ALGEBRAIC COMBINATORICS

Bob Lutz

Higher discrete homotopy groups of graphs

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ABSTRACT This paper studies a discrete homotopy theory for graphs introduced by Barcelo et al. We prove two main results. First we show that if G is a graph containing no 3- or 4-cycles, then the nth discrete homotopy group $A_n(G)$ is trivial for all $n \geqslant 2$. Second we exhibit for each $n \geqslant 1$ a natural homomorphism $\psi : A_n(G) \to \mathcal{H}_n(G)$, where $\mathcal{H}_n(G)$ is the nth discrete cubical singular homology group, and an infinite family of graphs G for which $\mathcal{H}_n(G)$ is nontrivial and ψ is surjective. It follows that for each $n \geqslant 1$ there are graphs G for which $A_n(G)$ is nontrivial.

1. INTRODUCTION

In [6] a new homotopy theory for simplicial complexes was introduced, motivated by a search for qualitative invariants in the study of complex systems and their dynamics [7]. Given a simplicial complex K, an integer $0 \leqslant q \leqslant \dim K$ and a simplex $\sigma_0 \in K$ of dimension at least q, one defines a family of groups

$$(1) \qquad A_n^q(K, \sigma_0), \quad n \geqslant 1,$$

called the *discrete homotopy groups* of K. In contrast to classical homotopy theory, the groups (1) are defined combinatorially.

For $n = 1$, the group (1) is called the *discrete fundamental group*, and is well understood. There is a discrete analog of the Seifert-van Kampen theorem, and in fact one can construct a cell complex X such that

$$(2) \qquad A_1^q(K, \sigma_0) \cong \pi_1(X, x_0),$$

where $\pi_1(X, x_0)$ is the classical fundamental group for some $x_0 \in X$. Results like these have enabled computations of $A_1^q(K, \sigma_0)$ for interesting simplicial complexes, including Coxeter complexes of finite Coxeter groups [8, 9].

For $n > 1$, the situation is less clear. The basic tools for computing classical higher homotopy groups, already a difficult problem in general, have no known discrete analogs. In [1] a higher-dimensional version of (2) was obtained, assuming a "plausible" cubical analog of the simplicial approximation theorem. Despite its plausibility, however, the proposed approximation theorem has resisted all attempts at a proof.

This paper studies the problem of computing higher discrete homotopy groups. To begin, we reduce this problem to considering only graphs instead of general simplicial

complexes. Given a graph G and a vertex $v_0 \in G$, one can define a family of groups $A_n(G, v_0)$ similarly to (1). For any triple (K, q, σ_0) as above, there is a pair (G, v_0) such that

$$A_n^q(K, \sigma_0) \cong A_n(G, v_0).$$

Namely, G is the graph whose vertices correspond to the maximal simplices of K of dimension at least q, and whose edges correspond to the pairs of such simplices sharing a q-face. We can therefore restrict our attention to the groups $A_n(G, v_0)$ without loss of generality. In particular, we will consider only connected graphs, so we can ignore the base vertex and simply write $A_n(G) = A_n(G, v_0)$.

We prove two main results. The first is motivated by a singular homology theory for graphs, introduced in [3] as a companion to the discrete homotopy theory described above. The relevant groups $\mathcal{H}_n(G)$, called the *discrete singular cubical homology groups* of G, are defined combinatorially. It can be shown that if G contains no 3- or 4-cycles, then $\mathcal{H}_n(G)$ is trivial for all $n \geqslant 2$. This was the main result of [5]. We prove a similar theorem for discrete homotopy.

THEOREM 1.1. *If G contains no 3- or 4-cycles, then $A_n(G)$ is trivial for all $n \geqslant 2$.*

If G contains no 3- or 4-cycles, then it can be shown that $A_1(G) \cong \pi_1(G)$, where we regard G as both a graph and a topological space. Thus Theorem 1.1 gives a complete picture of the discrete homotopy of such graphs.

Our second result makes progress toward a discrete Hurewicz theorem. Recall that the classical Hurewicz theorem [12] describes, for any path-connected topological space X, a map from $\pi_n(X)$ to the nth homology group of X. If $n = 1$, then this map is surjective, and its kernel is the commutator subgroup $[\pi_1(X), \pi_1(X)]$. If $n > 1$ and $\pi_k(X)$ is trivial for all $k < n$, then the map is an isomorphism.

In [3] it was shown that there is a surjective map

$$(3) \qquad\qquad A_1(G) \to \mathcal{H}_1(G)$$

with kernel $[A_1(G), A_1(G)]$, giving a discrete Hurewicz theorem in dimension 1. We generalize the map (3) to higher dimensions and show that it is surjective in a large number of cases.

THEOREM 1.2. *For any graph G and positive integer n, there is a natural homomorphism*

$$(4) \qquad\qquad \psi : A_n(G) \to \mathcal{H}_n(G).$$

For each $n \geqslant 1$, there is an infinite family of graphs G for which ψ is surjective and $\mathcal{H}_n(G)$ is nontrivial.

An important consequence of Theorem 1.2 is the existence of graphs G with nontrivial groups $A_n(G)$ for $n \geqslant 2$. To date, no examples of such graphs have appeared in the literature.

COROLLARY 1.3. *For each $n \geqslant 1$ there are graphs G for which $A_n(G)$ is nontrivial.*

The paper is organized as follows. In Section 2 we give preliminary definitions and results concerning discrete homotopy theory. In Section 3 we prove Theorem 1.1. In Section 4 we describe the basic constructions needed for Theorem 1.2. In Section 5 we recall the definition of discrete cubical singular homology groups and prove Theorem 1.2. In Section 6 we suggest directions for future work.

2. Basic definitions

2.1. Discrete homotopy. In this paper, the term *homotopy* refers to the discrete homotopy theory for graphs introduced in [6]. By a *graph* we will mean one that is simple, connected and undirected. We write $v \in G$ if v is a vertex of a graph G. If G_1 and G_2 are graphs, then we write $f : G_1 \to G_2$ when f is a function from the vertex set of G_1 to the vertex set of G_2. If two vertices $u, v \in G$ are adjacent or equal, then we write $u \simeq v$.

DEFINITION 2.1. *A function $f : G_1 \to G_2$ is a* graph map *if $f(u) \simeq f(v)$ whenever $u \simeq v$.*

Let $\mathbb{Z}$ denote the graph with vertex set $\{\ldots, -1, 0, 1, \ldots\}$ and an edge ij if and only if $|i - j| = 1$. Given an integer $m \geqslant 0$, we write I_m for the subgraph of $\mathbb{Z}$ induced by $\{0, \ldots, m\}$. We let $G_1 \times G_2$ denote the Cartesian product of graphs G_1 and G_2. The following notion of homotopy defines an equivalence relation on graph maps $G_1 \to G_2$.

DEFINITION 2.2. *Two graph maps $f, g : G_1 \to G_2$ are* homotopic *if for some m there is a graph map $h : G_1 \times I_m \to G_2$ such that $h(-, 0) = f$ and $h(-, m) = g$. The map h is a* homotopy *from f to g. We write $h_i = h(-, i)$ for all i, so $h_0 = f$ and $h_m = g$.*

Let $\mathbb{Z}^n$ denote the n-fold Cartesian product of $\mathbb{Z}$, whose vertices are n-tuples of integers. For any $r \geqslant 0$, let $\mathbb{Z}^n_{\geqslant r} \subseteq \mathbb{Z}^n$ be the subgraph induced by all vertices $x \in \mathbb{Z}^n$ with $|x_i| \geqslant r$ for some i.

Given a function $f : G_1 \to G_2$, and subgraphs $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$, we write

$$(5) \qquad\qquad f : (G_1, H_1) \to (G_2, H_2)$$

if $v \in H_1$ implies that $f(v) \in H_2$. We will abuse this notation by writing

$$(6) \qquad\qquad f : (\mathbb{Z}^n, \partial\mathbb{Z}^n) \to (G, v_0)$$

if there exists an $r \geqslant 0$ such that f is a function $(\mathbb{Z}^n, \mathbb{Z}^n_{\geqslant r}) \to (G, v_0)$. The minimum such r is called the *radius* of f. We emphasize that $\partial\mathbb{Z}^n$ is not defined by itself, but is shorthand for some $\mathbb{Z}^n_{\geqslant r}$. The following notion of homotopy, more restrictive than the last, defines an equivalence relation on graph maps $(\mathbb{Z}^n, \partial\mathbb{Z}^n) \to (G, v_0)$.

DEFINITION 2.3. *Two graph maps $f, g : (\mathbb{Z}^n, \partial\mathbb{Z}^n) \to (G, v_0)$, are* based homotopic *if there is a homotopy $h : \mathbb{Z}^n \times I_m \to G$ from f to g such that h_i is a graph map $(\mathbb{Z}^n, \partial\mathbb{Z}^n) \to (G, v_0)$ for all i. The function h is a* based homotopy *from f to g.*

DEFINITION 2.4. *Fix $v_0 \in G$. Let $A_n(G)$ denote the set of based homotopy classes $[f]$ of graph maps $f : (\mathbb{Z}^n, \partial\mathbb{Z}^n) \to (G, v_0)$. We endow $A_n(G)$ with a group structure as follows. If f and g are graph maps $(\mathbb{Z}^n, \partial\mathbb{Z}^n) \to (G, v_0)$ of radii r_f and r_g, respectively, then the product $[f] \cdot [g]$ in $A_n(G)$ is the based homotopy class of the map $p : (\mathbb{Z}^n, \partial\mathbb{Z}^n) \to (G, v_0)$ given by*

$$(7) \qquad p(x_1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, x_n) & \text{if } x_1 \leqslant r_f, \\ g(x_1 - (r_f + r_g), x_2, \ldots, x_n) & \text{if } x_1 > r_f. \end{cases}$$

The identity of this operation is the based homotopy class of the constant map $\mathbb{Z}^n \to v_0$. We call $A_n(G)$ the nth discrete homotopy group *of G.*

REMARK 2.5. In [6], working in the more general setting of simplicial complexes, it is shown that the operation in $A_n(G)$ is well defined, that it satisfies the group axioms, and that the definition of $A_n(G)$ is independent of the base vertex v_0 when G is connected. It is also shown that the groups $A_n(G)$ are abelian for all $n \geqslant 2$. We will not prove these facts here.
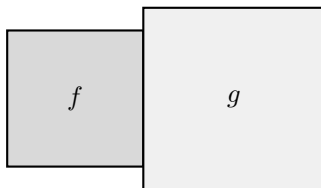
Figure 1. Illustrating the group operation in $A_2(G)$.

Proposition 2.6 ([6, Proposition 5.12]). *Let G be a graph, and let X_G be the cell complex obtained by regarding G as a 1-complex and attaching a 2-cell along the boundary of each 3- and 4-cycle. We have $A_1(G) \cong \pi_1(X_G)$.*

Example 2.7. Let G be a cycle graph of length m, i.e. $G = \mathbb{Z}_m$. If $m = 3$ or 4, then X_G is simply connected, so $A_1(G)$ is trivial by Proposition 2.6. If $m \geqslant 5$, then $X_G \approx S^1$, so $A_1(G) \cong \mathbb{Z}$. Label the vertices of G in a cycle as $v_0, \ldots, v_{m-1}$. An explicit generator of $A_1(G)$ is $[f]$, where f is given by $f(i) = v_i$ if $0 \leqslant i < m$ and $f(i) = v_0$ otherwise.

More generally, suppose that G contains no 3- or 4-cycles. For each cycle Z of G we obtain a generator $[f_Z]$ of $A_1(Z)$ as above. A set of generators of $A_1(G)$, although not necessarily a minimal one, is the set of $[f_Z]$ for all Z.

Example 2.8. Let K be a finite simplicial 2-complex. Suppose, for any vertices i, j and k, that if the edges ij, ik and jk all belong to K, then so does the triangle ijk. Suppose also that the 1-skeleton K^1 is a *chordal* graph, i.e. that if a cycle of G is an induced subgraph, then it contains exactly 3 vertices. These conditions ensure that if $G = K^1$, then $X_G = K$ in the notation of Proposition 2.6, so $A_1(G) \cong \pi_1(K)$. For example, if K is the triangulation of the real projective plane in Figure 2, then $A_1(G) \cong \mathbb{Z}/2\mathbb{Z}$. This example generalizes the remark following [6, Proposition 5.12].
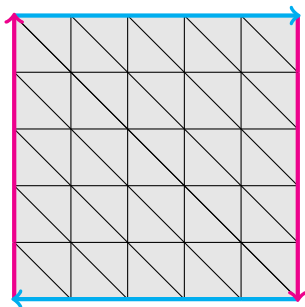
Figure 2. A triangulation of the projective plane.

2.2. Contractibility. We define a notion of a contractible graph as a discrete analog of a contractible topological space. We show that all discrete homotopy groups of a contractible graph are trivial. This fact will be used in proving the main results.

Definition 2.9. *A contraction of G is a homotopy from the identity map $G \to G$ to a constant map. A graph G is* contractible *if it admits a contraction.*

By our definition, infinite graphs are not contractible. We caution that this notion of contractibility does not imply that $\pi_1(G)$ is trivial when G is regarded as a topological space. For instance, the next example shows that the square graph $\mathbb{Z}_4$ is contractible, but $\pi_1(\mathbb{Z}_4) \cong \mathbb{Z}$.

EXAMPLE 2.10 (Grid graphs). A *grid graph* is any subgraph L of $\mathbb{Z}^n$ induced by a vertex set of the form

$$\{a_1, \ldots, b_1\} \times \cdots \times \{a_n, \ldots, b_n\}$$

for some $a_i, b_i \in \mathbb{Z}$ with $a_i \leqslant b_i$ for all i. The *boundary* of L is the subgraph ∂L induced by all vertices $x \in L$ with $x_i \in \{a_i, b_i\}$ for some i. Its complement $L^\circ = L \smallsetminus \partial L$ is the *interior* of L. Let $m = \max_i b_i - a_i$, and let $c : L \times I_m \to L$ be given by

$$c(v, i) = (\max\{v_1 - i, 0\}, \ldots, \max\{v_n - i, 0\})$$

for all $(v, i) \in L \times I_m$. It is routine to check that c is a homotopy. In particular, c_0 is the identity map $L \to L$, and c_m is the constant map $L \to (0, \ldots, 0)$. Hence L is contractible.

EXAMPLE 2.11 (Finite trees). Suppose that T is a finite tree, and let $v_0 \in T$. For every $v \in T \smallsetminus v_0$, let $\rho(v)$ denote the unique neighbor of v that lies on the simple path from v_0 to v. Set $\rho(v_0) = v_0$. This defines a graph map $\rho : G \to G$. Let m be the diameter of G. We define a function $c : G \times I_m \to G$ by setting

$$c(v, i) = \begin{cases} v & \text{if } i = 0 \\ \rho(c(v, i - 1)) & \text{if } 1 \leqslant i \leqslant m \end{cases}$$

for all $v \in T$. It is routine to check that c is a homotopy. In particular, c_0 is the identity map $T \to T$ and $c_m = v_0$, so G is contractible.

PROPOSITION 2.12. *If G is contractible, then $A_n(G)$ is trivial for all n.*

Proof. Let $c : G \times I_m \to G$ be a contraction of G to a vertex v_0, and let $f : (\mathbb{Z}^n, \partial \mathbb{Z}^n) \to (G, v_0)$ be a graph map of radius r. If $c(v_0, -)$ is constant, then there is an obvious based homotopy $\mathbb{Z}^n \times I_m \to G$ from f to the constant map v_0, given by

$$(8) \qquad\qquad\qquad (x, i) \mapsto c(f(x), i)$$

for all $(x, i) \in \mathbb{Z}^n \times I_m$. However, it need not be the case that $c(v_0, -)$ is constant. In general, the map (8) is a homotopy but not a based homotopy, and hence does not preserve the equivalence class $[f] \in A_n(G)$.

To get around this, we define a based homotopy $h : \mathbb{Z}^n \times I_{2m} \to G$ from f to v_0 in two steps. We first define h_i for $i = 0, \ldots, m$, and then for $i = m + 1, \ldots, 2m$. Let $\mathbb{Z}^n_{\leqslant r} \subseteq \mathbb{Z}^n$ denote the grid graph

$$\mathbb{Z}^n_{\leqslant r} = \{-r, \ldots, r\} \times \cdots \times \{-r, \ldots, r\}.$$

Let $B_0 = \partial \mathbb{Z}^n_{\leqslant r}$. For $k \geqslant 1$, let B_k denote the boundary of the grid graph whose interior is $\mathbb{Z}^n_{\leqslant r} \cup B_0 \cup \cdots \cup B_{k-1}$. Note that $\{\mathbb{Z}^n_{\leqslant r}, B_1, B_2, \ldots\}$ partitions $\mathbb{Z}^n$. Let $h_0 = f$. For $i = 1, \ldots, m$ let $h_i(x) = c(f(x), i)$ for all $x \in \mathbb{Z}^n_{\leqslant r}$, and let h_i be constant on B_k for each $k \geqslant 1$, taking the same value that h_{i-1} takes on B_{k-1}. The result is that h_m is constant on $\mathbb{Z}^n_{\leqslant r}$ and each B_k.
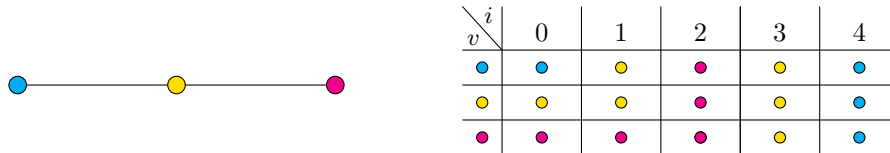

FIGURE 3. A graph G and a table of values of a contraction $c : G \times I_4 \to G$.

For example, let G be the graph on the left side of Figure 3, whose vertices are labeled by colors. Let $c : G \times I_4 \to v_0$ be the contraction of G with values $c(v, i)$ given

in the table on the right side of Figure 3, so that $m = 4$. We can represent a graph map $\mathbb{Z}^2 \to G$ as an infinite 2-dimensional array of colored dots, where the color of the dot at $x \in \mathbb{Z}^2$ corresponds to the value of the map at x. Several such maps appear in Figure 4, where we have restricted our attention to the domain $\mathbb{Z}^2_{\leqslant 5}$. We assume that each of these maps takes the color blue everywhere outside the pictured region. Let f be the map on the top left of Figure 4, and note that $r = 2$ is the radius of f. Thus $\mathbb{Z}^n_{\leqslant r}$ is outlined by dashes. The figure illustrates the maps h_i from above for $i = 0, \ldots, 4$.
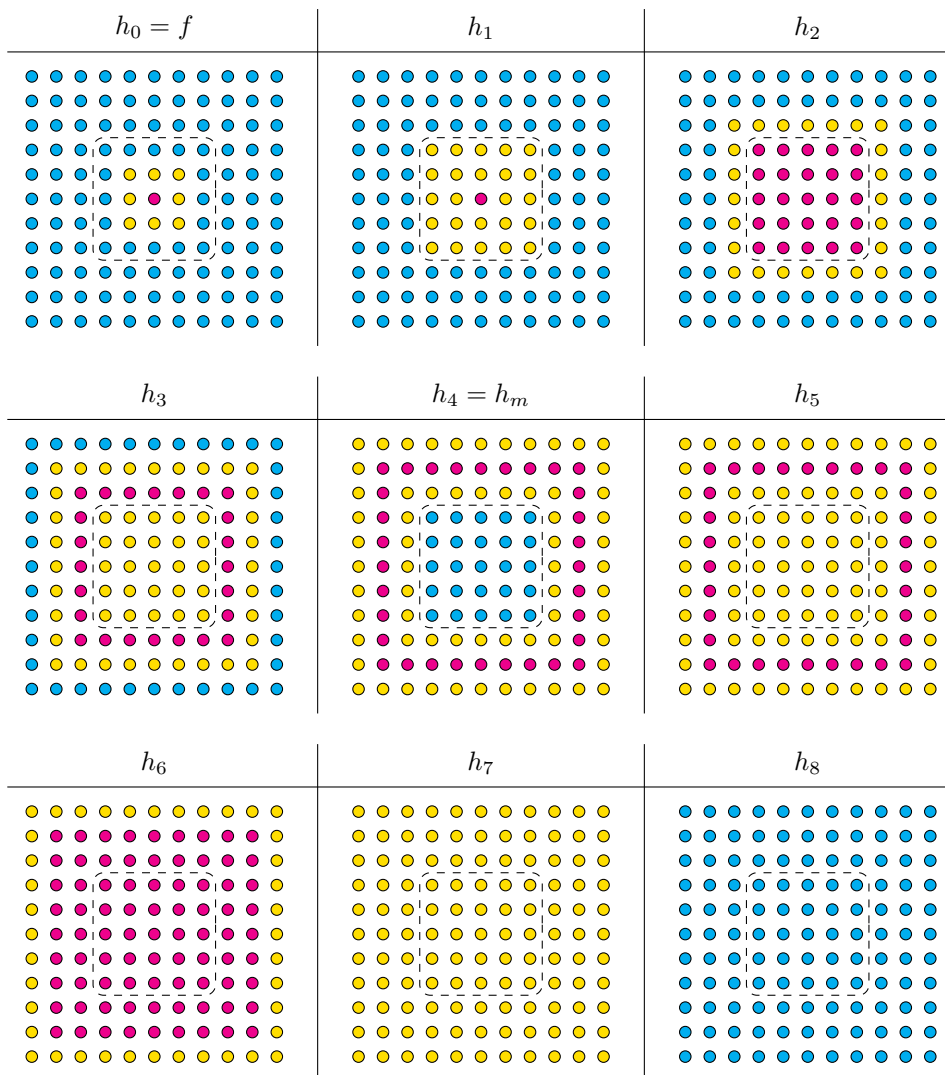
FIGURE 4. The stages of a homotopy $h : \mathbb{Z}^2 \times I_8 \to G$.

Returning to the general case, we define h_i for the remaining values of i. Suppose that $i \in \{m+1, \ldots, 2m\}$. Let h_i be constant on $\mathbb{Z}^n_{\leqslant r} \cup B_{m+1} \cup \cdots \cup B_{i-m}$, taking the same value that h_m takes on B_{i-m}. For $k > i-m$, let h_i equal h_m on B_k. Continuing the example from Figure 3, these maps h_i are illustrated in Figure 4 for $i = 5, \ldots, 8$. In the general case, it is routine to verify that $h : \mathbb{Z}^n \times I_{2m} \to G$ is a based homotopy; in particular, we have $h_0 = f$ and $h_{2m} = v_0$. Hence $[f] = 0$ in $A_n(G)$, as desired. $\square$

If there exists a contraction c of G to v_0 such that $c(v_0, -)$ is constant, then we say that G *deformation retracts* onto v_0. In classical topology, there are examples of contractible spaces that do not deformation retract onto a point (see e.g. [11, Chapter 0, Exercise 6] and [2, Section 2]). However, these examples rely on behaviors that have no analogs in the discrete theory. We are led to ask the following.

QUESTION 2.13. *Suppose that G is contractible. Does G deformation retract onto a vertex?*

3. TRIANGLE- AND SQUARE-FREE GRAPHS

We prove Theorem 1.1. If G contains no 3- or 4-cycles, then $A_1(G) \cong \pi_1(G)$ by Proposition 2.6, so we obtain a complete description of the discrete homotopy groups in this case.

DEFINITION 3.1. *A path in G is a sequence $(p_1, \ldots, p_k)$ of vertices of G such that $p_i \simeq p_{i+1}$ for all $i = 1, \ldots, k - 1$.*

Proof of Theorem 1.1. Suppose that G contains no 3- or 4-cycles, and set $n \geqslant 2$. Let $f : (\mathbb{Z}^n, \partial \mathbb{Z}^n) \to (G, v_0)$ be a graph map of radius r. We show that $[f] = 0$ in $A_n(G)$. For this proof only, consider G as directed with edge set E, so that if $(u, v) \in E$, then $(v, u) \notin E$. Let F_E be the free group on E, whose elements are the identity 1 and reduced words in the letters e and e^{-1} for all $e \in E$.

We define a function τ from the set of paths in $\mathbb{Z}^n$ of length at least 2 to F_E as follows. Given adjacent vertices x ans y of $\mathbb{Z}^n$, define an element $\tau(x, y)$ of F_E by

$$\tau(x, y) = \begin{cases} (f(x), f(y)) & \text{if } (f(x), f(y)) \in E, \\ (f(y), f(x))^{-1} & \text{if } (f(y), f(x)) \in E, \\ 1 & \text{if } f(x) = f(y). \end{cases}$$

Given a path $P = (p_0, \ldots, p_\ell)$ in $\mathbb{Z}^n$ with $\ell > 1$, let

$$\tau(P) = \tau(p_0, p_1) \tau(p_1, p_2) \cdots \tau(p_{\ell-1}, p_\ell).$$

Let $g : \mathbb{Z}^n \to F_E$ be given by

$$(9) \qquad\qquad\qquad g(x) = \tau(P),$$

where P is any path in $\mathbb{Z}^n$ beginning in $\mathbb{Z}^n_{\geqslant r}$ and ending with x. For example, consider the graph G and the graph map $f : (\mathbb{Z}^2, \partial \mathbb{Z}^2) \to (G, a)$ illustrated in Figure 5, where $r = 4$ and $\mathbb{Z}^n_{\geqslant r}$ consists of all vertices outside the dashed line. Let $x = (0, -2)$, so that $f(x) = c$. The three highlighted paths from $\mathbb{Z}^n_{\geqslant r}$ to x give three ways to compute $g(x)$, all resulting in the same value. Using the green path, clearly $g(x) = (a, b)(b, c)$. For the blue path, we have

$$g(x) = (a, b)(b, c)(c, d)(c, d)^{-1}(b, c)^{-1}(b, c) = (a, b)(b, c).$$

A similar computation with the red path gives the same result. Returning to the general case, it is not clear a priori that g is well defined. We will briefly assume that g is well defined in order to finish the proof. We will then prove our assumption.

Assume that g is well defined, and consider the Cayley graph $\Gamma(F_E, E)$ of F_E with generating set E. Let G_E be the subgraph of $\Gamma(F_E, E)$ induced by

$$\{\tau(P) : P \text{ is a path in } \mathbb{Z}^n \text{ beginning in } \mathbb{Z}^n_{\geqslant r}\}.$$

Note that G_E is a finite tree, since it is a finite connected subgraph of the tree F_E. The distance between any two vertices u and v in G_E is given by $|u^{-1}v|$, where if $w \in F_E$, then $|w|$ denotes the length of the reduced word representing w. If $x \simeq y$ in $\mathbb{Z}^n$, then $|g(x)^{-1}g(y)| = |\tau(x, y)| \leqslant 1$, so $g(x) \simeq g(y)$ in G_E. Hence $g : (\mathbb{Z}^n, \partial \mathbb{Z}^n) \to (G_E, 1)$ is a
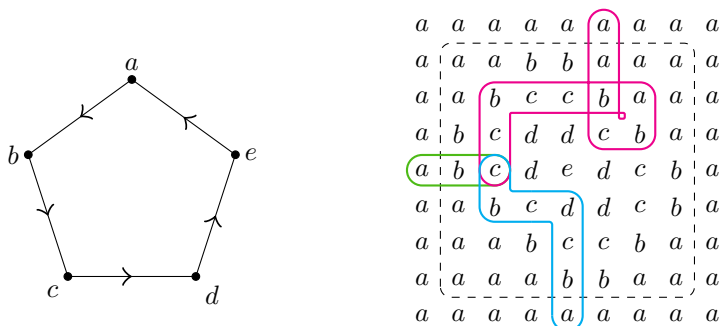
FIGURE 5. A graph $G = \mathbb{Z}_5$ and a graph map $f : (\mathbb{Z}^2, \partial \mathbb{Z}^2) \to (G, a)$.

graph map. Since finite trees are contractible by Example 2.11, Proposition 2.12 gives a based homotopy h from g to the constant map 1. Every vertex of G_E corresponds to a path in G from v_0 to some vertex; denote this vertex by $\pi(v)$. Clearly $\pi : (G_E, 1) \to (G, v_0)$ is a graph map and $\pi \circ g = f$, so the composition $\pi \circ h$ is a based homotopy from f to the constant map v_0. Hence $[f]$ is the identity element in $A_n(G)$, proving that $A_n(G)$ is trivial.

We now prove that g is well defined. Let $P = (p_0, \ldots, p_\ell)$ be a path in $\mathbb{Z}^n$. Suppose for some $0 < j < \ell$ that the vertices p_{j-1}, p_j and p_{j+1} are three corners of a square in $\mathbb{Z}^n$. Let q_j denote the fourth corner of this square. Consider the operation of replacing the term p_j in P with q_j. We call this operation a *corner swap*. Next suppose that $p_{k-1} = p_{k+1}$ for some $0 < k < \ell$. Consider the operation of removing the terms p_{k-1} and p_k from P. We call this operation a *backtrack deletion*.

We show that $\tau(P)$ is invariant under corner swaps and backtrack deletions. Let p_j and q_j be as above. The vertices p_{j-1}, p_j, p_{j+1} and q_j form a square in $\mathbb{Z}^n$. Since f is a graph map and G contains no squares, the image of this square under f is a path in G consisting of at most 3 vertices. The nontrivial possibilities are illustrated on the right side of Figure 6 up to relabeling. If the top case holds, then by using the definition of τ, we see that

$$\tau(p_{j-1}, p_j, p_{j+1}) = \tau(p_j, p_{j+1}) = \tau(p_j, q_j, p_{j+1}) = \tau(p_{j-1}, q_j, p_{j+1}).$$

If the middle case holds, then

$$\tau(p_{j-1}, p_j, p_{j+1}) = 1 = \tau(p_{j-1}, q_j, p_{j+1}).$$

If the bottom case holds, then $\tau(p_{j-1}, p_j, p_{j+1}) = \tau(p_{j-1}, q_j, p_{j+1})$ immediately. Hence in any case, we have

$$\begin{aligned}
\tau(P) &= \tau(p_0, \ldots, p_{j-1})\tau(p_{j-1}, p_j, p_{j+1})\tau(p_{j+1}, \ldots, p_\ell) \\
&= \tau(p_0, \ldots, p_{j-1})\tau(p_{j-1}, q_j, p_{j+1})\tau(p_{j+1}, \ldots, p_\ell) \\
&= \tau(p_0, \ldots, p_{j-1}, q_j, p_{j+1}, \ldots, p_\ell),
\end{aligned}$$

proving that $\tau(P)$ is invariant under corner swaps. To prove that $\tau(P)$ is invariant under backtrack deletions, let p_k be as in the previous paragraph, so that $p_{k-1} = p_{k+1}$. We have $\tau(p_{k-1}, p_k, p_{k+1}) = 1$, so

$$\begin{aligned}
\tau(P) &= \tau(p_0, \ldots, p_{k-1})\tau(p_{k-1}, p_k, p_{k+1})\tau(p_{k+1}, \ldots, p_\ell) \\
&= \tau(p_0, \ldots, p_{k-1})\tau(p_{k+1}, \ldots, p_\ell) \\
&= \tau(p_0, \ldots, p_{k-2}, p_{k+1})\tau(p_{k+1}, \ldots, p_\ell) \\
&= \tau(p_0, \ldots, p_{k-2}, p_{k+1}, \ldots, p_\ell),
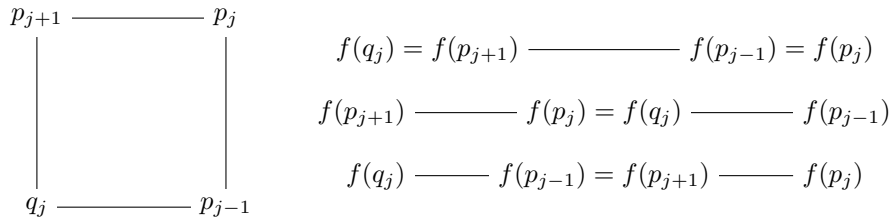\end{aligned}$$

as desired.

$$p_{j+1} \quad\text{------}\quad p_j$$

(figure)

FIGURE 6. A square in $\mathbb{Z}^n$ and three possibilities for its image under f.

If $P = (p_0, \ldots, p_\ell)$ is a closed path in $\mathbb{Z}^n$, then it is not hard to see that P can be transformed into the path (p_0, p_1, p_0) by a sequence of corner swaps and backtrack deletions. Since $\tau(P)$ is invariant under these operations, we must have

$$\tau(P) = \tau(p_0, p_1, p_0) = 1$$

whenever P is closed. Let $x \in \mathbb{Z}^n$, and let $p, q \in \mathbb{Z}^n_{\geqslant r}$. Let P be a path in $\mathbb{Z}^n$ from p to x and Q a path from q to x. Let R be a path from q to p with all vertices in $\mathbb{Z}^n_{\geqslant r}$. Since f is constant on $\mathbb{Z}^n_{\geqslant r}$, we have $\tau(R) = 1$. Write $PQ^{-1}R$ for the path obtained by concatenating P, the reverse of Q and R. Since $PQ^{-1}R$ is closed, we have $\tau(PQ^{-1}R) = 1$. Thus

$$1 = \tau(PQ^{-1}R) = \tau(P)\tau(Q)^{-1}\tau(R) = \tau(P)\tau(Q)^{-1},$$

giving $\tau(P) = \tau(Q)$. It follows that if a path begins in $\mathbb{Z}^n_{\geqslant r}$, its value under τ depends only on its endpoint. In other words, the function g from (9) is well defined. $\square$

4. Cones and suspensions

We define discrete analogs of the cone and suspension functors from classical topology. This material plays a crucial role in the proof of Theorem 1.2.

DEFINITION 4.1. *For $s \geqslant 1$, let $C_s G$ denote the graph obtained from $G \times I_s$ by contracting the subgraph $G \times \{0\}$. We call $C_s G$ a* cone *on G.*

PROPOSITION 4.2. *The cone $C_s G$ on G is contractible for any $s \geqslant 1$.*

Proof. Write $CG = C_s G$. For each $v \in G$ and $i = 1, \ldots, s$, there is a corresponding vertex v_i of CG contained in $G \times \{i\}$. We let v_0 denote the lone vertex in the image of $G \times \{0\}$ in CG. Thus the vertices of CG are indexed by v_i for $v \in G$ and $i = 0, \ldots, n$, with the understanding that $v_0 = w_0$ for all vertices v and w of G. This labeling scheme is illustrated in Figure 7. Let $h : CG \times I_s$ be given by

$$c(v_i, j) = \begin{cases} v_{s-j} & \text{if } i > s - j, \\ v_i & \text{if } i \leqslant s - j. \end{cases}$$

It is routine to check that c is a graph map. In addition, $c(-, 0)$ is the identity map on CG and $c(-, s) = v_0$. Hence c is a contraction of CG. $\square$

DEFINITION 4.3. *For $t \geqslant 2$, let $S_t G$ be the graph obtained from $G \times I_t$ by contracting $G \times \{0\}$ and $G \times \{t\}$ to single vertices. We call $S_t G$ a* suspension *of G.*

PROPOSITION 4.4. *If $t = 2$ or G is contractible, then $S_t G$ is contractible.*
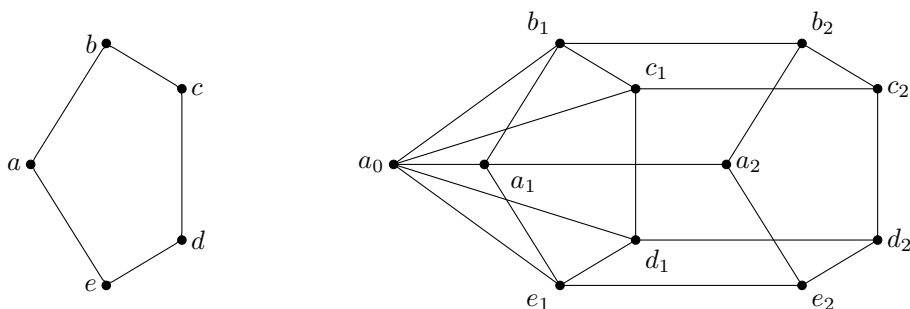
FIGURE 7. The pentagon $G = \mathbb{Z}_5$, left; and the cone $C_2 G$, right, with labeling scheme as in the proof of Proposition 4.2.

Proof. We will need a labeling scheme for the vertices of $S_t G$. For each $v \in G$ and $i = 0, \ldots, t$, we let v_i denote the corresponding vertex in the image of $G \times \{i\}$ in SG. Thus, for example, $u_0 = v_0$ and $u_t = v_t$ for all $u, v \in G$. This labeling scheme is illustrated in Figure 8, where the pentagon $G = \mathbb{Z}_5$ is labeled as in Figure 7.
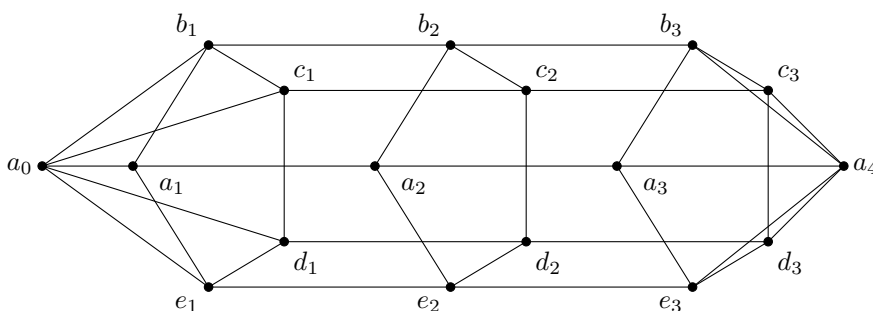
FIGURE 8. The labeling scheme of $S_4 \mathbb{Z}_5$.

Suppose that $t = 2$, and fix $w \in G$. There is contraction $c : S_t G \times I_2 \to S_t G$ of $S_t G$ to v_0 given by

$$c(v_i, 1) = \begin{cases} v_0 & \text{if } i < 2, \\ w_1 & \text{if } i = 2. \end{cases}$$

It is routine to verify that c is a graph map.

Suppose now that t is arbitrary and G is contractible. Let $C : G \times I_m \to G$ be a contraction of G to $w \in G$. Define a function $\kappa : S_t G \times I_{m+t} \to S_t G$ as follows:

$$\kappa(v_i, j) = \begin{cases} C(v, j)_i & \text{if } 0 \leqslant j \leqslant m, \\ w_{i+m-j} & \text{if } m < j \leqslant i + m, \\ v_0 & \text{if } i + m < j \leqslant m + t. \end{cases}$$

It is routine to verify that C is a contraction of $S_t G$ to v_0. □

5. Toward a discrete Hurewicz theorem

We prove Theorem 5.10 below, which implies Theorem 1.2 from the introduction. We first recall the definition of discrete singular cubical homology groups of graphs and define the map $\psi : A_n(G) \to \mathcal{H}_n(G)$ from (4).

5.1. DISCRETE SINGULAR CUBICAL HOMOLOGY OF GRAPHS. Let I_m^n denote the n-fold Cartesian product of I_m, regarded as a subgraph of $\mathbb{Z}^n$. For each $n \geqslant 0$, let Q_n be the graph given by

$$Q_n = \begin{cases} I_0 & \text{if } n = 0, \\ I_1^n & \text{if } n \geqslant 1. \end{cases}$$

An *n-cube of G* is a graph map $\sigma : Q_n \to G$. For $1 \leqslant i \leqslant n$, let $D_i^- \sigma$ and $D_i^+ \sigma$ be the $(n-1)$-cubes of G given by

$$D_i^- \sigma(a_1, \ldots, a_{n-1}) = \sigma(a_1, \ldots, a_{i-1}, 0, a_i, \ldots, a_{n-1})$$
$$D_i^+ \sigma(a_1, \ldots, a_{n-1}) = \sigma(a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_{n-1}).$$

We say that σ is *degenerate* if $D_i^- \sigma = D_i^+ \sigma$ for some i. Let $L_n(G)$ denote the free $\mathbb{Z}$-module generated by all n-cubes of G, and let $D_n(G)$ denote the submodule generated by the degenerate n-cubes of G. Let

$$C_n(G) = L_n(G)/D_n(G).$$

The elements of $C_n(G)$ are called *n-chains* of G.

Given an n-cube σ of G with $n \geqslant 1$, let

$$\partial_n(\sigma) = \sum_{i=1}^{n} (-1)^i (D_i^- \sigma - D_i^+ \sigma).$$

Extend linearly to obtain a map $\partial_n : C_n(G) \to C_{n-1}(G)$. It is routine to check that $(C_\bullet(G), \partial_\bullet)$ is a chain complex. Let

$$\mathcal{H}_n(G) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}).$$

The group $\mathcal{H}_n(G)$ is called the *nth discrete singular cubical homology group* of G. If $z \in \ker(\partial_n)$, then we write $\overline{z}$ for the equivalence class of z in $\mathcal{H}_n(G)$.

There is a discrete Hurewicz theorem in dimension 1:

THEOREM 5.1 ([3, Theorem 4.1]). *For any graph G, there is a surjective map*

(10) $$\psi : A_1(G) \to \mathcal{H}_1(G)$$

whose kernel is $[A_1(G), A_1(G)]$.

The map ψ can be described explicitly. For this, and for the remainder of this section, we will represent based homotopy classes $[f] \in A_n(G)$ as graph maps

$$(I_m^n, \partial I_m^n) \to (G, v_0).$$

This is justified by the following argument. Let $f : (\mathbb{Z}^n, \partial \mathbb{Z}^n) \to (G, v_0)$ be any graph map of radius r. Let $h : \mathbb{Z}^n \times I_r \to G$ be given by $h(x, i) = f(x_1 - i, \ldots, x_n - i)$ for all $(x, i) \in \mathbb{Z}^n \times I_r$. It is routine to check that h is a based homotopy from $h_0 = f$ to a graph map h_r that is constant outside the interior of I_m^n. By restricting our attention to the domain I_m^n, we obtain a map $(I_m^n, \partial I_m^n) \to (G, v_0)$ that is based homotopic to f.

Following the above, suppose that $f : (I_m, \partial I_m) \to (G, v_0)$ is a graph map, and for $i = 0, \ldots, m-1$ let $f^i : Q_1 \to G$ be given by $f^i(0) = f(i)$ and $f^i(1) = f(i+1)$. We have

$$\psi([f]) = \sum_{i=0}^{m-1} f^i.$$

For higher dimensions, we conjecture a direct analog of the classical theorem.

CONJECTURE 5.2. *If $n \geqslant 2$ and $A_k(G)$ is trivial for all $k < n$, then $A_n(G) \cong \mathcal{H}_n(G)$.*

Example 5.3. Let G_1 be a graph containing no 3- or 4-cycles. It follows from Proposition 2.6 and [11, Proposition 1A.2] that $A_1(G_1)$ is a free group of rank k, where k is the number of edges of G_1 not contained in a given spanning tree. Theorem 5.1 implies that

$$\mathcal{H}_1(G_1) \cong \mathbb{Z}^k,$$

where $\mathbb{Z}^k$ denotes a free abelian group of rank k. An explicit set of generators, although not always minimal one, is the set of $\psi([f_Z])$ for cycles Z of G_1, where the maps f_Z are defined as in Example 2.7. For each $n \geqslant 1$, let $G_{n+1} = S_{n+3}G_n$. It will follow from Proposition 5.12(ii) below that $\mathcal{H}_n(G_n) \cong \mathbb{Z}^k$ for all $n \geqslant 1$. This result originally appeared in [5].

5.2. A proposed discrete Hurewicz map. We define a map

$$\psi : A_n(G) \to \mathcal{H}_n(G)$$

that generalizes (10). Let $I_{m_1,\ldots,m_n} \subseteq \mathbb{Z}^n$ be the grid graph given by

$$(11) \qquad\qquad I_{m_1,\ldots,m_n} = I_{m_1} \times \cdots \times I_{m_n}.$$

For the following definitions, let $L = I_{m_1,\ldots,m_n}$ and $L' = I_{m_1-1,\ldots,m_n-1}$ for some $m_i \geqslant 1$, and let $f : L \to G$ be a graph map.

Definition 5.4. *For each $x \in L'$, let $f^x : Q_n \to G$ be the graph map given by*

$$(12) \qquad\qquad f^x(y) = f(x+y).$$

The maps f^x are called the n-cubes of f. If $x + Q_n \subseteq V$ for some $V \subseteq \mathbb{Z}^n$, then we say that f^x is contained in V.

Definition 5.5. *Define an element $\phi(f)$ of $C_n(G)$ by*

$$(13) \qquad\qquad \phi(f) = \sum_{x \in L'} f^x.$$

In other words, $\phi(f)$ is the sum of all n-cubes of f.

Lemma 5.6. *We have the following:*

(i) *If $[f] \in A_n(G)$, then $\phi(f) \in \ker(\partial_n)$,*
(ii) *If $[f] = [g]$ in $A_n(G)$, then $\overline{\phi(f)} = \overline{\phi(g)}$.*

Proof. First we prove (i). Let $f : (I_m^n, \partial I_m^n) \to (G, v_0)$ be a graph map. For $i = 1, \ldots, n$ let $\Omega_i = \{x \in I_{m-1}^n : x_i = 0\}$. Let $\epsilon_i \in \mathbb{Z}^n$ be the ith standard basis vector. We have

$$\partial_n \phi(f) = \sum_{i=1}^n (-1)^i \sum_{x \in \Omega_i} \sum_{r=0}^{m-1} D_i^- f^{x+r\epsilon_i} - D_i^+ f^{x+r\epsilon_i}.$$

For each i and each $x \in \Omega_i$, the innermost sum telescopes, and the leftover portion

$$D_i^- f^x - D_i^+ f^{x+(m-1)\epsilon_i}$$

is 0, since f is constant on ∂I_m^n. This proves (i).

To prove (ii), let h be a homotopy from f to g, say $h : I_m^{n+1} \to G$. By another telescoping argument, it is not hard to see that

$$\partial_{n+1} \phi(h) = (-1)^{n+1} (\phi(f) - \phi(g))$$

in $C_n(G)$, proving (ii). $\qquad\square$

Definition 5.7. *Let $\psi : A_n(G) \to \mathcal{H}_n(G)$ be the function given by $\psi([f]) = \overline{\phi(f)}$.*

Proposition 5.8. *The function $\psi : A_n(G) \to \mathcal{H}_n(G)$ is a group homomorphism.*

Proof. If $[f]$ and $[g]$ are any two elements of $A_n(G)$, and p is defined as in (7), then $\phi(p) - (\phi(f) + \phi(g))$ is a sum of constant n-cubes $Q^n \to v_0$, which are degenerate. Hence $\psi([f] \cdot [g]) = \psi([p]) = \psi([f]) + \psi([g])$ in $\mathcal{H}_n(G)$, as desired. $\square$

REMARK 5.9. If we regard A_n (resp. $\mathcal{H}_n$) as a functor from the category of connected simple graphs with graph maps to the category of groups (resp. abelian groups), then ψ is a natural transformation from A_n to $\mathcal{H}_n$.

5.3. SURJECTIVITY IN A SPECIAL CASE. Given any $n \geqslant 1$, we describe an infinite family of graphs G for which the map $\psi : A_n(G) \to \mathcal{H}_n(G)$ is surjective. From this we obtain an infinite class of G for which $A_n(G)$ is nontrivial. Throughout the section, we let G_1 be a graph and

$$(14) \qquad\qquad G_{n+1} = S_{n+3} G_n$$

for all $n \geqslant 1$, where S_{n+3} is the suspension functor from Definition 4.3. We think of G_{n+1} as consisting of $n + 2$ copies of G_n and two extra vertices, called the *north pole* and *south pole*. We now state the main result of this section, which implies Theorem 1.2.

THEOREM 5.10. *If G_1 contains no 3- or 4-cycles, then the map*

$$\psi : A_k(G_n) \to \mathcal{H}_k(G_n)$$

is surjective for all $1 \leqslant k \leqslant n$.

COROLLARY 5.11. *If G_1 contains a cycle but does not contain any 3- or 4-cycles, then $A_n(G_n)$ is infinite for all $n \geqslant 1$.*

Proof. This follows from Example 5.3, Theorem 5.10 and Proposition 5.12(ii) below. $\square$

The proof of Theorem 5.10 will proceed in several steps. We will focus on the case where G_1 is a cycle graph of length 5, i.e. $G_1 = \mathbb{Z}_5$, and then extend the proof to the general case. Example 5.3 says that if $G_1 = \mathbb{Z}_5$, then $\mathcal{H}_n(G_n) \cong \mathbb{Z}$ for all $n \geqslant 1$. The following proposition, which applies to any choice of G_1, will allow us to identify an explicit generator of $\mathcal{H}_n(G_n)$ when $G_1 = \mathbb{Z}_5$. Part (ii) first appeared as [5, Theorem 5.2].

PROPOSITION 5.12. *If $n \geqslant 2$, then*
  (i) *$\mathcal{H}_k(G_n) = 0$ for $k = 1, \ldots, n - 1$,*
  (ii) *there is an explicit isomorphism $\Delta : \mathcal{H}_n(G_n) \to \mathcal{H}_{n-1}(G_{n-1})$.*

Proof. We first prove (ii), adapting our argument from the proof of [5, Theorem 5.2]. Let $n \geqslant 2$. Let X (resp. Y) be the graph obtained from G_n by deleting the north (resp. south) pole. From the proof of [5, Theorem 5.2] we have the following exact sequences for $k = 1, \ldots, n$:

$$(15) \qquad \mathcal{H}_k(X) \oplus \mathcal{H}_k(Y) \to \mathcal{H}_k(G_n) \xrightarrow{\partial_*} \mathcal{H}_{k-1}(X \cap Y) \to \mathcal{H}_{k-1}(X) \oplus \mathcal{H}_{k-1}(Y).$$

The map ∂_* can be described explicitly: if $\overline{z} \in \mathcal{H}_k(G_n)$, then $z = x + y$ for some $x \in C_k(X)$ and $y \in C_k(Y)$, and $\partial_*(\overline{z}) = \overline{\partial_k(x)}$. Note that $G_{n-1} \times \{1\}$ is a deformation retract of $X \cap Y$ in the sense of [4, Section 4]. Hence if we identify G_{n-1} with $G_{n-1} \times \{1\}$, then the inclusion map $\iota : G_{n-1} \to X \cap Y$ induces an isomorphism $\iota_* : \mathcal{H}_{n-1}(G_{n-1}) \to \mathcal{H}_{n-1}(X \cap Y)$. Let

$$\Delta := (\iota_*)^{-1} \circ \partial_*.$$

Proposition 4.2 implies that X and Y are contractible, so $\mathcal{H}_k(X) = \mathcal{H}_k(Y) = 0$ for all $k \geqslant 1$. The exact sequences (15) now become

$$0 \to \mathcal{H}_k(G_n) \xrightarrow{\Delta} \mathcal{H}_{k-1}(G_{n-1}) \to \mathcal{H}_{k-1}(X) \oplus \mathcal{H}_{k-1}(Y).$$

For $k \geqslant 2$ we have $\mathcal{H}_{k-1}(X) \oplus \mathcal{H}_{k-1}(Y) = 0$, so Δ is an isomorphism, proving (ii).

We now prove (i). If $G = (V, E)$ is any connected graph, then $\mathcal{H}_0(G) \cong \mathbb{Z}$ has presentation

$$(16) \qquad \mathcal{H}_0(G) = \langle v \in V \mid u = v \text{ for all } u, v \in V \rangle.$$

Any $x \in C_1(X)$ can be written as $x = \sum_{i=1}^{\ell} m_i f_i$ for integers m_i and graph maps $f_i : Q_1 \to X$. We have $\partial_1(x) = \sum_{i=1}^{\ell} m_i(f_i(1) - f_i(0))$, so $\overline{\partial_1(x)} = \sum_{i=1}^{\ell} m_i(f_i(1) - f_i(1))) = 0$ in $\mathcal{H}_0(G_{n-1})$ by (16). Thus when $k = 1$, we have $\operatorname{im}\Delta = 0$. But also $\ker \Delta = 0$ by exactness, so $\mathcal{H}_1(G_n) = 0$ for all $n \geqslant 2$. Hence if $k < n$, then by applying (i) repeatedly we obtain an isomorphism $\mathcal{H}_k(G_n) \to \mathcal{H}_1(G_{n-k+1}) = 0$, proving (i). $\qquad \square$

We will need a labeling scheme for the vertices of G_n. The graph G_{n+1} consists of $n + 2$ isomorphic copies of G_n suspended between the north and south poles. For each $i = 0, \ldots, n + 3$ we obtain a graph map $\iota_n^i : G_n \to G_{n+1}$ taking each $v \in G_n$ to the corresponding vertex in the ith copy of G_n. Thus ι_n^0 and ι_n^{n+3} are the constant graph maps taking G_n to the north and south poles of G_{n+1}, respectively. For $v \in G_n$, we write $v_i = \iota_n^i(v)$ for all i, and we say that v_i is obtained from v by *adding the subscript i*. In general, if v is a vertex of G_k for $k < n$, then we write

$$(17) \qquad v_{i_k, \ldots, i_n} = \iota_n^{i_n} \cdots \iota_k^{i_k} v.$$

For example, $v_{i_k, \ldots, i_n}$ is the south pole of G_{n+1} if and only if $i_n = 0$.

If G is a graph and $L = I_{m_1, \ldots, m_n}$ is a grid graph, then we will think of a map $L \to G$ as an n-dimensional array of size $(m_1 + 1) \times \cdots \times (m_n + 1)$, whose entries are vertices of G. For example, when $n = 1$, a map $L \to G$ is a single column with $m_1 + 1$ entries. When $n = 2$, we get an $(m_1 + 1) \times (m_2 + 1)$ matrix. In both cases, we consider the top-left entry to be the image of the origin $(0, \ldots, 0) \in L$.

Suppose that $G_1 = \mathbb{Z}_5$. Our task now is to construct a generator of $\mathcal{H}_n(G_n) \cong \mathbb{Z}$ inductively. Define grid graphs J_n by $J_1 = I_5$ and

$$(18) \qquad J_{n+1} = J_n \times I_{n+3}$$

for all $n \geqslant 1$. Label the vertices of G_1 as in Figure 7. As just described, we will think of maps $J_n \to G$ as n-dimensional arrays of appropriate size. Let $\gamma_1 : J_1 \to G_1$ be given by

$$(19) \qquad \gamma_1 = \begin{pmatrix} a\ b\ c\ d\ e\ a \end{pmatrix}^T,$$

where T denotes the usual transpose. For $n \geqslant 1$ let $\gamma_{n+1} : J_{n+1} \to G_{n+1}$ be given by

$$(20) \qquad \gamma_{n+1}(v, i) = \gamma_n(v)_i$$

for all $(v, i) \in J_n \times I_{n+3}$, where we have added the subscript i to $\gamma_n(v)$ as defined in (17). For example, $\gamma_2 : I_5 \times I_4 \to G$ is given by

$$\gamma_2 = \begin{pmatrix} a_0\ a_0\ a_0\ a_0\ a_0\ a_0 \\ a_1\ b_1\ c_1\ d_1\ e_1\ a_1 \\ a_2\ b_2\ c_2\ d_2\ e_2\ a_2 \\ a_3\ b_3\ c_3\ d_3\ e_3\ a_3 \\ a_4\ a_4\ a_4\ a_4\ a_4\ a_4 \end{pmatrix}^T .$$

It is routine to verify that $\gamma_n : J_n \to G_n$ is a graph map for all n.

We record some basic properties of γ_n. First, since $g_1(0) = g_1(5)$, we have

(21) $$\gamma_n(0, v_2, \ldots, v_n) = \gamma_n(5, v_2, \ldots, v_n)$$

for all $v \in J_n$. We also have

(22) $$\gamma_n(v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_n) = \gamma_n(0, \ldots, 0, v_{i+1}, \ldots, v_n)$$

for all $i = 2, \ldots, n$, since adding the subscript 0 to any vertex of G_{i-1} gives the south pole of G_i. Similarly, adding the subscript $i + 2$ to any vertex of G_{i-1} gives the north pole of G_i, so

(23) $$\gamma_n(v_1, \ldots, v_{i-1}, i+2, v_{i+1}, \ldots, v_n) = \gamma_n(0, \ldots, 0, i+2, v_{i+1}, \ldots, v_n).$$

Finally, note that $\partial_n \phi(\gamma_n) = 0$ by a telescoping argument, so we can consider the homology class $\overline{\phi(\gamma_n)}$.

LEMMA 5.13. *If $G_1 = \mathbb{Z}_5$ and γ_n is defined as above for $n \geqslant 1$, then*

(i) $\mathcal{H}_n(G_n) = \langle \overline{\phi(\gamma_n)} \rangle$
(ii) $\Delta(\overline{\phi(\gamma_{n+1})}) = (-1)^n \overline{\phi(\gamma_n)}$, *where $\Delta : \mathcal{H}_{n+1}(G_{n+1}) \to \mathcal{H}_n(G_n)$ is the isomorphism from Proposition 5.12(ii).*

Proof. We prove (ii). Example 5.3 implies (i) for $n = 1$, so the general case of (i) will follow from (ii). Suppose that $n \geqslant 2$. We retain the definitions of X, Y, ∂_*, ι and ι_* from the proof of Proposition 5.12. Thus X (resp. Y) is obtained from G_n by removing the north (resp. south) pole. We can write $\phi(\gamma_n) = x + y$, where $x \in C_n(X)$ and $y \in C_n(Y)$. In particular, if $U = \{u \in J_n : 1 \leqslant u_n \leqslant n + 1\}$, then we can take

$$y = \sum_{u \in U} \gamma_n^u.$$

Let $V = \{v \in J_n : v_n = 1\}$ and $W = \{w \in J_n : w_n = n + 1\}$. The only terms in the sum $\partial_n(y)$ that do not cancel by telescoping are

$$\partial_n(y) = (-1)^n \left(\sum_{v \in V} D_n^- \gamma_n^v - \sum_{w \in W} D_n^+ \gamma_n^w \right).$$

By construction, each $D_n^+ \gamma_n^w$ is the constant map $Q^{n-1} \to a_{n+2,\ldots,n+2}$. This is a degenerate $(n-1)$-cube of G_n, so we can ignore it. We are left with the sum

$$\partial_n(y) = (-1)^n \sum_{v \in V} D_n^- \gamma_n^v,$$

but this is precisely $\phi(\iota \circ \gamma_{n-1})$. Hence

$$\partial_*(\overline{\phi(\gamma_n)}) = \overline{\partial_n(x)} = -\overline{\partial_n(y)} = (-1)^{n-1} \overline{\phi(\iota \circ \gamma_{n-1})} = (-1)^{n-1} (\iota_*(\overline{\phi(\gamma_{n-1})})),$$

from which we deduce that $\Delta(\overline{\phi(\gamma_n)}) = (\iota_*)^{-1}(\partial_*(\overline{\phi(\gamma_n)})) = (-1)^{n-1} \overline{\phi(\gamma_{n-1})}$, proving (ii). $\square$

LEMMA 5.14. *If $G_1 = \mathbb{Z}_5$, then the image of $\psi : A_n(G_n) \to \mathcal{H}_n(G_n)$ contains $\overline{\phi(\gamma_n)}$.*

We illustrate the proof of Lemma 5.14 for $n = 2$ before arguing the general case. Let $L_2 = I_{13,8}$, and let $M_2 = (4, 0) + I_{5,4} \subseteq L_2$. Let $f_2 : (L_2, \partial L_2) \to (G_2, a_0)$ be the graph map in Figure 9. Here f_2 is constant on every outlined region.

Let $g_2 : (L_2, \partial L_2) \to (G_2, a_0)$ be the graph map in Figure 10. Here M_2 is the region outlined in red. Note that $f_2(x) = g_2(x)$ for all $x \in L_2 \smallsetminus M_2^\circ$. Hence if $\Omega_2 = \{x \in \mathbb{Z}^2 : x + Q_2 \subseteq M_2\}$, then

$$\phi(g_2) - \phi(f_2) = \sum_{x \in \Omega_2} g_2^x - \sum_{x \in \Omega_2} f_2^x.$$
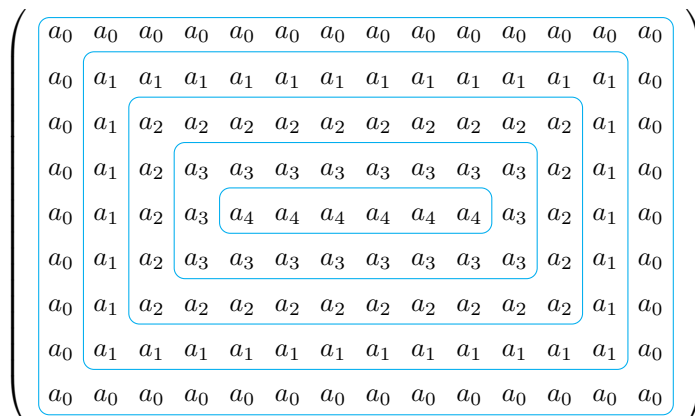
$$\begin{pmatrix} a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 \\ a_0 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_4 & a_4 & a_4 & a_4 & a_4 & a_3 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_0 \\ a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 \end{pmatrix}$$

FIGURE 9. The transpose of f_2 with its fibers $B_{2,k}$ outlined in blue.

The first sum on the right hand side is easily seen to be $\phi(\gamma_2)$ by inspection. Each term in the second sum is a degenerate 2-cube of f_2. Hence $\phi(g_2) - \phi(f_2) = \phi(\gamma_2)$, proving that

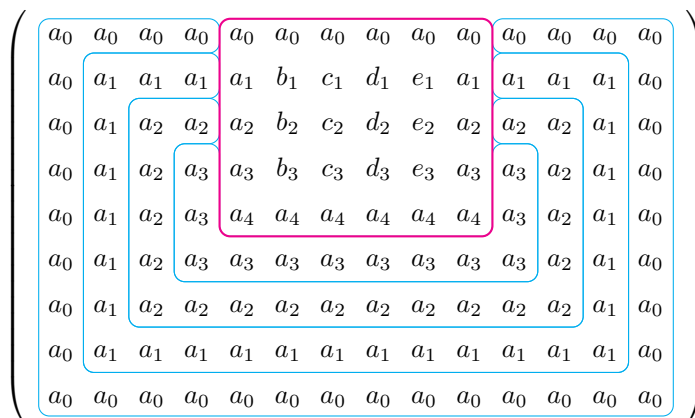$$\psi([g_2] - [f_2]) = \overline{\phi(\gamma_2)},$$

as desired.

$$\begin{pmatrix} a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 \\ a_0 & a_1 & a_1 & a_1 & a_1 & b_1 & c_1 & d_1 & e_1 & a_1 & a_1 & a_1 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_2 & a_2 & b_2 & c_2 & d_2 & e_2 & a_2 & a_2 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_3 & a_3 & b_3 & c_3 & d_3 & e_3 & a_3 & a_3 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_4 & a_4 & a_4 & a_4 & a_4 & a_3 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_1 & a_0 \\ a_0 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_0 \\ a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 \end{pmatrix}$$

FIGURE 10. The transpose of g_2 with M_2 outlined in red and each $B_{2,k} \smallsetminus M_2$ outlined in blue.

Proof of Lemma 5.14. The case $n = 1$ is trivial; simply note that

$$\psi([\gamma_1]) = \overline{\phi(\gamma_1)}.$$

Suppose now that $n \geqslant 2$. We will construct an n-dimensional grid graph L_n and graph maps

$$f_n, g_n : (L_n, \partial L_n) \to (G_n, a_{0,\dots,0})$$

such that

(24) $$\psi([g_n] - [f_n]) = \overline{\phi(\gamma_n)},$$

proving the result.

For $n \geqslant 2$ let $L_n = I_{m_1,\ldots,m_n}$, where

$$m_i = \begin{cases} (n-i+1)(n+i+4) - 1 & \text{if } i = 1, \\ (n-i+1)(n+i+4) & \text{if } i > 1. \end{cases}$$

For example, the first several L_n are

$$L_2 = I_{13} \times I_8$$
$$L_3 = I_{23} \times I_{18} \times I_{10}$$
$$L_4 = I_{35} \times I_{30} \times I_{22} \times I_{12}.$$

Also let

$$M_n = c + J_n \subseteq L_n,$$

where $c = (n+2,\ldots,n+2,0) \in \mathbb{Z}^n$ and J_n is defined as in (18). Let $L_{n,0} = \partial L_n$, and for $k = 1,\ldots,n+2$ let $L_{n,k} = L_{n,k-1}^\circ$. For each $k = 0,\ldots,n+2$, let $B_{n,k} = \partial L_{n,k}$. For example, the graphs $B_{2,k}$ are outlined in Figure 9, and M_2 is the large outlined region in Figure 10. Note that the $B_{n,k}$ partition L_n in a particular way; there is a "central" grid graph $B_{n,n+2}$ induced by all vertices of M_n with nth coordinate $n+2$, and $B_{n,k}$ is "wrapped around" $B_{n,k+1}$ for $k = 0,\ldots,n+1$.

Suppose that $x \in B_{n,k}$. Let $f_n : (L_n, \partial L_n) \to (G_n, a_{0,\ldots,0})$ be given by

$$f_n(x) = \gamma_n(0, x_2^*, x_3^*, \ldots, x_{n-1}^*, k),$$

where for $1 < i < n$ we set

$$x_i^* = \begin{cases} x_i - (n+2) & \text{if } n+2 \leqslant x_i \leqslant n+i+4, \\ 0 & \text{if } x_i \leqslant n+2, \\ i+2 & \text{if } x_i \geqslant n+i+4. \end{cases}$$

For example, f_2 is illustrated in Figure 9.

We claim that f_n is a graph map. Let $x,y \in L_n$ with $x \simeq y$. We must show that $f_n(x) \simeq f_n(x+y)$. If $x = y$, then this is immediate. Suppose that $x \neq y$ with $x \in B_{n,k}$. The vertices x and y differ in exactly one coordinate; call it j. We must have either $y \in B_{n,k}$ or $y \in B_{n,k\pm1}$. If $j \in \{1,n\}$ and $y \in B_{n,k}$, then $f_n(x) = f_n(y)$ by definition. If $j \in \{1,n\}$ and $y \in B_{n,k\pm1}$, then

$$f_n(x) = \gamma_n(0, x_2^*, \ldots, x_{n-1}^*, k) \simeq \gamma_n(0, x_2^*, \ldots, x_{n-1}^*, k\pm1) = f_n(y).$$

If $1 < j < n$ and $y \in B_{n,k}$, then $|x_j^* - y_j^*| \leqslant 1$, so

(25)
$$f_n(x) = \gamma_n(0, x_2^*, \ldots, x_{n-1}^*, k) \simeq \gamma_n(0, x_2^*, \ldots, x_{j-1}^*, y_j^*, x_{j+1}^*, \ldots, x_{n-1}^*, k) = f_n(y).$$

If $1 < j < n$ and $y \in B_{n,k\pm1}$, then either $x_j, y_j \leqslant n+2$ or $x_j, y_j \geqslant n+j+4$, so $x_j^* = y_j^*$ and (25) holds again. Thus in any case we have $f_n(x) \simeq f_n(y)$, so f_n is a graph map.

We claim that the following defines a function $g_n : (L_n, \partial L_n) \to (G_n, a_{0,\ldots,0})$:

$$g_n(x) = \begin{cases} \gamma_n(x-c) & \text{if } x \in M_n, \\ f_n(x) & \text{if } x \in L_n \smallsetminus M_n^\circ. \end{cases}$$

For example, g_2 is illustrated in Figure 10 with M_2 boxed in red and $B_{2,k} \smallsetminus M_2$ boxed in blue for $k = 0,\ldots,3$. To prove that g_n is well defined in general, we must show that

(26)
$$\gamma_n(x-c) = f_n(x)$$

for all $x \in \partial M_n$. It will follow that g_n is a graph map, since γ and f_n are graph maps.

We have $x \in M_n$ if and only if the following hold:

    (i) $n+2 \leqslant x_1 \leqslant n+7$

(ii) $n + 2 \leqslant x_i \leqslant n + i + 4$ for $i = 2, \ldots, n - 1$

(iii) $0 \leqslant x_n \leqslant n + 2$.

Thus $x \in \partial M_n$ if and only if (i)–(iii) hold with at least one of these inequalities being an equality. Note that the graphs $B_{n,k} \cap \partial M_n$ partition ∂M_n. Suppose that $x \in B_{n,k} \cap \partial M_n$. We have $x_n = k$ and $x_i^* = x_i - (n + 2)$ for $1 < i < n$, so

$$\gamma_n(x - c) = \gamma_n(x_1 - (n + 2), x_2^*, \ldots, x_{n-1}^*, k).$$

If $x_1 = n + 2$, then

$$\gamma_n(x - c) = \gamma_n(0, x_2^*, \ldots, x_{n-1}^*, k) = f_n(x)$$

by definition. If $x_1 = n + 7$, then

$$\gamma_n(x - c) = \gamma_n(5, x_2^*, \ldots, x_{n-1}^*, k) = \gamma_n(0, x_2^*, \ldots, x_{n-1}^*, k) = f_n(x),$$

where we have used (21). If $x_i = n + 2$ for some $1 < i < n$, then $x_i^* = 0$, so

$$\begin{aligned}
\gamma_n(x - c) &= \gamma_n(x_1 - (n + 2), x_2^*, \ldots, x_{i-1}^*, 0, x_{i+1}^*, \ldots, x_{n-1}^*, k) \\
&= \gamma_n(0, \ldots, 0, x_{i+1}^*, \ldots, x_{n-1}^*, k) \\
&= \gamma_n(0, x_2^*, \ldots, x_{i-1}^*, 0, x_{i+1}^*, \ldots, x_{n-1}^*, x_n) \\
&= \gamma_n(0, x_2^*, \ldots, x_{n-1}^*, k) \\
&= f_n(x),
\end{aligned}$$

where we have used (22). The argument is similar for $x_i = n + i + 4$, using (23) instead. If $x_n = 0$ (resp. $x_n = n + 2$), then (22) (resp. (23)) again implies (26). Therefore (26) holds for all $x \in \partial M_n$, proving the claim that g_n is well defined. It follows that g_n is a graph map.

We now prove (24). Let

$$\Omega_n = \{x \in \mathbb{Z}^n : x + Q_n \subseteq M_n\}.$$

Since $g_n(x) = f_n(x)$ for all $x \in L_n \setminus M_n^\circ$, we have

$$\phi([g_n] - [f_n]) = \sum_{x \in \Omega_n} g_n^x - \sum_{x \in \Omega_n} f_n^x.$$

The first sum on the right hand side is easily seen to be $\phi(\gamma_n)$. We claim that every term of the second sum is a degenerate n-cube of f_n. If $x \in \Omega_n$, then $n + 2 \leqslant x_i < n + i + 4$ for all $i < n$, so for any $q \in Q_n$ we have

$$\begin{aligned}
f_n^x(0, q_2, \ldots, q_n) &= \gamma_n(0, x_2 + q_2 - (n + 2), \ldots, x_{n-1} + q_{n-1} - (n + 2), x_n + q_n) \\
&= f_n^x(1, q_2, \ldots, q_n).
\end{aligned}$$

Hence $D_1^- f_n^x = D_1^+ f_n^x$, proving the claim. It follows that

$$\phi([g_n] - [f_n]) = \phi(\gamma_n),$$

proving (24). $\qquad\square$

REMARK 5.15. Close inspection reveals that the image of f_n is the set of all vertices of G_n of the form $a_{i_1, \ldots, i_{n-1}}$, where we have added subscripts to the vertex $a \in G_1$. This set induces a subgraph of G_n isomorphic to the graph U_n, where U_1 consists of a single vertex and $U_{n+1} = S_{n+3} U_n$ for all $n \geqslant 1$. Proposition 4.4 implies that U_n is contractible for all n. Hence $[f_n] = 0$ in $A_n(G_n)$, so in fact $\psi([g_n]) = \overline{\phi(\gamma_n)}$.

Proof of Theorem 5.10. Let G_1 be any graph containing no 3- or 4-cycles. If G_1 contains no cycles, then G_n is contractible for all n by Example 2.11 and Proposition 4.4, so $\mathcal{H}_n(G_n)$ is trivial for all $n \geqslant 1$ by [4, Lemma 4.2], and the theorem is immediate.

Suppose that G_1 contains a cycle. If $k < n$, then $\mathcal{H}_k(G_n) = 0$ by Proposition 5.12(i), and the theorem is immediate. We prove the case $k = n$. Let $\{Z_{1,1}, \ldots, Z_{1,\ell}\}$ be the set of cycles of G_1, considered as subgraphs of G_1. For $i = 1, \ldots, \ell$ and $n \geqslant 1$, let $Z_{n+1,i} = S_{n+3} Z_{n,i}$, considered as a subgraph of G_{n+1}. Lemma 5.13 can be easily generalized to obtain grid graphs $J_{n,i}$ and graph maps $\gamma_{n,i} : J_{n,i} \to Z_{n,i}$ satisfying the following for all i and $n \geqslant 1$:

(i) $\mathcal{H}_n(Z_{n,i}) = \langle \overline{\phi(\gamma_{n,i})} \rangle$

(ii) $\Delta(\overline{\phi(\gamma_{n+1,i})}) = (-1)^n \overline{\phi(\gamma_{n,i})}$, where $\Delta : \mathcal{H}_{n+1}(G_{n+1}) \to \mathcal{H}_n(G_n)$ is the isomorphism from Proposition 5.12(ii).

Let
$$\Upsilon_n = \{\overline{\phi(\gamma_{n,i})} : i = 1, \ldots, \ell\}.$$
Example 5.3 and item (i) together say that $\mathcal{H}_1(G_1)$ is generated by Υ_1. Item (ii) then implies that $\mathcal{H}_n(G_n)$ is generated by Υ_n for all $n \geqslant 1$. Lemma 5.13 can be easily generalized to show that the image of $\psi : A_n(G_n) \to \mathcal{H}_n(G_n)$ contains Υ_n. Hence ψ is surjective. $\qquad\square$

6. Final remarks

Our work leaves open several important questions. Let G_1 be a cycle graph of length 5, and define the graphs G_n as in (14). Our most immediate goal is to show that the map $\psi : A_n(G_n) \to \mathcal{H}_n(G_n)$ is an isomorphism for all n. Theorem 5.10 brings us halfway there; we leave the remaining half as a conjecture.

CONJECTURE 6.1. *The map $\psi : A_n(G_n) \to \mathcal{H}_n(G_n)$ is injective for all n.*

The main analogy guiding our intuition is to think of the graphs G_n as playing the role of the n-sphere S^n in classical topology. An obvious comparison to make is that $S^{n+1} \approx SS^n$, where S is the usual suspension functor, since this mirrors the construction of G_n via the discrete suspension functor S_t. Our analogy is strengthened by the fact that
$$\mathcal{H}_i(G_n) \cong H_i(S^n)$$
for all $i \leqslant n$. An important property of S^n, however, is $(n-1)$-connectedness. This is what allows one to apply the Hurewicz theorem and conclude that $\pi_n(S^n) \cong H_n(S^n)$. We suspect that G_n is $(n-1)$-connected in a discrete sense.

CONJECTURE 6.2. *If $i < n$, then $A_i(G_n)$ is trivial.*

The usual way to prove that S^n is $(n-1)$-connected is to invoke the cellular approximation theorem. A discrete analog of cellular approximation was proposed in [1]. An alternative proof of $(n-1)$-connectedness, such as the one in [13, Theorem 6.4.4], uses the *homotopy excision theorem* of Blakers and Massey.

THEOREM 6.3 (Blakers–Massey [10]). *Suppose that a topological space X is the union of open subspaces A and B with nonempty intersection $C = A \cap B$. If (A, C) is m-connected and (B, C) is n-connected, then the map*
$$\pi_k(B, C) \to \pi_k(X, A),$$
induced by the inclusion $(B, C) \to (X, A)$, is surjective if $k \leqslant m + n$ and bijective if $k < m + n$.

This result is a powerful means of computing higher classical homotopy groups. In particular, it is a key ingredient in the proof of the Hurewicz theorem. We therefore expect some discrete version of Theorem 6.3 to appear in a proof of Conjecture 5.2, should it hold. At the very least, a discrete homotopy excision theorem would enable us to perform more elegant computations than are currently possible.

QUESTION 6.4. *Is there a discrete homotopy excision theorem?*

A famous corollary of homotopy excision is the *Freudenthal suspension theorem*, which states that if X is n-connected, then the map

$$\pi_k(X) \to \pi_{k+1}(\Sigma X),$$

induced by suspension, is an isomorphism for $k < 2n+1$ and surjective if $k = 2n+1$. This gives, for example, a sequence of maps

$$\pi_1(S^1) \to \pi_2(S^2) \to \pi_3(S^3) \to \cdots$$

in which the first map is surjective and all subsequent maps are isomorphisms. Ideally, a discrete homotopy excision theorem would give such a sequence for the graphs G_n, i.e. a sequence of maps

$$A_1(G_1) \to A_2(G_2) \to A_3(G_3) \to \cdots$$

in which the first map is surjective and the rest are isomorphisms. Since $A_1(G_1) \cong \mathbb{Z}$ and $A_2(G_2)$ is infinite by Corollary 5.11, the first map must be an isomorphism as well. Thus it would follow that $A_n(G_n) \cong \mathbb{Z}$ for all n, confirming Conjecture 6.1 and further reinforcing the analogy between G_n and S^n.

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BOB LUTZ, Life Cycle Engineering, Inc., 4900 S. Broad St., Philadelphia, PA 19112, USA
E-mail : `rlutz@lce.com`