be too restrictive for q-PMs: in many q-PMs the only subspace satisfying $\rho(V) = \dim V$ is the zero space (e.g. Example 6.2). Nonetheless, the property $\rho(V) = \dim V$ turns out to play a conceptual role (see also [1]), and we will return to it in Section 5, where we will call such spaces strongly independent.

The following definition of independence is inspired by [8, Cor. 11.1.2], which deals with classical polymatroids.

Definition 3.1. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ (which need not be principal). A space $I \in \mathcal{V}(E)$ is called* μ-independent *if*

$$\rho(J) \geqslant \frac{\dim J}{\mu} \text{ for all subspaces } J \leqslant I.$$

I is called μ-dependent *if it is not μ-independent. A* μ-circuit *is a μ-dependent space for which all proper subspaces are μ-independent. A 1-dimensional μ-dependent space is called a* μ-loop. *We define $\mathcal{I}_\mu = \mathcal{I}_\mu(\mathcal{M}) = \{I \in \mathcal{V}(E) \mid I \text{ is } \mu\text{-independent}\}$. If μ is the principal denominator of $\mathcal{M}$, we may skip the quantifier μ and simply use independent, dependent, loop, circuit, and $\mathcal{I}$.*

It is easy to see that the inequality $\rho(I) \geqslant \dim I/\mu$ is not preserved under taking subspaces (take for instance the subspace $I = V$ from Example 6.3), which is why the condition for subspaces is built into our definition. Clearly, if $\hat{\mu}$ is the principal denominator of $\mathcal{M}$, then $\hat{\mu}$-independence implies μ-independence for any denominator μ of $\mathcal{M}$. Furthermore, the zero subspace of E is μ-independent, and every dependent space V contains a circuit: take any subspace W of V of smallest dimension satisfying $\rho(W) < \dim W/\mu$ (which clearly exists).

Let us consider the independent spaces of the q-PMs induced by MRD codes.

Example 3.2. Let $\mathcal{C} \leqslant \mathbb{F}^{n\times m}$ be an MRD code with rank distance d.

(a) If $m \geqslant n$ then $\mathcal{I}_1(\mathcal{M}_{\rm c}(\mathcal{C})) = \{V \in \mathcal{V}(\mathbb{F}^n) \mid \dim V \leqslant n-d+1\}$. This follows from Proposition 2.9(a) and the fact that the independent spaces of uniform q-matroid $\mathcal{U} = \mathcal{U}_k(E)$ are exactly the space of dimension at most k.
(b) If $m \leqslant n$ then $\mathcal{I}_m(\mathcal{M}_{\rm c}(\mathcal{C})) = \mathcal{V}(\mathbb{F}^n)$, which can be verified with the aid of Proposition 2.9(b).

For q-matroids our notion of independence coincides with independence in [7, Def. 2].

Proposition 3.3. *Let (E, ρ) be a q-PM. Then for all $V \in \mathcal{V}(E)$*

$$\rho(V) = \dim V \implies \rho(W) = \dim W \text{ for all } W \leqslant V.$$

As a consequence, if (E, ρ) is a q-matroid, then V is 1-independent iff $\rho(V) = \dim V$.

Proof. Writing $V = W \oplus Z$ for some complement Z of W, we obtain $\dim V = \rho(V) \leqslant \rho(W) + \rho(Z) \leqslant \dim W + \dim Z = \dim V$, and thus we have equality everywhere. □

We continue with discussing basic properties of independent spaces, thereby focusing on the differences to q-matroids. Supporting examples are given in Section 6.

Remark 3.4. (a) While in a q-matroid a space is independent iff its rank value assumes the maximal possible value, this is not the case for q-PMs. More precisely, independent spaces of the same dimension need not have the same rank value. This is illustrated by Example 6.2.
(b) Dependent spaces may have a larger rank value than independent spaces of the same dimension; see Example 6.3.

(c) Let $V \in \mathcal{V}(E)$ be a μ-circuit. Then $\mu\rho(V) = \dim V - 1 = \mu\rho(W)$ for all hyperplanes W in V. Indeed, independence of W along with (R2) tells us that $\dim V - 1 = \dim W \leqslant \mu\rho(W) \leqslant \mu\rho(V) < \dim V$. Thus we have equality since $\mu\rho$ takes integer values. While in a q-matroid a subspace V satisfying $\mu\rho(V) = \dim V - 1 = \mu\rho(W)$ for all its hyperplanes W is a circuit, this is not the case for q-PMs; see Example 6.4.

(d) A q-PM with principal denominator μ is not uniquely determined by its collection of μ-independent spaces. For instance, in either of the non-equivalent q-PMs in Example 6.1 all subspaces are 2-independent. This example also shows that — different from q-matroids — a q-PM in which all spaces are μ-independent need not be a uniform q-matroid.

Independence behaves well under scaling-equivalence if the denominator is taken into account.

Remark 3.5. Let $\dim E_1 = \dim E_2$ and $\mathcal{M}_i = (E_i, \rho_i), i = 1, 2$, be q-PMs with principal denominators μ_i. Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are scaling-equivalent, say $\rho_2(\alpha(V)) = a\rho_1(V)$ for all $V \in \mathcal{V}(E_1)$, where $a \in \mathbb{Q}_{>0}$ and $\alpha : E_1 \longrightarrow E_2$ an isomorphism. Then $a^{-1}\mu_1\rho_2(\alpha(V)) = \mu_1\rho_1(V) \in \mathbb{N}$ and thus $a^{-1}\mu_1$ is a denominator of $\mathcal{M}_2$. Hence $a^{-1}\mu_1 = k\mu_2$ for some $k \in \mathbb{N}$; see Remark 2.2. Similarly, $a\mu_2 = \hat{k}\mu_1$ for some $\hat{k} \in \mathbb{N}$. Thus $k = \hat{k} = 1$ and $a\mu_2 = \mu_1$. Now we have $\mu_2\rho_2(\alpha(V)) = \mu_1\rho_1(V)$ for all $V \in \mathcal{V}(E)$ and therefore

$$\begin{aligned} V \text{ is } \mu_1\text{-independent in } \mathcal{M}_1 & \\ &\iff \mu_1\rho_1(W) \geqslant \dim W \text{ for all subspaces } W \leqslant V \\ &\iff \mu_2\rho_2(\alpha(W)) \geqslant \dim \alpha(W) \text{ for all subspaces } \alpha(W) \leqslant \alpha(V) \\ &\iff \alpha(V) \text{ is } \mu_2\text{-independent in } \mathcal{M}_2. \end{aligned}$$

In order to derive our main result about the collection of μ-independent spaces, we will make use of an auxiliary q-matroid. The following construction mimics the corresponding one in [8, Prop. 11.1.7] for classical polymatroids.

Theorem 3.6. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ. Define the map*

$$r_{\rho,\mu} : \mathcal{V}(E) \longrightarrow \mathbb{N}_0, \quad V \longmapsto \min\{\mu\rho(W) + \dim V - \dim W \mid W \leqslant V\}.$$

Then $\mathcal{Z} := \mathcal{Z}_{\mathcal{M},\mu} := (E, r_{\rho,\mu})$ is a q-matroid, and the independent spaces of $\mathcal{Z}$ coincide with the μ-independent spaces of $\mathcal{M}$, i.e.,

$$\mathcal{I}_\mu(\mathcal{M}) = \mathcal{I}(\mathcal{Z}) = \{I \in \mathcal{V}(E) \mid r_{\rho,\mu}(I) = \dim I\}.$$

Proof. Recall the induced integer ρ-function $\tau = \mu\rho$. Thus $\tau(V) = \mu\rho(V) \leqslant \mu \dim V$ for all $V \in \mathcal{V}(E)$. Clearly $r := r_{\rho,\mu}$ takes integer values. We now verify (R1)–(R3) of Definition 2.1 for r.

(R1) Obviously $r(V) \geqslant 0$ for all V. Furthermore, $r(V) \leqslant \tau(0) + \dim(V) - \dim(0) = \dim V$.

(R2) Let $V \leqslant V'$. It suffices to consider the case $\dim V' = \dim V + 1$ and thus $V' = V \oplus \langle x\rangle$ for some $x \in E$. Assume by contradiction that $r(V) > r(V')$. Then there exists $W' \leqslant V'$ such that

$$\tau(W') + \dim V' - \dim W' < \tau(W) + \dim V - \dim W \quad \text{for all } W \leqslant V. \tag{3}$$

Clearly $W' \not\leqslant V$ and thus we may write $W' = X \oplus \langle y\rangle$ for some $X \leqslant V$ and $y \notin V$. Then $\dim X = \dim V - \dim V' + \dim W'$ and (3) leads to

$$\tau(W) - \dim W > \tau(W') + \dim V' - \dim W' - \dim V = \tau(W') - \dim X \text{ for all } W \leqslant V.$$

Choosing $W = X$, we arrive at $\tau(X) > \tau(W')$ and thus $\rho(X) > \rho(W')$. Since $X \leqslant W'$ this contradicts that ρ is a rank function. All of this establishes (R2) for the map r.
(R3) Let $V, V' \in \mathcal{V}(E)$. Choose $W \leqslant V$, $W' \leqslant V'$ such that

$$r(V) = \tau(W) + \dim V - \dim W \quad \text{and} \quad r(V') = \tau(W') + \dim V' - \dim W'.$$

Then $W + W' \leqslant V + V'$ and $W \cap W' \leqslant V \cap V'$ and therefore

$$\begin{aligned}
&r(V + V') + r(V \cap V') \\
&\qquad \leqslant \tau(W + W') + \dim(V + V') - \dim(W + W') \\
&\qquad\qquad + \tau(W \cap W') + \dim(V \cap V') - \dim(W \cap W') \\
&\qquad = \tau(W + W') + \tau(W \cap W') + \dim V - \dim W + \dim V' - \dim W' \\
&\qquad \leqslant \tau(W) + \tau(W') + \dim V - \dim W + \dim V' - \dim W' \\
&\qquad = r(V) + r(V'),
\end{aligned}$$

where the second inequality follows from (R3) for ρ. This establishes (R3) for the map r.
It remains to investigate the μ-independent spaces. From Definition 3.1 and (R1) we obtain

$$\begin{aligned}
V \text{ is } \mu\text{-independent } &\iff \tau(W) \geqslant \dim W \text{ for all } W \leqslant V \\
&\iff \tau(W) + \dim V - \dim W \geqslant \dim V \text{ for all } W \leqslant V \\
&\iff r(V) \geqslant \dim V \\
&\iff r(V) = \dim V.
\end{aligned}$$

Together with Proposition 3.3 this establishes the stated result. ☐

It should be noted that the auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$ does not uniquely determine the q-PM $\mathcal{M}$, even if μ is the principal denominator. This can be seen from Example 6.1: both $\mathcal{M}_1$ and $\mathcal{M}_2$ have principal denominator 2 and in either q-PM all spaces are 2-independent (see also Example 3.2(b)). Hence Theorem 3.6 implies $\mathcal{Z}_{\mathcal{M}_1,2} = \mathcal{Z}_{\mathcal{M}_2,2} = \mathcal{U}_5(\mathbb{F}_2^5)$.

As we show next, a q-matroid $\mathcal{M}$ coincides with its auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},1}$.

Remark 3.7. Let $\mathcal{M} = (E, \rho)$ be a q-matroid, thus ρ takes only integer values.

(a) We show that $\mathcal{Z}_{\mathcal{M},1} = \mathcal{M}$. The auxiliary rank function is $r_{\rho,1}(V) = \min\{\rho(W) + \dim V - \dim W \mid W \leqslant V\}$ for $V \in \mathcal{V}(E)$. Choosing $W = V$ we obtain $r_{\rho,1}(V) \leqslant \rho(V)$. For the opposite inequality, choose $W \leqslant V$. Then there exists $Z \leqslant V$ such that $W \oplus Z = V$ and submodularity (R3) yields

$$\rho(V) = \rho(W + Z) \leqslant \rho(W) + \rho(Z) \leqslant \rho(W) + \dim Z = \rho(W) + \dim V - \dim W.$$

Since W is arbitrary, this shows $\rho(V) \leqslant r_{\rho,1}(V)$ and thus $\mathcal{Z}_{\mathcal{M},1} = \mathcal{M}$.

(b) If we choose $\mu > 1$, then there is in general no obvious relation between $\mathcal{Z}_{\mathcal{M},\mu}$ and $\mathcal{M}$; see Example 6.5.

Theorem 3.6 shows that the μ-independent spaces of the q-PM $\mathcal{M}$ coincide with the independent spaces of the auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$. Therefore, all properties of independent spaces of q-matroids that do not involve the value of the rank function hold true for q-PMs as well. Such properties have been derived in [7, Thm. 8]. Before formulating our result we cast the following important notions.

Definition 3.8. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ. For $V \in \mathcal{V}(E)$ we define*

$$\mathcal{I}_\mu(V) = \{I \in \mathcal{I}_\mu(\mathcal{M}) \mid I \leqslant V\}.$$

A subspace $\hat{I} \in \mathcal{I}_\mu(V)$ *is said to be a* μ-basis *of* V *if there exists no* $J \in \mathcal{I}_\mu(V)$ *such that* $\hat{I} \lneq J$. *We denote by* $\mathcal{B}_\mu(V)$ *the set of all* μ*-bases of* V. *The* μ*-bases of* E *are called the* μ-bases *of* $\mathcal{M}$.

The μ-bases of V are thus the inclusion-maximal μ-independent subspaces of V (i.e., the maximal elements of the poset $(\mathcal{I}_\mu(V), \leqslant)$). Their rank values will be discussed in the next section. Note that the sets $\mathcal{I}_\mu(V)$ and $\mathcal{B}_\mu(V)$ are non-empty for every $V \in \mathcal{V}(E)$ since $\{0\}$ is μ-independent.

We are now ready to present the following properties of the collection of μ-independent spaces of a q-PM. The result is an immediate consequence of Theorem 3.6 together with [7, Thm. 8].

Corollary 3.9. *Let* $\mathcal{M} = (E, \rho)$ *be a* q*-PM with denominator* μ *and set* $\mathcal{I}_\mu := \mathcal{I}_\mu(\mathcal{M})$. *Then*

- (I1) $\mathcal{I}_\mu \neq \varnothing$, *in fact* $\{0\} \in \mathcal{I}_\mu$.
- (I2) *If* $I \in \mathcal{I}_\mu$ *and* $J \leqslant I$, *then* $J \in \mathcal{I}_\mu$.
- (I3) *If* $I, J \in \mathcal{I}_\mu$ *and* $\dim I < \dim J$, *then there exists* $x \in J \smallsetminus I$ *such that* $I \oplus \langle x \rangle \in \mathcal{I}_\mu$.
- (I4) *Let* $V, W \in \mathcal{V}(E)$ *and* $I \in \mathcal{B}_\mu(V)$, $J \in \mathcal{B}_\mu(W)$. *Then there exists a basis* $K \in \mathcal{B}_\mu(V + W)$ *that is contained in* $I + J$.

Note that (I3) implies that for any $V \in \mathcal{V}(E)$ we have

$$\mathcal{B}_\mu(V) = \{\hat{I} \in \mathcal{I}_\mu(V) \mid \hat{I} \text{ is dimension-maximal in } \mathcal{I}_\mu(V)\}. \tag{4}$$

Since the independent spaces of the q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$ coincide with those of the q-PM $\mathcal{M}$, the same is true for the dependent spaces, circuits, and bases. As a consequence, any property about the collection of these spaces in q-matroids holds true for q-PMs as well — as long as it does not involve the rank value. Let us illustrate this for the dependent spaces and bases. The following properties have been established in [3, Thm. 71] and [7, Thm. 37] for q-matroids and therefore apply to q-PMs as well.

Corollary 3.10. *Let* $\mathcal{M} = (E, \rho)$ *be a* q*-PM with denominator* μ. *Let* $\mathcal{D}_\mu$ *and* $\mathcal{B}_\mu$ *be the collection of* μ*-dependent spaces and* μ*-bases of* $\mathcal{M}$, *respectively. Then* $\mathcal{D}_\mu$ *and* $\mathcal{B}_\mu$ *satisfy*

- (D1) $\{0\} \notin \mathcal{D}_\mu$.
- (D2) *If* $D_1 \in \mathcal{D}_\mu$ *and* $D_2 \in \mathcal{V}(E)$ *such that* $D_1 \subseteq D_2$, *then* $D_2 \in \mathcal{D}_\mu$.
- (D3) *Let* $D_1, D_2 \in \mathcal{D}_\mu$ *be such that* $D_1 \cap D_2 \notin \mathcal{D}_\mu$. *Then every subspace of* $D_1 + D_2$ *of codimension* 1 *is in* $\mathcal{D}_\mu$.
- (B1) $\mathcal{B}_\mu \neq \varnothing$.
- (B2) *Let* $B_1, B_2 \in \mathcal{B}_\mu$ *be such that* $B_1 \leqslant B_2$. *Then* $B_1 = B_2$.
- (B3) *Let* $B_1, B_2 \in \mathcal{B}_\mu$ *and* A *be a subspace of* B_1 *of codimension* 1 *such that* $B_1 \cap B_2 \leqslant A$. *Then there exists a* 1*-dimensional subspace* Y *of* B_2 *such that* $A + Y \in \mathcal{B}_\mu$.
- (B4) *Let* $A_1, A_2 \in \mathcal{V}(E)$ *and* I_1, I_2 *be maximal dimensional intersections of some members of* $\mathcal{B}_\mu$ *with* A_1 *and* A_2, *respectively. Then there exist a maximal dimensional intersection of a member of* $\mathcal{B}_\mu$ *with* $A_1 + A_2$ *that is contained in* $I_1 + I_2$.

In [3, Thm. 72] and [7, Thm. 37] it has been shown that any collection of subspaces satisfying (D1)–(D3) (resp. (B1)–(B4)) is the collection of dependent spaces (resp. bases) of a unique q-matroid. Similar statements hold true for circuits in q-matroids (see [3, Cor. 76]). None of these characterizations extend to q-PMs — even if we take the rank values into account. This can be seen from the two non-equivalent q-PMs in Example 6.1: In both cases, the only 2-basis is $\mathbb{F}^5$ and has rank value $5/2$. Trivially, this example also shows that the circuits and dependent spaces along with their rank values do not determine the q-PM. Example 6.6 is a non-trivial example for the same phenomenon.

On the positive side, in the next section we will show that we can fully recover a q-PM from its independent spaces and their rank values. Recall from Remark 3.4(d) that the independent spaces alone (without their rank values) do not uniquely determine the q-PM.

4. The Rank Function on Independent Spaces

We begin by showing that for a q-PM the rank function is fully determined by its values on the independent spaces. We then go on to prove that all bases of a given subspace have the same rank value, and this value coincides with the rank value of the subspace. This result allows us to investigate whether a collection of spaces satisfying (I1)–(I4) from Corollary 3.9 gives rise to a q-PM whose collection of independent spaces is exactly the initial collection. Since the rank value of independent spaces in a q-PM is not as rigid as in a q-matroid, we also need to specify a meaningful rank function on the collection of spaces. All of this results in Theorems 4.4 and 4.5.

Theorem 4.1. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ and let $V \in \mathcal{V}(E)$. Then*

$$\rho(V) = \max\{\rho(I) \mid I \in \mathcal{I}_\mu(V)\}.$$

Proof. Set $\rho'(V) = \max\{\rho(I) \mid I \in \mathcal{I}_\mu(V)\}$. Thanks to (R2), $\rho'(V) \leqslant \rho(V)$, and it remains to establish $\rho(V) \leqslant \rho'(V)$. Let $\hat{I} \in \mathcal{I}_\mu(V)$ be of maximal possible dimension such that $\rho(\hat{I}) = \rho'(V)$. If V is μ-independent, then $\hat{I} = V$ and we are done. Thus let V be μ-dependent.

<u>Case 1:</u> $\dim \hat{I} = \dim V - 1$.
Then $V = \hat{I} \oplus \langle x \rangle$ for any $x \in V \smallsetminus \hat{I}$ and submodularity of ρ implies $\rho(V) \leqslant \rho(\hat{I}) + \rho(\langle x \rangle)$. As before, we use the integer ρ-function $\tau = \mu\rho$. Let s be minimal such that there exists an s-dimensional μ-circuit of V, say W. Such space exists by μ-dependence of V. Then Remark 3.4(c) implies $\tau(W) = \dim W - 1$. By (I2) W is not contained in $\hat{I}$ and thus $W \cap \hat{I}$ is a hyperplane of W thanks to $\dim \hat{I} = \dim V - 1$. Hence Remark 3.4(c) yields $\tau(W \cap \hat{I}) = \tau(W)$. Using that $V = W + \hat{I}$, we obtain by submodularity of τ

$$\tau(V) \leqslant \tau(W) + \tau(\hat{I}) - \tau(W \cap \hat{I}) = \tau(\hat{I}) = \mu\rho'(V).$$

All of this shows that $\rho(V) = \rho'(V)$, as desired.

<u>Case 2:</u> $\dim \hat{I} < \dim V - 1$.
Let $x \in V \smallsetminus \hat{I}$. Using that $\rho'(W) \leqslant \rho'(Z)$ for any subspaces W, Z such that $W \leqslant Z$, we obtain

$$\rho(\hat{I}) = \rho'(\hat{I}) \leqslant \rho'(\hat{I} \oplus \langle x \rangle) \leqslant \rho'(V) = \rho(\hat{I}),$$

and hence $\rho(\hat{I}) = \rho'(W)$, where $W := \hat{I} \oplus \langle x \rangle$. Note that W is μ-dependent thanks to the maximality of $\hat{I}$. Furthermore, $\dim \hat{I} = \dim W - 1$. Therefore Case 1 yields $\rho'(W) = \rho(W)$. Now we arrived at $\rho(\hat{I}) = \rho(\hat{I} + \langle x \rangle)$ for all $x \in V$, and [5, Prop. 2.5(a)] (based on [7, Prop. 6]) tells us that $\rho(\hat{I}) = \rho(V)$. Since $\rho(\hat{I}) = \rho'(V)$, this concludes the proof. $\square$

Corollary 3.9 and Theorem 4.1 generalize one direction of [7, Thm. 8] where the same properties are proven for the independent spaces of q-matroids. Our next goal is to generalize the other direction of [7, Thm. 8], namely to characterize the collections of spaces plus rank values that give rise to a q-PM having those spaces as independent spaces. The following result will be crucial. It shows that the rank value of any μ-basis of a subspace V equals the rank value of V.

THEOREM 4.2. *Let $\mathcal{M}=(E,\rho)$ be a q-PM with denominator μ. Let $V\in\mathcal{V}(E)$. Then*

$$\rho(I)=\rho(V) \text{ for all } I\in\mathcal{B}_\mu(V).$$

In particular, all μ-bases of V have the same rank value.

Proof. Throughout the proof we will omit the subscript μ. The result is clearly true if V is independent. Thus, let V be dependent. Set $t=\dim V$. In order to avoid denominators we use again the integer ρ-function $\tau:=\mu\rho$. First of all, there exists

$$J\in\mathcal{B}(V) \text{ such that } \tau(J)=\tau(V). \tag{5}$$

Indeed, by Theorem 4.1 there exists $J\in\mathcal{I}(V)$ such that $\tau(J)=\tau(V)$, and by Property (I2) along with the monotonicity of τ we may assume that $J\in\mathcal{B}(V)$. Note that by (4) all spaces in $\mathcal{B}(V)$ have the same dimension, which we denote by s.

Case 1: $s=t-1$. Let $I\in\mathcal{B}(V)$. We want to show that $\tau(I)=\tau(V)$. Choose a circuit, say C, in V. Then $\tau(C)=\dim C-1$ (see Remark 3.4(c)). Clearly, $C\nsubseteq I$ by Property (I2) and thus $C+I=V$ thanks to $\dim I=\dim V-1$. Furthermore, $C\cap I$ is independent, being a subspace of I, and thus $\tau(C\cap I)\geqslant\dim(C\cap I)$. Using submodularity, we obtain

$$\begin{aligned}\tau(V)=\tau(C+I)&\leqslant\tau(C)+\tau(I)-\tau(C\cap I)\leqslant\dim C-1+\tau(I)-\dim(C\cap I)\\&=\tau(I)+\dim(C+I)-(\dim I+1)=\tau(I),\end{aligned}$$

where the last step follows from $C+I=V$ and $\dim I+1=\dim V$. All of this shows $\tau(I)\geqslant\tau(V)$, and thus $\tau(I)=\tau(V)$ thanks to (R2). Hence all bases of V have the same rank value.

Case 2: $s<t-1$. We will show that

$$\tau(I)=\tau(J) \text{ for all } I\in\mathcal{B}(V), \tag{6}$$

where J is as in (5). We induct on the codimension of $I\cap J$ in I. Let $\dim(I\cap J)=s-r$, thus $0\leqslant r\leqslant s$. The case $r=0$ is trivial.

i) Let $r=1$. Then $I=(I\cap J)\oplus\langle x\rangle$ for some $x\in I\setminus J$. Set $W=J\oplus\langle x\rangle$. Then $W\leqslant V$ and $\dim W=\dim J+1$. Thus W is dependent by maximality of J. Hence I and J are elements of $\mathcal{B}(W)$, and Case 1 implies $\tau(I)=\tau(J)$.

ii) Assume now $\tau(I)=\tau(J)$ for all $I\in\mathcal{B}(V)$ such that $\dim(I\cap J)\geqslant s-(r-1)$ for some $r\geqslant 2$. Let $I\in\mathcal{B}(V)$ be such that $\dim(I\cap J)=s-r$. Choose $K\leqslant I$ and $x\in I\setminus J$ such that $I=(I\cap J)\oplus K\oplus\langle x\rangle$ and set $I_1=(I\cap J)\oplus K$. Then I_1 is independent and $\dim I_1=\dim I-1=\dim J-1$. Thanks to Property (I3) there exists $y\in J\setminus I_1$ such that

$$I':=I_1\oplus\langle y\rangle\in\mathcal{B}(V).$$

Now we have three bases, I', I, J, of V. We show first $\tau(I)=\tau(I')$. Since $y\notin I$ we have the subspace $W:=I\oplus\langle y\rangle$ of V, which must be dependent due to maximality of I. Furthermore, I, $I'\leqslant W$ and $\dim I'=\dim I=\dim W-1$, and therefore $\tau(I)=\tau(I')$ thanks to Case 1. Next, we show $\tau(I')=\tau(J)$. In order to do so, note that $I'=(I\cap J)\oplus K\oplus\langle y\rangle$, where $y\in J$. Thus $\dim(I'\cap J)\geqslant s-(r-1)$ and the induction hypothesis yields $\tau(I')=\tau(J)$. All of this establishes (6) and concludes the proof. □

REMARK 4.3. In a q-matroid $\mathcal{M}=(E,\rho)$ a subspace $V\in\mathcal{V}(E)$ satisfies

$$V \text{ is independent and } \rho(V)=\rho(E)\iff V \text{ is a basis of } \mathcal{M}.$$

The forward direction is the definition of basis in [7, Def. 2]. By Theorem 4.2 the direction "$\Longleftarrow$" holds true for q-PMs as well. However, "$\Longrightarrow$" is not true, as the q-PMs in Examples 6.1 and 6.6 show. In other words, in a q-PM not every $I \in \mathcal{I}_\mu(V)$ satisfying $\rho(I) = \rho(V)$ is a μ-basis of V.

We are now ready to provide a characterization of the pairs $(\mathcal{I}, \tilde{\rho})$ of collections $\mathcal{I}$ of subspaces and rank functions $\tilde{\rho}$ on $\mathcal{I}$ that give rise to a q-PM whose collection of independent spaces is $\mathcal{I}$ and whose rank function restricts to $\tilde{\rho}$. Clearly, $\mathcal{I}$ has to satisfy (I1)–(I4) from Corollary 3.9, and $\tilde{\rho}$ must satisfy (R1)–(R3). However, for independence we also need the rank condition from Definition 3.1. This leads to (R1′) in Theorem 4.4 below. Furthermore, since the sum of independent spaces need not be independent, we have to adjust (R3) and replace $\tilde{\rho}(I + J)$ by $\max\{\tilde{\rho}(K) \mid K \in \mathcal{I},\, K \leqslant I + J\}$, thereby accounting for Theorem 4.1. This results in the submodularity condition (R3′) below. Since one can easily find examples showing that (R1′)–(R3′) are not sufficient to guarantee submodularity of the extended rank function (defined in (7) below), we also have to enforce Theorem 4.2. This leads to condition (R4′), which states that for any space V all maximal subspaces that are contained in $\mathcal{I}$ have the same rank value. As we will see, all these conditions together guarantee submodularity of the extended rank function, and the spaces in $\mathcal{I}$ are independent in the resulting q-PM. However, the q-PM may have additional independent subspaces; see Example 6.7. In order to prevent this, we need a natural closure property. This will be spelled out in Theorem 4.5.

Theorem 4.4. *Let $\mathcal{I}$ be a subset of $\mathcal{V}(E)$. For $V \in \mathcal{V}(E)$ set $\mathcal{I}(V) = \{I \in \mathcal{I} \mid I \leqslant V\}$ and denote by $\mathcal{I}_{max}(V)$ the set of inclusion-maximal subspaces in $\mathcal{I}(V)$. Suppose $\mathcal{I}$ satisfies the following.*

- (I1) $\{0\} \in \mathcal{I}$.
- (I2) *If $I \in \mathcal{I}$ and $J \leqslant I$, then $J \in \mathcal{I}$.*
- (I3) *If $I, J \in \mathcal{I}$ and $\dim I < \dim J$, then there exists $x \in J \smallsetminus I$ such that $I \oplus \langle x\rangle \in \mathcal{I}$.*
- (I4) *Let $V, W \in \mathcal{V}(E)$ and $I \in \mathcal{I}_{max}(V)$, $J \in \mathcal{I}_{max}(W)$. Then there exists a space $K \in \mathcal{I}_{max}(V + W)$ that is contained in $I + J$.*

Furthermore, let $\tilde{\rho} : \mathcal{I} \longrightarrow \mathbb{Q}$ and $\mu \in \mathbb{Q}_{>0}$ such that $\mu\tilde{\rho}(I) \in \mathbb{Z}$ for all $I \in \mathcal{I}$. Suppose $\tilde{\rho}$ satisfies the following.

- (R1′) $0 \leqslant \mu^{-1} \dim I \leqslant \tilde{\rho}(I) \leqslant \dim I$ *for all $I \in \mathcal{I}$.*
- (R2′) *If $I, J \in \mathcal{I}$ such that $I \leqslant J$, then $\tilde{\rho}(I) \leqslant \tilde{\rho}(J)$.*
- (R3′) *For all $I, J \in \mathcal{I}$ we have $\max\{\tilde{\rho}(K) \mid K \in \mathcal{I}(I + J)\} + \tilde{\rho}(I \cap J) \leqslant \tilde{\rho}(I) + \tilde{\rho}(J)$.*
- (R4′) *For all $V \in \mathcal{V}(E)$ and $I, J \in \mathcal{I}_{max}(V)$ we have $\tilde{\rho}(I) = \tilde{\rho}(J)$.*

Define the map

$$\rho : \mathcal{V}(E) \longrightarrow \mathbb{Q}, \quad V \longmapsto \max\{\tilde{\rho}(I) \mid I \in \mathcal{I}(V)\}. \tag{7}$$

Then $\mathcal{M} = (E, \rho)$ is a q-PM with denominator μ, and $\mathcal{I} \subseteq \mathcal{I}_\mu(\mathcal{M})$.

Note that thanks to (I3) the set $\mathcal{I}_{\max}(V)$ equals the set of maximal-dimensional spaces in $\mathcal{I}(V)$. Furthermore, by (R2′) and (R4′) every $V \in \mathcal{V}(E)$ satisfies $\rho(V) = \tilde{\rho}(I)$ for each $I \in \mathcal{I}_{\max}(V)$.

Proof. It is clear that μ is a denominator of ρ. We have to show that ρ satisfies (R1)–(R3) from Definition 2.1.
(R1) Let $V \in \mathcal{V}(E)$ and $I \in \mathcal{I}$ such that $I \leqslant V$ and $\tilde{\rho}(I) = \rho(V)$. Then $0 \leqslant \tilde{\rho}(I) \leqslant \dim I \leqslant \dim V$, which establishes (R1).
(R2) Let $V, W \in \mathcal{V}(E)$ be such that $V \leqslant W$. Let $I \in \mathcal{I}$ be such that $I \leqslant V$ and $\tilde{\rho}(I) = \rho(V)$. Then $I \leqslant W$ and the definition of ρ implies $\rho(W) \geqslant \tilde{\rho}(I) = \rho(V)$, as desired.

(R3) Let $V, W \in \mathcal{V}(E)$. Choose $K \in \mathcal{I}_{\max}(V \cap W)$. Then (7) implies $\tilde{\rho}(K) = \rho(V \cap W)$. Applying (I3) repeatedly, we can find $I \in \mathcal{I}_{\max}(V)$ and $J \in \mathcal{I}_{\max}(W)$ such that $K \leqslant I$ and $K \leqslant J$. By (I4) there exists $H \in \mathcal{I}_{\max}(V + W)$ such that $H \leqslant I + J$. Now (R2′) and (R4′) imply

$$\begin{aligned}\tilde{\rho}(I) &= \rho(V), & \tilde{\rho}(H) &= \rho(I + J) = \rho(V + W),\\ \tilde{\rho}(J) &= \rho(W), & \tilde{\rho}(K) &= \tilde{\rho}(I \cap J) = \rho(V \cap W).\end{aligned}$$

From (R3′) we obtain $\rho(I + J) + \tilde{\rho}(I \cap J) \leqslant \tilde{\rho}(I) + \tilde{\rho}(J)$, and we finally arrive at

$$\begin{aligned}\rho(V + W) + \rho(V \cap W) &= \tilde{\rho}(H) + \tilde{\rho}(K) = \rho(I + J) + \tilde{\rho}(I \cap J)\\ &\leqslant \tilde{\rho}(I) + \tilde{\rho}(J) = \rho(V) + \rho(W),\end{aligned}$$

as desired. Finally, (R1′) shows that the spaces in $\mathcal{I}$ are μ-independent, thus $\mathcal{I} \subseteq \mathcal{I}_\mu(\mathcal{M})$. □

The q-PM $\mathcal{M}$ from the last theorem has in general more independent spaces than $\mathcal{I}$; see Example 6.7. We can easily force equality $\mathcal{I} = \mathcal{I}_\mu(\mathcal{M})$ by adding the following natural closure property.

Theorem 4.5. *Let the pair $(\mathcal{I}, \tilde{\rho})$ be as in Theorem* 4.4. *Suppose $(\mathcal{I}, \tilde{\rho})$ satisfies (I1)–(I4) and (R1′)–(R4′) as well as the following closure property:*

(C) *If $V \in \mathcal{V}(E)$ is such that*
  (a) *all proper subspaces of V are in $\mathcal{I}$,*
  (b) $\max\{\tilde{\rho}(I) \mid I \in \mathcal{I}(V)\} \geqslant \mu^{-1} \dim V$,
  then V is in $\mathcal{I}$.

Then $\mathcal{I} = \mathcal{I}_\mu(\mathcal{M})$ for the q-PM $\mathcal{M}$ from Theorem 4.4.

Note that by (I2) and (R1′), any subspace $V \in \mathcal{I}$ satisfies the properties in (a) and (b).

Proof. Thanks to Theorem 4.4 it remains to show that any $V \in \mathcal{I}_\mu(\mathcal{M})$ is in $\mathcal{I}$. Recall that $\rho(V) = \max\{\tilde{\rho}(I) \mid I \in \mathcal{I}(V)\}$. We induct on $\dim V$.
i) Let $\dim V = 1$. Then $\rho(V) \geqslant \mu^{-1} \dim V$ holds true by the definition of μ-independence, hence (b) is satisfied. Property (a) is trivially satisfied by (I1). Now (C) implies $V \in \mathcal{I}$.
ii) Let $\dim V = r$ and assume that all subspaces $V \in \mathcal{I}_\mu(\mathcal{M})$ of dimension at most $r - 1$ are in $\mathcal{I}$. Since $V \in \mathcal{I}_\mu(\mathcal{M})$, the same is true for all its subspaces. Hence all proper subspaces are in $\mathcal{I}$ by induction hypothesis. Again, $\rho(V) \geqslant \mu^{-1} \dim V$ is true by μ-independence and thus Property (C) implies that $V \in \mathcal{I}$. □

5. Spanning Spaces and Strongly Independent Spaces

In this section, we introduce (minimal) spanning spaces and (maximally) strongly independent subspaces. While in q-matroids the notions ‘minimal spanning space’, ‘maximally strongly independent space’, and ‘basis’ coincide, they are distinct for q-PMs. However, in q-PMs spanning spaces turn out to be the dual notion to strongly independent spaces. This result may be regarded as the generalization of the duality result for bases in q-matroids. The latter states that for a q-matroid $\mathcal{M}$ a space B is a basis of $\mathcal{M}$ if and only if $B^\perp$ is a basis of $\mathcal{M}^*$. We show that, in fact, this equivalence characterizes q-matroids within the class of q-PMs.

Definition 5.1. *Let $\mathcal{M} = (E, \rho)$ be a q-PM and let $V \in \mathcal{V}(E)$.*

(a) *V is called a* spanning space *if $\rho(V) = \rho(E)$ and V is a* minimal spanning space *if it is a spanning space and no proper subspace is a spanning space.*

(b) *V is* strongly independent *if* $\rho(V) = \dim V$ *and it is* maximally strongly independent *if it is strongly independent and not properly contained in a strongly independent subspace.*

Clearly, strongly independent subspaces are μ-independent for every denominator μ of $\mathcal{M}$. Furthermore, in q-matroids strong independence coincides with independence. We remark that strongly independent subspaces of q-PMs play a crucial role in [1] for the construction of subspace designs. For q-matroids the following notions coincide.

Proposition 5.2. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $V \in \mathcal{V}(E)$. Then*

$$\begin{aligned} V \text{ is maximally strongly independent} &\Longleftrightarrow V \text{ is a basis} \\ &\Longleftrightarrow V \text{ is a minimal spanning space.} \end{aligned}$$

Proof. The first equivalence is clear since for q-matroids strong independence coincides with independence (see Proposition 3.3). We turn to the second equivalence. "$\Rightarrow$" Let V be a basis of $\mathcal{M}$. Then $\dim V = \rho(V) = \rho(E)$. For every proper subspace $W \lneq V$ we have $\rho(W) \leqslant \dim W < \dim V = \rho(E)$, hence W is not a spanning space. This proves minimality of V. "$\Leftarrow$" Let now V be a minimal spanning space. Then $\rho(V) = \rho(E)$. Suppose V is dependent. Then there exists a basis W of V, and Theorem 4.2 implies $\rho(W) = \rho(V) = \rho(E)$. This contradicts minimality of V. Hence V is independent and thus a basis thanks to Remark 4.3. $\square$

The last result is not true for q-PMs. For instance, it can be verified that for either q-PM in Example 6.1 the basis has dimension 5, the minimal spanning spaces have dimension 3, and the maximally strongly independent spaces have dimension 2. On the other hand, there exist q-PMs that are not q-matroids and yet the bases coincide with the minimal spanning spaces. Thus the second equivalence in Proposition 5.2 does not characterize q-matroids. As for the first equivalence, note that if a μ-basis of a q-PM is strongly independent, then this is true for all bases (because they all have the same dimension by (4) and the same rank by Theorem 4.2). Thus all independent spaces are strongly independent thanks to Proposition 3.3 and the rank function is integer-valued by Theorem 4.1. This shows that the first equivalence does characterize q-matroids.

The following describes the relation between bases and minimal spanning spaces in a q-PM.

Proposition 5.3. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ.*

(a) *A minimal spanning space is μ-independent.*
(b) *Every μ-basis of $\mathcal{M}$ contains a minimal spanning space and every minimal spanning space is contained in a μ-basis.*

Proof. (a) Let V be a minimal spanning space. If V is μ-dependent, then V contains a μ-basis W, and Theorem 4.2 implies $\rho(W) = \rho(V) = \rho(E)$. This contradicts minimality of V. (b) is clear. $\square$

Recall duality from Theorem 2.4. Our next result shows that bases are compatible with duality in "the expected way" if and only if the q-PM is a q-matroid. Part (a) has been established in [7].

Proposition 5.4. *Let the q-PMs $\mathcal{M} = (E, \rho)$ and $\mathcal{M}^* = (E, \rho^*)$ be as in Theorem 2.4.*

(a) *If $\mathcal{M}$ is a q-matroid, then for every basis B of $\mathcal{M}$ the orthogonal space $B^\perp$ is a basis of $\mathcal{M}^*$.*
(b) *Let μ be a denominator of $\mathcal{M}$. Suppose there exists a μ-basis B of $\mathcal{M}$ such that the orthogonal space $B^\perp$ is a μ-basis of $\mathcal{M}^*$. Then $\mathcal{M}$ is a q-matroid.*

Proof. (a) has been proven in [7, Thm. 45].
(b) Let B be a μ-basis of $\mathcal{M}$ and $B^\perp$ be a μ-basis of $\mathcal{M}^*$. Then $\rho(B) = \rho(E)$ and thus $\rho^*(B^\perp) = \dim B^\perp + \rho(B) - \rho(E) = \dim B^\perp$. Theorem 4.2 implies that every basis $\hat{B}$ of $\mathcal{M}^*$ satisfies $\rho^*(\hat{B}) = \rho^*(B^\perp) = \dim B^\perp = \dim \hat{B}$. Now Proposition 3.3 yields $\rho^*(I) = \dim I$ for all μ-independent spaces I of $\mathcal{M}^*$. Hence the dual rank function ρ^* is integer-valued on the μ-independent spaces. But then the entire rank function ρ^* is integer-valued thanks to Theorem 4.1. Now $\rho = \rho^{**}$ is also integer-valued, which means that $\mathcal{M}$ is a q-matroid. □

The above result has an interesting consequence. Recall from Theorem 3.6 the auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$. Part (b) above implies that if $\mathcal{M}$ is a q-PM with denominator μ and $\mathcal{M}$ is not a q-matroid, then $\mathcal{Z}_{\mathcal{M}^*,\mu} \not\approx \mathcal{Z}^*_{\mathcal{M},\mu}$. Indeed, Theorem 3.6 implies that a subspace $B \in \mathcal{V}(E)$ is a μ-basis in $\mathcal{M}$ if and only if it is a basis in $\mathcal{Z}_{\mathcal{M},\mu}$. Thanks to Proposition 5.4(a) the latter is equivalent to $B^\perp$ being in basis in $\mathcal{Z}^*_{\mathcal{M},\mu}$. But by Proposition 5.4(b) $B^\perp$ is not a basis of $\mathcal{M}^*$, and thus not of $\mathcal{Z}_{\mathcal{M}^*,\mu}$.

Spanning spaces and strongly independent spaces are mutually dual, as one can see immediately with (1). This may be regarded a generalization of [3, Prop. 91] and [7, Thm. 45] (i.e., Proposition 5.4(a)), where the same results have been established for q-matroids.

Proposition 5.5. *Let $\mathcal{M}$ and $\mathcal{M}^*$ be as in Proposition* 5.4 *and let $V \in \mathcal{V}(E)$. Then V is a (minimal) spanning space in $\mathcal{M}$ if and only if $V^\perp$ is (maximally) strongly independent in $\mathcal{M}^*$.*

We close the section with a few remarks on the properties — or rather lack thereof — of strongly independent spaces and spanning spaces in q-PMs. Neither maximally strongly independent spaces nor minimal spanning spaces are as well-behaved as bases. This is not surprising since neither collection consists of subspaces of constant dimension (which can be verified with Example 6.6).

Remark 5.6.
(a) Let $\mathcal{M}$ be a q-PM and $\tilde{\mathcal{I}}$ be its collection of strongly independent subspaces. Thanks to Proposition 3.3 $\tilde{\mathcal{I}}$ satisfies (I2) of Corollary 3.9. It is not hard to find (sufficiently large) examples showing that $\tilde{\mathcal{I}}$ does not satisfy (I3) and (I4).
(b) Bases in a q-PM satisfy conditions (B1)–(B4) in Corollary 3.10. But neither the maximally strongly independent subspaces nor the minimal spanning spaces satisfy (B3) or (B4).

6. Examples

Example 6.1 ([5, Ex. 3.13]). We present two MRD codes in $\mathbb{F}_2^{5\times 2}$ with the same rank distance whose column q-PMs are not equivalent. Consider $\mathcal{C}_1 = \langle A_1, \ldots, A_5\rangle$ and $\mathcal{C}_2 = \langle B_1, \ldots, B_5\rangle$, where

$$A_1 = \begin{pmatrix}1&1\\1&0\\0&0\\1&0\\0&0\end{pmatrix},\ A_2 = \begin{pmatrix}1&1\\1&1\\1&0\\0&1\\0&0\end{pmatrix},\ A_3 = \begin{pmatrix}0&0\\0&0\\1&1\\0&0\\0&1\end{pmatrix},\ A_4 = \begin{pmatrix}0&0\\0&1\\0&0\\0&0\\1&1\end{pmatrix},\ A_5 = \begin{pmatrix}1&0\\0&1\\1&1\\0&0\\0&1\end{pmatrix},$$

and

$$B_1 = \begin{pmatrix} 1\,0\\ 0\,1\\ 0\,0\\ 0\,0\\ 0\,0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0\,0\\ 1\,0\\ 0\,1\\ 0\,0\\ 0\,0 \end{pmatrix},\ B_3 = \begin{pmatrix} 0\,0\\ 0\,0\\ 1\,0\\ 0\,1\\ 0\,0 \end{pmatrix},\ B_4 = \begin{pmatrix} 0\,0\\ 0\,0\\ 0\,0\\ 1\,0\\ 0\,1 \end{pmatrix},\ B_5 = \begin{pmatrix} 0\,1\\ 0\,0\\ 0\,1\\ 0\,0\\ 1\,0 \end{pmatrix}.$$

Both codes are MRD with rank distance $d = 2$, and $\mathcal{C}_2$ is actually a ($\mathbb{F}_{2^5}$-linear) Gabidulin code. Consider the q-PMs $\mathcal{M}_i := \mathcal{M}_{\rm c}(\mathcal{C}_i) = (\mathbb{F}^5, \rho_{\rm c}^i)$, $i = 1, 2$. From Proposition 2.9(b) we know that $\rho_{\rm c}^1(V) = \rho_{\rm c}^2(V) = \dim V$ for $\dim V \leqslant 1$ and $\rho_{\rm c}^1(V) = \rho_{\rm c}^2(V) = 5/2$ if $\dim V \geqslant 4$. As for the 2-dimensional subspaces of $\mathbb{F}_2^5$, it turns out that the map $\rho_{\rm c}^1$ assumes the value 1 exactly once and the values 3/2 and 2 exactly 28 and 126 times, respectively, whereas $\rho_{\rm c}^2$ assumes the values 3/2 and 2 exactly 31 and 124 times, respectively, and never takes the value 1. Similar differences occur for the 3-dimensional subspaces. Thus $\mathcal{M}_1$ and $\mathcal{M}_2$ are not equivalent.

Example 6.2. Independent spaces of the same dimension need not have the same rank value. Let $\mathbb{F} = \mathbb{F}_2$ and consider the code $\mathcal{C} \leqslant \mathbb{F}^{3\times 3}$ generated by

$$\begin{pmatrix} 0\,1\,0\\ 0\,0\,1\\ 0\,0\,1 \end{pmatrix},\ \begin{pmatrix} 0\,1\,1\\ 0\,0\,0\\ 0\,0\,1 \end{pmatrix},\ \begin{pmatrix} 0\,1\,1\\ 1\,0\,0\\ 0\,1\,0 \end{pmatrix}.$$

Let $\mathcal{M}_{\rm c}(\mathcal{C}) = (\mathbb{F}^3, \rho_{\rm c})$ be the associated column q-PM. Then for all $V \in \mathcal{V}(\mathbb{F}^3) \smallsetminus \{0\}$

$$\rho_{\rm c}(V) = \begin{cases} 2/3 & \text{if } \dim V = 1 \text{ or } V = \langle e_1 + e_2,\, e_3\rangle,\\ 1 & \text{otherwise.} \end{cases}$$

Thus 3 is the principal denominator and all spaces are independent. In particular, all 2-dimensional spaces are independent, but they do not assume the same rank value.

Example 6.3. A dependent space may have a larger rank value than an independent space of the same dimension. Let $\mathbb{F} = \mathbb{F}_2$ and $\mathcal{C} \leqslant \mathbb{F}^{5\times 3}$ be the code generated by the standard basis matrices $E_{11}, E_{12}, E_{23}, E_{32}, E_{41}, E_{42}$. In the column q-PM $\mathcal{M}_{\rm c}(\mathcal{C}) = (\mathbb{F}^5, \rho_{\rm c})$ the subspace $I = \langle e_2, e_3\rangle$ is independent with $\rho_{\rm c}(I) = 2/3$, while the subspace $V = \langle e_1 + e_2, e_5\rangle$ satisfies $\rho_{\rm c}(V) = 1$ and is dependent (because $\langle e_5\rangle$ is a loop).

Example 6.4. A subspace V satisfying $\mu\rho_{\rm c}(W) = \mu\rho_{\rm c}(V) = \dim V - 1$ for all its hyperplanes W need not be a circuit. Let $\mathbb{F} = \mathbb{F}_2$ and

$$\Delta = \begin{pmatrix} 0\,1\,0\,0\\ 0\,0\,1\,0\\ 0\,0\,0\,1\\ 1\,1\,0\,0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0\,1\,1\,0\\ 0\,0\,0\,0\\ 0\,0\,0\,0\\ 1\,0\,0\,1\\ 0\,0\,1\,1\\ 1\,0\,0\,1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1\,0\,0\,0\\ 0\,0\,0\,0\\ 1\,0\,0\,0\\ 1\,0\,0\,1\\ 0\,1\,1\,1\\ 0\,0\,0\,1 \end{pmatrix},\ A_3 = \begin{pmatrix} 1\,1\,0\,1\\ 0\,0\,1\,0\\ 0\,0\,1\,0\\ 1\,0\,0\,0\\ 0\,0\,0\,1\\ 1\,0\,1\,0 \end{pmatrix}.$$

Set $\mathcal{C} = \langle A_1, A_2, A_3, A_1U, A_2U, A_3U\rangle$, where $U = \Delta^5$. Then $\mathcal{C}$ is a right $\mathbb{F}_{2^2}$-linear rank-metric code of dimension 6. Indeed, Δ is the companion matrix of the primitive polynomial $f := x^4 + x + 1 \in \mathbb{F}_2[x]$ and since $5 = (2^4-1)/(2^2-1)$, any root ω of f leads to a primitive element ω^5 of the subfield $\mathbb{F}_{2^2}$. Thus $\Psi_{\mathcal{B}}^{-1}(\mathcal{C})$, where $\mathcal{B} = (1, \omega, \omega^2, \omega^3)$, is an $\mathbb{F}_{2^2}$-subspace of $\mathbb{F}_{2^4}^6$. The principal denominator of $\mathcal{M}_{\rm c}(\mathcal{C})$ is $\mu = 2$. There exist 497 μ-circuits, one of which has dimension 1 and all others have dimension 4. An additional 169 spaces V satisfy $\mu\rho_{\rm c}(V) = \dim V - 1$, and 97 of them also satisfy $\mu\rho_{\rm c}(W) = \dim V - 1$ for all its hyperplanes W.

Example 6.5. There is no obvious relation between the auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$ of a q-matroid $\mathcal{M}$ and $\mathcal{M}$ itself if $\mu > 2$. Let $n \geqslant 3$ and fix a 2-dimensional subspace $X \in \mathcal{V}(\mathbb{F}^n)$. Set $\rho(X) = 1$ and $\rho(V) = \min\{\dim V, 2\}$ for $V \neq X$. One can check straightforwardly that $\mathcal{M} = (\mathbb{F}^n, \rho)$ is a q-matroid (this also follows from [5, Prop. 4.6]). Choosing $\mu = 2$, one verifies that $r_{\rho,2} = \min\{\dim V, 4\}$, and thus the q-matroids $\mathcal{M}$ and $\mathcal{Z}_{\mathcal{M},2}$ are not equivalent.

Example 6.6. Let $\mathbb{F} = \mathbb{F}_2$ and consider the codes $\mathcal{C} = \langle A_1, A_2, A_3\rangle, \mathcal{C}' = \langle A_1, A_2, A_3'\rangle \leqslant \mathbb{F}^{4\times 3}$, where

$$A_1 = \begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&1\\ 0&1&0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1&0&1\\ 1&0&0\\ 0&0&0\\ 1&1&1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0&1&0\\ 0&1&1\\ 0&1&0\\ 0&1&1 \end{pmatrix}, \quad A_3' = \begin{pmatrix} 1&0&0\\ 0&0&0\\ 1&0&1\\ 1&0&1 \end{pmatrix}.$$

Both the associated q-PMs $\mathcal{M} = \mathcal{M}_{\mathrm{c}}(\mathcal{C}) = (\mathbb{F}^4, \rho_{\mathrm{c}})$ and $\mathcal{M}' = \mathcal{M}_{\mathrm{c}}(\mathcal{C}') = (\mathbb{F}^4, \rho_{\mathrm{c}}')$ have principal denominator 3, and the space $\mathbb{F}^4$ is the only dependent space. Hence $\mathcal{M}$ and $\mathcal{M}'$ share the same bases, namely all 3-dimensional spaces. Moreover, $\rho_{\mathrm{c}}(V) = 1 = \rho_{\mathrm{c}}'(V)$ for all bases V. Yet, $\mathcal{M}$ and $\mathcal{M}'$ are not equivalent: in $\mathcal{M}$ the rank value 1 is assumed by 33 subspaces of dimension 2, whereas in $\mathcal{M}'$ it is assumed by 32 subspaces of dimension 2 (in both q-PMs 4 subspaces of dimension 1 have rank value 1 as well).

Example 6.7. Consider the q-PM $\mathcal{M} = (\mathbb{F}^3, \rho_{\mathrm{c}})$ from Example 6.2. We have seen already that $\mathcal{I}_3(\mathcal{M}) = \mathcal{V}(\mathbb{F}^3)$. Define the set $\mathcal{I} = \{V \in \mathcal{V}(\mathbb{F}^3) \mid V \neq \langle e_1 + e_2, e_3\rangle \text{ and } V \neq \mathbb{F}^3\}$ and let $\tilde{\rho} = \rho_{\mathrm{c}}|_{\mathcal{I}}$. One easily verifies that $(\mathcal{I}, \tilde{\rho})$ satisfies (I1)–(I4) and (R1′)–(R4′). Furthermore, the extension ρ defined in Theorem 4.4 equals ρ_{c} and thus the induced q-PM $(\mathbb{F}^3, \rho)$ equals $\mathcal{M}$. Now we have $\mathcal{I} \subsetneq \mathcal{I}_3(\mathcal{M})$.

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