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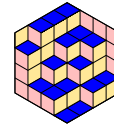
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ABSTRACT For a finite rank one simple Lie-type group acting 2-transitively on a set Ω and $k \in \mathbb{N}$ we derive formulae for the number of G -orbits on the set of all k -subsets of Ω .

1. INTRODUCTION

Suppose G is a permutation group acting upon a set Ω . Then G has an induced action upon $\mathcal{P}(\Omega)$, the power set of Ω . For $k \in \mathbb{N}$, let $\mathcal{P}_k(\Omega)$ denote all the k -subsets of Ω , clearly G also has an induced action upon $\mathcal{P}_k(\Omega)$. In this paper we shall assume G and Ω are finite. Let $\sigma_k(G, \Omega)$ denote the number of G -orbits on $\mathcal{P}_k(\Omega)$. Questions as to the behaviour of $\sigma_k(G, \Omega)$ as k varies and the relationship between G -orbit lengths for various k arise naturally. A contribution to the former type of question is given by the venerable theorem of Livingstone and Wagner [12] which states that for $k \in \mathbb{N}$ with $2 \leq k \leq |\Omega|/2$,

$$\sigma_{k-1}(G, \Omega) \leq \sigma_k(G, \Omega).$$

Generalizations of this theorem have been obtained by Mnukhin and Siemons [15]. While Bundy and Hart [3] have considered the situation when

$$\sigma_{k-1}(G, \Omega) = \sigma_k(G, \Omega).$$

Results on G -orbit lengths are somewhat patchy at present - see Siemons and Wagner [18, 19] and Mnukhin [14]. For the record, we remark there is a considerable literature concerning the infinite case; for a small selection see Cameron [4].

The main aim here is to establish formulae for $\sigma_k(G, \Omega)$ when G is a rank one simple Lie-type group acting 2-transitively on Ω . Thus set $q = p^a$ where p is a prime and $a \in \mathbb{N}$. Then the possibilities for G are $L_2(q)$ ($q > 3$), the 2-dimensional projective special linear groups, $Sz(q)$ ($q = 2^{2n+1} > 2$), the Suzuki groups, $U_3(q)$ ($q > 2$), the 3-dimensional projective special unitary groups and $R(q)$ ($q = 3^{2n+1} > 3$), the Ree groups. First we consider the exceptional 2-transitive representations of $L_2(7)$ of degree 7, $L_2(9)$ of degree 6 and $L_2(11)$ of degree 11. Here writing $\sigma_k(G, \Omega)$ as σ_k , we have for $L_2(7)$, $\sigma_1 = \sigma_2 = 1, \sigma_3 = 2$; for $L_2(9)$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$; and for $L_2(11)$, $\sigma_1 = \sigma_2 = 1, \sigma_3 = 2, \sigma_4 = 3, \sigma_5 = 4$. For the remaining 2-transitive representations, the corresponding sets Ω are the projective line (with $|\Omega| = q + 1$), the Suzuki ovoid (with $|\Omega| = q^2 + 1$), the isotropic 1-spaces of a 3-dimensional unitary space (with $|\Omega| = q^3 + 1$) and the Steiner system $S(2, q + 1, q^3 + 1)$ (with $|\Omega| = q^3 + 1$).

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Before stating our main results we must introduce some notation. For $b, c \in \mathbb{N}$, we use (b, c) to denote the greatest common divisor of b and c . For $\ell \in \mathbb{N}$ we let

$$\mathcal{D}(\ell) = \{n \in \mathbb{N} \mid n \text{ divides } \ell\}$$

and $\mathcal{D}^*(\ell) = \mathcal{D}(\ell) \setminus \{1\}$. Euler's phi function ϕ (see [17]) will feature in our results. Our final piece of notation concerns partitions. Let $n \in \mathbb{N}$, and let $\pi = \lambda_1 \lambda_2 \dots \lambda_r$ where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$ and $\sum_{i=1}^r \lambda_i = n$. Though we will frequently use the more compressed notation $\pi = \mu_1^{a_1} \mu_2^{a_2} \dots$ where $\mu_1 < \mu_2 < \dots$ (a_i being the multiplicity of μ_i in the partition π). For $k \in \mathbb{N}$, $\eta_k(\pi)$ is defined to be the number of subsequences $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_s}$ of $\lambda_1, \dots, \lambda_r$ which form a partition of k . Since G acts 2-transitively on Ω , we have $\sigma_1(G, \Omega) = \sigma_2(G, \Omega) = 1$, so we shall assume $k \geq 3$. Now we come to our first theorem.

THEOREM 1.1. *Suppose that $G \cong L_2(q)$ ($q > 3$) acts upon the projective line Ω , and let $k \in \mathbb{N}$ with $k \geq 3$. Set $d = (2, q - 1)$. Then*

$$\begin{aligned} \sigma_k(G, \Omega) &= \frac{d}{q(q+1)(q-1)} \eta_k(1^{q+1}) + \frac{d}{q} \eta_k(1^1 p^{\frac{q}{p}}) \\ &\quad + \frac{d}{2(q+1)} \sum_{m \in \mathcal{D}^*(\frac{q+1}{d})} \phi(m) \eta_k\left(m^{\frac{q+1}{m}}\right) \\ &\quad + \frac{d}{2(q-1)} \sum_{m \in \mathcal{D}^*(\frac{q-1}{d})} \phi(m) \eta_k\left(1^2 m^{\frac{q-1}{m}}\right). \end{aligned}$$

In [5] Cameron, Maimani, Omid and Teyfeh-Rezaie give methods for calculating $\sigma_k(L_2(q), \Omega)$, where Ω is the projective line and $q \equiv 3 \pmod{4}$, and Cameron, Omid and Teyfeh-Rezaie [6] follow a similar approach for $\sigma_k(PGL(2, q), \Omega)$, their interest being motivated by a search for 3-designs. Further work is presented by Liu, Tang and Wu [11] and also Chen and Liu [7], in the case when $q \equiv 1 \pmod{4}$.

THEOREM 1.2. *Suppose $G \cong Sz(q)$ ($q = 2^{2n+1} > 2, n \in \mathbb{N}$) acts upon the Suzuki ovoid Ω . Let $r \in \mathbb{N}$ be such that $r^2 = 2q$, and let $k \in \mathbb{N}$ with $k \geq 3$. Then*

$$\begin{aligned} \sigma(G, \Omega) &= \frac{1}{q^2(q-1)(q^2+1)} \eta_k(1^{q^2+1}) + \frac{1}{q^2} \eta_k(1^1 2^{\frac{q^2}{2}}) \\ &\quad + \frac{1}{q} \eta_k(1^1 4^{\frac{q^2}{4}}) + \frac{1}{2(q-1)} \sum_{m \in \mathcal{D}^*(q-1)} \phi(m) \eta_k(1^2 m^{\frac{q^2-1}{m}}) \\ &\quad + \frac{1}{4(q+r+1)} \sum_{m \in \mathcal{D}^*(q+r+1)} \phi(m) \eta_k(m^{\frac{q^2+1}{m}}) \\ &\quad + \frac{1}{4(q-r+1)} \sum_{m \in \mathcal{D}^*(q-r+1)} \phi(m) \eta_k(m^{\frac{q^2+1}{m}}). \end{aligned}$$

The corresponding result for $U_3(q)$ is more complicated than for $L_2(q)$ and $Sz(q)$ – see Definitions 3.1, 3.2, 3.3 and 3.4 for an explanation of the notation in the next theorem.

THEOREM 1.3. *Suppose $G \cong U_3(q)$ ($q > 2$) acts upon Ω , the set of isotropic points of a 3-dimensional unitary space. Let $k \in \mathbb{N}$ with $k \geq 3$, and set $d = (3, q + 1)$ and*

$\ell = \frac{q+1}{d}$. Then

$$\begin{aligned} \sigma_k(G, \Omega) &= \frac{d(\eta_k(\pi_1) + \mu_k)}{q^3(q^3 + 1)(q^2 - 1)} + \frac{d}{q(q+1)(q^2 - 1)} \sum_{m \in \mathcal{D}^*(\ell)} \phi(m)\eta_k(\pi_4^{(m)}) \\ &\quad + \frac{d}{q(q+1)(p-1)} \left(\sum_{\substack{m=pj \\ j \in \mathcal{D}^*(\ell)}} \phi(m)\eta_k(\pi_5^{(m)}) \right) + \frac{d\sigma_k(E_0^*, \Omega)}{6(q+1)^2} \\ &\quad + \frac{d}{2(q^2 - 1)} \sum_{\substack{m \in \mathcal{D}(\frac{q^2-1}{d}) \\ m \notin \mathcal{D}(\ell)}} \phi(m)\eta_k(\pi_7^{(m)}) \\ &\quad + \frac{d(q+1)}{3(q^3 + 1)} \sum_{m \in \mathcal{D}^*(\frac{q^2-q+1}{d})} \phi(m)\eta_k(\pi_8^{(m)}). \end{aligned}$$

Lemma 3.10 gives formulae for $\sigma_k(E_0^*, \Omega)$.

A corresponding result for the Ree groups, though a little more involved, is similar to Theorems 1.1 and 1.2 and may be found in Bradley [1]. MAGMA code based on the formula in Theorem 1.1 is given in Section 4, and similar routines are also available in [1] for the Suzuki, 3-dimensional unitary and characteristic 3 Ree groups.

The proofs of Theorems 1.1, 1.2 and 1.3 all rely upon the orbit counting result commonly referred to as Burnside’s Lemma, though this result can be traced back to earlier work of Cauchy and Frobenius. For a historical excursion see [16]. Thus our main task here is to enumerate the cycle types of elements in G in their action on the various Ω . For $G \cong L_2(q)$ or $Sz(q)$ this is straightforward, particularly as G may be partitioned by certain subgroups (see Lemmas 2.5 and 2.6). For $G \cong U_3(q)$ we make use of the description of conjugacy classes obtained in [20]. There the conjugacy classes are divided into a number of types – those of the same type have the same size and fix the same number of points of Ω , Ω being the set of isotropic 1-spaces of a 3-dimensional unitary space (see Table 2). Using information about the subgroup structure of G , in Lemma 3.6 we determine the cycle structures on Ω of elements in all conjugacy class types except for $\mathcal{C}_6 \cup \mathcal{C}'_6$. This is relatively straightforward. However $\mathcal{C}_6 \cup \mathcal{C}'_6$ is a different kettle of fish – see Table 1 which details the varied cycle structures for such elements in the case of $U_3(71)$ (so $|\Omega| = 357, 912$).

Representatives for all classes of type $\mathcal{C}_6 \cup \mathcal{C}'_6$ are to be found in an abelian subgroup E of G where E is isomorphic to a direct product of cyclic groups of order $(q+1)/d$ and $q+1$ ($d = (3, q+1)$). The key result in enumerating these possible cycle types (and their multiplicities) is Lemma 3.9. With this result to hand, Lemma 3.10 then determines the contribution of the $\mathcal{C}_6 \cup \mathcal{C}'_6$ conjugacy classes to the sum $\sum_{g \in G} |\text{fix}_{\Omega_k}(g)|$ (where $\Omega_k = \mathcal{P}_k(\Omega)$). Combining Lemmas 3.6 and 3.10 yields Theorem 1.3.

It is of interest to examine the number of orbits for these groups on subsets of size 3 (and in the case of $L_2(q)$, size 4), of their respective G -sets. The following corollaries can all be derived by appropriate substitutions into Theorems 1.1, 1.2, 1.3 and the corresponding result for the Ree groups as found in [1]. For full details of the proofs see [1].

COROLLARY 1.4. *Let $q = p^a > 2$ where p is a prime and let Ω be the projective line with $q + 1$ points. Then*

$$\sigma_3(L_2(q), \Omega) = \begin{cases} 2, & \text{if } q \equiv 1 \pmod{4}, \\ 1, & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

TABLE 1. Cycle structure for $\mathcal{C}_6 \cup \mathcal{C}'_6$ class types in $U_3(71)$

Cycle Type of Conjugacy Class Representative	Number of Classes	Cycle Type of Conjugacy Class Representative	Number of Classes
$2^{36}4^{89460}$	1	$6^{12}12^{29820}$	2
$2^{36}8^{44730}$	2	$6^{12}24^{14910}$	4
$2^{36}12^{29820}$	2	$6^{12}8^9 24^{14907}$	8
$2^{36}24^{14910}$	4	$8^9 12^6 24^{14907}$	16
$2^{36}3^{24}6^{59628}$	2	9^{39768}	3
$3^{24}6^{59640}$	1	$9^8 18^{19880}$	9
$3^{24}12^{29820}$	2	$9^8 36^{9940}$	18
$3^{24}24^{14910}$	4	$9^8 72^{4970}$	36
$3^{24}4^{18}12^{29814}$	4	$12^6 24^{14910}$	8
$3^{24}8^9 24^{14907}$	8	$18^4 36^{9940}$	18
$4^{18}8^{44730}$	4	$18^4 72^{4970}$	36
$4^{18}24^{14910}$	8	$36^2 72^{4970}$	72
$4^{18}6^{12}12^{29814}$	4	$3^{11}9^{304}$	1

COROLLARY 1.5. Let $L_n = L_2(2^n)$ acting on the projective line Ω_n , and put $a_n = \sigma_4(L_n, \Omega_n)$. Setting $a_1 = a_2 = 1$, for $n \geq 3$ we have

$$a_n = a_{n-1} + 2a_{n-2}.$$

COROLLARY 1.6. Let $q = 2^{2n+1}$ and Ω_n be the Suzuki ovoid with $q^2 + 1$ points. Set $a_n = \sigma_3(Sz(q), \Omega_n)$. Then

$$a_n = \frac{4^n + 2}{3}.$$

COROLLARY 1.7. Let $q = 3^n$ and Ω_n be the $q^3 + 1$ isotropic points of a unitary 3-space. Set $a_n = \sigma_3(U_3(q), \Omega_n)$. Then

$$a_n = \frac{3^n + 3}{2}.$$

COROLLARY 1.8. Let $q = 3^{2n+1}$ and Ω_n be the Ree unital on $q^3 + 1$ points. Set $a_n = \sigma_3(R(q), \Omega_n)$. Then

$$a_n = \frac{(3^{2n+1} + 3)^2}{6}.$$

Two of the sequences found are of interest more generally, the sequence $\{a_n\}$ appearing in Corollary 1.5 is known as the Jacobsthal sequence [21], and the sequence $\{a_n\}$ given in Corollary 1.7 is associated to Sierpinski’s Triangle, see [22].

2. BACKGROUND RESULTS

First we recall some background results, starting with the frequently mis-attributed Burnside’s Lemma.

LEMMA 2.1. Let H be a finite group and Ω a finite H -set. If t is the number of H -orbits on Ω , then

$$t = \frac{1}{|H|} \sum_{h \in H} |\text{fix}_\Omega(h)|.$$

For a partition of n , $\pi = \lambda_1 \lambda_2 \dots \lambda_r$ (such that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$ and $\sum_{i=1}^r \lambda_i = n$) we recall from Section 1 that, for $k \in \mathbb{N}$, $\eta_k(\pi)$ is the number of sub-sequences $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_s}$ of $\lambda_1, \dots, \lambda_r$ which form a partition of k . As an example consider $\pi = 11444$ ($= 1^2 4^3$ in compressed form), a partition of $n = 14$. Then $\eta_5(\pi) = 6$, $\eta_6(\pi) = 3$, $\eta_7(\pi) = 0$ and $\eta_8(\pi) = 3$. Our interest in $\eta_k(\pi)$ is because of the following two simple lemmas.

LEMMA 2.2. *Suppose $g \in \text{Sym}(\Lambda)$ where $|\Lambda| = n$, and let $k \in \mathbb{N}$. If g has cycle type π on Λ (viewed as a partition of n), then g fixes (set-wise) $\eta_k(\pi)$ k -subsets of Λ .*

LEMMA 2.3. *Let $\pi = \mu_1^{\alpha_1} \mu_2^{\alpha_2} \dots \mu_s^{\alpha_s}$ be a partition of n (in compressed form). Then*

$$\eta_k(\pi) = \prod_{(k_1, k_2, \dots, k_s)} \binom{\alpha_1}{k_1} \binom{\alpha_2}{k_2} \dots \binom{\alpha_s}{k_s},$$

running over all s -tuples (k_1, k_2, \dots, k_s) with $k_i \geq 0$ and $\mu_1 k_1 + \mu_2 k_2 + \dots + \mu_s k_s = k$.

Another elementary result we require is in the next lemma.

LEMMA 2.4. *Let $H \cong \mathbb{Z}_n$ and let $m \in \mathbb{N}$, $m \neq 1$, be such that $m \mid n$. If p_1, p_2, \dots, p_r are the distinct prime divisors of m , then the number of elements in H of order m is*

$$\phi(m) = \frac{m}{p_1 p_2 \dots p_r} (p_1 - 1)(p_2 - 1) \dots (p_r - 1).$$

A few well known facts regarding $L_2(q)$ and $Sz(q)$ are given in the next two lemmas.

LEMMA 2.5. *Let $q = p^a > 3$ where p is a prime and $a \in \mathbb{N}$. Suppose $G \cong L_2(q)$, and let $P \in \text{Syl}_p(G)$.*

- (i) $|G| = \frac{q(q+1)(q-1)}{d}$ where $d = (2, q-1)$.
- (ii) G acts 2-transitively upon the projective line Ω and $G_\alpha = N_G(P)$ for some $\alpha \in \Omega$.
- (iii) G contains cyclic subgroups $H_- = \langle h_- \rangle$ and $H_+ = \langle h_+ \rangle$ where $|H_-| = \frac{q-1}{d}$ and $|H_+| = \frac{q+1}{d}$. Set $\mathcal{S} = \{P^g, H_-^g, H_+^g \mid g \in G\}$. Then every non-identity element of G belongs to a unique subgroup in \mathcal{S} .
- (iv) h_- has cycle type $1^2 \left(\frac{q-1}{d}\right)^d$ on Ω and h_+ has cycle type $\left(\frac{q+1}{d}\right)^d$ on Ω . For $1 \neq x \in P$, x has cycle type $1^1 p^{\frac{a}{p}}$ on Ω .
- (v) The number of elements of order p is $(q-1)(q+1)$.
- (vi) $[G : N_G(H_-)] = \frac{q(q+1)}{2}$ and $[G : N_G(H_+)] = \frac{q(q-1)}{2}$.

Proof. For parts (i)–(iv) see 8.1 Hilfssatz and for (iii), (iv) and (vi) consult 8.3, 8.4, 8.5 Satz of [9]. Part (v) is given in 8.2 Satz (b) and (c) of [9]. \square

LEMMA 2.6. *Let $q = 2^{2n+1}$ where $n \in \mathbb{N}$ and set $r = 2^{n+1}$. Suppose $G \cong Sz(q)$ and let $P \in \text{Syl}_2 G$.*

- (i) $|G| = q^2(q-1)(q^2+1)$.
- (ii) G acts 2-transitively on the Suzuki ovoid Ω and $G_\alpha = N_G(P)$ for some $\alpha \in \Omega$.
- (iii) G contains cyclic subgroups $H_0 = \langle h_0 \rangle$, $H_- = \langle h_- \rangle$ and $H_+ = \langle h_+ \rangle$ where $|H_0| = q-1$, $|H_-| = q-r+1$ and $|H_+| = q+r+1$. Set $\mathcal{S} = \{P^g, H_0^g, H_-^g, H_+^g \mid g \in G\}$. Then every non-identity element of G belongs to a unique subgroup in \mathcal{S} .
- (iv) h_0 has cycle type $1^2(q-1)^{q+1}$, h_- has cycle type $(q-r+1)^{\frac{q^2+1}{q-r+1}}$ and h_+ has cycle type $(q+r+1)^{\frac{q^2+1}{q+r+1}}$ on Ω . For $x \in P$, x has cycle type $1^1 2^{\frac{q^2}{2}}$, respectively $1^1 4^{\frac{q^2}{4}}$, on Ω if it has order 2, respectively 4.
- (v) $|N_G(H_0)| = 2(q-1)$, $|N_G(H_-)| = 4(q-r+1)$ and $|N_G(H_+)| = 4(q+r+1)$.

(vi) P contains $q - 1$ elements of order 2 and $q^2 - q$ elements of order 4.

Proof. Consult [23, Theorem 9]. □

Next we give a compendium of facts about $U_3(q)$.

LEMMA 2.7. *Let $q = p^a > 2$ where p is a prime and $a \in \mathbb{N}$, and suppose $G \cong U_3(q)$. Let $P \in \text{Syl}_p G$ and set $d = (q + 1, 3)$.*

- (i) $|G| = q^3 \frac{(q^2-1)}{d} (q^3 + 1)$.
- (ii) G acts 2-transitively on Ω , the set of isotropic 1-spaces of a unitary 3-space, $|\Omega| = q^3 + 1$ and $G_\alpha = N_G(P)$ for some $\alpha \in \Omega$. Further for $\beta \in \Omega \setminus \{\alpha\}$, $N_G(P) = PG_{\alpha,\beta}$ with $G_{\alpha,\beta}$ cyclic of order $\frac{q^2-1}{d}$ and $|C_{G_{\alpha,\beta}}(Z(P))| = \frac{(q+1)}{d}$.
- (iii) P has class 2 with $|Z(P)| = q$. If p is odd, then P has exponent p and if $p = 2$ then P has exponent 4 with the set of involutions of P being $Z(P)^\#$.
Let $\hat{}$ denote the image of subgroups of $SU_3(q)$ in $U_3(q)$ ($\cong G$).
- (iv) G has a maximal subgroup M isomorphic to $\hat{GU}_2(q)$.
- (v) M has a subgroup E_0 of shape $\hat{(q+1)^2}$ for which $N_G(E_0) \sim \hat{(q+1)^2}.\text{Sym}(3)$ and any subgroup of G of shape $\hat{(q+1)^2}.\text{Sym}(3)$ is conjugate to $N_G(E_0)$.
- (vi) G has a cyclic subgroup C of order $\frac{q^2-q+1}{d}$ for which $N_G(C) \sim C.3$ is a Frobenius group.

Proof. For parts (i) to (iii) see Suzuki [23] and Huppert [9]. Part (iv) follows from Mitchell [13] (or Bray, Holt, Roney-Dougal [2]). Also from either [2] or [13] G has one conjugacy class of maximal subgroups of shape $\hat{(q+1)^2}.\text{Sym}(3)$ (except when $q = 5$) and (vi) holds (except when $q = 3, 5$). Using the Atlas [8] for these exceptional cases we obtain (v) and (vi). □

From Table 2 in [20] we can extract details of the conjugacy classes of $G = U_3(q)$ as well as some supplementary information which we display in Table 2 ($d = (3, q+1), \ell = \frac{q+1}{d}$ and Ω as in Lemma 2.7 (ii)).

TABLE 2. Conjugacy classes of $U_3(q)$

Class Type	Number of Classes of each Type	Centralizer Order	$ \text{fix}_\Omega(g) , g^G \in \mathcal{C}_i$
\mathcal{C}_1	1	$ G $	$q^3 + 1$
\mathcal{C}_2	1	$\frac{q^3(q+1)}{d}$	1
\mathcal{C}_3	d	q^2	1
\mathcal{C}_4	$\ell - 1$	$\frac{q(q+1)(q^2-1)}{d}$	$q + 1$
\mathcal{C}_5	$\ell - 1$	$\frac{q(q+1)}{d}$	1
\mathcal{C}_6	$\frac{q^2-q+1-d}{6d}$	$\frac{(q+1)^2}{d}$	0
\mathcal{C}'_6	0 if $d = 1, 1$ if $d = 3$	$(q + 1)^2$	0
\mathcal{C}_7	$\frac{q^2-q-2}{2d}$	$\frac{q^2-1}{d}$	2
\mathcal{C}_8	$\frac{q^2-q+1-d}{3d}$	$\frac{q^2-q+1}{d}$	0

We have used the same notation for the classes as in [20] except we have omitted the superscripts used there as they are of no importance here. Because $U_3(q)$ acts 2-transitively on Ω , by page 69 of [10] the permutation character must be $\chi_1 + \chi_{q^3}$ (as in Table 2 of [20]) which then yields the last column of the table.

3. THE NUMBER OF ORBITS ON $\mathcal{P}_k(\Omega)$

We begin with the

Proof of Theorem 1.1. Put $\Omega_k = \mathcal{P}_k(\Omega)$. Suppose $k \in \mathbb{N}$ with $k \geq 3$, and let $1 \neq g \in G (\cong L_2(q))$ be an element of order m . Then by Lemma 2.5 (iii) g must be contained (uniquely) in a conjugate of one of P , H_- and H_+ . Since we seek to determine $|\text{fix}_{\Omega_k}(g)|$ we may suppose that g is contained in one of P , H_- and H_+ .

First we consider the case when $g \in H_+$. Since g is some power of h_+ , by Lemma 2.5 (iv), g has cycle type $m^{\frac{q+1}{m}}$. By Lemmas 2.4 and 2.5 (iv) H_+ contains $\phi(m)$ elements of order m and there are $\frac{q(q-1)}{2}$ conjugates of H_+ , whence these elements contribute

$$\frac{q(q-1)}{2} \sum_{m \in \mathcal{D}^*(\frac{q+1}{d})} \phi(m) \eta_k \left(m^{\frac{q+1}{m}} \right)$$

to the sum $\sum_{g \in G} |\text{fix}_{\Omega_k}(g)|$. Now consider the case when $g \in H_-$. As g is a power of h_- , by Lemma 2.5 (iv), g has cycle type $1^2 m^{\frac{q-1}{m}}$. Employing Lemmas 2.4 and 2.5 (iv) we obtain

$$\frac{q(q+1)}{2} \sum_{m \in \mathcal{D}^*(\frac{q-1}{d})} \phi(m) \eta_k \left(1^2 m^{\frac{q-1}{m}} \right)$$

in the sum $\sum_{g \in G} |\text{fix}_{\Omega_k}(g)|$. Similar considerations for $g \in P$, using Lemma 2.5, yield

$$(q-1)(q+1) \eta_k(1^1 p^{\frac{q}{p}}).$$

Combining the above with Lemmas 2.1 and 2.5 (i) we obtain the expression for $\sigma_k(G, \Omega)$. □

The proof of Theorem 1.2 is similar to that of Theorem 1.1 except we use Lemma 2.6 in place of Lemma 2.5. The remainder of this section is devoted to establishing Theorem 1.3. So we assume $G \cong U_3(q)$, ($q = p^a > 2$) and Ω is the set of isotropic 1-spaces of a 3-dimensional unitary space.

Before beginning the proof of Theorem 1.3, there are a number of preparatory definitions, notation and lemmas we need. First we describe the partitions that arise in Theorem 1.3. Before we can do that, and introduce μ_k , we require a barrage of notation associated with pairs of natural numbers dividing ℓ' where $\ell' \in \mathcal{D}(\ell)$, $\ell = \frac{q+1}{d}$. So let $\ell' \in \mathcal{D}(\ell)$ and

$$(\ell_1, \ell_2) \in \mathcal{D}(\ell') \times \mathcal{D}(\ell'),$$

and let p_1, \dots, p_r be prime numbers such that $\ell_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ and $\ell_2 = p_1^{\beta_1} p_1^{\beta_2} \dots p_r^{\beta_r}$ where for $i = 1, \dots, r$ at least one of the the α_i and β_i is non-zero. If $\alpha_i \neq \beta_i$, then define $\gamma_i = \max\{\alpha_i, \beta_i\}$. Without loss of generality we shall assume that $\alpha_i = \beta_i$ for $1 \leq i \leq s$ and $\alpha_i \neq \beta_i$ for $s < i \leq r$. Set

$$\begin{aligned} \ell_0 &= p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s} (= p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}), \\ m_1 &= p_{s+1}^{\alpha_{s+1}} p_{s+2}^{\alpha_{s+2}} \dots p_r^{\alpha_r} \end{aligned}$$

and

$$m_2 = p_{s+1}^{\beta_{s+1}} p_{s+2}^{\beta_{s+2}} \dots p_r^{\beta_r}.$$

Also set $l_* = p_{s+1}^{\gamma_{s+1}} p_{s+2}^{\gamma_{s+2}} \dots p_r^{\gamma_r}$ and note that $l_1 = \ell_0 m_1$ and $l_2 = \ell_0 m_2$. We remark that we do not exclude the possibilities $l_1 = \ell_0 = l_2$ or $l_1 = m_1$ and $l_2 = m_2$. Put $\ell_{12} = \text{lcm}\{\ell_1, \ell_2\}$. If $\ell_i = 1$ then $\ell_0 = 1 = m_i$ and if $l_1 = l_2 = 1$ then we set $l_* = 1$.

DEFINITION 3.1.

- (i) $\pi_1 = 1^{q^3+1}$.
- (ii) $\pi_2 = 1^1 p^{\frac{q^3}{p}}$.
- (iii) $\pi_3 = 1^1 4^{\frac{q^3}{4}}$ (only defined for $p = 2$).
- (iv) $\pi_4^{(m)} = 1^{q+1} m^{\frac{q^3-q}{m}}$ where $m \in \mathcal{D}^*(\ell)$.
- (v) $\pi_5^{(m)} = 1^1 p^{\frac{q}{p}} m^{\frac{q^3-q}{m}}$ where $m = pj$ and $j \in \mathcal{D}^*(\ell)$.
- (vi) $\pi_6^{(\ell_1, \ell_2, n)} = \ell_1^{\frac{q+1}{\ell_1}} \ell_2^{\frac{q+1}{\ell_2}} n^{\frac{q+1}{n}} \ell_{12}^{\frac{q^3-3q-2}{\ell_{12}}}$ where $(\ell_1, \ell_2) \in \mathcal{D}^*(\ell) \times \mathcal{D}^*(\ell)$ and $n = n_* l_*$, $n_* \in \mathcal{D}(\ell_0)$.
- (vii) $\pi_7^{(m)} = 1^2 j^{\frac{q-1}{j}} m^{\frac{q^3-q}{m}}$ where $m \in \mathcal{D}(\frac{q^2-1}{d})$, $m \notin \mathcal{D}(\ell)$, $j = \frac{m}{(m, \ell)}$.
- (viii) $\pi_8 = m^{\frac{q+1}{m}}$ where $m \in \mathcal{D}^*(q^2 - q + 1)$.
- (ix) When $3^i | q + 1$ with $i \in \mathbb{N}$,

$$3^i \pi_6^{(\ell_1, \ell_2, n)} = (3^i \ell_1)^{\frac{q+1}{3^i \ell_1}} (3^i \ell_2)^{\frac{q+1}{3^i \ell_2}} (3^i n)^{\frac{q+1}{3^i n}} (3^i \ell_{12})^{\frac{q^3-3q-2}{3^i \ell_{12}}}$$

where $(\ell_1, \ell_2) \in \mathcal{D}(\ell) \times \mathcal{D}(\ell)$ and $n = n_* l_*$, $n_* \in \mathcal{D}(\ell_0)$.

DEFINITION 3.2.

$$\mu_k = (q^3 + 1)(q^3 - 1)\eta_k(\pi_2)$$

if $p \neq 2$ and

$$\mu_k = (q^3 + 1)((q - 1)\eta_k(\pi_2) + (q^3 - q)\eta_k(\pi_3))$$

if $p = 2$.

DEFINITION 3.3. Let $k \in \mathbb{N}$, and we continue to set $\ell = \frac{q+1}{d}$. For $(\ell_1, \ell_2) \in \mathcal{D}(q+1) \times \mathcal{D}(q+1)$ we use the notation ℓ_0, m_1, m_2, l_* as defined earlier.

- (i) Let $(\ell_1, \ell_2) \in \mathcal{D}(q+1) \times \mathcal{D}(q+1)$, $n = l_* n_*$ with $n_* \in \mathcal{D}(\ell_0)$ and $n_* = p_1^{\delta_1} p_2^{\delta_2} \dots p_s^{\delta_s}$. If $\ell_0 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$, then we define

$$f(\ell_1, \ell_2, n) = \phi(m_1)\phi(m_2)\phi(\ell_0) \prod_{\substack{\alpha_j = \delta_j \\ 1 \leq j \leq s}} p_j^{\alpha_j - 1} (p_j - 2) \phi \left(\prod_{\substack{\alpha_j \neq \delta_j \\ 1 \leq j \leq s}} p_j^{\delta_j} \right).$$

$$(ii) \lambda_k^*(\ell, \ell) = \sum_{(\ell_1, \ell_2) \in \mathcal{D}^*(\ell) \times \mathcal{D}^*(\ell)} \sum_{\substack{1 \neq n = l_* n_* \\ n_* \in \mathcal{D}(\ell_0)}} f(\ell_1, \ell_2, n) \eta_k(\pi_6^{(\ell_1, \ell_2, n)}).$$

- (iii) For $\ell' \in \mathcal{D}^*(q+1)$ and $i \in \mathbb{N}$ such that $3^i | q + 1$ set

$$\lambda_k(\ell', \ell'; i) = \sum_{(\ell_1, \ell_2) \in \mathcal{D}(\ell') \times \mathcal{D}(\ell')} \sum_{\substack{n = l_* n_* \\ n_* \in \mathcal{D}(\ell_0)}} f(\ell_1, \ell_2, n) \eta_k(3^i \pi_6^{(\ell_1, \ell_2, n)}).$$

DEFINITION 3.4. Let E_0 be an abelian subgroup of G isomorphic to the direct product of two cyclic groups of order $\frac{(q+1)}{d}$ and $(q+1)$ with $N_G(E_0) \sim \frac{(q+1)}{d}(q+1)$. $\text{Sym}(3)$. Put $E_0^* = (C_6 \cup C_6') \cap E_0$, and define

$$\sigma_k(E_0^*, \Omega) = \sum_{g \in E_0^*} |\text{fix}_{\Omega_k}(g)|,$$

where $\Omega_k = \mathcal{P}_k(\Omega)$.

REMARK 3.5. Subgroups such as E_0 in Definition 3.4 exist by Lemma 2.7 (v), and $\sigma_k(E_0^*, \Omega)$ is the contribution of E_0^* to the sum $\sum_{g \in G} |\text{fix}_{\Omega_k}(g)|$.

LEMMA 3.6. *The cycle type for elements in the classes of type \mathcal{C}_i for $i \neq 6$ are given in Table 3.*

TABLE 3. Cycle Types

Class Type	Order of g	Cycle type of g
\mathcal{C}_1	1	π_1
\mathcal{C}_2	p	π_2
\mathcal{C}_3	4 ($p = 2$)	π_3
	p ($p \neq 2$)	π_2
\mathcal{C}_4	$m, (m \in \mathcal{D}^*(\ell))$	$\pi_4^{(m)}$
\mathcal{C}_5	$px, (x \in \mathcal{D}^*(\ell))$	$\pi_5^{(m)}$
\mathcal{C}_7	$m = js, (j \in \mathcal{D}^*(q - 1),$ $s \in \mathcal{D}(\ell))$	$\pi_7^{(m)}$
\mathcal{C}_8	$m, (m \in \mathcal{D}^*(q^2 - q + 1))$	π_8

Proof. Let $X = g^G$ be a conjugacy class of type $\mathcal{C}_i, i \in \{1, \dots, 8\} \setminus \{6\}$, and let $P \in \text{Syl}_p(G)$. If $i = 1$, then clearly $g = 1$. From the centralizer sizes in Table 2, for $i = 2$ we must have X is the conjugacy class containing $Z(P)^\#$, while classes of type \mathcal{C}_3 are the conjugacy classes of elements in $P \setminus Z(P)$. If p is odd, then by Lemma 2.7 (iii) the elements in classes of type \mathcal{C}_3 all have order p whence, as elements in classes of types \mathcal{C}_2 and \mathcal{C}_3 fix just one element of Ω , their cycle type is π_2 . If $p = 2$, then for $i = 2$, we also have cycle type π_2 and for $i = 3$ as the elements of order 4 must square to involutions in $Z(P)$, their cycle type must be π_3 .

From Lemma 2.7 (iv) G possesses a maximal subgroup $M \cong \widehat{GU}_2(q)$. Further, $Z(M)$ is cyclic of order $\ell = \frac{q+1}{d}$. It is straightforward to check that no two elements in $Z(M)^\#$ are G -conjugate. So we see, using the centralizer sizes in Table 2, $Z(M)^\#$ supplies representatives for all the conjugacy classes of type \mathcal{C}_4 . Choose h to be an element of order p in M (in fact h is G -conjugate to an element in $Z(P)$, this can be seen as these elements must have centralizer with order divisible by $|Z(M)| = \frac{q+1}{d}$, hence cannot be in classes \mathcal{C}_3). By the structure of M , for any $h' \in Z(M)^\#, |C_G(hh')| = \frac{q(q+1)}{d}$. (We note that no element of $G \setminus M$ centralizes M and that $h, h' \in M$, both are centralized by $Z(P) < M$ and $Z(M) < M$.) Also for $h', h'' \in Z(M)^\#$ with $h' \neq h''$ we see that hh' and hh'' are not G -conjugate. Therefore $\{hh' \mid h' \in Z(M)^\#\}$ gives representatives for all conjugacy classes of type \mathcal{C}_5 . Now $|\text{fix}_\Omega(g')| = q + 1$ for all $g' \in Z(M)^\#$. So for $h' \in Z(M)^\#$ of order m, h' will have cycle type $\pi_4^{(m)}$ on Ω . Similarly for $hh', h' \in Z(M)^\#$ with h' of order m , as $|\text{fix}_\Omega(h)| = 1$, we infer that hh' has cycle type $\pi_5^{(m)}$. So we have dealt with classes of type \mathcal{C}_4 and \mathcal{C}_5 .

Next we look at classes of type \mathcal{C}_7 . Since $|\text{fix}_\Omega(g)| = 2$, we may suppose $g \in G_{\alpha, \beta} (\cong \frac{q^2-1}{d})$, where $\alpha, \beta \in \Omega, \alpha \neq \beta$. We may also suppose $Z(M) (\cong \frac{q+1}{d}) \leq G_{\alpha, \beta}$ where M is a maximal subgroup of G isomorphic to $\widehat{GU}_2(q)$. Since $|\text{fix}_\Omega(g')| = q + 1$ for all $g' \in Z(M)^\#, g \in G_{\alpha, \beta} \setminus Z(M)$. Let g have order m and let j be the smallest natural number such that $g^j \in Z(M)$. Then $m = (m, \ell)j$ where $j \in \mathcal{D}^*(q - 1)$ (note that $j = 1$, as $G_{\alpha, \beta} \cong \frac{(q+1)}{d}$ contains a unique subgroup, $Z(M)$, of order $\frac{q+1}{d} = \ell$, would mean $g \in Z(M)$). So $j = \frac{m}{(m, \ell)}$ and hence g has cycle type $\pi_7^{(m)}$.

Finally we look at those of type C_8 . From Lemma 2.7 (iv) G has a cyclic subgroup $C \cong \frac{(q^2-q+1)}{d}$ with $N_G(C) \sim C.3$ being a Frobenius group. Now $C^\#$ has $(|C| - 1)/3$ $N_G(C)$ -conjugacy classes and it can be seen that no two of these are G -conjugate. Hence we have that the $N_G(C)$ -conjugacy class representatives of $C^\#$ give us representatives for all of the classes of type C_8 . So, as $|\text{fix}_\Omega(h)| = 0$ for all $h \in C^\#$, the elements in $C^\#$ has cycle type π_8 on Ω , which completes the proof of Lemma 3.6. \square

We now turn our attention to the delicate process of dissecting the classes of type $C_6 \cup C'_6$.

LEMMA 3.7. *Suppose $A \cong \mathbb{Z}_e \times \mathbb{Z}_e$ with A containing three subgroups $A_i \cong \mathbb{Z}_e$ ($i = 1, 2, 3$), such that $A_i \cap A_j = 1$ for $i \neq j$. Then there exists $a_1 \in A_1, a_2 \in A_2$ such that $A_1 = \langle a_1 \rangle, A_2 = \langle a_2 \rangle$ and $A_3 = \langle a_1 a_2 \rangle$.*

Proof. Since $A_1 \cap A_2 = 1$ and $A_1 \cong \mathbb{Z}_e, A = A_1 A_2$. Let $A_3 = \langle c \rangle$. Then $c = a_1 a_2$ where $a_i \in A_i, i = 1, 2$. Suppose a_i has order $e_i, i = 1, 2$ and, without loss that, $e_1 \leq e_2$. Then $c^{e_1} = a_1^{e_1} a_2^{e_1} = a_2^{e_1} \in A_2 \cap A_3 = 1$. So $e_2 \leq e_1$ and hence $e_1 = e_2$. Since $c = a_1 a_2$ has order e , we must have that $e_1 = e_2 = e$, so proving the lemma. \square

HYPOTHESIS 3.8. *Suppose that A_0 is an abelian group containing a subgroup A with $[A_0 : A] = 1$ or 3 . Also suppose that A has order e^2 and contains three subgroups A_1, A_2, A_3 with $A_i \cong \mathbb{Z}_e$ ($i = 1, 2, 3$).*

Further suppose that Ω is an A_0 -set such that

- (i) *for $1 \neq g \in A_0, \text{fix}_\Omega(g) = \emptyset$ if $g \notin A_1 \cup A_2 \cup A_3$ and $\text{fix}_\Omega(g) = \text{fix}_\Omega(A_i)$ if $g \in A_i$;*
- (ii) *$\text{fix}_\Omega(A_i) \cap \text{fix}_\Omega(A_j) = \emptyset$ for $1 \leq i \neq j \leq 3$; and*
- (iii) *for $i = 1, 2, 3, |\text{fix}_\Omega(A_i)| = q + 1$.*

Set $\Lambda_i = \text{fix}_\Omega(A_i)$ for $i = 1, 2, 3$ and $\Lambda = \Omega \setminus (\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)$ and so, by (ii), Ω is the disjoint union

$$\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda.$$

Moreover, by (i), A_0 acts regularly on Λ and for $i = 1, 2, 3, A_0/A_i$ acts regularly on Λ_i .

We shall encounter Hypothesis 3.8 both in a recursive setting and in the group $A_0 = \mathbb{Z}_{q+1} \times \mathbb{Z}_\ell$ (where $\ell = \frac{q+1}{d}, d = (3, q + 1)$). For this A_0 we have $e = \ell$ and A would be the subgroup of A_0 generated by the elements of A_0 of order ℓ . Then $[A_0 : A] = d$.

LEMMA 3.9. *Assume Hypothesis 3.8 holds and use the notation A_0, A and A_i in the hypothesis. Let $(\ell_1, \ell_2) \in \mathcal{D}(e) \times \mathcal{D}(e)$ where $e \in \mathcal{D}(\ell)$. The cycle structure on Ω of the elements $g = g_1 g_2 \in A$ where $g_i \in A_i$ ($i = 1, 2$) with g_i of order ℓ_i is*

$$(\ell_1)^{\frac{q+1}{\ell_1}} (\ell_2)^{\frac{q+1}{\ell_2}} (n)^{\frac{q+1}{n}} (\ell_{12})^{\frac{|\Lambda|}{\ell_{12}}},$$

where $n = \ell_ n_*$ with $n_* \in \mathcal{D}(\ell_0)$ and $\ell_{12} = \text{lcm}(\ell_1, \ell_2)$. This cycle structure, as g_1 and g_2 ranges over the elements of order (respectively) ℓ_1 and ℓ_2 occurs*

$$\phi(m_1)\phi(m_2)\phi(\ell_0) \prod_{\substack{\alpha_j = \delta_j \\ 1 \leq j \leq s}} p_j^{\alpha_j - 1} (p_j - 2) \phi \left(\prod_{\substack{\alpha_j \neq \delta_j \\ 1 \leq j \leq s}} p_j^{\delta_j} \right)$$

times, where $n_ = p_1^{\delta_1} \dots p_s^{\delta_s}$.*

Proof. By Hypothesis 3.8 (i) and (ii) $A_i \cap A_j = 1$ for $i \neq j$. Hence, by Lemma 3.6 we may select $a_1 \in A_1, a_2 \in A_2$ so as to have $A_1 = \langle a_1 \rangle, A_2 = \langle a_2 \rangle$ and $A_3 = \langle a_1 a_2 \rangle$. Additionally we may identify A with $A_1 A_2$. Let $g = g_1 g_2$ where $g_i \in A_i$ and g_i has

order $\ell_i, i = 1, 2$. The smallest $k \in \mathbb{N}$ such that $g^k \in A_2$ is clearly ℓ_1 and, likewise, the smallest $k \in \mathbb{N}$ such that $g^k \in A_1$ is clearly ℓ_2 . Hence, as A/A_2 acts regularly on Λ_2 , g in its action on Λ_2 must be the product of disjoint cycles of length ℓ_1 . Similarly g acts upon Λ_1 as a product of disjoint cycles each of length ℓ_2 . Concerning the action of g on Λ , as A_0 acts regularly on Λ and $\ell_{12} = lcm\{\ell_1, \ell_2\}$ is the order of g , Λ is a disjoint union of $\frac{|\Lambda|}{\ell_{12}}$ length cycles of g .

Since $A_3 = \langle a_1 a_2 \rangle$ to find the lengths of g 's cycles on Λ_3 , we must determine the smallest $k \in \mathbb{N}$ such that $g^k \in \langle a_1 a_2 \rangle$. For $i = 1, 2$ let $k_i \in \mathbb{N}$ with $k_i \leq e$ be such that $g_i = a_i^{k_i}$. So $g = a_1^{k_1} a_2^{k_2}$ and, we recall, $\ell_i = e/(e, k_i)$ for $i = 1, 2$. Thus we seek the smallest $k \in \mathbb{N}$ for which

$$g^k = (a_1^{k_1} a_2^{k_2})^k = a_1^{k_1 k} a_2^{k_2 k} = (a_1 a_2)^j$$

for some $j, 0 \leq j < e$. This is the smallest $k \in \mathbb{N}$ such that $k_1 k \equiv k_2 k \pmod e$ which is $k = \frac{e}{(k_1 - k_2, e)}$.

Let C be a cyclic group isomorphic to \mathbb{Z}_e with generator c . Now for $i = 1, 2$ the order of c^{k_i} is $\frac{e}{(e, k_i)} = \ell_i$ and the order of $c^{k_1} (c^{k_2})^{-1}$ is k . Thus to enumerate the possibilities for k (recall (ℓ_1, ℓ_2) is a fixed ordered pair) we look at the order of $c^{k_1} (c^{k_2})^{-1}$ as we run through the ordered pairs (c^{k_1}, c^{k_2}) of elements of C of order, respectively, ℓ_1 and ℓ_2 . In doing this there is no loss in supposing $C = \langle c \rangle$ has order $lcm\{\ell_1, \ell_2\}$.

For $i = 1, \dots, r$, let $P_i \in Syl_{p_i}(C)$. Since the order of elements in C is the product of their orders in the projections into P_i for $i = 1, \dots, r$, we first consider the special case when $C = P_i \neq 1$, for some $i \in \{1, \dots, r\}$. So $\ell_1 = p_i^{\alpha_i}$ and $\ell_2 = p_i^{\beta_i}$.

(3.9.1) If $\alpha_i \neq \beta_i$, then for all choices of (c^{k_1}, c^{k_2}) , of which there are $\phi(p_i^{\alpha_i})\phi(p_i^{\beta_i})$, the order of $c^{k_1} (c^{k_2})^{-1}$ is $p_i^{\gamma_i}$.

Recalling that by definition $\gamma_i = \max\{\alpha_i, \beta_i\}$ we see that the order of $c^{k_1} (c^{k_2})^{-1}$ is $p_i^{\gamma_i}$ as asserted.

Now we turn to the case when $\alpha_i = \beta_i$. Here we have $\phi(p_i^{\alpha_i})^2$ possible choices for (c^{k_1}, c^{k_2}) . Let C_1 be the unique subgroup of C of order $p_i^{\alpha_i - 1}$ (and note c^{k_1} and c^{k_2} are in $C \setminus C_1$). Should c^{k_1} and c^{k_2} be in different C_1 cosets of C , then $c^{k_1} (c^{k_2})^{-1}$ is not in C_1 whence $c^{k_1} (c^{k_2})^{-1}$ has order $p_i^{\alpha_i}$. This will happen $\phi(p_i^{\alpha_i})p_i^{\alpha_i - 1}(p_i - 2)$ times. The number of ordered pairs (c^{k_1}, c^{k_2}) for which c^{k_1} and c^{k_2} are in the same C_1 coset of C is $\phi(p_i^{\alpha_i})p_i^{\alpha_i - 1}$. In this situation, for a fixed c^{k_1} , $c^{k_1} (c^{k_2})^{-1}$ runs through all the elements of C_1 thus yielding $\phi(p_i^{\alpha_i - 1})$ of order $p_i^{\alpha_i - 1}$, $\phi(p_i^{\alpha_i - 2})$ of order $p_i^{\alpha_i - 2}$, and so on. To summarize we have the following.

(3.9.2) Suppose $\alpha_i = \beta_i$. Then for $\phi(p_i^{\alpha_i})p_i^{\alpha_i - 1}(p_i - 2)$ of the ordered pairs (c^{k_1}, c^{k_2}) the order of $c^{k_1} (c^{k_2})^{-1}$ is $p_i^{\alpha_i}$ and, for $j = 1, \dots, \alpha_i$, $\phi(p_i^{\alpha_i})\phi(p_i^{\alpha_i - j})$ of the ordered pairs (c^{k_1}, c^{k_2}) the order of $c^{k_1} (c^{k_2})^{-1}$ is $p_i^{\alpha_i - j}$.

We now consider the general situation for $\ell_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ and $\ell_2 = p_1^{\beta_1} p_1^{\beta_2} \dots p_r^{\beta_r}$. Looking at all those i for which $\alpha_i \neq \beta_i$ (just for the moment considering the projections onto P_{s+1}, \dots, P_r) we obtain that $c^{k_1} (c^{k_2})^{-1}$ has order $p_{s+1}^{\gamma_{s+1}} \dots p_r^{\gamma_r} = \ell_*$ for

$$\prod_{i=s+1}^r \phi(p_i^{\alpha_i})\phi(p_i^{\beta_i}) = \phi(m_1)\phi(m_2)$$

pairs (c^{k_1}, c^{k_2}) by (3.9.1) We now wish to enumerate the pairs (c^{k_1}, c^{k_2}) for which the order of $c^{k_1}(c^{k_2})^{-1}$ is n , where $n = \ell_* n_*$, $n_* \in \mathcal{D}(\ell_0)$ and $n_* = p_1^{\delta_1} p_2^{\delta_2} \dots p_s^{\delta_s}$. Using (3.9.2) and by just considering the projections on P_1, \dots, P_s we see this occurs for

$$\prod_{\alpha_i=\delta_i} \phi(p_i^{\alpha_i}) p_i^{\alpha_i-1} (p_i-2) \prod_{\alpha_i \neq \delta_i} \phi(p_i^{\alpha_i}) \phi(p_i^{\delta_i}) = \prod_{1 \leq i \leq s} \phi(p_i^{\alpha_i}) \prod_{\alpha_i=\delta_i} p_i^{\alpha_i-1} (p_i-2) \prod_{\alpha_i \neq \delta_i} \phi(p_i^{\delta_i})$$

$$= \phi(\ell_0) \prod_{\alpha_i=\delta_i} p_i^{\alpha_i-1} (p_i-2) \phi \left(\prod_{\alpha_i \neq \delta_i} p_i^{\delta_i} \right)$$

pairs. Combining this with the projection onto P_{s+1}, \dots, P_r yields Lemma 3.9. \square

LEMMA 3.10. *Let E_0 be an abelian subgroup of G isomorphic to the direct product of two cyclic groups of order $\frac{(q+1)}{d}$ and $(q+1)$ with $N_G(E_0) \sim \frac{(q+1)}{d}(q+1) \cdot \text{Sym}(3)$. Also let E be the subgroup of E_0 generated by the elements of E_0 of order $\ell = \frac{(q+1)}{d}$. Then*

- (i) $\sigma_k(E_0^*, \Omega) = \lambda_k^*(\ell, \ell)$ if $d = 1$;
- (ii) $\sigma_k(E_0^*, \Omega) = \lambda_k^*(\ell, \ell) + 2\lambda_k(\ell, \ell; 1)$ if $d = 3$ and $3 \nmid |E|$; and
- (iii) $\sigma_k(E_0^*, \Omega) = \lambda_k^*(\ell, \ell) + 2 \cdot 9^{b-1} \lambda_k(\frac{q+1}{3^b}, \frac{q+1}{3^b}; b)$ if $d = 3$, $3 \mid |E|$ and 3^b is the largest power of 3 dividing $q+1$.

Proof. By Lemma 2.7 (iv), (v) G contains a subgroup M with $M \sim \widehat{GU}_2(q)$ and $E_0 \leq M$. From the structure of $N_G(E_0)$ and $\widehat{GU}_2(q)$, $Z(M) \leq E_0$ with $Z(M) \cong \frac{q+1}{d} (= \ell)$. Let $h \in N_G(E_0)$ be an element of order 3. Because $[N_M(E_0) : E_0] = 2$, $h \notin M = N_G(Z(M))$. In order to key in with the notation of Hypothesis 3.8, principally as we shall employ Lemma 3.9, we set $A_0 = E_0$ and $A = E$. Further, we set $A_1 = Z(M)$, $A_2 = Z(M)^h$ and $A_3 = Z(M)^{h^2}$. Since $\text{fix}_\Omega(g) = \text{fix}_\Omega(Z(M))$ for all $g \in Z(M)^\#$, it follows that $\text{fix}_\Omega(A_i) \cap \text{fix}_\Omega(A_j) = \emptyset$ for $1 \leq i \neq j \leq 3$. We also have $|\text{fix}_\Omega(A_i)| = q+1$ and, by Table 2, $\text{fix}_\Omega(g) = \emptyset$ if $g \in A_0 \setminus (A_1 \cup A_2 \cup A_3)$. Now A is the subgroup of A_0 generated by the elements of A_0 of order ℓ and $A_i \cong \ell$. So we have $A_i \leq A$, $i = 1, 2, 3$ and $[A : A_0] = d (= 1 \text{ and } 3)$. Hence Hypothesis 3.8 holds with $e = \ell$.

Suppose $d = 1$. Then $A = A_0$. Using Lemma 3.9 and Definition 3.3 (ii) we obtain $\sigma_k(E_0^*, \Omega) = \lambda_k^*(\ell, \ell)$. (Note the condition in Definition 3.3 (ii) on the outer sum that $(\ell_1, \ell_2) \in \mathcal{D}^*(\ell) \times \mathcal{D}^*(\ell)$ and on the inner sum that $n \neq 1$ prevents the counting of elements in \mathcal{C}_4 .) So Lemma 3.10 holds in this case.

So we now investigate the case when $d = 3$. Hence $[A_0 : A] = 3$. Let $\theta : A_0 \mapsto A_0$ be defined by $\theta : g \mapsto g^3$. Then, as A_0 is abelian, θ is a homomorphism with $\text{im } \theta \leq A$ and $\ker \theta = \{x \in A_0 \mid \text{order of } x \text{ is } 1 \text{ or } 3\}$.

Further assume that $3 \nmid |E|$ (so $3 \nmid \ell$). Then, as $|A| = \ell^2$, $3 \nmid |A|$. Also we have $|\ker \theta| = 3$ and therefore, by orders, $\text{im } \theta = A$. For every $g \in A_0 \setminus A$, the smallest power of g contained in A_i ($i = 1, 2, 3$) will be three times the corresponding power for $h = g^3 = \theta(g)$. Now $3 \nmid |A|$ means that θ restricted to A is a one-to-one map, and so the inverse image of h contains two elements of $A_0 \setminus A$. Hence, using Lemma 3.9, the elements of $A_0 \setminus A$ contribute $2\lambda_k(\ell, \ell; 1)$ to the sum $\sum_{g \in G} |\text{fix}_{\Omega_k}(g)|$. Thus, using Lemma 3.9 again, $\sigma_k(E_0^*, \Omega) = \lambda_k^*(\ell, \ell) + 2\lambda_k(\ell, \ell; 1)$, as stated.

The last case to be considered is when, as well as $d = 3$, we have $3 \mid |E|$. So $3 \mid \ell$. As a consequence $|\ker \theta| = 3^2$ and thus $[A : \text{im } \theta] = 3$. We seek to pinpoint $\text{im } \theta$. Now let A_i^3 denote the unique subgroup of A_i of index 3 ($i = 1, 2, 3$). Let B be the subgroup of A generated by the elements of A of order $\ell/3$. Then $[A : B] = 3^2$ and, for $1 \leq i < j \leq 3$, $B = A_i^3 A_j^3$. Also observe that $\text{im } \theta \geq B$. Set $N = N_G(A_0)$, and recall that $N \sim \frac{(q+1)}{d}(q+1) \cdot \text{Sym}(3)$. Also the N -conjugacy class of A_1 is $\{A_1, A_2, A_3\}$. Hence N normalizes $B (= A_i^3 A_j^3, 1 \leq i < j \leq 3)$ and the N -conjugacy of $A_1 B$ is $\{A_1 B, A_2 B, A_3 B\}$. Moreover $A_i B \neq A_j B$ for $1 \leq i < j \leq 3$ (as $A = A_i A_j$). Evidently

$\text{im } \theta$ is a normal subgroup of N and therefore $\{\text{im } \theta, A_1B, A_2B, A_3B\}$ comprise the four subgroups of index 3 in A which contain B . Observe that the inverse image under θ of B is A . Since, by Lemma 3.9, $\lambda_k^*(\ell, \ell)$ is the count for the contribution of elements in A , we are looking to determine the contribution from the elements in $A_0 \setminus A$. Thus for $h \in \text{im } \theta \setminus B$, $\theta^{-1}(\{h\}) \subseteq A_0 \setminus A$ with $|\theta^{-1}(\{h\})| = 9$. For $g \in \theta^{-1}(\{h\})$, $h = g^3 \in \text{im } \theta$ and so we must multiply the cycle lengths of h by 3 and their multiplicities by 9 to count the contribution of $\theta^{-1}(\{h\})$.

Now $\text{im } \theta$ contains $B = A_i^{(3)}A_j^{(3)}$ ($1 \leq i \neq j \leq 3$) as a subgroup of index 3, and clearly $A_i \cap \text{im } \theta \geq A_i^{(3)}$, ($1 \leq i \leq 3$). If $A_i \cap \text{im } \theta \neq A_i^{(3)}$, then, as $[A_i : A_i^{(3)}] = 3$, we get $A_i \leq \text{im } \theta$. But then

$$\text{im } \theta \geq A_iA_j^{(3)} = A_iA_i^{(3)}A_j^{(3)} = A_iB$$

whence $\text{im } \theta = A_iB$. This is impossible as $\text{im } \theta \neq A_iB$ and so we conclude that $A_i \cap \text{im } \theta = A_i^{(3)}$ for $1 \leq i \leq 3$. Hence we have that $\text{im } \theta$ satisfies Hypothesis 3.8 with B playing the role of A and $A_i \cap \text{im } \theta$ ($1 \leq i \leq 3$) the role of the A_i . Further B itself satisfies Hypothesis 3.8 with B playing the role also of A and the $A_i^{(3)}$ ($1 \leq i \leq 3$) the role of the A_i . We may repeat this process for $\text{im } \theta \setminus B$ (note $b \geq 2$ in this case), each time we multiply cycle lengths by 3 and the multiplicity by 9. Eventually we arrive at $\text{im}(\theta^{b-1})$, containing a subgroup B^* of index 3. Observe that the count for $\text{im}(\theta^{b-1}) \cong \frac{q+1}{3^{b-1}} \times \frac{q+1}{3^b}$ is given by part (ii) (with $\ell = \frac{q+1}{3^{b-1}}$) and the count for $B^* \cong \frac{q+1}{3^b} \times \frac{q+1}{3^b}$ with ($\ell = \frac{q+1}{3^b}$). Keeping track of changes in cycle length and multiplicity we obtain

$$\begin{aligned} 9^{b-1} \left(\lambda_k^* \left(\frac{q+1}{3^b}, \frac{q+1}{3^b} \right) + 2\lambda_k \left(\frac{q+1}{3^b}, \frac{q+1}{3^b}; b \right) - \lambda_k^* \left(\frac{q+1}{3^b}, \frac{q+1}{3^b} \right) \right) \\ = 2 \cdot 9^{b-1} \lambda_k \left(\frac{q+1}{3^b}, \frac{q+1}{3^b}; b \right) \end{aligned}$$

which is the contribution for $A_0 \setminus A$. Consequently

$$\sigma_k(E_0^*, \Omega) = \lambda_k^*(\ell, \ell) + 2 \cdot 9^{b-1} \lambda_k \left(\frac{q+1}{3^b}, \frac{q+1}{3^b}; b \right),$$

and the proof of Lemma 3.10 is complete. □

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Again we apply Lemma 2.1 using the information on cycle types given in Lemmas 3.6 and 3.10. The size of a given conjugacy class is obtained using the centralizer orders displayed in Table 2. So the conjugacy classes in \mathcal{C}_i for a fixed i all have the same size hence using their multiplicity in each \mathcal{C}_i together with Lemmas 3.6 and 3.10 we obtain $\sigma_k(G, \Omega)$, so proving Theorem 1.3. □

REMARK 3.11. Type \mathcal{C}'_6 classes only arise when $d = 3$ (and then there is only one conjugacy class of this type) and consists of elements of order 3 with no fixed points, and is one third the size of the type \mathcal{C}_6 classes. When $d = 3$ we include this class along with the other type \mathcal{C}_6 classes in $\sigma_k(E_0^*, \Omega)$. It occurs as the case $\ell_1 = \ell_2 = n = 3$ and appears $f(3, 3, 3) = 2$ times. As the size of this class is divided by 6, this corrects the size of this class in our count.

4. MAGMA CODE FOR $L_2(q)$

In this section we give a MAGMA implementation for the formula in Theorem 1.1.

```

PSLsig:=procedure(q,k,~sigma);
Z:=Integers();d:=GreatestCommonDivisor(q-1,2);
p:=Factorisation(q)[1,1];sig:=0;
I:=(d/(q*(q+1)*(q-1)))*Binomial(Z!(q+1),k);
CC:=[]; Append(~CC,[<(d/q),1>,<p,Z!(q/p)>,<1,1>]);
for m in Divisors(Z!((q+1)/d)) do if m ne 1 then
Append(~CC,[<(d/(2*(q+1))),EulerPhi(m)>,<m,Z!((q+1)/m)>]);
end if;end for;
for m in Divisors(Z!((q-1)/d)) do if m ne 1 then
Append
(~CC,[<(d/(2*(q-1))),EulerPhi(m)>,<m,Z!((q-1)/m)>,<1,2>]);
end if;end for; a:=0;
for i:=1 to #CC do Cg:=CC[i];S:={Z!(Cg[i][1]): i in [2..#Cg]};
RPg:=RestrictedPartitions(k,S);
for l:=1 to #RPg do p:=RPg[l];np:=1;
for j:=2 to #Cg do
pj:={m:m in [1..#p] | p[m] eq Cg[j][1]};
np:=np*Binomial(Cg[j][2],pj); end for;
a:=a + np*(Cg[1][1]*Cg[1][2]);end for;end for;
sigma:=a+I;end procedure;

```

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