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# A construction of pairs of non-commutative rank 8 association schemes from non-symmetric rank 3 association schemes

Akihide Hanaki & Masayoshi Yoshikawa

**ABSTRACT** We construct a pair of non-commutative rank 8 association schemes from a rank 3 non-symmetric association scheme. For the pair, two association schemes have the same character table but different Frobenius–Schur indicators. This situation is similar to that for the dihedral and quaternion groups of order 8. We also determine the structures of adjacency algebras of the rank 8 schemes over the rational number field.

## 1. INTRODUCTION

From a rank 3 non-symmetric association scheme of order  $n - 1$ , we construct a pair of association schemes  $(\mathcal{D}, \mathcal{Q})$  with the following properties.

- $\mathcal{D}$  and  $\mathcal{Q}$  are non-commutative, of order  $4n$ , and of rank 8.
- $\mathcal{D}$  and  $\mathcal{Q}$  have the same character tables but their Frobenius–Schur indicators are different.

These properties are similar to the pair of the dihedral group  $D_8$  and the quaternion group  $Q_8$  of order 8.

In the theory of association schemes, the adjacency algebras are mainly considered over the complex number field. In this paper, we determine the structures of adjacency algebras of  $\mathcal{D}$  and  $\mathcal{Q}$  over the rational number field  $\mathbb{Q}$ . We prove

$$\begin{aligned}\mathbb{Q}\mathcal{D} &\cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q}), \\ \mathbb{Q}\mathcal{Q} &\cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}(-1, -n+1),\end{aligned}$$

where  $M_2(\mathbb{Q})$  is the full matrix algebra of degree 2 and  $\mathbb{Q}(-1, -n+1)$  is the quaternion division algebra.

It is known that a rank 3 non-symmetric association scheme of order  $n - 1$  exists if and only if there exists a skew-Hadamard matrix of order  $n$  with  $n \equiv 0 \pmod{4}$ . There is a conjecture that such a matrix of order  $n$  exists for an arbitrary  $n \equiv 0 \pmod{4}$ .

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**KEYWORDS.** Character table, quaternion algebras, association schemes.

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2. PRELIMINARIES

For a field  $K$ , we denote by  $M_n(K)$  the full matrix algebra of degree  $n$  over  $K$ . For a matrix  $M$ , the transposed matrix of  $M$  will be denoted by  ${}^tM$ . By  $I_n$ , we denote the identity matrix of degree  $n$ . By  $J_n$ , we denote the  $n \times n$  matrix with all entries 1.

We define an association scheme in matrix form. Let  $\mathcal{S} = \{A_0, \dots, A_d\}$  be a set of non-zero  $n \times n$  matrices with entries in  $\{0, 1\}$ . Then the set  $\mathcal{S}$  is called an *association scheme of order  $n$  and rank  $d + 1$*  if

- (1)  $A_0 = I_n$ ,
- (2)  $\sum_{i=0}^d A_i = J_n$ , and
- (3) for any  $0 \leq i, j \leq d$ ,  ${}^tA_i$  and  $A_iA_j$  are linear combinations of  $A_0, \dots, A_d$ .

By definition, all rows of  $A_i$  contain the same number of 1. We call this number the *valency* of  $A_i$  and denote it by  $n_i$ . The association scheme  $\mathcal{S}$  is said to be *symmetric* if  ${}^tA_i = A_i$  for all  $0 \leq i \leq d$  and *non-symmetric* otherwise. The association scheme  $\mathcal{S}$  is said to be *commutative* if  $A_iA_j = A_jA_i$  for all  $0 \leq i, j \leq d$  and *non-commutative* otherwise. For a field  $K$ ,  $K\mathcal{S} := \bigoplus_{i=0}^d KA_i$  is a  $(d + 1)$ -dimensional  $K$ -algebra by the condition (3). We call  $K\mathcal{S}$  the *adjacency algebra* of  $\mathcal{S}$  over  $K$ . It is known that  $K\mathcal{S}$  is semisimple if the characteristic of  $K$  is 0 [5, Theorem 4.1.3 (ii)]. A *representation* of  $\mathcal{S}$  over  $K$  means a  $K$ -algebra homomorphism from  $K\mathcal{S}$  to a full matrix algebra  $M_t(K)$  for some positive integer  $t$ .

A subset  $\mathcal{T}$  of an association scheme  $\mathcal{S}$  is called a *closed subset* if  $e_{\mathcal{T}} := n_{\mathcal{T}}^{-1} \sum_{A_i \in \mathcal{T}} A_i$  is an idempotent, where  $n_{\mathcal{T}} := \sum_{A_i \in \mathcal{T}} n_i$ . A closed subset  $\mathcal{T}$  of  $\mathcal{S}$  is said to be *normal* if  $e_{\mathcal{T}}$  is a central element of the complex adjacency algebra  $\mathbb{C}\mathcal{S}$ . A closed subset  $\mathcal{T}$  of  $\mathcal{S}$  is said to be *strongly normal* if  ${}^tA_jA_iA_j \in \bigoplus_{A_\ell \in \mathcal{T}} \mathbb{C}A_\ell$  for  $A_i \in \mathcal{T}$  and  $A_j \in \mathcal{S}$ . It is known that a strongly normal closed subset is normal. The *thin radical* of  $\mathcal{S}$  is  $O_\theta(\mathcal{S}) = \{A_i \mid n_i = 1\}$  and is known to be a closed subset of  $\mathcal{S}$ . The *thin residue*  $O^\theta(\mathcal{S})$  of  $\mathcal{S}$  is the intersection of all strongly normal closed subsets of  $\mathcal{S}$  and is known to be a strongly normal closed subset of  $\mathcal{S}$ . For details, the reader is referred to [5] or [2].

Now we consider the adjacency algebra over the complex number field  $\mathbb{C}$ . By Wedderburn’s theorem [4, 3.5 Theorem],  $\mathbb{C}\mathcal{S} \cong \bigoplus_{i=1}^\ell M_{t_i}(\mathbb{C})$  for some  $t_1, \dots, t_\ell$ . The set of projections  $\mathbb{C}\mathcal{S} \rightarrow M_{t_i}(\mathbb{C})$  ( $i = 1, \dots, \ell$ ) is a complete set of representatives of equivalence classes of irreducible representations of  $\mathbb{C}\mathcal{S}$ . The matrix trace of a representation is called the *character* of the representation. By  $\text{Irr}(\mathcal{S}) = \{\chi_1, \dots, \chi_\ell\}$ , we denote the set of irreducible characters of  $\mathbb{C}\mathcal{S}$ . The  $\ell \times (d + 1)$  matrix  $(\chi_i(A_j))$  is called the *character table* of  $\mathcal{S}$ . Since  $\mathbb{C}\mathcal{S}$  is defined as a matrix algebra, the map  $A_i \mapsto A_i$  is a representation. We call it the *standard representation* and its character the *standard character* of  $\mathcal{S}$ . The decomposition of the standard character into the irreducibles is written as  $\sum_{\chi \in \text{Irr}(\mathcal{S})} m_\chi \chi$ . We call the coefficient  $m_\chi$  the *multiplicity* of  $\chi$ .

The *Frobenius–Schur indicator*  $\nu(\chi)$  of  $\chi \in \text{Irr}(\mathcal{S})$  is defined by

$$\nu(\chi) := \frac{m_\chi}{n \chi(A_0)} \sum_{i=0}^d \frac{1}{n_i} \chi(A_i^2).$$

An irreducible character is said to be of the *first kind* if it is afforded by a real representation, of the *second kind* if it is afforded by a representation which is equivalent to its complex conjugate but is not of the first kind, and of the *third kind* if it is not of the first or second kind. The following theorem is known.

THEOREM 2.1 ([3, (7.5), (7.6)]).

(1) For  $\chi \in \text{Irr}(\mathcal{S})$ ,

$$\nu(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is of the first kind,} \\ -1 & \text{if } \chi \text{ is of the second kind,} \\ 0 & \text{if } \chi \text{ is of the third kind.} \end{cases}$$

(2)  $\sum_{\chi \in \text{Irr}(\mathcal{S})} \nu(\chi)\chi(A_0) = \#\{A_i \in \mathcal{S} \mid {}^t A_i = A_i\}$ .

### 3. CONSTRUCTION

Let  $\{A_0 = I_{n-1}, A_1, A_2 = {}^t A_1\}$  be a non-symmetric rank 3 association scheme of order  $n - 1$ . Remark that  $n \equiv 0 \pmod{4}$  for such an association scheme to exist. We will construct a pair  $(\mathcal{D}, \mathcal{Q})$  of association schemes with the properties described in the Introduction.

Set  $a := n - 1$  and  $b := (n - 2)/2$ .

The following lemma is well-known.

LEMMA 3.1.

- (1)  $A_1^2 = \frac{b-1}{2}A_1 + \frac{b+1}{2}A_2$ .
- (2)  $A_2^2 = \frac{b+1}{2}A_1 + \frac{b-1}{2}A_2$ .
- (3)  $A_1A_2 = A_2A_1 = bA_0 + \frac{b-1}{2}A_1 + \frac{b-1}{2}A_2$ .

Set  $x := (1, 2, 3, 4)$ ,  $y := (1, 2)(3, 4)$ , permutations of degree 4, and  $G := \langle x, y \rangle \cong D_8$ . We identify the elements of  $G$  with the corresponding permutation matrices, namely

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and so on. The next lemma is clear by definition.

LEMMA 3.2. As matrices,  $1 + x^2 = xy + x^3y$ .

We keep the above notations in the rest of this article.

3.1. THE ASSOCIATION SCHEME  $\mathcal{D}$ . We define  $n \times n$  matrices by

$$E := \left( \begin{array}{c|ccc} 0 & 1 & \dots & 1 \\ \hline 1 & & & \\ \vdots & & & \\ \hline & & O & \\ \hline 1 & & & \end{array} \right), \quad \tilde{A}_i := \left( \begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ \hline & & A_i & \\ \hline 0 & & & \end{array} \right) \quad (i = 1, 2).$$

By Lemma 3.1 and direct calculations, we have the following lemma.

LEMMA 3.3.

- (1)  $E^2 + \tilde{A}_1\tilde{A}_2 + \tilde{A}_2\tilde{A}_1 = aI_n + b\tilde{A}_1 + b\tilde{A}_2$ .
- (2)  $\tilde{A}_1^2 + \tilde{A}_2^2 = b\tilde{A}_1 + b\tilde{A}_2$ .
- (3)  $E\tilde{A}_1 + \tilde{A}_2E = E\tilde{A}_2 + \tilde{A}_1E = bE$ .

We consider a subgroup  $H := C_G(y) = \{1, x^2, y, x^2y\} \cong C_2 \times C_2$  of  $G$ . It is easy to see that  $\sum_{h \in H} h = \sum_{g \in G \setminus H} g = J_4$ . We define  $4n \times 4n$  matrices by

$$\sigma_h := I_n \otimes h \quad \text{for } h \in H$$

and

$$\begin{aligned} \mu_g &:= E \otimes g + \tilde{A}_1 \otimes gy + \tilde{A}_2 \otimes gx^2y \\ &= (I_n \otimes g)(E \otimes 1 + \tilde{A}_1 \otimes y + \tilde{A}_2 \otimes x^2y) \\ &= (E \otimes 1 + \tilde{A}_1 \otimes x^2y + \tilde{A}_2 \otimes y)(I_n \otimes g) \quad \text{for } g \in G \setminus H. \end{aligned}$$

We will show that  $\mathcal{D} := \{\sigma_h \mid h \in H\} \cup \{\mu_g \mid g \in G \setminus H\}$  forms an association scheme.

LEMMA 3.4. *The set  $\mathcal{D}$  is closed under the transposition and*

$$\sum_{h \in H} \sigma_h + \sum_{g \in G \setminus H} \mu_g = J_{4n}.$$

*Proof.* We have  ${}^t\sigma_h = \sigma_h$  for  $h \in H$ ,  ${}^t\mu_{xy} = \mu_{xy}$ ,  ${}^t\mu_{x^2y} = \mu_{x^2y}$  and  ${}^t\mu_x = \mu_{x^3}$ . Thus  $\mathcal{D}$  is closed under the transposition.

Since  $\sum_{h \in H} h = \sum_{g \in G \setminus H} g = J_4$  and  $E + \tilde{A}_1 + \tilde{A}_2 = J_n - I_n$ , we have  $\sum_{h \in H} \sigma_h + \sum_{g \in G \setminus H} \mu_g = J_{4n}$ . □

LEMMA 3.5.

- (1)  $\sigma_h \sigma_{h'} = \sigma_{hh'}$  for  $h, h' \in H$ .
- (2)  $\sigma_h \mu_g = \mu_{hg}$  and  $\mu_g \sigma_h = \mu_{gh}$  for  $h \in H$  and  $g \in G \setminus H$ .
- (3)  $\mu_g \mu_{g'} = a\sigma_{gg'} + b\mu_{gg'x} + b\mu_{gg'x^3}$  for  $g, g' \in G \setminus H$ .

*Proof.* It is easy to show that (1) and (2) hold.

Suppose  $g, g' \in G \setminus H$ . Remark that  $gg' \in H$ . By  $g'x^2 = x^2g'$ ,  $yg' = g'x^2y$ , Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} \mu_g \mu_{g'} &= (I_n \otimes g)(E \otimes 1 + \tilde{A}_1 \otimes y + \tilde{A}_2 \otimes x^2y)(E \otimes 1 + \tilde{A}_1 \otimes x^2y + \tilde{A}_2 \otimes y)(I_n \otimes g') \\ &= (I_n \otimes g)(aI_n \otimes 1 + b(\tilde{A}_1 + \tilde{A}_2) \otimes (1 + x^2) + bE \otimes (y + x^2y))(I_n \otimes g') \\ &= a\sigma_{gg'} + b(\tilde{A}_1 + \tilde{A}_2) \otimes g(1 + x^2)g' + bE \otimes g(y + x^2y)g' \\ &= a\sigma_{gg'} + b(\tilde{A}_1 + \tilde{A}_2) \otimes gg'(1 + x^2) + bE \otimes gg'x^2(y + x^2y) \\ &= a\sigma_{gg'} + b(\tilde{A}_1 + \tilde{A}_2) \otimes gg'(xy + x^3y) + bE \otimes gg'x^2y(xy + x^3y) \\ &= a\sigma_{gg'} + b(\tilde{A}_1 + \tilde{A}_2) \otimes gg'(xy + x^3y) + bE \otimes gg'(x + x^3) \\ &= a\sigma_{gg'} + b\mu_{gg'x} + b\mu_{gg'x^3}. \end{aligned}$$

Now (3) holds. □

THEOREM 3.6. *The set  $\mathcal{D} = \{\sigma_h \mid h \in H\} \cup \{\mu_g \mid g \in G \setminus H\}$  forms an association scheme.*

*Proof.* This is clear by Lemma 3.4 and Lemma 3.5. □

We summarize basic properties of the association scheme  $\mathcal{D}$  by Lemma 3.4 and Lemma 3.5.

PROPOSITION 3.7. *For the association scheme  $\mathcal{D} = \{\sigma_1, \sigma_{x^2}, \sigma_y, \sigma_{x^2y}, \mu_x, \mu_{x^3}, \mu_{xy}, \mu_{x^3y}\}$ , the following properties hold.*

- (1) *The valencies of  $\sigma_1, \sigma_{x^2}, \sigma_y, \sigma_{x^2y}$  are 1 and the valencies of  $\mu_x, \mu_{x^3}, \mu_{xy}, \mu_{x^3y}$  are  $n - 1$ .*
- (2) *The thin radical is  $O_\theta(\mathcal{D}) = \{\sigma_1, \sigma_{x^2}, \sigma_y, \sigma_{x^2y}\}$ , and is normal in  $\mathcal{D}$ .*
- (3) *The thin residue is  $O^\theta(\mathcal{D}) = \{\sigma_1, \sigma_{x^2}, \mu_x, \mu_{x^3}\}$ .*
- (4) *The matrices  $\sigma_1, \sigma_{x^2}, \sigma_y, \sigma_{x^2y}, \mu_{xy}, \mu_{x^3y}$  are symmetric and  $\mu_x, \mu_{x^3}$  are non-symmetric.*

3.2. THE ASSOCIATION SCHEME  $\mathcal{Q}$ . We define  $n \times n$  matrices by

$$\hat{A}_1 := \left( \begin{array}{c|ccc} 0 & 1 & \dots & 1 \\ \hline 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{array} \right), \quad \hat{A}_2 := \left( \begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 1 & & & \\ \vdots & & A_2 & \\ 1 & & & \end{array} \right).$$

By Lemma 3.1 and direct calculations, we have the following lemma.

LEMMA 3.8.

- (1)  $\hat{A}_1^2 + \hat{A}_2^2 = b(J_n - I_n)$ .
- (2)  $\hat{A}_1\hat{A}_2 + \hat{A}_2\hat{A}_1 = bJ_n + (a - b)I_n$ .

We consider a subgroup  $K := C_G(x) = \{1, x, x^2, x^3\} \cong C_4$  of  $G$ . It is easy to see that  $\sum_{k \in K} k = \sum_{g \in G \setminus K} g = J_4$ . We define  $4n \times 4n$  matrices by

$$\sigma_k := I_n \otimes k \quad \text{for } k \in K$$

and

$$\begin{aligned} \tau_g &:= \hat{A}_1 \otimes g + \hat{A}_2 \otimes gx^2 \\ &= (I_n \otimes g)(\hat{A}_1 \otimes 1 + \hat{A}_2 \otimes x^2) \\ &= (\hat{A}_1 \otimes 1 + \hat{A}_2 \otimes x^2)(I_n \otimes g) \quad \text{for } g \in G \setminus K. \end{aligned}$$

We will show that  $\mathcal{Q} := \{\sigma_k \mid k \in K\} \cup \{\tau_g \mid g \in G \setminus K\}$  forms an association scheme.

LEMMA 3.9. *The set  $\mathcal{Q}$  is closed under the transposition and  $\sum_{k \in K} \sigma_k + \sum_{g \in G \setminus K} \tau_g = J_{4n}$ .*

*Proof.* We have  ${}^t\sigma_1 = \sigma_1$ ,  ${}^t\sigma_{x^2} = \sigma_{x^2}$ ,  ${}^t\sigma_x = \sigma_{x^3}$ ,  ${}^t\tau_y = \tau_{x^2y}$ , and  ${}^t\tau_{xy} = \tau_{x^3y}$ . Thus  $\mathcal{D}$  is closed under the transposition.

Since  $\sum_{k \in K} k = \sum_{g \in G \setminus K} g = J_4$  and  $I_n + \hat{A}_1 + \hat{A}_2 = J_n$ , we have  $\sum_{k \in K} \sigma_k + \sum_{g \in G \setminus K} \tau_g = J_{4n}$ .  $\square$

LEMMA 3.10.

- (1)  $\sigma_k \sigma_{k'} = \sigma_{kk'}$  for  $k, k' \in K$ .
- (2)  $\sigma_k \tau_g = \tau_{kg}$  and  $\tau_g \sigma_k = \tau_{gk}$  for  $k \in K$  and  $g \in G \setminus K$ .
- (3)  $\tau_g \tau_{g'} = a\sigma_{gg'x^2} + b\tau_{gg'xy} + b\tau_{gg'x^3y}$  for  $g, g' \in G \setminus K$ .

*Proof.* It is easy to show that (1) and (2) hold.

Suppose  $g, g' \in G \setminus K$ . Remark that  $gg' \in K$ . By Lemma 3.2 and Lemma 3.8, we have

$$\begin{aligned} \tau_g \tau_{g'} &= (I_n \otimes g)(\hat{A}_1 \otimes 1 + \hat{A}_2 \otimes x^2)^2(I_n \otimes g') \\ &= (I_n \otimes gg')(\hat{A}_1^2 \otimes 1 + (\hat{A}_1\hat{A}_2 + \hat{A}_2\hat{A}_1) \otimes x^2 + \hat{A}_1^2 \otimes 1) \\ &= (I_n \otimes gg')(b(J_n - I_n) \otimes 1 + (bJ_n + (a - b)I_n) \otimes x^2) \\ &= (I_n \otimes gg')(aI_n \otimes x^2 + b(J_n - I_n) \otimes (1 + x^2)) \\ &= (I_n \otimes gg')(aI_n \otimes x^2 + b(\hat{A}_1 + \hat{A}_2) \otimes (1 + x^2)) \\ &= (I_n \otimes gg')(aI_n \otimes x^2 + b(\hat{A}_1 + \hat{A}_2) \otimes (xy + x^3y)) \\ &= a\sigma_{gg'x^2} + b\tau_{gg'xy} + b\tau_{gg'x^3y}. \end{aligned}$$

Now (3) holds.  $\square$

THEOREM 3.11. *The set  $\mathcal{Q} = \{\sigma_k \mid k \in K\} \cup \{\tau_g \mid g \in G \setminus K\}$  forms an association scheme.*

*Proof.* This is clear by Lemma 3.9 and Lemma 3.10. □

We summarize basic properties of the association scheme  $\mathcal{Q}$  by Lemma 3.9 and Lemma 3.10.

PROPOSITION 3.12. *For the association scheme  $\mathcal{Q} = \{\sigma_1, \sigma_{x^2}, \sigma_x, \sigma_{x^3}, \tau_{xy}, \tau_{x^3y}, \tau_y, \tau_{x^2y}\}$ , the following properties hold.*

- (1) *The valencies of  $\sigma_1, \sigma_{x^2}, \sigma_x, \sigma_{x^3}$  are 1 and the valencies of  $\tau_{xy}, \tau_{x^3y}, \tau_y, \tau_{x^2y}$  are  $n - 1$ .*
- (2) *The thin radical is  $O_\theta(\mathcal{Q}) = \{\sigma_1, \sigma_{x^2}, \sigma_x, \sigma_{x^3}\}$ , and is normal in  $\mathcal{Q}$ .*
- (3) *The thin residue is  $O^\theta(\mathcal{Q}) = \{\sigma_1, \sigma_{x^2}, \tau_{xy}, \tau_{x^3y}\}$ .*
- (4) *The matrices  $\sigma_1, \sigma_{x^2}$  are symmetric and  $\sigma_x, \sigma_{x^3}, \tau_{xy}, \tau_{x^3y}, \tau_y, \tau_{x^2y}$  are non-symmetric.*

#### 4. THE CHARACTER TABLES OF $\mathcal{D}$ AND $\mathcal{Q}$

In this section, we will determine the character tables of  $\mathcal{D}$  and  $\mathcal{Q}$ . Consequently, we can see that the tables are the same. Moreover, we will show that their Frobenius–Schur indicators are different.

PROPOSITION 4.1. *The character table of  $\mathcal{D}$  is*

	$\sigma_1$	$\sigma_{x^2}$	$\sigma_y$	$\sigma_{x^2y}$	$\mu_x$	$\mu_{x^3}$	$\mu_{xy}$	$\mu_{x^3y}$	$m_{\chi_i}$
$\chi_1$	1	1	1	1	$n - 1$	$n - 1$	$n - 1$	$n - 1$	1
$\chi_2$	1	1	-1	-1	$n - 1$	$n - 1$	$-n + 1$	$-n + 1$	1
$\chi_3$	1	1	1	1	-1	-1	-1	-1	$n - 1$
$\chi_4$	1	1	-1	-1	-1	-1	1	1	$n - 1$
$\chi_5$	2	-2	0	0	0	0	0	0	$n$

*The Frobenius–Schur indicators are  $\nu(\chi_i) = 1$  ( $i = 1, 2, 3, 4, 5$ ).*

*Proof.* By Proposition 3.7 (2), the thin radical  $O_\theta(\mathcal{D}) = \{\sigma_1, \sigma_{x^2}, \sigma_y, \sigma_{x^2y}\}$  is a normal closed subset of  $\mathcal{D}$ . By [1, Theorem 3.5], we can determine  $\chi_1$  and  $\chi_3$ . By Proposition 3.7 (3), the thin residue  $O^\theta(\mathcal{D}) = \{\sigma_1, \sigma_{x^2}, \mu_x, \mu_{x^3}\}$  is a strongly normal closed subset of  $\mathcal{D}$  and  $\chi_2$  is determined. By [2, Theorem 3.5],  $\chi_4 := \chi_2\chi_3$  is an irreducible character. Now, by  $\sum_{i=1}^5 m_{\chi_i}\chi_i(\rho) = 0$  for  $\sigma_1 \neq \rho \in \mathcal{D}$  [5, Chap. 4], we can determine  $\chi_5$ .

Since  $\chi_i$  ( $i = 1, 2, 3, 4$ ) are rational characters of degree 1, they are of the first kind. There are 6 symmetric matrices by Proposition 3.7 (4). By Theorem 2.1 (2),

$$6 = \sum_{i=1}^5 \nu(\chi_i)\chi_i(1) = 1 + 1 + 1 + 1 + 2\nu(\chi_5)$$

and we have  $\nu(\chi_5) = 1$ . □

PROPOSITION 4.2. *The character table of  $\mathcal{Q}$  is*

	$\sigma_1$	$\sigma_{x^2}$	$\sigma_x$	$\sigma_{x^3}$	$\tau_{xy}$	$\tau_{x^3y}$	$\tau_y$	$\tau_{x^2y}$	$m_{\varphi_i}$
$\varphi_1$	1	1	1	1	$n - 1$	$n - 1$	$n - 1$	$n - 1$	1
$\varphi_2$	1	1	-1	-1	$n - 1$	$n - 1$	$-n + 1$	$-n + 1$	1
$\varphi_3$	1	1	1	1	-1	-1	-1	-1	$n - 1$
$\varphi_4$	1	1	-1	-1	-1	-1	1	1	$n - 1$
$\varphi_5$	2	-2	0	0	0	0	0	0	$n$

*The Frobenius–Schur indicators are  $\nu(\varphi_i) = 1$  ( $i = 1, 2, 3, 4$ ) and  $\nu(\varphi_5) = -1$ .*

*Proof.* By Proposition 3.12 (2), the thin radical  $O_\theta(\mathcal{Q}) = \{\sigma_1, \sigma_{x^2}, \sigma_x, \sigma_{x^3}\}$  is a normal closed subset of  $\mathcal{Q}$ . We can determine  $\varphi_1$  and  $\varphi_3$ . By Proposition 3.12 (3), the thin residue  $O^\theta(\mathcal{Q}) = \{\sigma_1, \sigma_{x^2}, \tau_{xy}, \tau_{x^3y}\}$  is a strongly normal closed subset of  $\mathcal{Q}$  and  $\varphi_2$  is determined. Now the character table can be calculated in the same way as in Proposition 4.1.

Since  $\varphi_i$  ( $i = 1, 2, 3, 4$ ) are rational characters of degree 1, they are of the first kind. There are 2 symmetric matrices by Proposition 3.12 (4). By Theorem 2.1 (2),

$$2 = \sum_{i=1}^5 \nu(\varphi_i)\varphi_i(1) = 1 + 1 + 1 + 1 + 2\nu(\varphi_5)$$

and we have  $\nu(\varphi_5) = -1$ . □

### 5. IRREDUCIBLE REPRESENTATIONS AND RATIONAL ADJACENCY ALGEBRAS

We will determine irreducible representations and the structures of rational adjacency algebras of  $\mathcal{D}$  and  $\mathcal{Q}$ , respectively.

PROPOSITION 5.1. *The map  $T : \mathcal{D} \rightarrow M_2(\mathbb{C})$  defined by*

$$\sigma_{x^2} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mu_x \mapsto \begin{pmatrix} 0 & -1 \\ n-1 & 0 \end{pmatrix}$$

*is an irreducible representation of  $\mathcal{D}$  affording  $\chi_5$ .*

*Proof.* We can check all products in Lemma 3.5. □

Similarly, we have the following proposition.

PROPOSITION 5.2. *The map  $T' : \mathcal{Q} \rightarrow M_2(\mathbb{C})$  defined by*

$$\sigma_x \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \tau_y \mapsto \begin{pmatrix} 0 & -1 \\ n-1 & 0 \end{pmatrix}$$

*is an irreducible representation of  $\mathcal{Q}$  affording  $\varphi_5$ .*

Now we can determine the structures of rational adjacency algebras. To describe the result, we define (generalized) quaternion algebras [4, 1.6]. For a field  $F$  and  $r, s \in F \setminus \{0\}$ , a *quaternion algebra*  $F(r, s)$  is a four dimensional  $F$ -algebra with basis 1,  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  with the products

$$\mathbf{i}^2 = r, \quad \mathbf{j}^2 = s, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

If  $r$  and  $s$  are negative rational numbers, then  $\mathbb{Q}(r, s)$  is a division algebra.

PROPOSITION 5.3. *As  $\mathbb{Q}$ -algebras, we have the following isomorphisms.*

- (1)  $\mathbb{Q}\mathcal{D} \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q})$ .
- (2)  $\mathbb{Q}\mathcal{Q} \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}(-1, -n+1)$ . ( $\mathbb{Q}(-1, -n+1)$  is a division algebra.)

*Proof.* Since  $\chi_i$  ( $i = 1, 2, 3, 4$ ) have rational values on  $\mathcal{D}$ , we can determine  $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$ . For the representation  $T$  in Proposition 5.1, it is easy to see that  $T(\mathbb{Q}\mathcal{D}) = M_2(\mathbb{Q})$ . Thus (1) holds.

Since  $\varphi_i$  ( $i = 1, 2, 3, 4$ ) have rational values on  $\mathcal{Q}$ , we can determine  $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$ . For the representation  $T'$  in Proposition 5.2, the set consisting of

$$\begin{aligned} T'(\sigma_1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & T'(\sigma_x) &= \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \\ T'(\tau_y) &= \begin{pmatrix} 0 & -1 \\ n-1 & 0 \end{pmatrix}, & T'(\tau_{xy}) &= \begin{pmatrix} 0 & -\sqrt{-1} \\ -(n-1)\sqrt{-1} & 0 \end{pmatrix} \end{aligned}$$



is a  $\mathbb{Q}$ -basis of  $T'(\mathbb{Q}\mathcal{Q})$ . We can see that

$$\begin{aligned} T'(\sigma_x)^2 &= -T'(\sigma_1), & T'(\tau_y)^2 &= -(n-1)T'(\sigma_1), \\ T'(\sigma_x)T'(\tau_y) &= -T'(\tau_y)T'(\sigma_x) &= T'(\tau_{xy}). \end{aligned}$$

This shows that  $T'(\mathbb{Q}\mathcal{Q}) \cong \mathbb{Q}(-1, -n+1)$  and (2) holds.  $\square$

## 6. REMARK

Our pair  $(\mathcal{D}, \mathcal{Q})$  has similar properties to the pair  $(D_8, Q_8)$ , the dihedral and quaternion groups of order 8. Our association schemes have order  $4n$  and  $n \equiv 0 \pmod{4}$ , and thus  $(D_8, Q_8)$  is not obtained by our construction. If we set  $n = 2$  and  $A_1 = A_2 = O$  and apply our construction, then we can construct the pair  $(D_8, Q_8)$  and all arguments are valid for it.

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