ALGEBRAIC COMBINATORICS

Ronald C. King

Stretched Newell–Littlewood coefficients

Stretched Newell–Littlewood coefficients

Ronald C. King

*Dedicated to Ian Goulden and David Jackson in recognition of their
contributions to enumerative and algebraic combinatorics*

ABSTRACT Newell–Littlewood coefficients $n_{\mu,\nu}^{\lambda}$ are the multiplicities occurring in the decomposition of products of universal characters of the orthogonal and symplectic groups. They may also be expressed, or even defined directly in terms of Littlewood–Richardson coefficients, $c_{\mu,\nu}^{\lambda}$. Both sets of coefficients have stretched forms $c_{t\mu,t\nu}^{t\lambda}$ and $n_{t\mu,t\nu}^{t\lambda}$, where $t\kappa$ is the partition obtained by multiplying each part of the partition κ by the integer t. It is known that $c_{t\mu,t\nu}^{t\lambda}$ is a polynomial in t and here it is shown that $n_{t\mu,t\nu}^{t\lambda}$ is an Ehrhart quasi-polynomial in t with minimum quasi-period at most 2. The evaluation of $n_{t\mu,t\nu}^{t\lambda}$ is effected both by deriving its generating function and by establishing a hive model analogous to that used for the calculation of $c_{t\mu,t\nu}^{t\lambda}$. These two approaches lead to a whole battery of conjectures about the nature of the quasi-polynomials $n_{t\mu,t\nu}^{t\lambda}$. These include both positivity, stability and saturation conjectures that are supported by a significant amount of data from a range of examples.

1. INTRODUCTION

There exist irreducible representations, V_G^{λ}, of each of the classical Lie groups, G, whose characters, ch V_G^{λ}, are specified by their highest weights, λ, which take the form of partitions. The decomposition of the tensor product of such irreducible representations gives rise to multiplicities, $m_{\mu,\nu}^{\lambda}(G)$, that are defined at the level of characters by

$$
(1) \qquad \text{ch}\, V_G^{\mu}\ \text{ch}\, V_G^{\nu} = \sum_{\lambda} m_{\mu,\nu}^{\lambda}(G)\ \text{ch}\, V_G^{\lambda}\,.
$$

In the case of the general linear group, GL_r, these multiplicities are known as Littlewood–Richardson coefficients and are denoted here by $c_{\mu,\nu}^{\lambda}$. In the case of the orthogonal and symplectic groups, SO_{2r+1}, Sp_{2r} and SO_{2r}, with r sufficiently large, Newell [26, 27] and Littlewood [23], by means of quite different arguments, arrived at identical results allowing the corresponding tensor product multiplicities, $n_{\mu,\nu}^{\lambda}$, that we refer to as Newell–Littlewood coefficients, to be expressed in terms of Littlewood–Richardson coefficients as follows:

$$
(2) \qquad n_{\mu,\nu}^{\lambda} = \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta}^{\mu}\, c_{\alpha,\gamma}^{\nu}\, c_{\beta,\gamma}^{\lambda}\,,
$$

where the sum is over all partitions α, β and γ.

These Newell–Littlewood coefficients have been the subject of more recent interest in the hands of Gao *et al.* [11, 12]. They took (2) as their definition and made a systematic study of their properties that included some remarks and results on stretched Newell–Littlewood coefficients that stimulated the work presented here.

For each partition λ let its weight, i.e. the sum of its parts, be denoted by $|\lambda|$ and its length, i.e. the number of its non-zero parts, by $\ell(\lambda)$. It is well known that

$$(3) \qquad c_{\mu,\nu}^{\lambda} = c_{\nu,\mu}^{\lambda} \quad \text{and} \quad c_{\mu,\nu}^{\lambda} = 0 \text{ unless } |\lambda| = |\mu| + |\nu| \text{ and } \ell(\lambda) \leqslant \ell(\mu) + \ell(\nu).$$

It then follows from (2) that

$$(4) \qquad \begin{aligned} &n_{\mu,\nu}^{\lambda} = n_{\nu,\mu}^{\lambda} = n_{\nu,\lambda}^{\mu} = n_{\lambda,\nu}^{\mu} = n_{\lambda,\mu}^{\nu} = n_{\mu,\lambda}^{\nu} \text{ and} \\ &n_{\mu,\nu}^{\lambda} = 0 \text{ unless } |\lambda| + |\mu| + |\nu| \text{ is even, } |\lambda| \leqslant |\mu| + |\nu| \text{ and } \ell(\lambda) \leqslant \ell(\mu) + \ell(\nu). \end{aligned}$$

It might be noted here that it follows from (2) that

$$(5) \qquad n_{\mu,\nu}^{\lambda} = c_{\mu,\nu}^{\lambda} \quad \text{if} \quad |\lambda| = |\mu| + |\nu|,$$

since in this case $|\alpha| = 0$ and this implies in turn that $\alpha = (0)$, $\beta = \mu$ and $\gamma = \nu$.

For any $t \in \mathbb{Z}_{\geqslant 0}$ and partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ let $t\lambda$ denote the partition $(t\lambda_1, t\lambda_2, \ldots, t\lambda_p)$. Then $c_{t\mu,t\nu}^{t\lambda}$ and $n_{t\mu,t\nu}^{t\lambda}$ are referred to as stretched Littlewood–Richardson and Newell–Littlewood coefficients, respectively. Their generating functions take the form:

$$(6) \qquad C_{\mu,\nu}^{\lambda}(w) = \sum_{t=0}^{\infty} c_{t\mu,t\nu}^{t\lambda} \, w^t \quad \text{and} \quad N_{\mu,\nu}^{\lambda}(w) = \sum_{t=0}^{\infty} n_{t\mu,t\nu}^{t\lambda} \, w^t.$$

It has been established by Rassart [28] that $c_{t\mu,t\nu}^{t\lambda}$ is a polynomial in t, and as will be shown here, it follows from the work of De Loera and McAllister [7] that $n_{t\mu,t\nu}^{t\lambda}$ is a quasi-polynomial in t of minimum quasi-period at most 2, that is to say

$$(7) \qquad c_{t\mu,t\nu}^{t\lambda} = P(t) \quad \text{for all } t \quad \text{and} \quad n_{t\mu,t\nu}^{t\lambda} = \begin{cases} P_e(t) & \text{for } t \text{ even;} \\ P_o(t) & \text{for } t \text{ odd,} \end{cases}$$

where $P(t)$, $P_e(t)$ and P_o are all polynomials in t. In terms of generating functions this implies that [32]

$$(8) \qquad C_{\mu,\nu}^{\lambda}(w) = \frac{F(w)}{(1-w)^d} \quad \text{and} \quad N_{\mu,\nu}^{\lambda}(w) = \frac{G(w)}{(1-w)^{d_1}(1-w^2)^{d_2}}$$

with $F(w)$ and $G(w)$ polynomials in w of degrees less than d and $d_1 + 2d_2$, respectively.

For example, in the Littlewood–Richardson case

$$(9) \qquad c_{(6,5,3),(6,4,1)}^{(9,7,5,4)} = 7 \quad \text{and} \quad c_{t(6,5,3),t(6,4,1)}^{t(9,7,5,4)} = (t+1)(5t^2 + 10t + 6)/6$$

with

$$(10) \qquad C_{(6,5,3),(6,4,1)}^{(9,7,5,4)}(w) = \frac{w^2 + 3w + 1}{(1-w)^4}.$$

On the other hand, in the Newell–Littlewood case

$$(11) \qquad n_{(5,3),(4,1)}^{(5,2)} = 6 \quad \text{and} \quad n_{t(5,3),t(4,1)}^{t(5,2)} = \begin{cases} (t+2)(14t^2 + 23t + 12)/24 & t \text{ even;} \\ (t+1)(14t^2 + 37t + 21)/24 & t \text{ odd,} \end{cases}$$

with

$$(12) \qquad N_{(5,3),(4,1)}^{(5,2)}(w) = \frac{3w^2 + 3w + 1}{(1-w)^3(1-w^2)},$$

while

$$(13) \qquad n^{(4,2)}_{(5,3),(4,1)} = 0 \quad \text{and} \quad n^{t(4,2)}_{t(5,3),t(4,1)} = \begin{cases} (t+2)(19t^2+40t+24)/48 & t \text{ even}; \\ 0 & t \text{ odd}, \end{cases}$$

with

$$(14) \qquad N^{(4,2)}_{(5,3),(4,1)}(w) = \frac{7w^4+11w^2+1}{(1-w^2)^4}.$$

The two Newell–Littlewood examples illustrate some of the properties we wish to explore in an attempt to find analogues of the following statements that apply to the Littlewood–Richardson case:

LR(i) Theorem [Knutson–Tao] [19]: $c^\lambda_{\mu,\nu} > 0 \iff c^{t\lambda}_{t\mu,t\nu} > 0$, for all integers $t > 0$;

LR(ii) Theorem [Fulton] [20]: $c^\lambda_{\mu,\nu} = 1 \iff c^{t\lambda}_{t\mu,t\nu} = 1$, for all integers $t > 0$;

LR(iii) Theorem [Rassart] [28]: $c^{t\lambda}_{t\mu,t\nu}$ is a polynomial in t with rational coefficients;

LR(iv) Conjecture [18]: All coefficients of the polynomial $c^{t\lambda}_{t\mu,t\nu}$ are non-negative.

LR(v) Conjecture [18]: $F(w)$ is a polynomial in w all of whose coefficients are non-negative integers.

In the Newell–Littlewood case it is helpful, as in the above examples, to distinguish between those stretched coefficients $n^{t\lambda}_{t\mu,t\nu}$ for which $|\lambda|+|\mu|+|\nu|$ is either even or odd. Then the closest analogues of the above statements that we offer as conjectures about stretched Newell–Littlewood coefficients are as follows:

CONJECTURE 1.1. *If $|\lambda|+|\mu|+|\nu|$ is even then*

   E(i) *$n^\lambda_{\mu,\nu} > 0 \iff n^{t\lambda}_{t\mu,t\nu} > 0$, for all integers $t > 0$;*

   E(ii) *$n^\lambda_{\mu,\nu} = 1$ and $n^{2\lambda}_{2\mu,2\nu} = 1 \iff n^{t\lambda}_{t\mu,t\nu} = 1$, for all integers $t > 1$;*

  E(iii) *$n^{t\lambda}_{t\mu,t\nu}$ is a quasi-polynomial in t of minimum quasi-period at most 2 with rational coefficients;*

  E(iv) *All coefficients of the quasi-polynomial $n^{t\lambda}_{t\mu,t\nu}$ are non-negative;*

   E(v) *$G(w)$ is a polynomial in w all of whose coefficients are non-negative integers.*

If $|\lambda|+|\mu|+|\nu|$ is odd then $n^{t\lambda}_{t\mu,t\nu} = 0$ for all odd integers t and

   O(i) *$n^{2\lambda}_{2\mu,2\nu} > 0 \iff n^{2t\lambda}_{2t\mu,2t\nu} > 0$, for all integers $t > 0$;*

  O(ii) *$n^{2\lambda}_{2\mu,2\nu} = 1 \iff n^{2t\lambda}_{2t\mu,2t\nu} = 1$, for all integers $t > 0$;*

  O(iii) *$n^{t\lambda}_{t\mu,t\nu}$ is a polynomial in t^2 with rational coefficients;*

  O(iv) *All coefficients of the polynomial $n^{t\lambda}_{t\mu,t\nu}$ are non-negative;*

   O(v) *$G(w)$ is a polynomial in w^2 all of whose coefficients are non-negative integers, and $d_1 = 0$.*

Of these, the validity of the quasi-polynomial and polynomial conjectures E(iii) and O(iii) is established here in Corollary 2.2 of Proposition 2.1. It will then be noted that E(i) and E(ii) are corollaries of E(iv), while O(i) and O(ii) are corollaries of O(iv), with the validity of both E(iv) and O(iv) supported through the accumulation of a considerable amount of data that also supports both E(v) and O(v). Independently of this, the validity of O(i) is established in Corollary 2.4 of Proposition 2.3, along with a slightly weaker form of E(i).

The key point to recognise is that Newell–Littlewood coefficients are nothing other than Clebsch–Gordan coefficients that govern the decomposition of products of universal characters of the classical orthogonal and symplectic groups, or equivalently their corresponding simple Lie algebras. This relationship between coefficients is explained in Section 2. This allows us to conclude immediately that stretched Newell–Littlewood coefficients are quasi-polynomial in nature by virtue of a proposition to this effect established by De Loera and McAllister [7]. This applies to all stretched

Clebsch–Gordan coefficients of classical simple Lie algebras. They also established that the minimum quasi-period of all such quasi-polynomials is at most 2. This allows us to prove a Corollary 2.2 of their proposition that comprises parts E(iii) and O(iii) of Conjecture 1.1.

This same relationship between Clebsch–Gordan coefficients and Newell–Littlewood coefficients allows us to extract from the work of Kapovich and Millson [14], Belkale and Kumar [2], and Sam [29] on the saturation problem for the orthogonal and symplectic groups some conjectures on Newell–Littlewood coefficients. These are stated at the close of Section 2. They serve to establish the validity of part O(i) of Conjecture 1.1, and are consistent with, but weaker than part E(i).

Section 3 is concerned with a new method of calculating the generating function for stretched Newell–Littlewood coefficients that is based on the use of known generating functions for universal characters [9,16,22]. This is motivated by a desire to test further the validity of the positivity conjectures E(iv) and O(iv) that were first formulated in Conjecture 4.7 of [7] within the stretched Clebsch–Gordan coefficient context. The required formula applicable to stretched Newell–Littlewood coefficients is provided in Theorem 3.1 and is thereafter exploited by means of Xin's algorithm [34] to calculate both $N_{\mu,\nu}^{\lambda}(w)$ and $n_{t\mu,t\nu}^{t\lambda}$, first for all the examples considered by Gao et $al.$ [11], and in subsequent sections for many other examples.

In order to study the nature of the quasi-polynomials $n_{t\mu,t\nu}^{t\lambda}$ for various triples (μ,ν,λ), in particular their degrees, $\deg n_{t\mu,t\nu}^{t\lambda}$, it is helpful to construct a combinatorial model for the evaluation of Newell–Littlewood coefficients $n_{\mu,\nu}^{\lambda}$ and to examine the impact of scaling the parts of μ, ν and λ by t. Such a model was used by Gao et $al.$ [11] for their evaluation of Clebsch–Gordan coefficients. It expressed the required coefficients as the number of integer points in certain BZ-polytopes that had been defined for each of the classical simple Lie algebras by Berenstein and Zelevinsky [3,4]. These BZ-polytopes are defined by a set of linear inequalities and equalities involving the parts of μ, ν and λ. Scaling all these parts simultaneously by the stretching parameter t allows one to identify stretched Clebsch–Gordan coefficients with the Ehrhart quasi-polynomial [8,32] of the BZ-polytope.

In the case of the general linear algebra, the BZ-polytope is equivalent to the more simply defined hive polytope of Knutson and Tao [19]. This is introduced here in Section 4 as a hive model for the evaluation of Littlewood–Richardson coefficients. Then, rather than using the BZ-polytopes for the orthogonal and symplectic algebras to evaluate Newell–Littlewood coefficients, we introduce from first principles a new hive model for their evaluation. The corresponding polytopes, which we refer to as K-polytopes, are again defined by a set of linear inequalities and equalities involving the parts of μ, ν and λ. Scaling by the stretching parameter t in the usual way then allows one to identify stretched Newell–Littlewood coefficients with the Ehrhart quasi-polynomial of these K-polytopes. This hive model enables one to see that

$$(15) \qquad \deg n_{t\mu,t\nu}^{t\lambda} \leqslant 3n(n-1)/2 \quad \text{where } n = \max\{\ell(\mu),\ell(\nu),\ell(\lambda)\}.$$

The use of skeletal graphs [18] in obtaining lower upper bounds for particular triple (μ,ν,λ) is illustrated in three examples, the first of which sheds some light on parts E(ii) and O(ii) of Conjecture 1.1 regarding the conditions under which $n_{t\mu,t\nu}^{t\lambda} = 1$.

These bounds are satisfied by explicit calculations of $n_{t\mu,t\nu}^{t\lambda}$ in the following sections which all, more importantly, support the positivity conjectures of Conjecture 1.1 as well as giving rise to certain new stability conjectures. These arise in Section 5 which commences with an example in which the quasi-polynomials $n_{t\mu,t\nu}^{t\lambda}$ are evaluated for fixed $\mu = \nu$ but arbitrary λ. One stability phenomenon takes the form of the

independence of $n_{t\mu,t\nu}^{t\lambda}$ on a in the case $\mu = (a,\sigma)$, $\nu = (a,\tau)$ and $\lambda = (a,\rho)$ for a sufficiently large. Section 6 involves a further exploration of both the positivity and stability conjectures in the case of Newell–Littlewood cubes for which $\mu = \nu = \lambda$, with some further results relegated to Appendix A.

The concluding Section 7 includes some remarks about the connections between the various conjectures.

2. Universal characters

The classical Lie groups, G, of interest here are the general linear groups, GL_r, the odd orthogonal group, SO_{2r+1}, the symplectic group, Sp_{2r}, and the even orthogonal group, SO_{2r}, for all $r \in \mathbb{N}$. Let $\mathcal{P}_r$ be the set of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_k \in \mathbb{Z}_{\geqslant 0}$ for $k = 1, 2, \ldots, r$ and $\lambda_k \geqslant \lambda_{k+1}$ for $k = 1, 2, \ldots, r-1$. The partition λ is said to have length $\ell(\lambda) = p$ if $\lambda_k > 0$ and $\lambda_k = 0$ for all $k > p$. In such a case, one often drops trailing zeros and writes $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$. Each of the groups G possesses a finite-dimensional irreducible representation, V_G^λ, of highest weight λ for each $\lambda \in \mathcal{P}_r$.

The characters may be evaluated through the use of Weyl's character formula:

$$(16) \qquad \operatorname{ch} V_G^\lambda = \sum_{w \in W_{\mathfrak{g}}} \operatorname{sgn}(w) e^{w(\lambda+\rho)} \Big/ \prod_{\alpha \in \Delta_{\mathfrak{g}}^+} (e^{\alpha/2} - e^{-\alpha/2}),$$

where $W_{\mathfrak{g}}$ is the Weyl group of the Lie algebra $\mathfrak{g}$ corresponding to the group G, $\operatorname{sgn}(w) = \pm 1$ is the signature or parity of w, $\Delta_{\mathfrak{g}}^+$ is the set of positive roots of $\mathfrak{g}$ and $\rho_{\mathfrak{g}}$ is half the sum of the positive roots. It follows that

$$(17) \qquad \prod_{\alpha \in \Delta_{\mathfrak{g}}^+} (1 - e^{-\alpha}) \operatorname{ch} V_G^\lambda = \sum_{w \in W_{\mathfrak{g}}} \operatorname{sgn}(w) e^{w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}} = e^\lambda + \sum_\kappa (\pm e^\kappa),$$

where all terms in the sum on the right are distinct. Each $\kappa = w(\lambda+\rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}}$ with $w \neq id$, the identity element of $W_{\mathfrak{g}}$, does not, unlike λ, lie in the fundamental, positive Weyl chamber. Hence, for any coefficients m_μ

$$(18) \qquad [e^\lambda] \prod_{\alpha \in \Delta_{\mathfrak{g}}^+} (1 - e^{-\alpha}) \sum_\mu m_\mu \operatorname{ch} V_G^\mu = m_\lambda$$

where $[e^\lambda](\cdots)$ is the coefficient of e^λ in the expansion of $(\cdots)$. This allows one to evaluate the coefficients appearing in (1) by means of the formula

$$(19) \qquad m_{\mu,\nu}^\lambda(G) = [e^\lambda] \prod_{\alpha \in \Delta_{\mathfrak{g}}^+} (1 - e^{-\alpha}) \operatorname{ch} V_G^\mu \operatorname{ch} V_G^\nu.$$

In each case of interest here the character $\operatorname{ch} V_G^\lambda$ may be expressed in terms of a sequence of indeterminates $\mathbf{x} = (x_1, x_2, \ldots)$ whose non-vanishing components may be identified either with eigenvalues x_i of group elements or with formal exponentials $x_i = e^{\epsilon_i}$ in the Euclidean basis of the root space of the algebra $\mathfrak{g}$. For each of our reductive Lie groups, G, the corresponding simple classical Lie algebra, $\mathfrak{g}$, and their positive roots are given in the (20).

$$(20)$$

G	$\mathfrak{g}$	$\Delta_{\mathfrak{g}}^+$
GL_r	$A_{r-1}\dot{+}D_1$	$\{\epsilon_i - \epsilon_j \mid 1 \leqslant i < j \leqslant r\}$
SO_{2r+1}	B_r	$\{\epsilon_i \pm \epsilon_j \mid 1 \leqslant i < j \leqslant r\} \cup \{\epsilon_i \mid 1 \leqslant i \leqslant r\}$
Sp_{2r}	C_r	$\{\epsilon_i \pm \epsilon_j \mid 1 \leqslant i < j \leqslant r\} \cup \{2\epsilon_i \mid 1 \leqslant i \leqslant r\}$
SO_{2r}	D_r	$\{\epsilon_i \pm \epsilon_j \mid 1 \leqslant i < j \leqslant r\}$

It is convenient to set $x_i = e^{\epsilon_i}$ and $\overline{x}_i = x_i^{-1} = e^{-\epsilon_i}$ for $i = 1, 2, \ldots, r$. In this Euclidean basis $\lambda = \lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \cdots + \lambda_r \epsilon_r$ and $e^\lambda = \mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_r^{\lambda_r}$ for any partition λ of length $\ell(\lambda) \leqslant r$, with the inclusion where necessary of trailing zeros. The products in (18) take the form shown in (21).

(21)

G	$\prod_{\alpha \in \Delta_{\mathfrak{g}}^+} (1 - e^{-\alpha})$
GL_r	$\prod_{1 \leqslant i < j \leqslant r} (1 - \overline{x}_i x_j)$
SO_{2r+1}	$\prod_{1 \leqslant i < j \leqslant r} (1 - \overline{x}_i x_j)(1 - \overline{x}_i \overline{x}_j) \ \prod_{1 \leqslant i \leqslant r} (1 - \overline{x}_i)$
Sp_{2r}	$\prod_{1 \leqslant i < j \leqslant r} (1 - \overline{x}_i x_j)(1 - \overline{x}_i \overline{x}_j) \prod_{1 \leqslant i \leqslant r} (1 - \overline{x}_i^2)$
SO_{2r}	$\prod_{1 \leqslant i < j \leqslant r} (1 - \overline{x}_i x_j)(1 - \overline{x}_i \overline{x}_j)$

The connection between complex Lie algebras and compact Lie groups is such that the parameters x_i and $\overline{x}_i$, together with 1, may also be interpreted as eigenvalues of group elements, in which case the relevant characters can be specified in the following way:

(22)
$$\text{ch} \, V_{GL_r}^\lambda = gl_\lambda(\mathbf{x}) \quad \text{with} \quad \mathbf{x} = (x_1, x_2, \ldots, x_r, 0, 0, \ldots)$$
$$\text{ch} \, V_{SO_{2r+1}}^\lambda = oo_\lambda(\mathbf{x}) \quad \text{with} \quad \mathbf{x} = (x_1, x_2, \ldots, x_r, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_r, 1, 0, 0, \ldots)$$
$$\text{ch} \, V_{Sp_{2r}}^\lambda = sp_\lambda(\mathbf{x}) \quad \text{with} \quad \mathbf{x} = (x_1, x_2, \ldots, x_r, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_r, 0, 0, \ldots)$$
$$\text{ch} \, V_{SO_{2r}}^\lambda = eo_\lambda(\mathbf{x}) \quad \text{with} \quad \mathbf{x} = (x_1, x_2, \ldots, x_r, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_r, 0, 0, \ldots)$$

Here each character is expressed as a specialisation of an appropriate universal character $\mathfrak{g}_\lambda(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2, \ldots)$ a countably infinite sequence. The corresponding universal characters themselves, without any restrictions on $\mathbf{x}$ are defined by means of the generating functions [9, 16, 22]:

(23)
$$\prod_{i,a}(1 - x_i y_a)^{-1} = \sum_\lambda gl_\lambda(\mathbf{x}) \, gl_\lambda(\mathbf{y}) \, ;$$
$$\prod_{i,a}(1 - x_i y_a)^{-1} \prod_{a \leqslant b}(1 - y_a y_b) = \sum_\lambda oo_\lambda(\mathbf{x}) \, gl_\lambda(\mathbf{y}) \, ;$$
$$\prod_{i,a}(1 - x_i y_a)^{-1} \prod_{a < b}(1 - y_a y_b) = \sum_\lambda sp_\lambda(\mathbf{x}) \, gl_\lambda(\mathbf{y}) \, ;$$
$$\prod_{i,a}(1 - x_i y_a)^{-1} \prod_{a \leqslant b}(1 - y_a y_b) = \sum_\lambda eo_\lambda(\mathbf{x}) \, gl_\lambda(\mathbf{y}) \, ,$$

where $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ and the sum is over all partitions λ.

The universal character $gl_\lambda(\mathbf{x})$ is nothing other than the Schur function $s_\lambda(\mathbf{x})$ whose product rule [24] defines the Littlewood–Richardson coefficients $c_{\mu,\nu}^\lambda$. The product rules for the universal characters take the form [9, 16, 23, 26]

(24)
$$gl_\mu(\mathbf{x}) \, gl_\nu(\mathbf{x}) = \sum_\lambda c_{\mu,\nu}^\lambda \, gl_\lambda(\mathbf{x}) \, ;$$
$$oo_\mu(\mathbf{x}) \, oo_\nu(\mathbf{x}) = \sum_\lambda n_{\mu,\nu}^\lambda \, oo_\lambda(\mathbf{x}) \, ;$$
$$sp_\mu(\mathbf{x}) \, sp_\nu(\mathbf{x}) = \sum_\lambda n_{\mu,\nu}^\lambda \, sp_\lambda(\mathbf{x}) \, ;$$
$$eo_\mu(\mathbf{x}) \, eo_\nu(\mathbf{x}) = \sum_\lambda n_{\mu,\nu}^\lambda \, eo_\lambda(\mathbf{x}) \, ,$$

where the coefficients $n_{\mu,\nu}^\lambda$ are precisely the Newell–Littlewood coefficients given by (2).

The product rules are universal in that they apply in the case of all countably infinite sequences $\mathbf{x} = (x_1, x_2, \ldots)$. However, under the rank-dependent specialisations of (22) the characters $\mathfrak{g}_\lambda(\mathbf{x})$ appearing on the right in (24) may not all be independent. In such a case those for which $\ell(\lambda) > r$ are subject to modification rules that can be expressed in a variety of different ways [5, 15, 21, 26]. However, if $r \geqslant \ell(\mu) + \ell(\nu)$ then no modifications are required so that the product rules (24) apply directly to products of characters of GL_r, SO_{2r+1}, Sp_{2r} and SO_{2r} for all $r \geqslant \ell(\mu) + \ell(\nu)$. As we shall see in Section 3 this observation gives us a way not only of evaluating the quasi-polynomial stretched Newell–Littlewood coefficients but also of evaluating their generating functions.

The coefficients $m_{\mu,\nu}^\lambda(G)$ appearing (1) are referred to variously as Clebsch–Gordan coefficients, Kronecker product coefficients or tensor product coefficients in those cases for which G is a Lie group with corresponding Lie algebra $\mathfrak{g}$. To distinguish various cases of interest we adopt the following notation:

(25)

G	$\mathfrak{g}$	$m_{\mu,\nu}^\lambda(G)$
GL_r	$A_{r-1} \dotplus D_1$	$m_{\mu,\nu}^\lambda(gl_r)$
SO_{2r+1}	B_r	$m_{\mu,\nu}^\lambda(oo_r)$
Sp_{2r}	C_r	$m_{\mu,\nu}^\lambda(sp_r)$
SO_{2r}	D_r	$m_{\mu,\nu}^\lambda(eo_r)$

where r is the rank of the relevant finite-dimensional complex simple Lie algebra.

Following the observation made previously,

$$(26) \qquad m_{\mu,\nu}^\lambda(oo_r) = m_{\mu,\nu}^\lambda(sp_r) = m_{\mu,\nu}^\lambda(eo_r) = n_{\mu,\nu}^\lambda$$

for all $r \geqslant \ell(\mu) + \ell(\nu)$. That is to say, as the rank r increases, each of these Clebsch–Gordan coefficients attains a stable limit and this limit coincides with the corresponding Newell–Littlewood coefficient specified by the same triple of partition labels (μ, ν, λ). This result extends to the stretched case:

$$(27) \qquad m_{t\mu,t\nu}^{t\lambda}(oo_r) = m_{t\mu,t\nu}^{t\lambda}(sp_r) = m_{t\mu,t\nu}^{t\lambda}(eo_r) = n_{t\mu,t\nu}^{t\lambda}$$

for all $t \in \mathbb{Z}_{\geqslant 0}$ and all $r \geqslant \ell(\mu) + \ell(\nu)$.

The behaviour with respect to rank r of stretched Clebsch–Gordan coefficients for each of the Lie algebras B_r, C_r and D_r is illustrated in Table 1 in the case $\mu = \nu = \lambda = (2, 1, 1)$ for which it is required that $r \geqslant 3$.

The data underlying this tabulation were compiled both by using the software package SCHUR [30] to evaluate the decomposition of products of characters in accordance with (1) and by using the well-known character formulae appearing for example in [10] and exploiting (19). In this way Clebsch–Gordan coefficients were explicitly calculated for various stretching parameters t for each of the Lie algebras of rank r, with r ranging from its minimum possible value 3 to the value 6 where stability is known to set in. The expressions displayed in Table 1 were then obtained by fitting the data to quasi-polynomials of quasi-period 2. The results illustrate the fact that the stable limits of the stretched Clebsch–Gordan coefficients of all three Lie algebras coincide, as claimed in (27). The fact, that these coefficients differ for lower values of the rank r is a consequence of the modification rules applying to universal characters being different for each of the three families of Lie algebras [5, 15, 21, 26].

This data fitting approach is justified by the fact that De Loera and McAllister [7] have already established in their Proposition 1.2 the quasi-polynomial nature of stretched Clebsch–Gordan coefficients for all classical Lie algebras, and shown that the minimum quasi-period is at most 2. By taking r sufficiently large, that is to say

$\mathfrak{g}$	$G(w)/(1-w)^{d_1}(1-w^2)^{d_2}$ $\begin{cases} P_e(t) & t \text{ even} \\ P_o(t) & t \text{ odd} \end{cases}$
$B_r \simeq so(2r+1)$ $r = 3$	$(w^2+w+1)/(1-w)^3(1-w^2)$ $\begin{cases} (t+2)(2t^2+5t+4)/8 & t \text{ even} \\ (t+1)(2t^2+7t+7)/8 & t \text{ odd} \end{cases}$
$B_r \simeq so(2r+1)$ $r \geqslant 4$	$(w^6+w^5+8w^4+4w^3+5w^2+w+1)/(1-w)^3(1-w^2)^4$ $\begin{cases} (t+2)^2(t+4)(7t^3+43t^2+126t+240)/3840 & t \text{ even} \\ (t+1)(t+3)(7t^4+71t^3+305t^2+697t+840)/3840 & t \text{ odd} \end{cases}$
$C_r \simeq sp(2r)$ $r = 3$	$(w^4+w^2+1)/(1-w)(1-w^2)^3$ $\begin{cases} (t+2)(t^2+4t+8)/16 & t \text{ even} \\ (t+1)(t^2+2t+5)/16 & t \text{ odd} \end{cases}$
$C_r \simeq sp(2r)$ $r \geqslant 4$	$(w^6+w^5+8w^4+4w^3+5w^2+w+1)/(1-w)^3(1-w^2)^4$ $\begin{cases} (t+2)^2(t+4)(7t^3+43t^2+126t+240)/3840 & t \text{ even} \\ (t+1)(t+3)(7t^4+71t^3+305t^2+697t+840)/3840 & t \text{ odd} \end{cases}$
$D_r \simeq so(2r)$ $r = 3$	$1/(1-w)$ $\begin{cases} 1 & t \text{ even} \\ 1 & t \text{ odd} \end{cases}$
$D_r \simeq so(2r)$ $r = 4$	$(w^5+3w^4+5w^3+5w^2+3w+1)/(1-w)^4(1-w^2)^2$ $\begin{cases} (t+2)(6t^4+33t^3+104t^2+152t+80)/160 & t \text{ even} \\ (t+1)(6t^4+39t^3+131t^2+229t+155)/160 & t \text{ odd} \end{cases}$
$D_r \simeq so(2r)$ $r \geqslant 5$	$(w^6+w^5+8w^4+4w^3+5w^2+w+1)/(1-w)^3(1-w^2)^4$ $\begin{cases} (t+2)^2(t+4)(7t^3+43t^2+126t+240)/3840 & t \text{ even} \\ (t+1)(t+3)(7t^4+71t^3+305t^2+697t+840)/3840 & t \text{ odd} \end{cases}$

TABLE 1. The rank dependence of stretched Clebsch–Gordan coefficients

$r \geqslant \ell(\mu) + \ell(\nu)$ as in (27), their Proposition 1.2 as applied to Newell–Littlewood coefficients takes the form:

PROPOSITION 2.1. *For all triples of partitions (λ, μ, ν) and all $t \in \mathbb{Z}_{\geqslant 0}$ the stretched Newell–Littlewood coefficients $n^{t\lambda}_{t\mu,t\nu}$ are quasi-polynomials in t with minimum quasi-period at most 2.*

As a consequence of this we have

COROLLARY 2.2.
- E(iii) If $|\lambda|+|\mu|+|\nu|$ is even and $n^{\lambda}_{\mu,\nu} > 0$ then $n^{t\lambda}_{t\mu,t\nu}$ is a quasi-polynomial in t with minimum quasi-period at most 2 with rational coefficients.
- O(iii) If $|\lambda|+|\mu|+|\nu|$ is odd and $n^{2\lambda}_{2\mu,2\nu} > 0$ then $n^{t\lambda}_{t\mu,t\nu}$ is a polynomial in t^2 with rational coefficients.

Proof. As pointed out in the Introduction, Proposition 2.1 is equivalent to the statement that the generating function of (6) for Newell–Littlewood coefficients takes the

form

$$(28) \quad \begin{aligned} N_{\mu,\nu}^{\lambda}(w) &= \sum_{t=0}^{\infty} n_{t\mu,t\nu}^{t\lambda}\, w^t = \frac{G(w)}{(1-w)^{d_1}\,(1-w^2)^{d_2}} \\ &= \sum_{t\equiv 0 (\mathrm{mod}\,2)} P_e(t)\, w^t \;+\; \sum_{t\equiv 1 (\mathrm{mod}\,2)} P_o(t)\, w^t, \end{aligned}$$

where $G(w)$, $P_e(t)$ and $P_o(t)$ are all polynomials. The fact that $n_{t\mu,t\nu}^{t\lambda}$ is a non-negative integer for all t ensures that the coefficients in $G(w)$, $P_e(t)$ and $P_o(t)$ are rational.

If $|\lambda|+|\mu|+|\nu|$ is even this completes the proof of E(iii). If $|\lambda|+|\mu|+|\nu|$ is odd then $n_{t\mu,t\nu}^{t\lambda} = 0$ for all odd t, that is to say $P_o(t) = 0$. It follows that $d_1 = 0$ and $G(w)$ must be a polynomial in w^2, thereby completing the proof of O(iii). □

The fact that Newell–Littlewood coefficients are special cases of Clebsch–Gordan coefficients or tensor product multiplicities of characters of complex reductive Lie groups or their associated Lie algebras also allows one to make considerable progress towards the verification of the saturation conjectures E(i) and O(i). In particular, Theorem 1.1 of Kapovich and Millson in [14], Theorems 6 and 7 of Belkale and Kumar [2], and Theorem 1.1 of Sam in [29] reveal, when applied to our universal characters of the orthogonal and symplectic groups the validity of the following:

PROPOSITION 2.3. *For all partitions μ, ν and λ, and integer t*

$$(29) \qquad n_{t\mu,t\nu}^{t\lambda} > 0 \quad \text{for some } t \geqslant 1 \implies \begin{cases} n_{4\mu,4\nu}^{4\lambda} > 0, & [14], \\ n_{2\mu,2\nu}^{2\lambda} > 0, & [2,29]. \end{cases}$$

To justify this one notes that by taking the rank r of the Lie algebra $\mathfrak{g}$ of the relevant group G to be such that $r \geqslant \ell(\mu) + \ell(\nu) \geqslant \ell(\lambda)$ then $n_{\mu,\nu}^{\lambda} = m_{\mu,\nu}^{\lambda}(\mathfrak{g})$, as explained previously. Moreover, by also taking $r > \ell(\lambda)$ the irreducible representation of highest weight λ will be self-contragredient so that $m_{\mu,\nu}^{\lambda}(\mathfrak{g}) = m_{\mu,\nu,\lambda}^{0}(\mathfrak{g})$, where the latter is the multiplicity of the identity representation in the product of the three irreducible representations of highest weights μ, ν and λ. This allows a direct connection to be made with the three-fold case of the tensor products encountered in [2,14,29]. That the result in [14] may be applied to Newell–Littlewood coefficients follows from the fact that for any three dominant integral weights μ, ν and λ, their sum lies in the root lattice of B_r, even though that may not be the case for C_r or D_r.

As a consequence of this we have

COROLLARY 2.4.
 • E′(i) For $|\lambda|+|\mu|+|\nu|$ even, if $n_{t\mu,t\nu}^{t\lambda} > 0$ for some $t \geqslant 1$, then $n_{2\mu,2\nu}^{2\lambda} > 0$:
 • O′(i) For $|\lambda|+|\mu|+|\nu|$ odd, if $n_{t\mu,t\nu}^{t\lambda} > 0$ for some $t \geqslant 1$, then $n_{2\mu,2\nu}^{2\lambda} > 0$.

and for $|\lambda|+|\mu|+|\nu|$ odd, we have
 • O(i) $n_{2\mu,2\nu}^{2\lambda} > 0 \iff n_{2t\mu,2t\nu}^{2t\lambda} > 0$, for all integers $t > 0$.

Proof. Both E′(i) and O′(i) are immediate consequences of Proposition 2.3. To derive O(i) from O′(i) one first notes that for $|\lambda|+|\mu|+|\nu|$ odd, $n_{t\mu,t\nu}^{t\lambda}$ may only be > 0 if t is even. This implies the right to left implication of O(i). The left to right implication follows from the fact that $n_{2\mu,2\nu}^{2\lambda} > 0$ implies that there exist α, β and γ such that $c_{\alpha,\beta}^{2\mu}\, c_{\alpha,\gamma}^{2\nu}\, c_{\beta,\gamma}^{2\lambda} > 0$. This implies in turn, by virtue of LR(i), that $c_{t\alpha,t\beta}^{2t\mu}\, c_{t\alpha,t\gamma}^{2t\nu}\, c_{t\beta,t\gamma}^{2t\lambda} > 0$. □

This still leaves open the conjecture E(i) that was first proposed by Gao *et al.* [11]. To test this and all remaining unproven parts of Conjecture 1.1 we proceed by gathering together more data.

3. GENERATING FUNCTION APPROACH

Although the data of the previous section also supports the positivity conjectures E(iv) and O(iv) on the coefficients of $P_e(t)$ and $P_o(t)$, as well as the positivity conjectures E(v) and O(v) on the coefficients of $G(w)$, these are far from being proved as yet. In fact the positivity of the coefficients of the polynomials $P_e(t)$ and $P_o(t)$ were first proposed by De Loera and McAllister in Conjecture 4.7 of [7] within the context of their study of stretched Clebsch–Gordan coefficients. This conjecture was based on an extraordinary wealth of data accumulated by the random selection of hundreds of triples (λ, μ, ν) involving partitions of impressively high weight. However, the calculations were only feasible for Lie algebras of comparatively low rank. As a result their tabulation of results in support of the positivity conjecture in the Clebsch–Gordan context does not include any Newell–Littlewood examples. It therefore seems worthwhile calculating some more Newell–Littlewood quasi-polynomials from first principles. This may be done by means of the following

THEOREM 3.1. *Let λ, μ and ν be partitions of lengths m, p and q, respectively, with $m \leqslant p + q$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_r)$, $\mathbf{y} = (y_1, y_2, \ldots, y_p)$ and $\mathbf{z} = (z_1, z_2, \ldots, z_q)$ with $r = p + q$, and let $\overline{x}_i = x_i^{-1}$, $\overline{y}_i = y_i^{-1}$ and $\overline{z}_i = z_i^{-1}$ for all i. Then*

$$(30) \qquad n_{\mu,\nu}^\lambda = [\mathbf{x}^\lambda \mathbf{y}^\mu \mathbf{z}^\nu] \; K(\mathbf{x},\mathbf{y}) K(\mathbf{x},\mathbf{z}) A(\mathbf{y}) A(\mathbf{z}) C(\overline{\mathbf{x}}) V(\mathbf{x}) V(\mathbf{y}) V(\mathbf{z})$$

and

$$(31) \qquad N_{\mu,\nu}^\lambda(w) = [\mathbf{x}^0 \mathbf{y}^0 \mathbf{z}^0] \; \frac{K(\mathbf{x},\mathbf{y}) K(\mathbf{x},\mathbf{z}) A(\mathbf{y}) A(\mathbf{z}) C(\overline{\mathbf{x}}) V(\mathbf{x}) V(\mathbf{y}) V(\mathbf{z})}{(1 - w/(\mathbf{x}^\lambda \mathbf{y}^\mu \mathbf{z}^\nu))}$$

where

$$\begin{aligned}
K(\mathbf{x},\mathbf{y}) &= \prod_{(i,a)=(1,1)}^{(r,p)} (1 - x_i y_a)^{-1} (1 - \overline{x}_i y_a)^{-1}; \\[6pt]
K(\mathbf{x},\mathbf{z}) &= \prod_{(i,c)=(1,1)}^{(r,p)} (1 - x_i z_c)^{-1} (1 - \overline{x}_i z_c)^{-1}; \\[6pt]
(32) \qquad C(\overline{\mathbf{x}}) &= \prod_{1 \leqslant i \leqslant j \leqslant r} (1 - \overline{x}_i \overline{x}_j); \\[6pt]
A(\mathbf{y}) &= \prod_{1 \leqslant a < b \leqslant p} (1 - y_a y_b); \quad A(\mathbf{z}) = \prod_{1 \leqslant c < d \leqslant q} (1 - z_c z_d); \\[6pt]
V(\mathbf{x}) &= \prod_{1 \leqslant i < j \leqslant r} (1 - \overline{x}_i x_j); \\[6pt]
V(\mathbf{y}) &= \prod_{1 \leqslant a < b \leqslant p} (1 - \overline{y}_a y_b); \quad V(\mathbf{z}) = \prod_{1 \leqslant c < d \leqslant q} (1 - \overline{z}_c z_d);
\end{aligned}$$

and $[\mathbf{x}^\lambda \mathbf{y}^\mu \mathbf{z}^\mu](\cdots)$ means the coefficient of $x_1^{\lambda_1} \cdots x_s^{\lambda_r} y_1^{\mu_1} \cdots y_p^{\mu_p} z_1^{\nu_1} \cdots z_q^{\nu_q}$ in $(\cdots)$.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_r, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_r, 0, 0, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots, y_p, 0, 0, \ldots)$ and $\mathbf{z} = (z_1, z_2, \ldots, z_q, 0, 0, \ldots)$ for the moment. Then the symplectic case of (23) yields, with the notation of (32),

$$(33) \quad K(\mathbf{x},\mathbf{y}) A(\mathbf{y}) = \sum_\sigma sp_\sigma(\mathbf{x}) \, gl_\sigma(\mathbf{y}) \quad \text{and} \quad K(\mathbf{x},\mathbf{z}) A(\mathbf{z}) = \sum_\tau sp_\tau(\mathbf{x}) \, gl_\tau(\mathbf{z}) \,.$$

The choice of $p = \ell(\mu)$ and $q = \ell(\nu)$ in defining $\mathbf{y}$ and $\mathbf{z}$ are the lowest possible values so as to ensure that the sums over σ and τ include the cases $\sigma = \mu$ and $\tau = \nu$ by virtue of the non-vanishing of $gl_\mu(\mathbf{y}) = s_\mu(\mathbf{y})$ and $gl_\nu(\mathbf{z}) = s_\nu(\mathbf{z})$. This choice also

ensures that $\ell(\sigma) \leqslant p$ and $\ell(\tau) \leqslant q$ in the case of all non-vanishing terms. Taking the product of these two expressions yields

$$(34) \qquad K(\mathbf{x}, \mathbf{y})\, K(\mathbf{x}, \mathbf{z})\, A(\mathbf{y})\, A(\mathbf{z}) = \sum_{\rho, \sigma, \tau} n_{\sigma, \tau}^{\rho}\; sp_\rho(\mathbf{x})\; gl_\sigma(\mathbf{y})\; gl_\tau(\mathbf{z})\,.$$

This time the choice of $r = p + q \geqslant \ell(\sigma) + \ell(\tau)$ in defining $\mathbf{x}$ ensures by virtue of (4) and (24) that all possible terms $sp_\rho(\mathbf{x})$ appearing in the product of the universal characters $sp_\sigma(\mathbf{x})$ and $sp_\tau(\mathbf{x})$ also appear in (34). This includes $sp_\lambda(\mathbf{x})$ since $r = p + q > m$, the length of λ.

To project out the coefficients $n_{\mu, \nu}^{\lambda}$ it is necessary to pick out of this expression the coefficient of $sp_\lambda(\mathbf{x})\, gl_\mu(\mathbf{y})\, gl_\nu(\mathbf{z})$. This can be done through the use of (18) and the products taken from (21) for Sp_{2r}, GL_p and GL_q. Again in the notation of (32), these take the form $C(\overline{\mathbf{x}})\, V(\mathbf{x})$, $V(\mathbf{y})$ and $V(\mathbf{z})$. Putting these results together and taking the coefficient of $[\mathbf{x}^\lambda\, \mathbf{y}^\mu\, \mathbf{z}^\nu]$ yields $n_{\mu, \nu}^{\lambda}$ as in (30), where it is important to note that the reversion to $\mathbf{x} = (x_1, x_2, \ldots, x_r)$, $\mathbf{y} = (y_1, y_2, \ldots, y_p)$ and $\mathbf{z} = (z_1, z_2, \ldots, z_q)$ is allowed since λ, μ and ν are partitions of lengths $m \leqslant p + q = r$, p and q.

To evaluate $n_{t\mu, t\nu}^{t\lambda}$ for any stretching parameter t, one merely has to replace λ, μ and ν in (30) by the stretched partitions $t\lambda$, $t\mu$ and $t\nu$, respectively, The fact that there is no change in the lengths of these partitions obviates the necessity of making any changes to the parameters r, p and q in (32).

The generating function $N_{\mu, \nu}^{\lambda}(w)$ of (31) then follows immediately by expanding $1/(1 - w/Z)$ in the form $1 + w/Z + w^2/Z^2 + \cdots + w^t/Z^t + \cdots$ with $Z^t = \mathbf{x}^{t\lambda}\, \mathbf{y}^{t\mu}\, \mathbf{z}^{t\nu}$. $\qquad \square$

Similar, but distinct formulae of the same type may be obtained by exploiting the orthogonal cases of (23) and the $SO(2r)$ and $SO(2r + 1)$ cases of (21). However, in each case the result is slightly less efficient as a means of calculating $n_{\mu, \nu}^{\lambda}$ and $N_{\mu, \nu}^{\lambda}(w)$. By exploiting the self-contragredient nature of the relevant irreducible representations it is also straightforward to write down expressions for $n_{\mu, \nu}^{\lambda}$ and $N_{\mu, \nu}^{\lambda}(w)$, analogous to those in Theorem 3.1, but that are manifestly symmetric in μ, ν and λ, namely

$$n_{\mu, \nu}^{\lambda} = [\mathbf{v}^0 \mathbf{x}^\lambda \mathbf{y}^\mu \mathbf{z}^\nu]\; K(\mathbf{v}, \mathbf{x}) K(\mathbf{v}, \mathbf{y}) K(\mathbf{v}, \mathbf{z}) A(\mathbf{x}) A(\mathbf{y}) A(\mathbf{z}) C(\overline{\mathbf{v}}) V(\mathbf{v}) V(\mathbf{x}) V(\mathbf{y}) V(\mathbf{z})$$

and

$$N_{\mu, \nu}^{\lambda}(w) = [\mathbf{v}^0 \mathbf{x}^0 \mathbf{y}^0 \mathbf{z}^0]\; \frac{K(\mathbf{v}, \mathbf{x}) K(\mathbf{v}, \mathbf{y}) K(\mathbf{v}, \mathbf{z}) A(\mathbf{x}) A(\mathbf{y}) A(\mathbf{z}) C(\overline{\mathbf{v}}) V(\mathbf{v}) V(\mathbf{x}) V(\mathbf{y}) V(\mathbf{z})}{(1 - w/(\mathbf{v}^0 \mathbf{x}^\lambda \mathbf{y}^\mu \mathbf{z}^\nu))}$$

where $\mathbf{v} = (v_1, v_2, \ldots, v_s)$ with $s = p + q + r$. However, the use of these expressions is again computationally more demanding than the use of those in Theorem 3.1.

To exploit Theorem 3.1 it is convenient to use Xin's algorithm [34] as implemented in the Maple package Ell2.mpl. This proceeds by successively picking out the constant terms in each x_i, each y_a and each z_c. To do this it is necessary to multiply x_i and $\overline{x}_i$ in $K(\mathbf{x}, \mathbf{y})$ by a parameter u, and in $K(\mathbf{x}, \mathbf{z})$ by a parameter v, only setting $u = v = 1$ after the elimination of x_i. The result is an explicit expression for $N_{\mu, \nu}^{\lambda}(w)$ in the form $G(w)/(1 - w)^{d_1}(1 - w^2)^{d_2}$ from which can be extracted polynomial expressions $P_e(t)$ and $P_o(t)$ for $n_{t\mu, t\nu}^{t\lambda}$ in the cases t even and odd, respectively.

The results of Gao *et al.* in section 5.4 of [11], displayed here in Table 2, have been confirmed by this means.

To illustrate cases in which μ and ν are fixed and λ varies over all partitions of weight $|\lambda|$ ranging from 0 to $|\mu| + |\nu|$ it is instructive to consider, for example, the case $\mu = \nu = (3, 1)$ with λ arbitrary. The results are tabulated in Tables 3 and 4, stratified by $|\lambda|$ which varies from 8 down to 0. One can see that in each case the positivity of all coefficients in $P_e(t)$, $P_o(t)$ and $G(w)$ is confirmed, thereby supporting the parts E(iv), O(iv), E(v) and O(v) of Conjecture 1.1.

μ	$G(w)/(1-w)^{d_1}(1-w^2)^{d_2}$
ν	$\begin{cases} P_e(t) & t \text{ even} \\ P_o(t) & t \text{ odd} \end{cases}$
λ	
$(1,1)$	$1/(1-w)(1-w^2)$
$(1,1)$	$\begin{cases} (t+2)/2 & t \text{ even} \\ (t+1)/2 & t \text{ odd} \end{cases}$
$(1,1)$	
$(2,1,1)$	$1/(1-w)^2(1-w^2)^2$
$(2,1,1)$	$\begin{cases} (t+2)(t+3)(t+4)/24 & t \text{ even} \\ (t+1)(t+3)(t+5)/24 & t \text{ odd} \end{cases}$
$(1,1,1,1)$	
$(2,1,1)$	$(w^6+w^5+8w^4+4w^3+5w^2+w+1)/(1-w)^3(1-w^2)^4$
$(2,1,1)$	$\begin{cases} (t+2)^2(t+4)(7t^3+43t^2+126t+240)/3840 & t \text{ even} \\ (t+1)(t+3)(7t^4+71t^3+305t^2+697t+840)/3840 & t \text{ odd} \end{cases}$
$(2,1,1)$	

TABLE 2. Examples of Gao *et al.*

4. Hive model

In the case of the Lie algebra $gl(n)$ the Littlewood–Richardson coefficients may be evaluated using the hive model introduced by Knutson and Tao [19], with properties described in more detail by Buch [6]. An integer n-hive is a labelling of the vertices of a planar, equilateral triangular graph of side length n with integers a_{ij}, for $0 \leqslant i \leqslant j \leqslant n$, satisfying certain rhombus inequalities which are to be applied to each elementary rhombus formed from the union of any pair of elementary triangles having a common edge whatever their orientation. One such hive is illustrated below on the left in the case $n = 4$. The elementary triangles and rhombi are shown on the right. They are provided with both vertex and edge labels [18, 33].

(35)
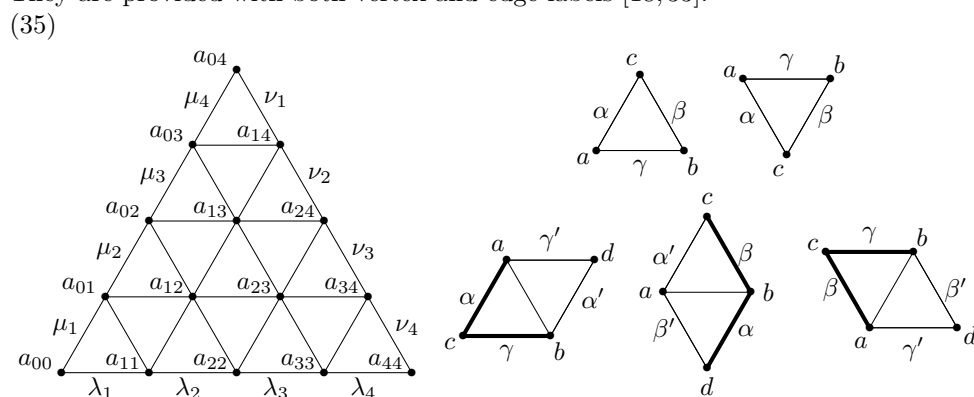
An edge between vertices labelled a and b is given the label $b - a$ if b is to the right of a. Thus in each of the two elementary triangles we have

$$(36) \qquad \alpha = c - a, \ \beta = b - c, \ \gamma = b - a \quad \text{so that} \quad \gamma = \alpha + \beta.$$

For all three elementary rhombi, the rhombus constraints take the form

$$(37) \qquad a + b \geqslant c + d \quad \text{so that} \quad \alpha \geqslant \alpha', \ \beta \geqslant \beta', \ \gamma \geqslant \gamma'.$$

This is indicated in the above diagram by making the edge with the potentially larger edge label thicker than the other for each pair of parallel edges.

| λ	$N^{\lambda}_{(3,1),(3,1)}(w)$	$n^{t\lambda}_{t(3,1),t(3,1)}$
$(62), (61^2), (53), (51^3), (4^2),$ $(42^2), (421^2), (3^22), (3^21^2)$	$\dfrac{1}{(1-w)}$	1
$(521), (431)$	$\dfrac{1}{(1-w)^2}$	$(t+1)$
$(61), (41^3), (32^2), (321^2)$	$\dfrac{1}{(1-w^2)}$	1 t even $\quad$ 0 t odd
$(52), (51^2), (43), (3^21)$	$\dfrac{1}{(1-w^2)^4}$	$(t+2)(t+4)(t+6)/48$ t even $\qquad 0 \qquad\qquad t$ odd
(421)	$\dfrac{(1+3w^2)}{(1-w^2)^5}$	$(t+2)^2(t+4)(t+6)/96$ t even $\qquad 0 \qquad\qquad t$ odd
$(6), (2^3), (2^21^2)$	$\dfrac{1}{(1-w)}$	1
$(51), (41^2)$	$\dfrac{1}{(1-w)^3(1-w^2)}$	$(t+2)(t+4)(2t+3)/24$ t even $(t+1)(t+3)(2t+7)/24$ t odd
(42)	$\dfrac{1}{(1-w)^4}$	$(t+1)(t+2)(t+3)/6$
(3^2)	$\dfrac{1}{(1-w)^2(1-w^2)}$	$(t+2)^2/4 \qquad t$ even $(t+1)(t+3)/4$ t odd
(321)	$\dfrac{1}{(1-w)^4(1-w^2)}$	$(t+2)(t+4)(t^2+6t+6)/48$ t even $(t+1)(t+3)^2(t+5)/48 \qquad t$ odd
$(5), (21^3)$	$\dfrac{1}{(1-w^2)^2}$	1 t even $\quad$ 0 t odd
(41)	$\dfrac{(1+6w^2+4w^4)}{(1-w^2)^4}$	$(t+2)(11t^2+26t+24)/48$ t even $\qquad 0 \qquad\qquad t$ odd
(32)	$\dfrac{(1+7w^2+5w^4)}{(1-w^2)^4}$	$(t+2)(13t^2+28t+24)/48$ t even $\qquad 0 \qquad\qquad t$ odd
(31^2)	$\dfrac{(1+3w^2)}{(1-w^2)^4}$	$(t+2)(t+4)(2t+3)/24$ t even $\qquad 0 \qquad\qquad t$ odd
(2^21)	$\dfrac{(1+w^2)}{(1-w^2)^4}$	$(t+2)(t+3)(t+4)/24$ t even $\qquad 0 \qquad\qquad t$ odd

TABLE 3. All non-zero $N^{\lambda}_{(3,1),(3,1)}(w)$ and $n^{t\lambda}_{t(3,1),t(3,1)}$ with $|\lambda| > 4$.

As emphasised elsewhere [33], one consequence of the rhombus constraints is that if they are paired together as in the following three diagrams they imply the betweenness

λ	$N^\lambda_{(3,1),(3,1)}(w)$	$n^{t\lambda}_{t(3,1),t(3,1)}$
(4)	$\dfrac{1}{(1-w)^2}$	$(t+1)$
(31)	$\dfrac{(1+w+w^2)}{(1-w)^3(1-w^2)}$	$(t+2)(2t^2+5t+4)/8$ t even $(t+1)(2t^2+7t+7)/8$ t odd
(2^2)	$\dfrac{1}{(1-w)^3}$	$(t+1)(t+2)/2$
(21^2)	$\dfrac{1}{(1-w)^3(1-w^2)}$	$(t+2)(t+4)(2t+3)/24$ t even $(t+1)(t+3)(2t+7)/24$ t odd
(1^4)	$\dfrac{1}{(1-w)}$	1
(3)	$\dfrac{(1+w^2)}{(1-w^2)^2}$	$(t+1)$ t even 0 t odd
(21)	$\dfrac{(1+w^2)(1+4w^2)}{(1-w^2)^4}$	$(t+2)(5t^2+11t+12)/24$ t even 0 t odd
(1^3)	$\dfrac{1}{(1-w^2)^2}$	$(t+1)/2$ t even 0 t odd
(2)	$\dfrac{1}{(1-w)^2}$	$(t+1)$
(1^2)	$\dfrac{1}{(1-w)^2(1-w^2)}$	$(t+2)^2/4$ t even $(t+1)(t+3)/4$ t odd
(1)	$\dfrac{1}{(1-w^2)^2}$	$(t+2)/2$ t even 0 t odd
(0)	$\dfrac{1}{(1-w)}$	1

TABLE 4. All non-zero $N^\lambda_{(3,1),(3,1)}(w)$ and $n^{t\lambda}_{t(3,1),t(3,1)}$ with $|\lambda| \leqslant 4$.

conditions as specified below each diagram:

(38)

$$\alpha \geqslant \alpha' \geqslant \alpha'' \qquad \beta \geqslant \beta' \geqslant \beta'' \qquad \gamma \geqslant \gamma' \geqslant \gamma''$$
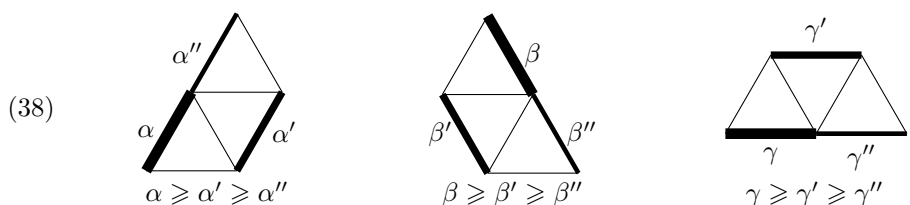
This immediately implies that the sequence of edge labels along any line parallel to one or other of the boundaries of the hive, including the boundaries themselves, constitute a partition if read from bottom left to top right parallel to the left-hand boundary, or from top left to bottom right parallel to the right-hand boundary or from left to right parallel to the bottom boundary.

DEFINITION 4.1. *For fixed integer n, and partitions μ, ν and λ of lengths $\ell(\mu), \ell(\nu), \ell(\lambda) \leqslant n$, let $\mathcal{H}^{(n)}(\mu, \nu; \lambda)$ be the set of integer n-hives H, satisfying the constraints* (36) *and* (37) *whose boundary edge labels are specified by the parts of the partitions μ, ν and λ in accordance with the formulae*

(39) $\quad \mu_i = a_{0,i} - a_{0,i-1}, \ \nu_i = a_{i,n} - a_{i-1,n}, \ \lambda_i = a_{i,i} - a_{i-1,i-1} \quad for \quad i = 1, 2, \ldots, n.$

This labelling of boundary edges is illustrated for the case $n = 4$ in the above diagram (35).

With this definition, Fulton has established in the Appendix of [6] a bijection between such hives and the tableaux whose enumeration serves to evaluate the Littlewood–Richardson coefficients $c_{\mu,\nu}^{\lambda}$. This implies the validity of the following hive model formula for these coefficients as given in Appendix 2 of [19].

PROPOSITION 4.2. *Let μ, ν and λ be partitions of lengths $\ell(\mu), \ell(\nu), \ell(\lambda) \leqslant n$. Then*

(40) $\qquad\qquad c_{\mu,\nu}^{\lambda} = \#\{H \in \mathcal{H}^{(n)}(\mu, \nu; \lambda)\}.$

It is then a simple matter to exploit the definition (2) of Newell–Littlewood coefficients in terms of Littlewood–Richardson coefficients to arrive at a hive model formula for the former. To this end, let K be the composite n-hive constructed from three standard n-hives as shown below.

(41)
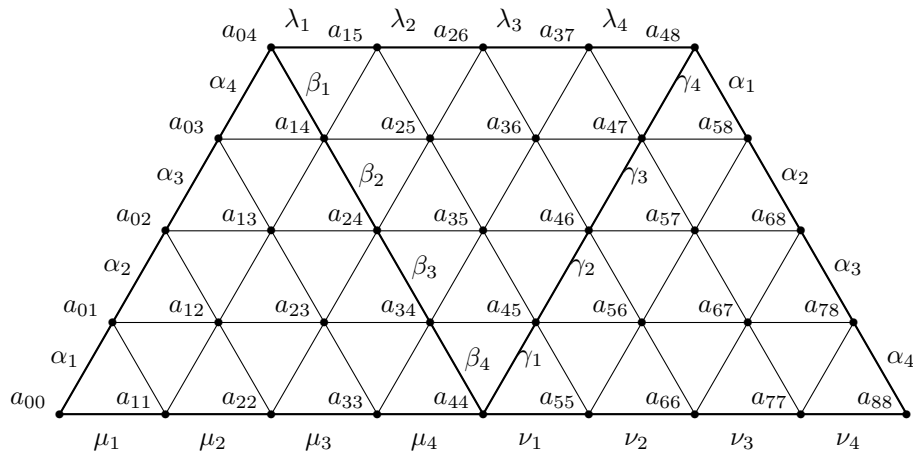
Such a trapezoidal composite n-hive is constructed from three constituent triangular integer n-hives, H_μ, H_ν and H^λ, with horizontal boundary edge labels specified by the parts of μ, ν and λ, respectively. The λ triangular n-hive has been turned upside down, so that it shares common boundaries with both the μ and ν triangular n-hives. We refer to these internal boundaries as the β and γ boundaries, respectively, and the sloping outer boundaries as left and right α-boundaries.

DEFINITION 4.3. *For some fixed positive integer n, and partitions μ, ν and λ of lengths $\ell(\mu)$, $\ell(\nu)$ and $\ell(\lambda)$ all $\leqslant n$, let $\mathcal{K}^{(n)}(\mu, \nu; \lambda)$ be the set of all composite n-hives K with horizontal lower boundary edge labels specified by the parts of μ followed by the parts of ν and horizontal upper boundary edge labels specified by the parts of λ, all read from left to right, such that:*

  (i) *the triangle rule* (36) *is satisfied by each elementary triangle;*
  (ii) *the rhombus constraints* (37) *apply to each elementary rhombus that does not cross either the β- or the γ-boundary;*
  (iii) *the bottom to top edge labels on the left α-boundary coincide with the top to bottom edge labels on the right α-boundary.*

With this definition we have:

PROPOSITION 4.4. *Let μ, ν and λ be partitions of lengths $\ell(\mu), \ell(\nu), \ell(\lambda) \leqslant n$. Then*

$$(42) \qquad n_{\mu,\nu}^{\lambda} = \#\{K \in \mathcal{K}^{(n)}(\mu,\nu;\lambda)\}.$$

Proof. First it should be noted that the sets of triangle conditions (36) and rhombus constraints (37) are unchanged by turning the triangles and rhombi upside down. In enumerating all possible composite hives K with horizontal boundary edges determined by μ, ν and λ the rhombus constraints alone on H_μ, H_ν and H^λ would lead to hives with α-, β- and γ-boundary edge labels specified by all possible partitions $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ such that $c_{\alpha,\beta'}^{\mu}$, $c_{\gamma,\alpha'}^{\nu}$ and $c_{\beta,\gamma'}^{\lambda}$ are non-zero. However, the juxtaposition of H_μ, H_ν and H^λ imposes the constraints $\beta' = \beta$ and $\gamma' = \gamma$, while the above requirement (iii) in the definition of a composite n-hive requires that $\alpha' = \alpha$. The enumeration of such composite n-hives therefore yields the product of the Littlewood–Richardson coefficients $c_{\alpha,\beta}^{\mu}$, $c_{\gamma,\alpha}^{\nu}$ and $c_{\beta,\gamma}^{\lambda}$, with of course $c_{\gamma,\alpha}^{\nu} = c_{\alpha,\gamma}^{\nu}$ for all possible partitions α, β and γ such that these coefficients are non-zero. The formula (2) for $n_{\mu,\nu}^{\lambda}$ involving a sum over partitions α, β and γ then immediately gives (42). $\qquad\square$

In this model, the enumeration of all composite hives $K \in \mathcal{K}^{(n)}(\mu,\nu;\lambda)$ involves edge labels specified by the parts of α, β and γ that are restricted in the first instance only by the constraints

$$(43) \qquad 2|\alpha| = |\mu| + |\nu| - |\lambda|, \quad 2|\beta| = |\lambda| + |\mu| - |\nu|, \quad 2|\gamma| = |\nu| + |\lambda| - |\mu|,$$

as can be established by repeated use of the triangle conditions (36), but also seen directly of course from (2). Since α, β and γ are partitions, these constraints (43) suffice to show that $n_{\mu,\nu}^{\lambda} = 0$ if $|\mu| + |\nu| + |\lambda|$ is odd.

DEFINITION 4.5. *For a triple of partitions (μ,ν,λ) with $\ell(\mu), \ell(\nu), \ell(\lambda) \leqslant n$, let $k = (3n+2)(n+1)/2$. Then k is the number of vertices of the composite n-hive K of edge length n with lower and upper edge labels specified by the parts of (μ,ν) and λ, respectively. Then the K-polytope $\mathcal{P}_{\mu,\nu}^{\lambda} \subset \mathbb{R}^k$ is the convex hull of the points $a_{i,j} \in \mathbb{R}^k$ for $j = 0, 1, \ldots, n$ and $i = 0, 1, \ldots, 2n - j$ subject to the linear equalities and inequalities:*

(i) $a_{0,0} = 0$, *and* $a_{0,n} - a_{0,0} = (|\mu| + |\nu| - |\lambda|)/2$;

(ii) $a_{n+j,2n} - a_{n,2n} = a_{0,j} - a_{0,0}$ *for* $j = 1, 2, \ldots, n-1$;

(iii) $\mu_i = a_{i,i} - a_{i-1,i-1}$, $\nu_i = a_{n+i,n+i} - a_{n+i-1,n+i-1}$, $\lambda_i = a_{i,n+i} - a_{i-1,n+i-1}$ *for* $i = 1, 2, \ldots, n$;

(iv) $R_{i,j}, U_{i,j}, L_{i,j}, R_{i+1,j+n}, U_{i,j+n-1}, L_{i,j+n-1}, R_{i+n,j+n}, U_{i+n,j+n}, L_{i+n,j+n}$ *all* $\geqslant 0$ *for* $1 \leqslant i < j \leqslant n$, *where* $R_{i,j} = a_{i-1,j-1} + a_{i,j-1} - a_{i,j} - a_{i-1,j-2}$, $U_{i,j} = a_{i-1,j-1} + a_{i,j} - a_{i-1,j} - a_{i,j-1}$, $L_{i,j} = a_{i-1,j-1} + a_{i+1,j} - a_{i,j} - a_{i,j-1}$.

The first part $a_{0,0} = 0$ of (i) is just an anchoring condition, with all other conditions involving differences of the form $a_{i,j} - a_{k,l}$. The second part of (i) is the constraint on $|\alpha|$ given in (43), while (ii) is the left and right boundary matching condition. Part (iii) corresponds to the lower and upper boundary conditions, while part (iv) corresponds to the rhombus conditions (37) applied to the three different orientations of a rhombus in each of the three constituent n-hives H_μ, H_ν and H^λ.

We then have:

THEOREM 4.6. *With the notation of Definition 4.5 we have*

$$(44) \quad n_{\mu,\nu}^{\lambda} = i(\mathcal{P}_{\mu,\nu}^{\lambda}, 1) := \#\{\mathcal{P}_{\mu,\nu}^{\lambda} \cap \mathbb{Z}^k\} \quad and \quad n_{t\mu,t\nu}^{t\lambda} = i(\mathcal{P}_{\mu,\nu}^{\lambda}, t) := \#\{\mathcal{P}_{t\mu,t\nu}^{t\lambda} \cap \mathbb{Z}^k\}$$

*where $i(\mathcal{P}, t)$ is the Ehrhart quasi-polynomial of the polytope $\mathcal{P}$.*

Proof. The first claim is just a restatement of the fact that $n_{\mu,\nu}^{\lambda}$ is the number of composite K-hives with vertices labelled by integers $a_{i,j}$ subject to the given equalities and inequalities applicable to $\mathcal{P}_{\mu,\nu}^{\lambda}$. All of these can be expressed in the form $E(\mathbf{a}) = \mathbf{b}$ and $I(\mathbf{a}) \leqslant \mathbf{c}$ for some rectangular matrices E and I with integer elements, where $\mathbf{a}$ is a sequence of length k of the labels $a_{i,j}$, and $\mathbf{b}$ and $\mathbf{c}$ are linear in the parts of μ, ν and λ. Scaling these parts by t serves to specify $\mathcal{P}_{t\mu,t\nu}^{t\lambda}$ whose intersection with $\mathbb{Z}^k$ necessarily yields $n_{t\mu,t\nu}^{t\lambda}$. $\qquad\square$

Propositions 4.2 and 4.4 imply that both Littlewood–Richardson and Newell–Littlewood coefficients are independent of the parameter n that determines the boundary edge lengths of the n-hives and the composite n-hives, respectively, provided that these edge lengths are large enough to accommodate the numbers of non-vanishing parts of the partitions μ, ν and λ. This can be seen within the context of the hive models themselves. In the case of any $c_{\mu,\nu}^{\lambda} > 0$, if λ has a trailing 0, then so must both μ and ν, and the corresponding hive takes the typical form:

(45)
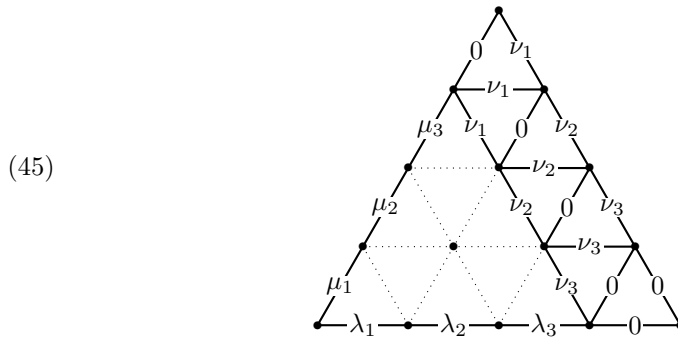
The fact that μ and ν must also have a trailing 0 is a consequence of the triangle and rhombus conditions (36) and (37), respectively. These conditions also suffice to fix all edge labels of the ladder on the right, with its rungs labelled 0 and its side and remaining interior labels specified by the parts of ν. This reduces the enumeration of 4-hives to that of 3-hives. In this way the evaluation of $c_{\mu,\nu}^{\lambda}$ can always be reduced to the enumeration of all n-hives with $n = \ell(\lambda)$.

The same type of argument applies to the evaluation of $n_{\mu,\nu}^{\lambda}$, which can be reduced to the enumeration of composite n-hives with $n = \max\{\ell(\mu), \ell(\nu), \ell(\lambda)\}$. The reduction process is illustrated in the following example.

(46)
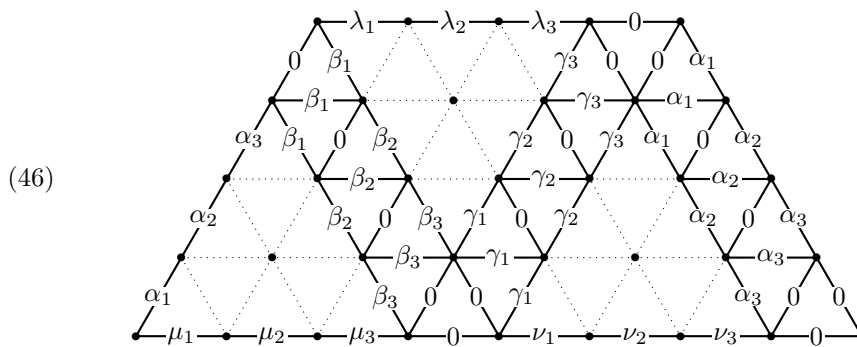
Removal of the three ladders, all of whose edges are fixed by a combination of triangle and rhombus conditions, reduces the composite 4-hive to a composite 3-hive.

Dealing with trailing zeros common to all three partitions μ, ν and λ by the iteration of this process enables any composite n-hive to be reduced to one for which $n = \max\{\ell(\mu), \ell(\nu), \ell(\lambda)\}$. This immediately implies that the degree of the corresponding Ehrhart quasi-polynomial $i(\mathcal{P}_{\mu,\nu}^\lambda, t)$ is bounded by k with $k = (3n+2)(n+1)/2$. This maximum degree can be reduced still further by using all the equalities in the definition of the K-polytope. This can be done by noting that, for given μ, ν and λ, the parts (i)–(iii) of Definition 4.5 fix the vertex labels on the top and bottom horizontal and the right hand diagonal boundary. It follows that the maximum degree of $n_{t\mu,t\nu}^{t\lambda}$ is $3n(n-1)/2$, that is to say $0, 3, 9, 18, \ldots$ for $n = 1, 2, 3, 4, \ldots$

That this upper bound on the degree of $n_{t\mu,t\nu}^{t\lambda}$ is not reached in any of the three examples of Table 2 is clear. However, a lower upper bound on the degree may found by exploiting the notion of skeletal graphs [18]. In the present context a skeletal graph is a partial labelling of the underlying graph of the appropriately shaped composite K-hive in which edges are identified for which the labels are fixed from a knowledge of the parts of μ, ν and λ, including trailing zeros. These are shown in the examples below as thick lines along with their labels as determined by the triangle conditions and rhombus constraints. Since the labels on the right-hand boundary are determined by those on the left-hand boundary, the edges of the former are shown as dashed lines. All remaining edges are represented as dotted lines to indicate that their labels are not fixed. The remaining degrees of freedom in enumerating the composite K-hives contributing to $n_{t\mu,t\nu}^{t\lambda}$ is then the number of components of the skeletal graph, including isolated vertices, that are disconnected from the lower and upper horizontal boundaries.

All this is illustrated in the case of the three examples of Table 2 as follows.

For $\mu = \nu = \lambda = (1,1)$ the skeletal K-hive takes the form:

(47)
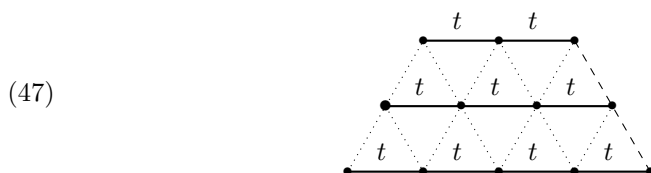
The skeletal graph contains a single component disconnected from the horizontal boundaries and this is signified by marking its left-most vertex by means of an enlarged bullet point. There is therefore just one degree of freedom in enumerating the corresponding composite K-hives, so the quasi-polynomial $n_{t(1,1),t(1,1)}^{t(1,1)}$ is of degree 1 in agreement with Table 2. In this particular case it is easy to go further. The left-hand boundary edge labels $\alpha = (\alpha_1, \alpha_2)$ must be such that with $\alpha_1 + \alpha_2 = 2t$ with $t \geqslant \alpha_1 \geqslant \alpha_2$, leading to the conclusion that $n_{t(1,1),t(1,1)}^{t(1,1)} = (t+2)/2$ if t is even, and $(t+1)/2$ if t is odd, as first found by Gao *et al.* [11]. It can also be seen in this case that $n_{\mu,\nu}^\lambda = 1$ and $n_{2\mu,2\nu}^{2\lambda} = 2$, thereby ruling out any notion that $n_{\mu,\nu}^\lambda = 1$ might always imply $n_{t\mu,t\nu}^{t\lambda} = 1$ for all $t > 0$, as in LR(ii). However, as yet, we have uncovered no counterexample to the conjecture that, taken together, $n_{\mu,\nu}^\lambda = 1$ and $n_{2\mu,2\nu}^{2\lambda} = 1$ imply $n_{t\mu,t\nu}^{t\lambda} = 1$ for all $t > 0$, as would be required by E(ii).

In the case $\mu = \nu = (2,1,1)$ and $\lambda = (1,1,1,1)$ the skeletal K-hive takes the form:

(48)

.
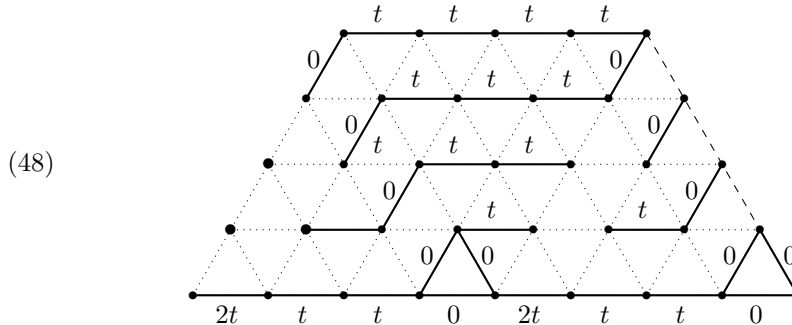
Here, there are 5 components disconnected from the horizontal boundaries, but only 3 of them have been signified by an enlarged bullet point at their left-hand extremity. This is because the skeletal upper bound of 5 in the degree of the quasi-polynomial $n^{t(1,1,1,1)}_{t(2,1,1),t(2,1,1)}$ can be lowered to 3 by noting the requirement that the edge labels on the right- and left-hand boundaries coincide. This leads to the expectation that the quasi-polynomial is of degree 3, as first established in [11] and confirmed here in Table 2.

For $\mu = \nu = \lambda = (2,1,1)$ the skeletal K-hive is given by:

(49)

.
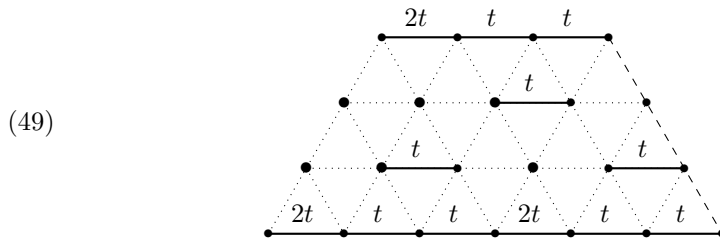
This time there are 8 components disconnected from the horizontal boundaries. However, the coincidence of right and left hand boundary edge labels means that there are just 6 degrees of freedom, leading to the expectation that $n^{t(2,1,1)}_{t(2,1,1),t(2,1,1)}$ is a quasi-polynomial of degree 6, as confirmed in Table 2 and first established in a footnote of [11].

5. Stability phenomena

While continuing to support the validity of the various positivity conjectures, an interesting new stability phenomenon shows itself by considering the effect of adding a fixed integer a to the first part of all three partitions μ, ν and λ. The results for three choices of μ, ν and λ are given in Tables 5, 6 and 7, in which the stability of the stretched Newell–Littlewood coefficients for sufficiently large a occurs for $a \geqslant 4$, 5 and 4, respectively, as tested by explicit calculation for $0 \leqslant a \leqslant 20$.

On the basis of this and many similar results, we offer the following

CONJECTURE 5.1. *For any partitions μ, ν and λ, let $\mu = (\mu_1, \sigma)$, $\nu = (\nu_1, \tau)$ and $\lambda = (\lambda_1, \rho)$. Then for integer a there exists $N \in \mathbb{N}$ such that $n^{(a+\lambda_1,\rho)}_{(a+\mu_1,\sigma),(a+\nu_1,\tau)}$, and more generally $n^{t(a+\lambda_1,\rho)}_{t(a+\mu_1,\sigma),t(a+\nu_1,\tau)}$ for all $t \in \mathbb{N}$, are independent of a for all $a \geqslant N$.*

A similar stability conjecture of the same type is supported by a good deal of evidence of the same type, namely

a	$N^{(a+3,2)}_{(a+3,3),(a+2,1)}(w)$	$n^{t(a+3,2)}_{t(a+3,3),t(a+2,1)}$	
$a = 0$	$\dfrac{1}{(1-w)^2(1-w^2)}$	$(t+2)^2/4$	t even
		$(t+1)(t+3)/4$	t odd
$a = 1$	$\dfrac{(6w^4+8w^2+1)}{(1-w^2)^4}$	$(t+2)(5t^2+10t+8)/16$	t even
		0	t odd
$a = 2$	$\dfrac{(3w^2+3w+1)}{(1-w)^3(1-w^2)}$	$(t+2)(14t^2+23t+12)/24$	t even
		$(t+1)(14t^2+37t+21)/24$	t odd
$a = 3$	$\dfrac{(17w^4+20w^2+1)}{(1-w^2)^4}$	$(t+2)(19t^2+28t+12)/24$	t even
		0	t odd
$a \geqslant 4$ a even	$\dfrac{(4w+1)}{(1-w)^4}$	$(t+1)(t+2)(5t+3)/6$	all t
$a \geqslant 5$ a odd	$\dfrac{(17w^4+22w^2+1)}{(1-w^2)^4}$	$(t+1)(t+2)(5t+3)/6$	t even
		0	t odd

TABLE 5. Cases $\mu = (a+3,3)$, $\nu = (a+2,1)$ and $\lambda = (a+3,2)$ showing stability for $a \geqslant 4$.

a	$N^{(a+2,2)}_{(a+3,3),(a+2,1)}(w)$	$n^{t(a+2,2)}_{t(a+3,3),t(a+2,1)}$	
$a = 0$	$\dfrac{1}{(1-w^2)}$	1	t even
		0	t odd
$a = 1$	$\dfrac{1}{(1-w)^3(1-w^2)}$	$(t+2)(t+4)(2t+3)/24$	t even
		$(t+1)(t+3)(2t+7)/24$	t odd
$a = 2$	$\dfrac{(7w^4+11w^2+1)}{(1-w^2)^4}$	$(t+2)(19t^2+40t+24)/48$	t even
		0	t odd
$a = 3$	$\dfrac{(3w+1)}{(1-w)^4}$	$(t+1)(t+2)(4t+3)/6$	for all t
$a = 4$	$\dfrac{(17w^4+21w^2+1)}{(1-w^2)^4}$	$(t+2)(13t^2+20t+8)/16$	t even
		0	t odd
$a \geqslant 5$ a odd	$\dfrac{(4w+1)}{(1-w)^4}$	$(t+1)(t+2)(5t+3)/6$	all t
$a \geqslant 6$ a even	$\dfrac{(17w^4+22w^2+1)}{(1-w^2)^4}$	$(t+1)(t+2)(5t+3)/6$	t even
		0	t odd

TABLE 6. Cases $\mu = (a+3,3)$, $\nu = (a+2,1)$ and $\lambda = (a+2,2)$ showing stability for $a \geqslant 5$.

CONJECTURE 5.2. *For integer a and any partitions σ, τ and ρ, let $\mu = (a,\sigma)$, $\nu = (a,\tau)$ and $\lambda = (a,\rho)$ with $a \geqslant \max\{\sigma_1, \tau_1, \rho_1\}$. Then there exists $N \in \mathbb{N}$ such that*

	$G(w)/(1-w)^{d_1}(1-w^2)^{d_2}$ $\begin{cases} P_e(t) & t \text{ even} \\ P_o(t) & t \text{ odd} \end{cases}$
$a=0$	$1/(1-w)^4(1-w^2)^4$ $\begin{cases} (t+2)(t+4)(t+6)(t+8)(t+10)(t^2+12t+21)/80840 \\ (t+1)(t+3)(t+5)(t+6)(t+7)(t+11)/80840 \end{cases}$
$a=1$	$(33w^{10}+378w^8+823w^6+449w^4+53w^2+1)/(1-w^2)^9$ $\begin{cases} (t+2)(t+4)(t+6)(1737t^5+2268t^4+113836t^3+277488t^2+339584t+215040)/10321920 \\ 0 \end{cases}$
$a=2$	$(20w^6+59w^5+105w^4+86w^3+43w^2+9w+1)/(1-w)^5(1-w^2)^4$ $\begin{cases} (t+2)(t+4)(t+6)(323t^5+3416t^4+13886t^3+27892t^2+29216t+13440)/645120 \\ (t+1)(t+3)(t+5)(323t^5+4385t^4+22196t^3+54868t^2+68777t+37611)/645120 \end{cases}$
$a=3$	$(161w^{10}+1471w^8+2676w^6+1208w^4+113w^2+1)/(1-w^2)^9$ $\begin{cases} (t+2)(t+4)(t+6)(2815t^5+30932t^4+129972t^3+260752t^2+252032t+107520)/5160960 \\ 0 \end{cases}$
$a \geqslant 4$ a even	$(20w^6+61w^5+113w^4+96w^3+50w^2+11w+1)/(1-w)^5(1-w^2)^4$ $\begin{cases} (t+2)^2(t+3)(t+4)(t+6)(22t^3+132t^2+229t+140)/40320 \\ (t+1)(t+2)(t+3)(t+5)(22t^4+264t^3+1087t^2+1910t+1197)/40320 \end{cases}$
$a \geqslant 5$ a odd	$(161w^{10}+1471w^8+2676w^6+1208w^4+115w^2+1)/(1-w^2)^9$ $\begin{cases} (t+2)^2(t+3)(t+4)(t+6)(22t^3+132t^2+229t+140)/40320 \\ 0 \end{cases}$

TABLE 7. Cases $\mu = (a+3,3,2)$, $\nu = (a+2,1,1)$ and $\lambda = (a+3,2,1)$ showing stability for $a \geqslant 4$.

$n^{(a,\rho)}_{(a,\sigma),(a,\tau)}$, *and more generally* $n^{t(a,\rho)}_{t(a,\sigma),t(a,\tau)}$ *for all* $t \in \mathbb{N}$, *are independent of* a *for all* $a \geqslant N$.

6. Newell–Littlewood cubes

Since the case $\mu = \nu = \lambda$ appears to be of particular interest [13], we gather together here some some results and observations about Newell–Littlewood stretched cubes.

For example, for $\mu = \nu = \lambda = (a,b)$ with $b = 1, 2, 3$ we find the data shown in Tables 8–10.

In these three tables, apart from the positivity of all coefficients in $G(w)$, $P_e(t)$ and $P_o(t)$, the other notable feature is that as a function of a the results for $\mu = \nu = \lambda = (a,b)$ are stable, that is to say independent of a, for all even and all odd $a \geqslant 3b > 0$.

This is borne out by further examples and these stable values may be summarised as follows.

(50)

(a,b) with $a \geqslant 3b > 0$	$N^{(a,b)}_{(a,b),(a,b)}(w)$	$n^{t(a,b)}_{t(a,b),t(a,b)}$
a even b even	$\dfrac{G_{ee}(w)}{(1-w)^4}$	$P_e(t)$ for all t
a even b odd a odd b even	$\dfrac{G_{eo}(w)}{(1-w^2)^4}$	$P_e(t)$ t even 0 t odd
a odd b odd	$\dfrac{G_{oo}(w)}{(1-w)^3(1-w^2)}$	$P_e(t)$ t even $P_o(t)$ t odd

λ	$N_{\lambda,\lambda}^{\lambda}(w)$	$P_e(t)\ t$ even $P_o(t)\ t$ odd
$(1,1)$	$\dfrac{1}{(1-w)(1-w^2)}$	$(t+2)/2\ t$ even $(t+1)/2\ t$ odd
$(2,1)$	$\dfrac{(1+2w^2)^2}{(1-w^2)^4}$	$(t+2)(3t^2+6t+8)/16\ t$ even $\qquad 0 \qquad t$ odd
$(a,1)$ $a\geqslant 3$ a odd	$\dfrac{(1+w+w^2)}{(1-w)^3(1-w^2)}$	$(t+2)(2t^2+5t+4)/8\ t$ even $(t+1)(2t^2+7t+7)/8\ t$ odd
$(a,1)$ $a\geqslant 4$ a even	$\dfrac{(1+7w^2+4w^4)}{(1-w^2)^4}$	$(t+2)(2t^2+5t+4)/8\ t$ even $\qquad 0 \qquad t$ odd

TABLE 8. Cubes $\mu = \nu = \lambda = (a,1)$ showing stability for all even and all odd $a \geqslant 3$.

λ	$N_{\lambda,\lambda}^{\lambda}(w)$	$P_e(t)\ t$ even $P_o(t)\ t$ odd
$(2,2)$	$\dfrac{1}{(1-w)^2}$	$(t+1)$
$(3,2)$	$\dfrac{(1+11w+14w^2+w^6)}{(1-w^2)^4}$	$(3t+4)(3t^2+4t+4)/16\ t$ even $\qquad 0 \qquad t$ odd
$(4,2)$	$\dfrac{(1+2w)^2}{(1-w)^4}$	$(t+1)(3t^2+3t+2)/2$
$(5,2)$	$\dfrac{(1+35w^2+53w^4+4w^6)}{(1-w^2)^4}$	$(31t^3+66t^2+48t+16)/16\ t$ even $\qquad 0 \qquad t$ odd
$(a,2)$ $a\geqslant 6$ a even	$\dfrac{(1+7w+4w^2)}{(1-w)^4}$	$(t+1)(4t^2+5t+2)/2$
$(a,2)$ $a\geqslant 7$ a odd	$\dfrac{(1+38w^2+53w^4+4w^6)}{(1-w^2)^4}$	$(t+1)(4t^2+5t+2)/2\ t$ even $\qquad 0 \qquad t$ odd

TABLE 9. Cubes $\mu = \nu = \lambda = (a,2)$ showing stability for all even and all odd $a \geqslant 6$.

where

$$P_e(t) = \frac{1}{8}(b\,t+2)(2b^2\,t^2+5b\,t+4)\,,$$

(51)

$$P_o(t) = \frac{1}{8}(b\,t+1)(2b^2\,t^2+7b\,t+7)\,,$$

λ	$N_{\lambda,\lambda}^{\lambda}(w)$	$P_e(t)$ t even $P_o(t)$ t odd
$(3,3)$	$\dfrac{(1+w+w^2)}{(1-w)(1-w^2)}$	$(3t+2)/2$ t even $(3t+1)/2$ t odd
$(4,3)$	$\dfrac{(1+18w^2+24w^4+2w^6)}{(1-w^2)^4}$	$(15t^3+36t^2+36t+16)/16$ t even 0 t odd
$(5,3)$	$\dfrac{(1+4w+w^2)^2}{(1-w)^3(1-w^2)}$	$(6t^3+9t^2+6t+2)/2$ t even $(6t^3+9t^2+6t+1)/2$ t odd
$(6,3)$	$\dfrac{(1+72w^2+150w^4+20w^6)}{(1-w^2)^4}$	$(3t+2)(27t^2+18t+8)/16$ t even 0 t odd
$(7,3)$	$\dfrac{(1+17w+36w^2+20w^3+w^4)}{(1-w)^3(1-w^2)}$	$(50t^3+69t^2+34t+8)/8$ t even $(50t^3+69t^2+34t+7)/8$ t odd
$(8,3)$	$\dfrac{(1+99w^2+198w^4+23w^6)}{(1-w^2)^4}$	$(107t^3+156t^2+76t+16)/16$ t even 0 t odd
$(a,3)$ $a\geqslant 9$ a odd	$\dfrac{(1+20w+39w^2+20w^3+w^4)}{(1-w)^3(1-w^2)}$	$(3t+2)(18t^2+15t+4)/8$ t even $(3t+1)(18t^2+21t+7)/8$ t odd
$(a,3)$ $a\geqslant 10$ a even	$\dfrac{(1+102w^2+198w^4+23w^6)}{(1-w^2)^4}$	$(3t+2)(18t^2+15t+4)/8$ t even 0 t odd

TABLE 10. Cubes $\mu=\nu=\lambda=(a,3)$ showing stability for all even and all odd $a\geqslant 9$.

and
(52)
$$G_{ee}(w)=1+(b^3/4+9b^2/8+7b/4-3)w+(b^3-7b/2+3)w^2+(b^3/4-9b^2/8+7b/4-1)w^3$$
$$G_{eo}(w)=1+(2b^3+9b^2/2+7b/2-3)\,w^2+(8b^3-7b+3)\,w^4+(2b^3-9b^2/2+7b/2-1)\,w^6$$
$$G_{oo}(w)=1+(b^3/4+9b^2/8+7b/4-17/8)\,w+(5b^3/4+9b^2/8-7b/4+3/8)\,w^2$$
$$+(5b^3/4-9b^2/8-7b/4+13/8)\,w^3+(b^3/4-9b^2/8+7b/4-7/8)\,w^4.$$

These formulae have been verified for all $a\geqslant 3b>0$ with $b\leqslant 9$ and $a\leqslant 31$. Quite why the results should be stable for all $a\geqslant 3b>0$ is not clear, but it may be checked for each $b>0$ that the stable value is not reached in the case $a=3b-1$.

Moving to length 3 partitions, in the simplest case we have

(53)

(a,a,a) $a>0$	$N_{(a,a,a),(a,a,a)}^{(a,a,a)}(w)$	$n_{t(a,a,a),t(a,a,a)}^{t(a,a,a)}$
a even	$\dfrac{((a/2-1)w+1)}{(1-w)^2}$	$(at+2)/2$ for all t
a odd	$\dfrac{((a-1)w^2+1)}{(1-w^2)^2}$	$(at+2)/2$ t even 0 t odd

.

As a less trivial example we offer:

(54)

(a,b,b) with $a \geqslant 3b > 0$	$N^{(a,b,b)}_{(a,b,b),(a,b,b)}(w)$	$n^{t(a,b,b)}_{t(a,b,b),t(a,b,b)}$
a even b even	$\dfrac{G_{ee}(w)}{(1-w)^7}$	$P_e(t)$ for all t
a even b odd	$\dfrac{G_{eo}(w)}{(1-w)^3(1-w^2)^4}$	$P_e(t)$ t even $P_o(t)$ t odd
a odd b even	$\dfrac{G_{oe}(w)}{(1-w^2)^7}$	$P_e(t)$ t even 0 t odd
a odd b odd	$\dfrac{G_{oo}(w)}{(1-w^2)^7}$	$P_e(t)$ t even $P_o(t)$ t odd

where

(55)
$$P_e(t) = \frac{1}{1920}(bt+2)(bt+4)(4b^4t^4 + 36b^3t^3 + 137b^2t^2 + 270bt + 240)\,,$$

$$P_o(t) = \frac{1}{1920}(bt+1)(bt+3)(4b^4t^4 + 44b^3t^3 + 197b^2t^2 + 400bt + 315)\,,$$

with

(56)

$G(w)$	Degree of $G(w)$	(d_1, d_2)
$G_{ee}(w)$	6	$(7,0)$
$G_{eo}(w)$	10	$(3,4)$
$G_{oe}(w)$	12	$(0,7)$
$G_{oo}(w)$	12	$(0,7)$

where $G_{ee}(w)$ and $G_{eo}(w)$ are polynomials in w, while $G_{oe}(w)$ and $G_{oo}(w)$ are polynomials in w^2, all with positive coefficients. Again, quite why the results should be stable for all $a \geqslant 3b > 0$ is not clear, but it may be verified for each $b > 0$ that the stable value is not reached in the case $a = 3b - 1$.

We can also offer a case for which we have a two-parameter explicit formula:

(57)

(a,a,b) with $a \geqslant 3b > 0$	$N^{(a,a,b)}_{(a,a,b),(a,a,b)}(w)$	$n^{t(a,a,b)}_{t(a,a,b),t(a,a,b)}$
a even b even	$\dfrac{G_{ee}(w)}{(1-w)^7}$	$P_e(t)$ for all t
a even b odd	$\dfrac{G_{eo}(w)}{(1-w^2)^7}$	$P_e(t)$ t even 0 t odd
a odd b even	$\dfrac{G_{oe}(w)}{(1-w)^3(1-w^2)^4}$	$P_e(t)$ t even $P_o(t)$ t odd
a odd b odd	$\dfrac{G_{oo}(w)}{(1-w^2)^7}$	$P_e(t)$ t even 0 t odd

where

(58)
$$P_e(t) = \frac{1}{7680}(bt+2)((108a-179)b^5t^5 + (864a/b-1412)b^4t^4 + (2592a/b-3916)b^3t^3$$
$$+ (3456a/b-3808)b^2t^2 + (1920a/b+960)bt + 3840),$$

$$P_o(t) = \frac{1}{7680}(bt+2)((108a-179)b^5t^5 + (864a/b-1412)b^4t^4 + (2592a/b-3916)b^3t^3$$
$$+ (3456a/b-3568)b^2t^2 + (1920a/b)bt + 1920),$$

with $G_{ee}(w)$ a polynomial in w of degree 6, $G_{eo}(w)$ and $G_{oo}(w)$ polynomials in w^2 of degree 5, and $G_{oe}(w)$ a polynomial in w of degree 9, all with positive coefficients. It can be checked that because $a \geqslant 3b \geqslant 0$, all coefficients in $P_e(t)$ and $P_o(t)$ are positive. These polynomials are of degree 6, as would be predicted by means of the hive model.

In the most general length 3 cubic case for which $\mu = \nu = \lambda = (a, b, c)$ we do not have any stable limit 3-parameter formula, but we gather together some data in Appendix A that give a striking demonstration of the positivity of coefficients in $P_e(t)$, $P_o(t)$ and $G(w)$.

7. Conclusion

By noting that the Newell–Littlewood coefficients $n_{\mu,\nu}^\lambda$ are nothing other than the Clebsch–Gordan coefficients $m_{\mu,\nu}^\lambda(\mathfrak{g})$ for orthogonal and symplectic Lie algebras $\mathfrak{g}$ of sufficiently high rank, and leaning on the work of De Loera and McAllister [7] it has been established in Proposition 2.1 that stretched Newell–Littlewood coefficients are quasi-polynomial of minimum quasi-period at most 2. Taking into account the evenness or oddness of $|\mu| + |\nu| + |\lambda|$ this has as corollary the validity of two of the items, E(iii) and O(iii), included in the list of items in Conjecture 1.1. In a somewhat similar manner, the prior work of Belkale and Kumar [2] and Sam [29] has led to a proof of the saturation property O(i), but not E(i) for which there is still room for a counterexample to arise.

Unfortunately, attempts to provide proofs of the validity of these and the other remaining items of Conjecture 1.1, are beyond the scope of this particular study. However, thanks to the use of universal characters, a constant term formula has been established in Theorem 3.1 for the generating function, $N_{\mu,\nu}^\lambda(w)$, of the stretched Newell–Littlewood coefficients $n_{t\mu,t\nu}^{t\lambda}$. This has enabled both the generating function itself and the corresponding quasi-polynomials to be evaluated explicitly without the need for any fitting of data. Beyond the confirmation of results both established and conjectured in [7], this has allowed the accumulation of a good deal of evidence in support of the positivity conjectures forming parts E(iv), O(iv), E(v) and O(v) of Conjecture 1.1.

The hive model of Section 4 offers an alternative method of calculating Newell–Littlewood coefficients using Proposition 4.4. This model has led to the proof in Theorem 4.6 that stretched Newell–Littlewood coefficients are the Ehrhart quasi-polynomials associated with a convex polytope specified by means of triangular and rhombus conditions, (36) and (37). This polytope is much simpler to specify than that used for stretched Clebsch–Gordan coefficients [4, 7]. In fact it is much more akin to the Newell–Littlewood polytope introduced by Gao *et al.* [11], differing only in its emphasis on vertex rather than edge labels in the underlying hive. No attempt has been made to establish the precise conditions on μ, ν and λ under which $n_{\mu,\nu}^\lambda$ is non-vanishing, but this has been pursued by Gao *et al.* [12].

Newell–Littlewood polytopes based on composite hives exemplified in (41) are in general rational, rather than integer. One way to see this is as follows. The hive conditions (36) and (37) are such that $\alpha = (0)$ implies $\beta = \mu$ and $\gamma = \nu$, so that the constituents H_μ and H_ν of K as exemplified in (41) are trivial, with the third constituent H^λ coinciding, when turned upside down, with a triangular Littlewood–Richardson hive whose boundary edge labels are specified by the parts of λ, μ and ν. By virtue of (43) it follows that in the case $|\lambda| = |\mu| + |\nu|$, the map from $K \in \mathcal{K}^{(n)}(\mu, \nu; \lambda)$ to $H \in \mathcal{H}^{(n)}(\mu, \nu; \lambda)$ by means of the deletion of H_μ and H_ν, and the inversion of H^λ is an isomorphism. This is of course all in accord with, and indeed implies the validity of (5), but more important is its implication that the polytopes $\mathcal{P}^\lambda_{\mu,\nu}$ appearing in Theorem 4.6 include polytopes $\mathcal{P}$ based on Littlewood–Richardson hives that are known to be rational but not integer. For example, in the case $\lambda = (6, 5, 4, 3, 2, 1)$, $\mu = (5, 4, 3, 2, 1)$ and $\nu = (3, 2, 1)$ the lowest common multiple p of the vertex denominators is 6. Unfortunately, this means that the Theorem, attributed to Stanley [31], but rederived in [1], on the positivity of coefficients in the numerator of $N^\lambda_{\mu,\nu}(w)$ when expressed in a form with denominator $(1 - w^p)^{\dim(\mathcal{P})+1}$, cannot be used to immediately assert the positivity of the coefficients of $G(w)$ in (6) when expressed in a form with denominator $(1 - w)^{d_1}(1 - w^2)^{d_2}$, as would be required to validate the conjectures E(v) and O(v). Nonetheless, these conjectures are supported by all explicit calculations made to date.

The accumulated data has also shown the existence of somewhat unexpected stability phenomena that form the basis of Conjectures 5.1 and 5.2. The latter is reminiscent of the stability property Kronecker of coefficients that led to the concept of reduced Kronecker coefficients [17, 25]. This can be seen by setting $|\mu| = |\nu| = |\lambda| = m$ in Conjecture 5.1. This then yields as a special case

CONJECTURE 7.1. *For any partitions σ, τ and ρ, and positive integer m, there exists $N \in \mathbb{N}$ such that both $n^{(m-|\rho|,\rho)}_{(m-|\sigma|,\sigma),(m-|\tau|,\tau)}$ and $n^{t(m-|\rho|,\rho)}_{t(m-|\sigma|,\sigma),t(m-|\tau|,\tau)}$ are independent of m for all $m > N$.*

The generating function approach by means of Theorem 3.1 offers the possibility of firmly establishing the notion of such m-independent reduced coefficients in the Newell–Littlewood context. The data gathered together on stretched Newell–Littlewood cubes in Section 6 and Appendix A points to the possibility of being able to fix the point at which stability is reached, at least in this case of cubes.

While no direct attack has been made on the analogs of the Fulton conjecture [20] as offered in parts E(ii) and O(ii) of Conjecture 1.1, the data in support of the various positivity conjectures also supports these. In fact, along with E(i) and O(i), they should be rather easy corollaries to the positivity conjectures. It is intended, and indeed hoped that both the methods described here and the accumulated data will serve as an impetus to further work on these problems.

APPENDIX A. CUBES FOR PARTITIONS OF LENGTH THREE

The following data on $n^{t\lambda}_{t\mu,t\nu}$ in the case $\mu = \nu = \lambda = (a, b, c)$ with $a > b > c > 0$ supports the validity of parts E(iv) and O(iv) of Conjecture 1.1 concerning the positivity of the quasi-polynomial coefficients, as well as the stability Conjecture 5.2 arising for all $a \geqslant 3b$. This data has been gathered by evaluating $N^\lambda_{\mu,\nu}(w)$ in the form $G(w)/(1 - w)^{d_1}(1 - w^2)^{d_2}$ through the use of Theorem 3.1, and in so doing checking that in every instance $G(w)$ is a polynomial with positive integer coefficients in accordance with E(v) and O(v) of Conjecture 1.1.

In the simplest case, that is $(a, 2, 1)$ with $a > 2$, the results are given in Table 11.

(a,b,c)	$P_e(t)$ t even / $P_o(t)$ t odd
$(3,2,1)$	$(t+2)(448t^8+5401t^7+28972t^6+91870t^5+191110t^4+272728t^3+272760t^2$ $+\ 183456t+80640)/161280$ $(t+3)(t+1)(448t^7+4505t^6+20410t^5+54659t^4+94984t^3+111035t^2$ $+\ 83454t+33705)/161280$
$(4,2,1)$	$(t+2)(55987t^8+703627t^7+3826570t^6+11870644t^5+23340040t^4$ $+\ 30448384t^3+26725248t^2+15344640t+5160960)/10321920$ 0
$(5,2,1)$	$(t+2)(5441t^8+69758t^7+389318t^6+1243412t^5+2507732t^4+3313136t^3$ $+\ 2898864t^2+1635264t+483840)/967680$ $(t+3)(t+1)(5441t^7+58876t^6+277007t^5+737392t^4+1213967t^3$ $+\ 1260556t^2+793761t+249480)/967680$
$(a,2,1)$ $a \geqslant 6$ even	$(t+2)(10883t^8+139559t^7+779384t^6+2493518t^5+5047520t^4+6700760t^3$ $+\ 5845248t^2+3212928t+967680)/1935360$ 0
$(a,2,1)$ $a \geqslant 7$ odd	$(t+2)(10883t^8+139559t^7+779384t^6+2493518t^5+5047520t^4+6700760t^3$ $+\ 5845248t^2+3212928t+967680)/1935360$ $(t+3)(t+1)(10883t^7+117793t^6+554681t^5+1480183t^4+2449781t^3$ $+\ 2556127t^2+1576527t+446985)/1935360$

TABLE 11. Quasi-polynomials $n^{t\lambda}_{t\mu,t\nu}$ in the case $\mu = \nu = \lambda = (a,2,1)$

For $\mu = \nu = \lambda = (a,b,c)$ with $a > b > c > 0$, the results in the stable cases $a \geqslant 3b$ take the general form:

(59)

(a,b,c) with $a \geqslant 3b$	$N^{(a,b,c)}_{(a,b,c),(a,b,c)}(w)$	$n^{t(a,b,c)}_{t(a,b,c),t(a,b,c)}$
$a+b+c$ even b,c both even	$\dfrac{G_{ee}(w)}{(1-w)^{10}}$	$P_e(t)$ for all t
$a+b+c$ even b,c not both even	$\dfrac{G_{eo}(w)}{(1-w)^6(1-w^2)^4}$	$P_e(t)$ t even $P_o(t)$ t odd
$a+b+c$ odd	$\dfrac{G_{oo}(w)}{(1-w^2)^{10}}$	$P_e(t)$ t even 0 t odd

with explicit expressions for $P_e(t)$ and $P_o(t)$ for all $b \leqslant 6$ given in Table 12.

(a,b,c)	Either $\dfrac{P_e(t)}{P_o(t)}$ or $P_e(t)$ if $P_e(t) = P_o(t)$
$(a,2,1)$	$(t+2)(10883t^8+139559t^7+779384t^6+2493518t^5+5047520t^4+6700760t^3+5845248t^2$ $+3212928t+967680)/1935360$ $(t+3)(t+1)(10883t^7+117793t^6+554681t^5+1480183t^4+2449781t^3+2556127t^2$ $+1576527t+446985)/1935360$
$(a,3,1)$	$(t+2)(7699t^8+88300t^7+421804t^6+1101286t^5+1742992t^4+1741588t^3+1095000t^2+410976t$ $+80640)/161280$ $(t+3)(t+1)(7699t^7+72902t^6+283699t^5+591392t^4+728899t^3+537800t^2+220119t$ $+37170)/161280$
$(a,3,2)$	$(t+1)(190727t^8+1525816t^7+5461022t^6+11404708t^5+15266888t^4+13518064t^3+7967808t^2$ $+3036672t+645120)/645120$ $(t+1)^3(190727t^6+1144362t^5+2981571t^4+4297204t^3+3690909t^2$ $+1839042t+452025)/645120$
$(a,4,1)$	$(t+2)(26011t^8+286576t^7+1288150t^6+3076144t^5+4302184t^4+3661564t^3+1884120t^2$ $+552096t+80640)/161280$ $(t+3)(t+1)(26011t^7+234554t^6+845053t^5+1568570t^4+1645033t^3+980090t^2$ $+308319t+37170)/161280$
$(a,4,2)$	$(t+1)(43532t^8+279118t^7+779384t^6+1246759t^5+1261880t^4+837595t^3+365328t^2$ $+100404t+15120)/15120$
$(a,4,3)$	$(1860257t^9+13261743t^8+42512874t^7+80444826t^6+99261036t^5+83191752t^4+48076336t^3$ $+19027584t^2+4891392t+645120)/645120$ $(t+1)^2(1860257t^7+9541229t^6+21570159t^5+27763279t^4+22164319t^3$ $+11099835t^2+3289617t+446985)/645120$
$(a,5,1)$	$(t+2)(61387t^8+661900t^7+2876944t^6+6523006t^5+8442784t^4+6438628t^3+2854680t^2$ $+693216t+80640)/161280$ $(t+3)(t+1)(61387t^7+539126t^6+1860079t^5+3219200t^4+3031759t^3+1539560t^2$ $+396519t+37170)/161280$
$(a,5,2)$	$(19719167t^9+137161701t^8+413898366t^7+714090510t^6+781934748t^5+568830024t^4$ $+278307184t^3+90083520t^2+18224640t+1935360)/1935360$ $(t+1)(19719167t^8+117442534t^7+296455832t^6+417634678t^5+364300070t^4$ $+204529954t^3+73658160t^2+16079490t+1696275)/1935360$
$(a,5,3)$	$(14374271t^9+78323328t^8+192117936t^7+278601120t^6+264107886t^5+170634492t^4$ $+76100344t^3+23213520t^2+4609728t+483840)/483840$ $(t+1)(14374271t^8+63949057t^7+128168879t^6+150432241t^5+113675645t^4$ $+56958847t^3+18294777t^2+3308463t+238140))/483840$
$(a,5,4)$	$(9374843t^9+57962079t^8+159266238t^7+255773490t^6+265478220t^5+185644536t^4$ $+88964272t^3+29037120t^2+6100992t+645120)/645120$ $(t+1)^2(9374843t^7+39212393t^6+71466609t^5+73627879t^4+46755853t^3+18504951t^2$ $+4298247t+452025)/645120$
$(a,6,1)$	$(t+2)(119371t^8+1269712t^7+5401126t^6+11832472t^5+14509528t^4+10193740t^3+4006680t^2$ $+834336t+80640)/161280$ $(t+3)(t+1)(119371t^7+1030970t^6+3458557t^5+5707586t^4+4968457t^3+2216210t^2$ $+484719t+37170))/161280$
$(a,6,2)$	$(t+1)(61592t^8+353200t^7+843608t^6+1101286t^5+871496t^4+435397t^3+136875t^2$ $+25686t+2520)/2520$
$(a,6,3)$	$(3t+2)(2644569t^8+11304279t^7+21043368t^6+22441662t^5+15142560t^4+6700760t^3$ $+1948416t^2+356992t+35840)/71680$ $(3t+1)(t+1)(2644569t^7+9541233t^6+14976387t^5+13321647t^4+7349343t^3$ $+2556127t^2+525509t+49665)/71680$
$(a,6,4)$	$(2t+1)(381454t^8+1525816t^7+2730511t^6+2851177t^5+1908361t^4+844879t^3+248994t^2$ $+47448t+5040)/5040$
$(a,6,5)$	$(33064469t^9+185217615t^8+455264850t^7+646494282t^6+587226444t^5+356073480t^4$ $+146931760t^3+41033088t^2+7310592t+645120)/645120$ $(t+1)^2(33064469t^7+119088677t^6+184023027t^5+159359551t^4+84484315t^3+27745299t^2$ $+5296797t+446985)/645120$

TABLE 12. Stable quasi-polynomials $n_{t\mu,t\nu}^{t\lambda}$ in the case $\mu = \nu = \lambda = (a,b,c)$ for all $a \geqslant 3b$ and $6 \geqslant b > c > 0$

References

[1] Matthias Beck and Frank Sottile, *Irrational proofs for three theorems of Stanley*, European J. Combin. **28** (2007), no. 1, 403–409.

[2] Prakash Belkale and Shrawan Kumar, *Eigencone, saturation and Horn problems for symplectic and odd orthogonal groups*, J. Algebraic Geom. **19** (2010), no. 2, 199–242.

[3] Arkady Berenstein and Andrei Zelevinsky, *Triple multiplicities for* sl($r + 1$) *and the spectrum of the exterior algebra of the adjoint representation*, J. Algebraic Combin. **1** (1992), no. 1, 7–22.

[4] _____, *Tensor product multiplicities, canonical bases and totally positive varieties*, Invent. Math. **143** (2001), no. 1, 77–128.

[5] G. R. E. Black, Ronald C. King, and Brian G. Wybourne, *Kronecker products for compact semisimple Lie groups*, J. Phys. A **16** (1983), no. 8, 1555–1589.

[6] Anders Skovsted Buch, *The saturation conjecture (after A. Knutson and T. Tao). With an appendix by William Fulton*, Enseign. Math. (2) **46** (2000), no. 1-2, 43–60.

[7] Jesús A. De Loera and Tyrrell B. McAllister, *On the computation of Clebsch–Gordan coefficients and the dilation effect*, Experiment. Math. **15** (2006), no. 1, 7–19.

[8] Eugène Ehrhart, *Polynômes arithmétiques et méthode des polyèdres en combinatoire*, International Series of Numerical Mathematics, vol. 35, Birkhäuser Verlag, Basel-Stuttgart, 1977.

[9] Bertfried Fauser, Peter D. Jarvis, and Ronald C. King, *The Hopf algebra structure of the character rings of classical groups*, J. Phys. A **46** (2013), no. 3, Paper no. 035205 (30 pages).

[10] William Fulton and Joe Harris, *Representation theory. A first course*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.

[11] Shiliang Gao, Gidon Orelowitz, and Alexander Yong, *Newell–Littlewood numbers*, Trans. Amer. Math. Soc. **374** (2021), no. 9, 6331–6366.

[12] _____, *Newell–Littlewood numbers II: extended Horn inequalities*, Algebr. Comb. **5** (2022), no. 6, 1287–1297.

[13] Heekyoung Hahn, *On classical groups detected by the triple tensor product and the Littlewood–Richardson semigroup*, Res. Number Theory **2** (2016), Paper no. 19 (12 pages).

[14] Michael Kapovich and John J. Millson, *A path model for geodesics in Euclidean buildings and its applications to representation theory*, Groups Geom. Dyn. **2** (2008), no. 3, 405–480.

[15] Ronald C. King, *Modification rules and products of irreducible representations of the unitary, orthogonal, and symplectic groups*, J. Mathematical Phys. **12** (1971), 1588–1598.

[16] _____, *S-functions and characters of Lie algebras and superalgebras*, in Invariant theory and tableaux (Minneapolis, MN, 1988), IMA Vol. Math. Appl., vol. 19, Springer, New York, 1990, pp. 226–261.

[17] _____, *Some remarks on characters of symmetric groups, Schur functions, Littlewood–Richardson and Kronecker coefficients*, 2009, Mathematical Foundations of Quantum Information Workshop. Sevilla, 23-27 de noviembre de 2009, `http://congreso.us.es/enredo2009/Workshop_files/Sevilla_King.pdf`.

[18] Ronald C. King, Christophe Tollu, and Frédéric Toumazet, *The hive model and the polynomial nature of stretched Littlewood–Richardson coefficients*, Sém. Lothar. Combin. **54A** (2005/07), Paper no. B54Ad (19 pages).

[19] Allen Knutson and Terence Tao, *The honeycomb model of* GL$_n$(**C**) *tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090.

[20] Allen Knutson, Terence Tao, and Christopher Woodward, *The honeycomb model of* GL$_n$(ℂ) *tensor products. II. Puzzles determine facets of the Littlewood–Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48.

[21] Kazuhiko Koike and Itaru Terada, *Young diagrammatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank*, Adv. Math. **79** (1990), no. 1, 104–135.

[22] Dudley E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford University Press, New York, 1940.

[23] _____, *Products and plethysms of characters with orthogonal, symplectic and symmetric groups*, Canadian J. Math. **10** (1958), 17–32.

[24] Dudley E. Littlewood and Archibald R. Richardson, *Group characters and algebra*, Phil. Trans. Roy. Soc. London A **233** (1934), 99–141.

[25] F. D. Murnaghan, *The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group*, Amer. J. Math. **60** (1938), no. 3, 761–784.

[26] Martin J. Newell, *Modification rules for the orthogonal and symplectic groups*, Proc. Roy. Irish Acad. Sect. A **54** (1951), 153–163.

[27] _____, *On the representations of the orthogonal and symplectic groups*, Proc. Roy. Irish Acad. Sect. A **54** (1951), 143–152.

[28] Etienne Rassart, *A polynomiality property for Littlewood-Richardson coefficients*, J. Combin. Theory Ser. A **107** (2004), no. 2, 161–179.

[29] Steven V. Sam, *Symmetric quivers, invariant theory, and saturation theorems for the classical groups*, Adv. Math. **229** (2012), no. 2, 1104–1135.

[30] SCHUR6.09, *Interactive Program For Calculating Properties Of Symmetric Functions*, 2014, http://sourceforge.net/projects/schur.

[31] Richard P. Stanley, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342.

[32] _____, *Enumerative Combinatorics. Vol. 1. With a foreword by Gian-Carlo Rota*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997.

[33] Itaru Terada, Ronald C. King, and OOlga Azenhas, *The symmetry of Littlewood-Richardson coefficients: a new hive model involutory bijection*, SIAM J. Discrete Math. **32** (2018), no. 4, 2850–2899.

[34] Guoce Xin, *A fast algorithm for MacMahon's partition analysis*, Electron. J. Combin. **11** (2004), Paper no. 58 (20 pages).

Ronald C. King, Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, UK
E-mail : r.c.king@soton.ac.uk