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Abstract We study a family of dissections of flow polytopes arising from the subdivision algebra. To each dissection of a flow polytope, we associate a polynomial, called the left-degree polynomial, which we show is invariant of the dissection considered (proven independently by Grinberg). We prove that left-degree polynomials encode integer points of generalized permutahedra. Using that certain left-degree polynomials are related to Grothendieck polynomials, we resolve special cases of conjectures by Monical, Tokcan, and Yong regarding the saturated Newton polytope property of Grothendieck polynomials.

1. Introduction

The flow polytope $\mathcal{F}_G$ associated to a directed acyclic graph G is the set of all flows $f : E(G) \to \mathbb{R}_{\geqslant 0}$ of size one. Flow polytopes are fundamental objects in combinatorial optimization [17], and in the past decade they were also uncovered in representation theory [1, 12], the study of the space of diagonal harmonics [8, 13], and the study of Schubert and Grothendieck polynomials [4, 5]. A natural way to analyze a convex polytope is to dissect it into simplices. The relations of the subdivision algebra, developed in a series of papers [9, 10, 11], encode dissections of a family of flow (and root) polytopes (see Section 2 for details).

Take any graph G with special source and sink vertices and fix a dissection $\mathcal{R}$ (into simplices) produced by the subdivision algebra. We study an invariant of $\mathcal{R}$ called the left-degree polynomial. Left-degree polynomials were introduced in [5] by Escobar and Mészáros. They showed that for a family of trees, the left-degree polynomial does not depend on the particular dissection considered. In Theorem A, we extend this result to any (not necessarily simple) graph. This was independently proven by Grinberg in [7] using algebraic techniques.

Our main technique is to connect left-degree polynomials to flow polytopes. We study the left-degree polynomial of a particular recursive dissection from [11]. In Corollary 3.16, we partition the support of this left-degree polynomial (with multiplicity) into blocks and show that the convex hull of each block is integrally equivalent to a flow polytope. Using this flow perspective, we give an inductive proof of Theorem A.

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Using the flow approach again, we connect the Newton polytopes of left-degree polynomials to generalized permutahedra. In Theorem B, we show that the Newton polytope of every homogeneous component of a left-degree polynomial is a generalized permutahedron. We also prove the *saturated Newton polytope* property (SNP) of Monical, Tokcan, and Yong [14]: every integer point in the Newton polytope is in the support of the polynomial.

We apply these results to Schubert and Grothendieck polynomials. Escobar and Mészáros showed in [5, Theorem 5.3] that a certain family of Grothendieck polynomials are related to left-degree polynomials. We conclude in Theorem C that this family of Grothendieck polynomials have SNP, and that the Newton polytopes of their homogeneous components are generalized permutahedra. We conjecture this holds for all Grothendieck polynomials (Conjecture 5.1).

The outline of this paper is as follows. Section 2 covers the necessary background. In Section 3, we study the support of left-degree polynomials (left-degree sequences) directly, and make the connection to flow polytopes. To maximize ease of reading, we restrict to the case of simple graphs. In Section 4 we introduce left-degree polynomials and describe their Newton polytopes. We apply this description to a family of Grothendieck polynomials in Section 5. In Section 6, we describe the technical modifications required to drop the simple graph assumption in the previous sections. We combinatorially prove left-degree polynomials are an invariant of the underlying graph.

2. Background

In this section, we summarize definitions, notations, and results that we use later. Throughout this paper, by *graph*, we mean a directed acyclic graph where multiple edges are allowed (as described below). Although we sometimes refer to edges by their endpoints, we allow that G may have multiple edges. We also adopt the convention of viewing each element of a multiset as being distinct, so that we may speak of subsets, though we will use the word submultiset interchangeably to highlight the multiplicity. Due to this convention, all unions in this paper are assumed to be disjoint multiset unions. For any integers m and n, we will frequently use the notation $[m,n]$ to refer to the set $\{m, m+1, \dots, n\}$ and $[n]$ to refer to the set $[1,n]$.

2.1. Flow Polytopes.

Let G be a graph on vertex set $[0,n]$ with edges directed from smaller to larger vertices. For each edge e, let $\mathrm{in}(e)$ denote the smaller (initial) vertex of e, and $\mathrm{fin}(e)$ the larger (final) vertex of e. Imagine fluid moving along the edges of G. At vertex i let there be an external inflow of fluid a_i (outflow of $-a_i$ if $a_i < 0$), and call the vector $\boldsymbol{a} = (a_0, \dots, a_n) \in \mathbb{R}^{n+1}$ the *netflow vector*. Formally, a *flow* on G with netflow vector $\boldsymbol{a}$ is an assignment $f : E(G) \to \mathbb{R}_{\geqslant 0}$ of nonnegative values to each edge such that fluid is conserved at each vertex. That is, for each vertex i

$$\sum_{\mathrm{in}(e)=i} f(e) - \sum_{\mathrm{fin}(e)=i} f(e) = a_i.$$

The *flow polytope* $\mathcal{F}_G(\boldsymbol{a})$ is the collection of all flows on G with netflow vector $\boldsymbol{a}$. Alternatively, let M_G denote the incidence matrix of G. That is, let the columns of M_G be the vectors $e_i - e_j$ for $(i,j) \in E(G)$, $i < j$, where e_i is the $(i+1)$-th standard basis vector in $\mathbb{R}^{n+1}$. Then,

$$\mathcal{F}_G(\boldsymbol{a}) = \{f \in \mathbb{R}^{E}_{\geqslant 0} \mid M_G f = \boldsymbol{a}\}. \tag{1}$$

From this perspective, note that the number of integer points in $\mathcal{F}_G(\boldsymbol{a})$ is exactly the number of ways to write $\boldsymbol{a}$ as a nonnegative integral combination of the vectors $e_i - e_j$

for edges (i,j) in G, $i<j$. This number is known as the *Kostant partition function* $K_G(\boldsymbol{a})$. For brevity, we write $\mathcal{F}_G$ to mean $\mathcal{F}_G(1,0,\dots,0,-1)$, and we refer to $\mathcal{F}_G$ as the flow polytope of G, since in this paper our primary focus is on studying these particular flow polytopes.

The following milestone result on volumes of flow polytopes was shown by Postnikov and Stanley in unpublished work.

Theorem 2.1 (Postnikov–Stanley). *Let G be a directed acyclic connected graph on vertex set $[0,n]$. Set $d_i = \operatorname{indeg}_G(i)-1$ for each vertex i, where $\operatorname{indeg}_G(i)$ is the number of edges incoming to vertex i in G. The normalized volume of the flow polytope of G is given by*

$$\operatorname{Vol}\mathcal{F}_G = K_G\left(0, d_1, \dots, d_n, -\sum_{i=1}^{n} d_i\right).$$

Baldoni and Vergne [1] generalized this result for flow polytopes with arbitrary netflow vectors. Theorem 2.1 beautifully connects the volume of the flow polytope of any graph to an evaluation of the Kostant partition function. We note that since the number of integer points of a flow polytope is already given by a Kostant partition function evaluation, the volume of any flow polytope is given by the number of integer points of another.

Recall that two polytopes $P_1 \subseteq \mathbb{R}^{k_1}$ and $P_2 \subseteq \mathbb{R}^{k_2}$ are *integrally equivalent* if there is an affine transformation $T:\mathbb{R}^{k_1}\to\mathbb{R}^{k_2}$ that is a bijection $P_1 \to P_2$ and a bijection $\operatorname{aff}(P_1)\cap\mathbb{Z}^{k_1} \to \operatorname{aff}(P_2)\cap\mathbb{Z}^{k_2}$. Integrally equivalent polytopes have the same face lattice, volume, and Ehrhart polynomial.

Given a graph G and a set S of its edges, we use the notation G/S to denote the graph obtained from G by contracting the edges in S (and deleting loops). We use the notation $G\backslash S$ to denote the graph obtained from G by deleting the edges in S. For a set V of vertices of G, we also use the notation $G\backslash V$ to denote the graph obtained from G by deleting the vertices in V together with all edges incident to them. When S or V consists of just one element, we simply write G/e or $G\backslash v$.

While simple to prove, the following lemma is important. We leave its proof to the reader.

Lemma 2.2. *Let G be a graph on $[0,n]$. Assume vertex j has only one outgoing edge e and netflow $a_j \geqslant 0$. If e is directed from j to k, then*

$$\mathcal{F}_G(a_0,\dots,a_n) \text{ and } \mathcal{F}_{G/e}(a_0,\dots,a_{j-1},a_{j+1},a_{j+2},\dots,a_{k-1},a_k+a_j,a_{k+1},\dots,a_n)$$

are integrally equivalent. An analogous result holds if j has only one incoming edge and $a_j \leqslant 0$.

2.2. Dissections of Flow Polytopes.

For graphs with a special source and sink, there is a systematic way to dissect the flow polytope $\mathcal{F}_{\widetilde{G}}$ studied in [11]. Let G be a graph on $[0,n]$, and define $\widetilde{G}$ on $[0,n]\cup\{s,t\}$ with s being the smallest vertex and t the biggest vertex by setting $E(\widetilde{G}) = E(G)\cup\{(s,i),(i,t) \mid i\in[0,n]\}$. Although we defined the flow polytope $\mathcal{F}_G(\boldsymbol{a})$ above only when G was a graph on $[0,n]$, the definition (1) makes sense with any totally ordered vertex set. For graphs $\widetilde{G}$, we take the ordering $s<0<1<\cdots<n<t$. The systematic dissections of $\mathcal{F}_{\widetilde{G}}$ can be expressed either in the language of the subdivision algebra or in terms of reduction trees [9, 10, 11]. We use the language of reduction trees.

Let G_0 be a graph on $[0,n]$ with edges (i,j) and (j,k) for some $i<j<k$. By a *reduction* on G_0, we mean the construction of three new graphs G_1, G_2 and G_3 on

$[0, n]$ given by

$$E(G_1) = E(G_0) \backslash \{(j, k)\} \cup \{(i, k)\}$$
$$E(G_2) = E(G_0) \backslash \{(i, j)\} \cup \{(i, k)\} \quad (2)$$
$$E(G_3) = E(G_0) \backslash \{(i, j), (j, k)\} \cup \{(i, k)\}.$$

See Figure 1 for an example reduction. We say G_0 *reduces* to G_1, G_2 and G_3. We also say that the above reduction is at vertex j, on the edges (i, j) and (j, k). The following proposition explains how the process of taking reductions dissects the flow polytope $\mathcal{F}_{G_0}$ into other flow polytopes.

Proposition 2.3. *Let G_0 be a graph on $[0, n]$ which reduces to G_1, G_2 and G_3 as above. Then for each $m \in [3]$, there is a polytope Q_m integrally equivalent to $\mathcal{F}_{\widetilde{G}_m}$ such that Q_1 and Q_2 subdivide $\mathcal{F}_{\widetilde{G}_0}$ and intersect in Q_3. That is, the polytopes Q_1, Q_2, and Q_3 satisfy*

$$\mathcal{F}_{\widetilde{G}_0} = Q_1 \bigcup Q_2 \text{ with } Q_1^o \bigcap Q_2^o = \varnothing \text{ and } Q_1 \bigcap Q_2 = Q_3.$$

Moreover, Q_1 and Q_2 have the same dimension as $\mathcal{F}_{\widetilde{G}_0}$, and Q_3 has dimension one less.

Proof. Let r_1 and r_2 denote the edges of G_0 from i to j and from j to k respectively that were used in the reduction. Viewing $\mathbb{R}^{\#E(\widetilde{G}_0)}$ as functions $f : E(\widetilde{G}_0) \to \mathbb{R}$, cut $\mathcal{F}_{\widetilde{G}_0}$ with the hyperplane H defined by the equation $f(r_1) = f(r_2)$. Let Q_1 be the intersection of $\mathcal{F}_{\widetilde{G}_0}$ with the positive half-space $f(r_1) \geqslant f(r_2)$, let Q_2 be the intersection of $\mathcal{F}_{\widetilde{G}_0}$ with the negative half-space $f(r_1) \leqslant f(r_2)$, and let Q_3 be the intersection of $\mathcal{F}_{\widetilde{G}_0}$ with the hyperplane H. See Figure 1 for an illustration of the integral equivalence between Q_m and $\mathcal{F}_{\widetilde{G}_m}$. Notice that since we are doing the reductions on the edges of G_0 (as opposed to on the edges incident to the source or sink in $\widetilde{G}_0$), it follows that the hyperplane H meets $\mathcal{F}_{\widetilde{G}_0}$ in its interior, giving the claims on the dimensions of each Q_m. □

Iterating this subdivision process will produce a dissection of $\mathcal{F}_{\widetilde{G}_0}$ into simplices. This process can be encoded using a reduction tree. A *reduction tree* of G is constructed as follows. Let the root node of the tree be labeled by G. If a node has any children, then it has three children obtained by performing a reduction on that node and labeling the children with the graphs defined in (2). Continue this process until the graphs labeling the leaves of the tree cannot be reduced. See Figure 2 for an example.

Fix a reduction tree $\mathcal{R}$ of G. Let L be a graph labeling one of the leaves in $\mathcal{R}$. Lemma 2.2 implies that $\mathcal{F}_{\widetilde{L}}$ is integrally equivalent to the standard simplex, so the flow polytopes of the graphs labeling the leaves of $\mathcal{R}$ dissect $\mathcal{F}_{\widetilde{G}}$ into unimodular simplices. Consequently, all dissections we consider in this paper will be dissections into unimodular simplices. By *full-dimensional leaves* of $\mathcal{R}$, we mean the leaves L with $\#E(L) = \#E(G)$. By *lower-dimensional leaves* we mean all other leaves L of $\mathcal{R}$. Note that the full-dimensional leaves correspond to top-dimensional simplices in the dissection of $\mathcal{F}_{\widetilde{G}}$, and the lower-dimensional leaves index intersections of the top-dimensional simplices. The dissections produced by a reduction tree are not generally triangulations, due to how leaves on different sides of the reduction tree can intersect.

Recall the *normalized volume* of a polytope is the usual Euclidean volume scaled by the volume of a unimodular simplex in the affine span of the polytope. Since all simplices $\mathcal{F}_{\widetilde{L}}$ of leaves in a reduction tree are unimodular, we have the following result.

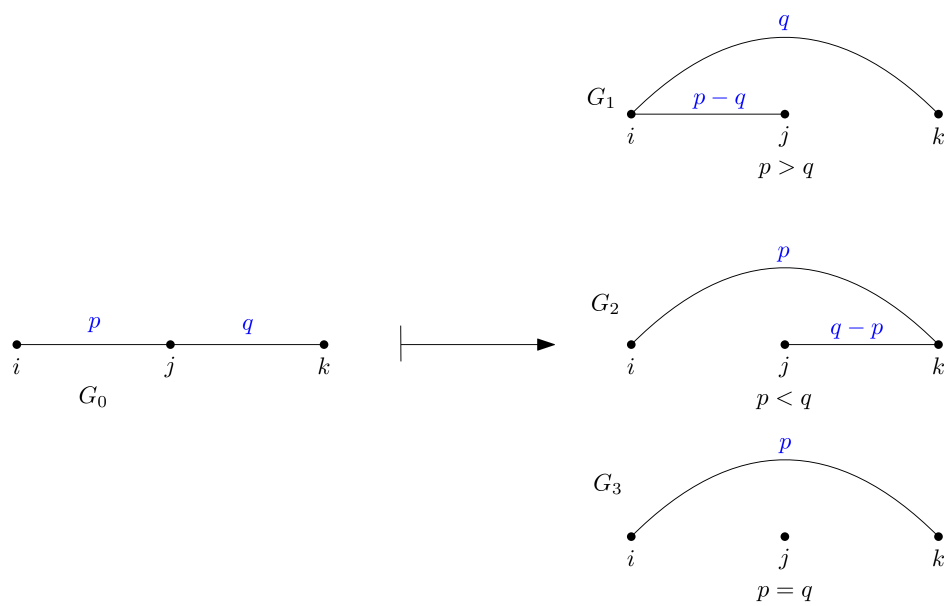


Figure 1. An illustration of the integral equivalence between Q_m and $\mathcal{F}_{\widetilde{G}_m}$ for $m \in [3]$ used Proposition 2.3.

Corollary 2.4. *The normalized volume of* $\mathcal{F}_{\widetilde{G}}$ *equals the number of full-dimensional leaves in any reduction tree of* G.

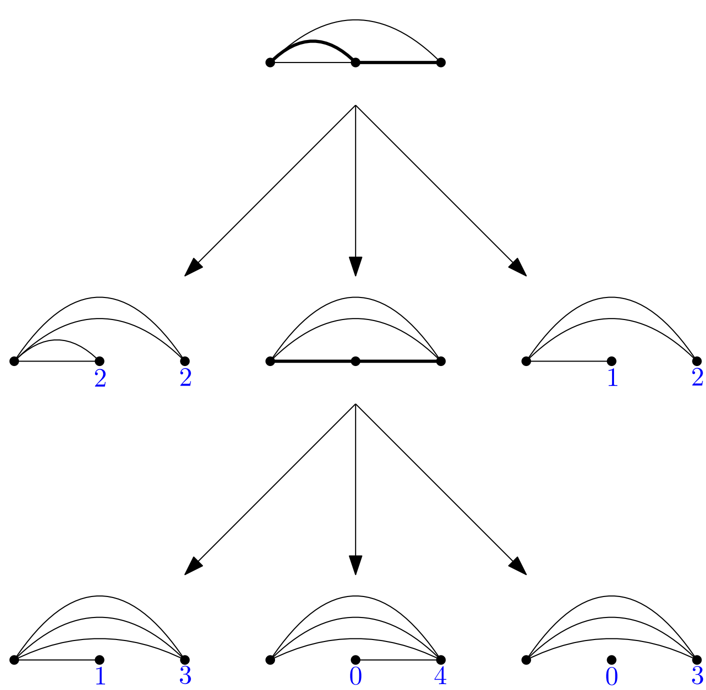


Figure 2. A reduction tree for a graph on three vertices. The edges involved in each reduction are shown in bold. The left-degree sequences of the leaves are displayed below each leaf.

2.3. Left-Degree Sequences. Let G be a graph on $[0,n]$, and let $\mathcal{R}$ be a reduction tree of G. For each leaf L of $\mathcal{R}$, consider the *left-degree sequence*

$$(\mathrm{indeg}_L(1), \mathrm{indeg}_L(2), \dots, \mathrm{indeg}_L(n))\,.$$

By *full-dimensional* sequences, we will mean left-degree sequences of full-dimensional leaves of $\mathcal{R}$. Although the actual leaves of a reduction tree are dependent on the individual reductions performed, we prove in Theorem A that the left-degree sequences are not.

Example 2.5. Any reduction tree of K_4 has the full-dimensional left-degree sequences

$$\{(0,0,6),(0,0,6),(0,1,5),(0,1,5),(0,2,4),(0,2,4),(0,3,3),(1,0,5),(1,1,4),(1,2,3)\}.$$

3. Triangular arrays and left-degree sequences

In this section, we expand the technique described in [11] that characterized left-degree sequences of full-dimensional leaves in a specific reduction tree of any graph. Given a graph G, we construct this reduction tree $\mathcal{T}(G)$. We give a characterization of the left-degree sequences of all leaves of this reduction tree, not just the full-dimensional ones. We then connect this characterization to flow polytopes. The main result of this section is Corollary 3.16, where we provide a partition of the left-degree sequences of $\mathcal{T}(G)$ and biject each block to the set of integer points in a flow polytope.

For simplicity, throughout this section we restrict to the case where G is a simple graph on the vertex set $[0,n]$. The set $\mathrm{Sol}_G(F)$ is defined in Definition 3.6 for simple graphs. We address the more technical general case in Section 6 and prove Theorem A.

We begin by generalizing [11, Lemma 3] to include the descriptions of the lower dimensional leaves of reductions performed at a special vertex v. The proof is a straightforward generalization of that of [11, Lemma 3], illustrated in Figure 3. The key to the proof is the special reduction order, whereby we always perform a reduction on the longest edges possible that are incident to the vertex at which we are reducing (the length of an edge being the absolute value of the difference of its vertex labels).

Lemma 3.1. *Assume G has a distinguished vertex v with p incoming edges and one outgoing edge (v,u). If we perform all reductions possible which involve only edges incident to v in the special reduction order, then we obtain graphs H_i for $i \in [p+1]$, and K_j for $j \in [p]$, with*

$$(\mathrm{indeg}_{H_i}(v), \mathrm{indeg}_{H_i}(u)) = (p+1-i,\, \mathrm{indeg}_G(u) - 1 + i),$$
$$(\mathrm{indeg}_{K_j}(v), \mathrm{indeg}_{K_j}(u)) = (p-j,\, \mathrm{indeg}_G(u) - 1 + j).$$

Note that the previous lemma vacuously yields only $H_1 = G$ if $p = 0$.

We now construct a specific reduction tree $\mathcal{T}(G)$ and characterize the left-degree sequences of its leaves. Denote by I_i the set of incoming edges to vertex i in G. Let V_i be the set of vertices k with $(k,i) \in I_i$, and let $G[0,i]$ be the restriction of G to the vertices $[0,i]$. For any reduction tree $\mathcal{R}$, by $\mathrm{InSeq}(\mathcal{R})$ we mean the multiset of left-degree sequences of the leaves of $\mathcal{R}$. Since we will build $\mathcal{T}(G)$ inductively from $\mathcal{T}(H)$ for smaller graphs H, it is convenient to let $\mathrm{InSeq}^n(\mathcal{R})$ denote the multiset $\mathrm{InSeq}(\mathcal{R})$ with each sequence padded on the right with zeros to have length n.

We proceed using the following algorithm, analogous to the one described in [11].

- For the base case, define the reduction tree $\mathcal{T}(G[0,1])$ to be the single leaf $G[0,1]$. Hence,

$$\mathrm{InSeq}(\mathcal{T}(G[0,1])) = \{(\mathrm{indeg}_G(1))\}.$$

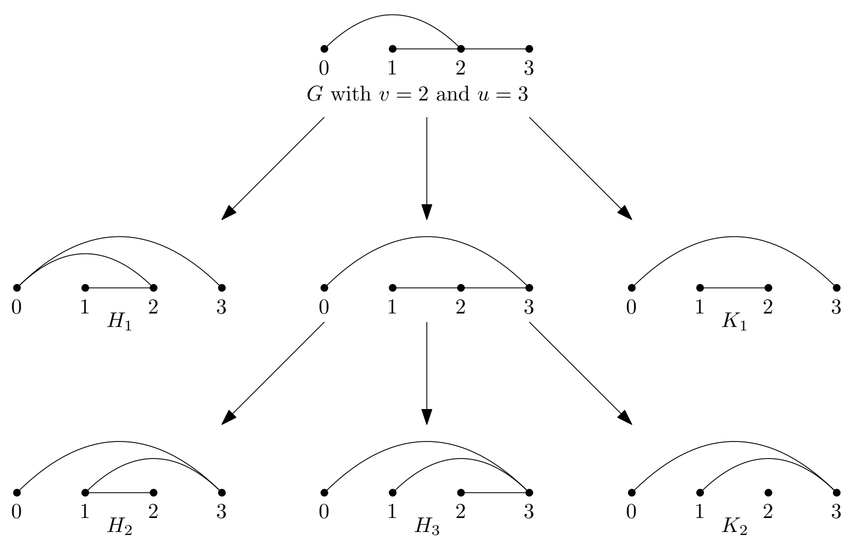


FIGURE 3. The graphs H_i and K_j of Lemma 3.1.

- Having built $\mathcal{T}(G[0,i])$, construct the reduction tree $\mathcal{T}(G[0,i+1])$ from $\mathcal{T}(G[0,i])$ by appending the vertex $i+1$ and the edges I_{i+1} to all graphs in $\mathcal{T}(G[0,i])$ and then performing reductions at each vertex in V_{i+1} on all graphs corresponding to the leaves of $\mathcal{T}(G[0,i])$ in the special reduction order as described above Lemma 3.1.
- Let $V_{i+1} = \{i_1 < i_2 < \cdots < i_k\}$ and let $(s_1,\dots,s_n) \in \mathrm{InSeq}^n(\mathcal{T}(G[0,i]))$. Applying Lemma 3.1 to each of the vertices $i_1,\dots,i_k$, we see that the leaves of $\mathcal{T}(G[0,i+1])$ which are descendants of the graph with n-left-degree sequence $(s_1,\dots,s_n)$ in $\mathcal{T}(G[0,i])$ will have n-left-degree sequences exactly given by

$$(s_1,\dots,s_n) + v^{i+1}[i_1] + \cdots + v^{i+1}[i_k]$$

where $v^{i+1}[i_l] \in S_1(i_l) \cup S_2(i_l)$ and S_1, S_2 are given by

$$S_1(i_l) = \left\{ (c_1,\dots,c_n) \mid c_j = \begin{cases} 0 & \text{if } j \notin \{i_l, i+1\}, \\ 1-s & \text{if } j = i_l, \\ s-1 & \text{if } j = i+1, \end{cases} \quad \text{for } s \in [s_{i_l}+1] \right\}$$

$$S_2(i_l) = \left\{ (c_1,\dots,c_n) \mid c_j = \begin{cases} 0 & \text{if } j \notin \{i_l, i+1\}, \\ -s & \text{if } j = i_l, \\ s-1 & \text{if } j = i+1, \end{cases} \quad \text{for } s \in [s_{i_l}] \right\}.$$

DEFINITION 3.2. *For a simple graph G on $[0,n]$, denote by $\mathcal{T}(G)$ the specific reduction tree constructed using the algorithm described above. Denote by $\mathrm{LD}(G)$ the multiset $\mathrm{InSeq}(\mathcal{T}(G))$.*

We prove the following surprising property of $\mathrm{LD}(G)$ in Section 6, where we drop the assumption that G be simple.

THEOREM A. *Let G be any (not necessarily simple) graph on $[0,n]$. Then for any reduction tree $\mathcal{R}$ of G,*

$$\mathrm{LD}(G) = \mathrm{InSeq}(\mathcal{R}).$$

Definition 3.3. *To each leaf L of $\mathcal{T}(G)$, associate the triangular array of numbers* $\mathrm{Arr}(L)$ *given by*

$$\begin{array}{llllll} a_{n1} & a_{n-1,1} & \cdots & a_{31} & a_{21} & a_{11} \\ a_{n2} & a_{n-1,2} & \cdots & a_{32} & a_{22} & \\ \vdots & \vdots & \iddots & & & \\ a_{n,n-1} & a_{n-1,n-1} & & & & \\ a_{nn} & & & & & \end{array}$$

where $(a_{i1}, a_{i2}, \ldots, a_{ii})$ is the left-degree sequence of the leaf of $\mathcal{T}(G[0,i])$ preceding (or equaling if $i = n$) L in the construction of $\mathcal{T}(G)$.

Theorem 3.4 ([11], Theorem 4). *The arrays* $\mathrm{Arr}(L)$ *for full-dimensional leaves L of $\mathcal{T}(G)$ are exactly the nonnegative integer solutions in the variables*

$$\{a_{ij} \mid 1 \leqslant j \leqslant i \leqslant n\}$$

to the constraints

- $a_{11} = \#E(G[0,1])$
- $a_{ij} \leqslant a_{i-1,j}$ *if* $(j,i) \in E(G)$
- $a_{ij} = a_{i-1,j}$ *if* $(j,i) \notin E(G)$
- $a_{ii} = \#E(G[0,i]) - \sum_{k=1}^{i-1} a_{ik}$.

Example 3.5. If G is the graph on $[0,4]$ with

$$E(G) = \{(0,1),(0,2),(1,2),(2,3),(2,4),(3,4)\},$$

then Theorem 3.4 gives the inequalities

$$\begin{aligned} 0 &\leqslant a_{41} = a_{31} = a_{21} \leqslant a_{11} = 1 \\ 0 &\leqslant a_{42} \leqslant a_{32} \leqslant a_{22} = 3 - a_{21} \\ 0 &\leqslant a_{43} \leqslant a_{33} = 4 - a_{31} - a_{32} \\ 0 &\leqslant a_{44} = 6 - a_{41} - a_{42} - a_{42}. \end{aligned}$$

The first columns

$$(a_{41}, a_{42}, a_{43}, a_{44})$$

of solutions to these inequalities are exactly the full-dimensional left-degree sequences of G.

Given a graph G, we write the constraints specified in Theorem 3.4 in the form shown in Example 3.5 and call them the *triangular constraint array* of G. We proceed by generalizing triangular constraint arrays to encode the lower-dimensional leaves of $\mathcal{T}(G)$ as well.

Definition 3.6. *Denote by* $\mathrm{Tri}_G(\varnothing)$*, or when the context is clear, by* $\mathrm{Tri}(\varnothing)$*, the triangular constraint array of G. For each subset $F \subseteq E(G\backslash 0)$ (recall that G is simple in this section), define a constraint array* $\mathrm{Tri}(F)$ *by modifying* $\mathrm{Tri}(\varnothing)$ *as follows: for each $(j,i) \in F$ and each ordered pair (m,j) with $n \geqslant m \geqslant i$, replace each occurrence (anywhere in the inequalities) of a_{mj} by $a_{mj} + 1$ and add 1 to the constant at the leftmost edge of row j. Denote by* $\mathrm{Sol}_G(F)$*, or when the context is clear, by* $\mathrm{Sol}(F)$*, the collection of all integer solution arrays to the constraints* $\mathrm{Tri}(F)$*.*

Example 3.7. With G as in Example 3.5 and $F = \{(2,3),(2,4),(3,4)\}$, we have

$$\mathrm{Tri}(F) : \begin{aligned} 0 &\leqslant a_{41} = a_{31} = a_{21} \leqslant a_{11} = 1 \\ 2 &\leqslant a_{42} + 2 \leqslant a_{32} + 1 \leqslant a_{22} = 3 - a_{21} \\ 1 &\leqslant a_{43} + 1 \leqslant a_{33} = 3 - a_{31} - a_{32} \\ 0 &\leqslant a_{44} = 3 - a_{41} - a_{42} - a_{43}. \end{aligned}$$

The characterization of $\mathrm{LD}(G) = \mathrm{InSeq}(\mathcal{T}(G))$ given in the construction of $\mathcal{T}(G)$ implies the following theorem.

Theorem 3.8. *The leaves of $\mathcal{T}(G)$ are in bijection with the multiset union of solutions to the arrays* $\mathrm{Tri}(F)$*, that is*

$$\{\mathrm{Arr}(L) \mid L \text{ is a leaf of } \mathcal{T}(G)\} = \bigcup_{F \subseteq E(G\backslash 0)} \mathrm{Sol}_G(F).$$

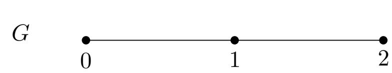
In particular, $\mathrm{LD}(G)$ *is the (multiset) image of the right-hand side under the map that takes a triangular array to its first column $(a_{n1}, \dots, a_{nn})$.*

Definition 3.9. *For any $F \subseteq E(G\backslash 0)$, denote by* $\mathrm{LD}(G, F)$ *the submultiset of* $\mathrm{LD}(G)$ *consisting of sequences occurring as the first column of an array in* $\mathrm{Sol}(F)$*.*

As a consequence of Theorem 3.8,

$$\mathrm{LD}(G) = \bigcup_{F \subseteq E(G\backslash 0)} \mathrm{LD}(G, F).$$

Remark 3.10. Combinatorially, we can think of $\mathrm{LD}(G, F)$ in the following way. Construct the reduction tree $\mathcal{T}(G)$ of G. Take any graph H appearing as a node of $\mathcal{T}(G)$. Let H have descendants H_1, H_2 and H_3 in $\mathcal{T}(G)$ obtained by the reduction on edges (i, j) and (j, k) in H with $i < j < k$, so that H_3 has edge set $E(H)\backslash\{(i, j), (j, k)\} \cup \{(i, k)\}$. Label the edge in $\mathcal{T}(G)$ between H and H_3 by (j, k). To each leaf L of $\mathcal{T}(G)$, associate the set of all labels on the edges of the unique path from L to the root G of $\mathcal{T}(G)$. The left-degree sequences of leaves assigned a set F in this manner are exactly the elements of the multiset $\mathrm{LD}(G, F)$.



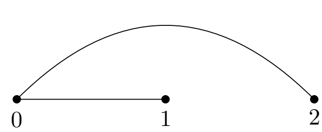
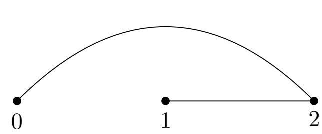
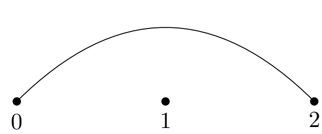
Leaves L of $\mathcal{T}(G)$	$F \subseteq E(G\backslash 0)$	$\mathrm{Tri}_G(F)$	$\mathrm{Sol}_G(F)$
 	$\emptyset$	$0 \leq a_{21} \leq a_{11} = 1$ $0 \leq a_{22} = 2 - a_{21}$	$\begin{matrix} 1 & 1 \\ 1 & \end{matrix}$ $\begin{matrix} 0 & 1 \\ 2 & \end{matrix}$
	$\{(1, 2)\}$	$1 \leq a_{21} + 1 \leq a_{11} = 1$ $0 \leq a_{22} = 2 - (a_{21} + 1)$	$\begin{matrix} 0 & 1 \\ 1 & \end{matrix}$

Figure 4. A small example demonstrating Theorem 3.8. In general, $\mathrm{Sol}_G(F)$ will be empty for many F.

To understand the multisets $\mathrm{Sol}(F)$ and $\mathrm{LD}(G,F)$, we connect the constraint arrays $\mathrm{Tri}(F)$ to flow polytopes. We begin by investigating the case where $G = K_{n+1}$ is the complete graph on $[0,n]$. Given $F \subseteq E(K_{n+1}\backslash 0)$, consider the numbers

$$f_{ij} = \#\{(j,k) \in F \mid k \leqslant i\}. \tag{3}$$

Observe that for each $F \subseteq E(K_{n+1}\backslash 0)$, $\mathrm{Tri}(F)$ is obtained from $\mathrm{Tri}(\varnothing)$ by replacing a_{ij} in $\mathrm{Tri}(\varnothing)$ by $a_{ij} + f_{ij}$ and replacing the 0 in the leftmost spot of row j by f_{nj}. Also note that $f_{jj} = 0$ for each j. Thus, $\mathrm{Tri}(F)$ is given by

$$\begin{aligned}
&f_{n1} \leqslant a_{n1} + f_{n1} \leqslant \cdots \leqslant a_{21} + f_{21} \leqslant a_{11} + f_{11} = \#E(K_{n+1}[0,1])\\
&f_{n2} \leqslant a_{n2} + f_{n2} \leqslant \cdots \leqslant a_{22} + f_{22} = \#E(K_{n+1}[0,2]) - a_{21} - f_{21}\\
&\quad\vdots \qquad\quad \vdots \qquad\quad \iddots\\
&f_{nn} \leqslant a_{nn} + f_{nn} = \#E(K_{n+1}) - \sum_{k=1}^{n-1} a_{nk} - \sum_{k=1}^{n-1} f_{nk}.
\end{aligned}$$

Note that the real solution set in variables $\{a_{ij}\}$ to $\mathrm{Tri}(F)$ is a polytope in $\mathbb{R}^{\binom{n+1}{2}}$. For any constraint array A, denote by $\mathrm{Poly}(A)$ the *polytope defined by the inequalities in A*. We now work toward showing that the polytopes $\mathrm{Poly}(\mathrm{Tri}_G(F))$ are integrally equivalent to flow polytopes. We first continue analyzing the case of the complete graph. Fix $F \subseteq E(K_{n+1}\backslash 0)$.

For $\{(i,j) \mid 1 \leqslant j < i \leqslant n\}$, we introduce (nonnegative) slack variables z_{ij} to convert the inequalities in $\mathrm{Poly}(\mathrm{Tri}(F))$ into equations Y_{ij}, given by

$$Y_{ij} : \begin{cases} a_{ij} + f_{ij} + z_{ij} = a_{i-1,j} + f_{i-1,j} & \text{if } i > j\\ \displaystyle\sum_{k=1}^{i} a_{ik} + \sum_{k=1}^{i} f_{ik} = \#E(K_{n+1}[0,i]) & \text{if } i = j. \end{cases}$$

Define an equivalent system of equations $\{Z'_{ij}\}$ by setting

$$Z'_{ij} : \begin{cases} Y_{ij} & \text{if } i > j \text{ or } i = j = 1\\ \displaystyle Y_{ij} - Y_{i-1,j-1} - \sum_{k=1}^{j-1} Y_{jk} & \text{if } i = j > 1. \end{cases}$$

We then modify each equation Z'_{ij} by rearranging negated terms to get equations Z_{ij} given by

$$Z_{ij} : \begin{cases} a_{ij} + z_{ij} = a_{i-1,j} + f_{i-1,j} - f_{ij} & \text{if } i > j\\ a_{ij} = \mathrm{indeg}_{K_{n+1}}(1) & \text{if } i = j = 1\\ \displaystyle a_{ij} = \mathrm{indeg}_{K_{n+1}}(j) + \sum_{k=1}^{j-1} z_{jk} & \text{if } i = j > 1. \end{cases}$$

We now construct a graph $\mathrm{Gr}(K_{n+1})$ whose flow polytope will be given by the equations Z_{ij} (plus the conditions $z_{ij} \geqslant 0$). Let the vertices of $\mathrm{Gr}(K_{n+1})$ be

$$\{v_{ij} \mid 1 \leqslant j \leqslant i \leqslant n\} \cup \{v_{n+1,n+1}\}$$

with the ordering $v_{11} < v_{21} < \cdots < v_{n1} < v_{22} < \cdots < v_{nn} < v_{n+1,n+1}$.

Let the edges of $\mathrm{Gr}(K_{n+1})$ be labeled by the flow variables a_{ij} and z_{ij}. Set $E(\mathrm{Gr}(K_{n+1})) = E_a \cup E_z$ where

E_a consists of edges $a_{ij} : v_{ij} \to v_{i+1,j}$ for $1 \leqslant j \leqslant i \leqslant n$ and

E_z consists of edges $z_{ij} : v_{ij} \to v_{ii}$ for $1 \leqslant j < i \leqslant n$

and we take indices $(n+1, j)$ to refer to $(n+1, n+1)$.

To define the netflow vector $\boldsymbol{a}^F_{K_{n+1}}$, we assign netflow $\mathrm{indeg}_{K_{n+1}}(j)$ to vertices v_{jj} with $j < n+1$, we assign netflow

$$-\#E(K_{n+1}) + \sum_{k=1}^{n-1} f_{nk}$$

to $v_{n+1,n+1}$, and we assign netflow $f_{i-1,j} - f_{ij}$ to each remaining vertex v_{ij}.

The netflow vector $\boldsymbol{a}^F_{K_{n+1}}$ is given by reading each row of the triangular array

$$\begin{array}{lllll} f_{n-1,1} - f_{n1} & f_{n-2,1} - f_{n-1,1} & \cdots & f_{11} - f_{21} & \mathrm{indeg}_{K_{n+1}}(1) \\ f_{n-1,2} - f_{n2} & \cdots & f_{22} - f_{32} & \mathrm{indeg}_{K_{n+1}}(2) & \\ \vdots & \iddots & & & \\ \mathrm{indeg}_{K_{n+1}}(n) & & & & \end{array}$$

right to left starting with the first row, moving top to bottom, and then appending $-\#E(K_{n+1}) + \sum_{k=1}^{n-1} f_{nk}$ at the end.

Lemma 3.11. *The polytopes*

$$\mathcal{F}_{\mathrm{Gr}(K_{n+1})}(\boldsymbol{a}^F_{K_{n+1}}) \text{ and } \mathrm{Poly}(\mathrm{Tri}(F))$$

are integrally equivalent.

Proof. By construction, the flow equation at vertex v_{ij} in $\mathrm{Gr}(K_{n+1})$ is exactly the equation Z_{ij} for $(i,j) \neq (n+1,n+1)$. At $v_{n+1,n+1}$, the flow equation is Y_{nn}, which follows from the equations Z_{ij} and adds no additional restrictions. The result now follows from the fact that the transformation from $\{Y_{ij}\}_{i,j}$ to $\{Z_{ij}\}_{i,j}$ was unimodular. □

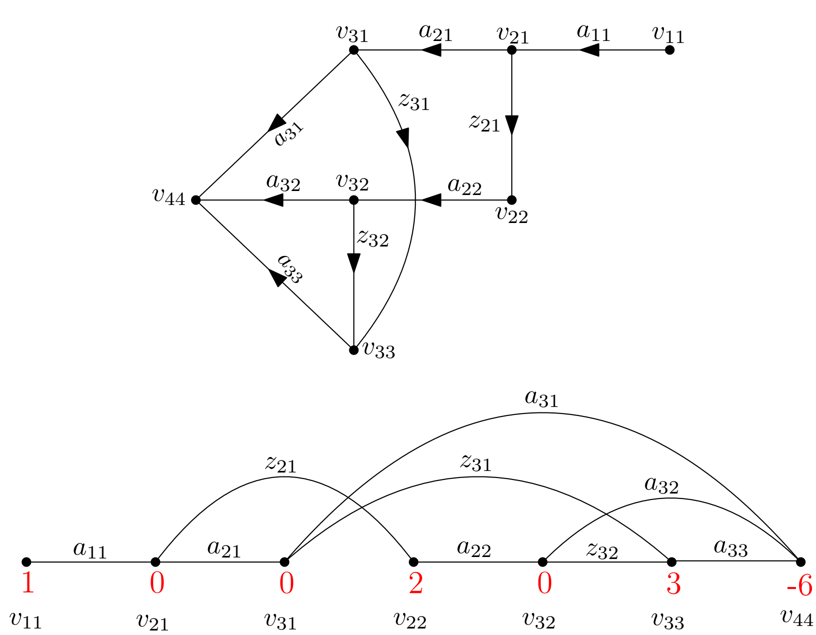


Figure 5. Two drawings of the graph $\mathrm{Gr}(K_{n+1})$ of Lemma 3.11. The lower drawing has the netflow vector $\boldsymbol{a}^{\varnothing}_{K_{n+1}}$.

We now generalize Lemma 3.11 to any simple graph G on $[0,n]$. Note that for $F \subseteq E(G\backslash 0)$, $\mathrm{Tri}_G(F)$ can be obtained from $\mathrm{Tri}_{K_{n+1}}(F)$ by turning certain inequalities into equalities and changing all occurrences of $\#E(K_{n+1}[0,j])$ to $\#E(G[0,j])$ for each j. In terms of $\{Z_{ij}\}_{i,j}$, this amounts to setting $z_{ij}=0$ whenever $(j,i)\notin E(G)$. Relative to the graph $\mathrm{Gr}(K_{n+1})$, this is equivalent to deleting the edges labeled z_{ij} for $(j,i)\notin E(G)$. Thus, we have the following extension of $\mathrm{Gr}(K_{n+1})$.

Definition 3.12. *For a simple graph G on $[0,n]$ define a graph $\mathrm{Gr}(G)$ on vertices*

$$\{v_{ij}\mid 1\leqslant j\leqslant i\leqslant n\}\cup\{v_{n+1,n+1}\}$$

ordered $v_{11}<v_{21}<\cdots<v_{n1}<v_{22}<\cdots<v_{nn}<v_{n+1,n+1}$ and with edges $E_a\cup E_z$ where

$$E_a \text{ consists of edges } a_{ij}: v_{ij}\to v_{i+1,j} \text{ for } 1\leqslant j\leqslant i\leqslant n \text{ and}$$
$$E_z \text{ consists of edges } z_{ij}: v_{ij}\to v_{ii} \text{ for } (j,i)\in E(G\backslash 0).$$

For any $F\subseteq E(G\backslash 0)$, we define a netflow vector $\boldsymbol{a}_G^F$ for $\mathrm{Gr}(G)$ by reading each row of the triangular array

$$\begin{array}{ccccc} f_{n-1,1}-f_{n1} & f_{n-2,1}-f_{n-1,1} & \cdots & f_{11}-f_{21} & \mathrm{indeg}_G(1)\\ f_{n-1,2}-f_{n2} & \cdots & f_{22}-f_{32} & \mathrm{indeg}_G(2) & \\ \vdots & \iddots & & & \\ \mathrm{indeg}_G(n) & & & & \end{array}$$

right to left starting with the first row, moving top to bottom, and then appending $-\#E(G)+\sum_{k=1}^{n-1}f_{nk}$ at the end, where again, $f_{ij}=\#\{(j,k)\in F\mid k\leqslant i\}$.

We now have the following extension of Lemma 3.11 to all simple graphs.

Proposition 3.13. *Let G be a simple graph on $[0,n]$ and $F\subseteq E(G\backslash 0)$. Then, $\mathrm{Poly}(\mathrm{Tri}_G(F))$ is integrally equivalent to $\mathcal{F}_{\mathrm{Gr}(G)}(\boldsymbol{a}_G^F)$. In particular, the multiset of solutions $\mathrm{Sol}_G(F)$ to $\mathrm{Tri}_G(F)$ consists precisely of the projections of integral flows on $\mathrm{Gr}(G)$ with netflow $\boldsymbol{a}_G^F$ onto the edges labeled $\{a_{ij}\}$.*

Example 3.14. Let G be the graph on $[0,4]$ with

$$E(G)=\{(0,1),(0,2),(1,2),(2,3),(2,4),(3,4)\}$$

and $F=\{(2,3)\}$. The graph $\mathrm{Gr}(G)$ and its netflow vector $\boldsymbol{a}_G^F$ are shown in Figure 6.

Observe that contracting the edges $\{a_{11},a_{21},a_{31},a_{22},a_{32},a_{33}\}$ in $\mathrm{Gr}(G)$ yields the graph shown in Figure 7, which is exactly $\widetilde{G}\backslash\{s,0\}$. The next result shows that this occurs in general.

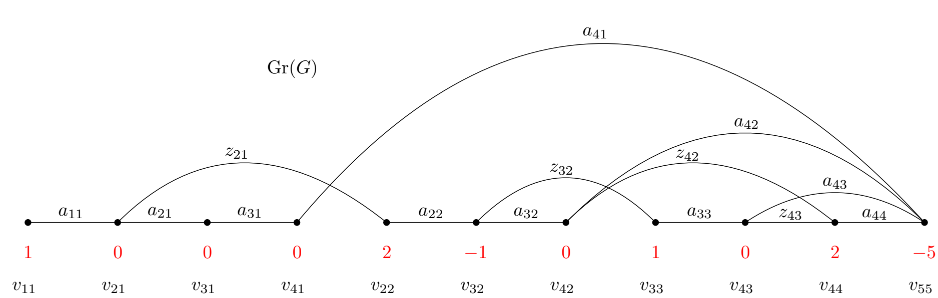


Figure 6. The graph $\mathrm{Gr}(G)$ when $E(G)=\{(0,1),(0,2),(1,2),(2,3),(2,4),(3,4)\}$.

For a graph G and a subset $F\subseteq E(G\backslash 0)$, view F as a subgraph of G on the same vertex set. Note that for each j,

$$f_{nj}=\#\{(j,k)\in F\mid k\leqslant n\}=\mathrm{outdeg}_F(j)$$

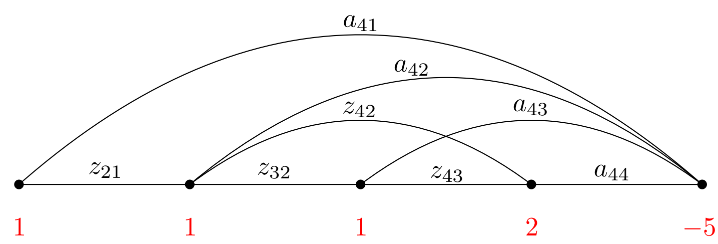


FIGURE 7. The graph $\mathrm{Gr}(G)/\{a_{11}, a_{21}, a_{31}, a_{22}, a_{32}, a_{33}\}$

and the number

$$-\#E(G) + \sum_{k=1}^{n-1} f_{nk}$$

appearing as the last entry of $\boldsymbol{a}_G^F$ equals $-\#E(G\backslash F)$.

THEOREM 3.15. *Let G be a simple graph on $[0, n]$ and $F \subseteq E(G\backslash 0)$. Then, the flow polytopes*

$$\mathcal{F}_{\mathrm{Gr}(G)}\left(\boldsymbol{a}_G^F\right)$$

and

$$\mathcal{F}_{\widetilde{G}\backslash\{s,0\}}\left(\mathrm{indeg}_G(1) - \mathrm{outdeg}_F(1), \dots, \mathrm{indeg}_G(n) - \mathrm{outdeg}_F(n), -\#E(G\backslash F)\right)$$

are integrally equivalent.

Proof. First, note that in $\mathrm{Gr}(G)$, the edges $\{a_{ij} \mid i < n\}$ are each the only edges incoming to their target vertex. Contracting these edges via Lemma 2.2 identifies vertices v_{ij} and $v_{i'j}$. Label the representative vertices v_{jj} by j for $j \in [n]$ and $v_{n+1,n+1}$ by t. The remaining edges are

$$z_{ij} : j \to i \text{ for } (j, i) \in E(G) \text{ and } a_{nj} : j \to t \text{ for } j \in [n],$$

which are exactly the edges of $\widetilde{G} - \{s, 0\}$.

Viewing the netflow vector $\boldsymbol{a}_G^F$ as the array

$$\begin{array}{lllll} f_{n-1,1} - f_{n1} & f_{n-2,1} - f_{n-1,1} & \cdots & f_{11} - f_{21} & \mathrm{indeg}_G(1) \\ f_{n-1,2} - f_{n2} & \cdots & f_{22} - f_{32} & \mathrm{indeg}_G(2) & \\ \quad\vdots & \iddots & & & \\ \mathrm{indeg}_G(n) & & & & \\ -\#E(G\backslash F), & & & & \end{array}$$

Lemma 2.2 implies the entries of the netflow vector after contracting are given by reading the sums of each row from top to bottom. □

Recall from Definition 3.9 that $\mathrm{LD}(G, F)$ is the multiset of left-degree sequences in $\mathrm{InSeq}(\mathcal{T}(G))$ occurring as the first column $(a_{n1}, \dots, a_{nn})$ of an array in $\mathrm{Sol}(F)$. We now arrive at the main result of this section.

COROLLARY 3.16. *Let G be a simple graph on $[0, n]$ and $F \subseteq E(G\backslash 0)$. If $\boldsymbol{b}_G^F$ is the vector*

$$\boldsymbol{b}_G^F = \left(\mathrm{indeg}_G(1) - \mathrm{outdeg}_F(1), \dots, \mathrm{indeg}_G(n) - \mathrm{outdeg}_F(n), -\#E(G\backslash F)\right)$$

and ψ is the map that takes a flow on $\widetilde{G}\backslash\{s, 0\}$ to the tuple of its values on the edges $\{(j, t) \mid j \in [n]\}$, then $\mathrm{LD}(G, F)$ equals the (multiset) image under ψ of all integral flows on $\widetilde{G}\backslash\{s, 0\}$ with netflow vector $\boldsymbol{b}_G^F$.

In particular, $\mathrm{LD}(G,F)$ *is in bijection with integral flows on* $\widetilde{G}\backslash\{s,0\}$ *with netflow* $\boldsymbol{b}_G^F$*.*

We note that the preceding result implies a formula for the Ehrhart polynomial of flow polytopes of graphs with special source and sink vertices. In particular, a special case of Theorem 2.1 follows readily.

Theorem 3.17. *Let* G *be a simple graph on* $[0,n]$ *and let* $d_i = \mathrm{indeg}_G(i)$*. Then, the normalized volume of the flow polytope of* $\widetilde{G}$ *is*

$$\mathrm{Vol}\,\mathcal{F}_{\widetilde{G}} = K_{\widetilde{G}\backslash\{s,0\}}\left(d_1,\dots,d_n,\,-\#E(G)\right). \tag{4}$$

Moreover, the Ehrhart polynomial of $\mathcal{F}_{\widetilde{G}}$ *is*

$$\mathrm{Ehr}(\mathcal{F}_{\widetilde{G}},t) = (-1)^d\sum_{i=0}^{d}(-1)^i\left(\sum_{\substack{F\subseteq E(G\backslash 0)\\ \#F=d-i}} K_{\widetilde{G}\backslash\{s,0\}}\left(\boldsymbol{b}_G^F\right)\right)\binom{t+i}{i}, \tag{5}$$

where $\boldsymbol{b}_G^F = (\mathrm{indeg}_G(1)-\mathrm{outdeg}_F(1),\dots,\,\mathrm{indeg}_G(n)-\mathrm{outdeg}_F(n),-\#E(G\backslash F))$ *and* $d=\#E(\widetilde{G})-\#V(\widetilde{G})+1$ *is the dimension of* $\mathcal{F}_{\widetilde{G}}$*.*

Proof. From the dissection of $\mathcal{F}_{\widetilde{G}}$ obtained via the reduction tree $\mathcal{T}(G)$, it follows that $\mathrm{Vol}\,\mathcal{F}_{\widetilde{G}}$ is the number of full-dimensional left-degree sequences. By Corollary 3.16, these are in bijection with the integer points in the flow polytope $\mathcal{F}_{\widetilde{G}\backslash\{s,0\}}(d_1,\dots,d_n,-\#E(G))$, proving (4).

To prove (5) note that $\mathcal{F}^\circ_{\widetilde{G}} = \bigsqcup_{\sigma^\circ\in D_{\mathcal{T}(G)}}\sigma^\circ$, where $D_{\mathcal{T}(G)}$ is the set of open simplices corresponding to the leaves of the reduction tree $\mathcal{T}(G)$. Then,

$$\mathrm{Ehr}(\mathcal{F}^\circ_{\widetilde{G}},t) = \sum_{\sigma^\circ\in D_{\mathcal{T}(G)}}\mathrm{Ehr}(\sigma^\circ,t).$$

Since all simplices in $D_{\mathcal{T}(G)}$ are unimodular, it follows that for a k-dimensional simplex $\sigma^\circ\in D_{\mathcal{T}(G)}$,

$$\mathrm{Ehr}(\sigma^\circ,t) = \mathrm{Ehr}(\Delta^\circ,t),$$

where Δ is the standard k-simplex. By [3, Theorem 2.2], $\mathrm{Ehr}(\Delta^\circ,t)=\binom{t-1}{k}$. Thus,

$$\mathrm{Ehr}(\mathcal{F}^\circ_{\widetilde{G}},t) = \sum_{i=0}^{\infty} f_i\binom{t-1}{i},$$

where f_i is the number of i-simplices in $D_{\mathcal{T}(G)}$. For $i\in[0,d]$,

$$f_i = \sum_{\substack{F\subseteq E(G\backslash 0)\\ \#F=d-i}}\#LD(G,F).$$

Corollary 3.16 then implies

$$f_i = \sum_{\substack{F\subseteq E(G\backslash 0)\\ \#F=d-i}} K_{\widetilde{G}\backslash\{s,0\}}\left(\boldsymbol{b}_G^F\right)\ \text{ for } i\in[0,d].$$

Therefore,

$$\mathrm{Ehr}(\mathcal{F}^\circ_{\widetilde{G}},t) = \sum_{i=0}^{d}\left(\sum_{\substack{F\subseteq E(G\backslash 0)\\ \#F=d-i}} K_{\widetilde{G}\backslash\{s,0\}}\left(\boldsymbol{b}_G^F\right)\right)\binom{t-1}{i}.$$

From the Ehrhart–Macdonald reciprocity [3, Theorem 4.1]

$$\mathrm{Ehr}(\mathcal{F}_{\widetilde{G}}, t) = (-1)^d \,\mathrm{Ehr}(\mathcal{F}^{\circ}_{\widetilde{G}}, -t),$$

it follows that

$$\begin{aligned}\mathrm{Ehr}(\mathcal{F}_{\widetilde{G}}, t) &= (-1)^d \sum_{i=0}^{d} \left(\sum_{\substack{F\subseteq E(G\backslash 0)\\ \#F=d-i}} K_{\widetilde{G}\backslash\{s,0\}}\left(\boldsymbol{b}_G^F\right)\right) \binom{-t-1}{i} \\ &= (-1)^d \sum_{i=0}^{d} (-1)^i \left(\sum_{\substack{F\subseteq E(G\backslash 0)\\ \#F=d-i}} K_{\widetilde{G}\backslash\{s,0\}}\left(\boldsymbol{b}_G^F\right)\right) \binom{t+i}{i}. \end{aligned}$$ □

4. Newton polytopes of left-degree polynomials

In this section, we study the Newton polytopes of polynomials $L_G(\boldsymbol{t})$ built from left-degree sequences (see Definition 4.2). We first show that each of these polynomials have SNP (Definition 4.1). Then, we investigate the Newton polytopes of their homogeneous components and certain homogeneous subcomponents. We prove that these Newton polytopes are generalized permutahedra. Our main results can be summarized as:

Theorem B. *Let G be a graph on $[0,n]$. Then the left-degree polynomial $L_G(\boldsymbol{t})$ has SNP, and the Newton polytope of each homogeneous component $L_G^k(\boldsymbol{t})$ of $L_G(\boldsymbol{t})$ of degree $\#E(G)-k$ is a generalized permutahedron.*

Theorems 4.8, 4.9 and 4.23 imply Theorem B, and contain more detail regarding the elements of Theorem B. Recall that for a polynomial $f = \sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} c_\alpha \boldsymbol{t}^\alpha$, the *Newton polytope* is

$$\mathrm{Newton}(f) = \mathrm{Conv}\left(\{\alpha \mid c_\alpha \neq 0\}\right).$$

Definition 4.1. *We say a polynomial f has* saturated Newton polytope *(SNP) if $c_\alpha \neq 0$ whenever $\alpha \in \mathrm{Newton}(f)$; that is, if the integer points of $\mathrm{Newton}(f)$ are exactly the exponents of monomials appearing in f with nonzero coefficients.*

The question of when a polynomial has SNP is a very natural one, and has recently been investigated for various polynomials from algebra and combinatorics by Monical, Tokcan and Yong in [14].

Recall from Definition 3.9 that for a simple graph G and a subset $F \subseteq E(G\backslash 0)$, $\mathrm{LD}(G,F)$ denotes the submultiset of $\mathrm{LD}(G)$ consisting of sequences occurring as the first column of an array in $\mathrm{Sol}(F)$. Just as in Section 3, for the remainder of this section we add the simplifying assumption that G has no multiple edges. All of the results of this section are also valid for graphs with multiple edges, with similar proof and notation modifications to those described in Section 6.

Definition 4.2. *Let G be a graph on $[0,n]$. For $\alpha \in \mathrm{LD}(G)$, let $\mathrm{codim}(\alpha) = \#E(G) - \sum_{i=1}^n \alpha_i$. Define the* left-degree polynomial *$L_G(\boldsymbol{t})$ in variables $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ by*

$$L_G(\boldsymbol{t}) = \sum_{\alpha\in\mathrm{LD}(G)} (-1)^{\mathrm{codim}(\alpha)} \boldsymbol{t}^\alpha.$$

Similarly, for $F \subseteq E(G\backslash 0)$, define $L_{G,F}(\boldsymbol{t})$ by

$$L_{G,F}(\boldsymbol{t}) = \sum_{\alpha\in\mathrm{LD}(G,F)} (-1)^{\mathrm{codim}(\alpha)} \boldsymbol{t}^\alpha = (-1)^{\#F} \sum_{\alpha\in\mathrm{LD}(G,F)} \boldsymbol{t}^\alpha.$$

Note that the $(-1)^{\operatorname{codim}(\alpha)}$ in Definition 4.2 has no effect on the Newton polytope. It is inherited from the definition of right-degree polynomials utilized in [5], which was designed to agree with Grothendieck polynomials.

Restating Theorem 3.8 in terms of left-degree sequences gives the multiset union decomposition

$$\mathrm{LD}(G) = \bigcup_{F \subseteq E(G\backslash 0)} \mathrm{LD}(G,F).$$

Relative to Newton polytopes, this implies

$$\text{Newton}(L_G(\boldsymbol{t})) = \text{Conv}\left(\bigcup_{F\subseteq E(G\backslash 0)} \text{Newton}\left(L_{G,F}(\boldsymbol{t})\right)\right). \tag{6}$$

We first study the polytope $\text{Newton}(L_G(\boldsymbol{t}))$, and then the component pieces $\text{Newton}\left(L_{G,F}(\boldsymbol{t})\right)$. To start, we define a new constraint array.

Definition 4.3. *Let G be a simple graph on $[0,n]$. Proceed as follows:*

- *Start with the triangular constraint array $\text{Tri}_G(\varnothing)$ of G as in Theorem 3.4.*
- *Replace the zero on the left of row j by $y_{nj}+y_{n-1,j}+\cdots+y_{j+1,j}$ for $j\in[n-1]$, so the zero on the left in row n is left unchanged.*
- *For each (i,j) with $n\geqslant i>j\geqslant 1$, replace all occurrences of a_{ij} in the array by $a_{ij}+\sum_{k=j+1}^{i} y_{kj}$.*
- *For every $(j,i)\notin E(G\backslash 0)$, set $y_{ij}=0$ throughout.*

We refer to this array as the augmented constraint array of G *and view it as having variables a_{ij} and y_{ij} subject to the additional constraints that for all $1\leqslant j<i\leqslant n$,*

$$0\leqslant y_{ij}\leqslant 1.$$

Example 4.4. If G is the graph on vertex set $[0,4]$ with

$$E(G)=\{(0,1),(0,2),(1,2),(2,3),(2,4),(3,4)\},$$

then we start with the constraints

$$\begin{aligned}
0 \leqslant a_{41} = a_{31} = a_{21} &\leqslant a_{11} = 1\\
0 \leqslant a_{42} \leqslant a_{32} &\leqslant a_{22} = 3 - a_{21}\\
0 \leqslant a_{43} &\leqslant a_{33} = 4 - a_{31} - a_{32}\\
0 &\leqslant a_{44} = 6 - a_{41} - a_{42} - a_{43}.
\end{aligned}$$

After performing the modifications, we arrive at

$$\begin{aligned}
y_{21} \leqslant a_{41}+y_{21} = a_{31}+y_{21} = a_{21}+y_{21} &\leqslant a_{11} = 1\\
y_{42}+y_{32} \leqslant a_{42}+y_{42}+y_{32} \leqslant a_{32}+y_{32} &\leqslant a_{22} = 3 - a_{21} - y_{21}\\
y_{43} \leqslant a_{43}+y_{43} &\leqslant a_{33} = 4 - a_{31} - y_{21} - a_{32} - y_{32}\\
0 &\leqslant a_{44} = 6 - a_{41} - y_{21} - a_{42} - y_{42} - y_{32} - a_{43} - y_{43}.
\end{aligned}$$

Analogous to Lemma 3.11, we now work toward showing that $\text{Poly}(A)$ is integrally equivalent to a flow polytope. We use the technique with which we constructed $\text{Gr}(G)$ in Lemma 3.11 together with the proof idea of Theorem 3.15. Begin with the case where G is a complete graph. By introducing slack variables z_{ij} for the inequalities in the augmented constraint array (not $0\leqslant y_{ij}\leqslant 1$), we get equations Y_{ij} given by

$$Y_{ij}:\begin{cases} a_{ij}+y_{ij}+z_{ij}=a_{i-1,j} & \text{if } i>j,\\ a_{ij}=\#E(G[0,1]) & \text{if } i=j=1,\\ \displaystyle\sum_{k=1}^{i} a_{ik}+\sum_{m=2}^{i}\sum_{k=1}^{m-1} y_{mk}=\#E(G[0,i]) & \text{if } i=j>1.\end{cases}$$

Applying the exact same transformation used in the proof of Lemma 3.11, we get equivalent equations Z_{ij} given by

$$Z_{ij}:\begin{cases} a_{ij}+y_{ij}+z_{ij}=a_{i-1,j} & \text{if } i>j,\\ a_{ij}=\mathrm{indeg}_G(1) & \text{if } i=j=1,\\ a_{ij}=\mathrm{indeg}_G(i)+\displaystyle\sum_{k=1}^{i-1} z_{ik} & \text{if } i=j>1.\end{cases}$$

To move from the complete graph to any simple graph, just set $y_{ij}=0$ and $z_{ij}=0$ whenever $(j,i)\notin E(G)$. We can realize the solutions to the Z_{ij} as points in a flow polytope of some graph. However, to account for the additional restrictions $0\leqslant y_{ij}\leqslant 1$, we view it as a *capacitated flow polytope.* This is for convenience and is not mathematically significant since any capacitated flow polytope is integrally equivalent to an uncapacitated flow polytope [2, Lemma 1].

Definition 4.5. *Define the* augmented constraint graph $\mathrm{Gr}^{\mathrm{aug}}(G)$ *to have vertex set* $\{v_{ij}\mid 1\leqslant j\leqslant i\leqslant n\}\cup\{v_{n+1,n+1}\}$ *with the ordering* $v_{11}<v_{21}<\cdots<v_{n1}<v_{22}<\cdots<v_{nn}<v_{n+1,n+1}$ *and edge set* $E_a\cup E_z\cup E_y$ *labeled by the variables* a_{ij}, z_{ij}, *and* y_{ij} *respectively, where*

E_a *consists of edges* $a_{ij}: v_{ij}\to v_{i+1,j}$ *for* $1\leqslant j\leqslant i\leqslant n$,

E_z *consists of edges* $z_{ij}: v_{ij}\to v_{ii}$ *for* $(j,i)\in E(G\backslash 0)$,

E_y *consists of edges* $y_{ij}: v_{ij}\to v_{n+1,n+1}$ *for* $(j,i)\in E(G\backslash 0)$,

and we take indices $(n+1,j)$ *to refer to* $(n+1,n+1)$. *Define a netflow vector* $\boldsymbol{a}_G^{\mathrm{aug}}$ *by reading each row of the array*

$$\begin{array}{lllll} 0 & 0 & \cdots & 0 & \mathrm{indeg}_G(1)\\ 0 & 0 & \cdots\ 0 & \mathrm{indeg}_G(2) & \\ \vdots & \iddots & & & \\ \mathrm{indeg}_G(n) & & & & \\ -\#E(G) & & & & \end{array}$$

from right to left and reading the rows from top to bottom.

Denote by $\mathcal{F}^{c}_{\mathrm{Gr}^{\mathrm{aug}}(G)}(\boldsymbol{a}_G^{\mathrm{aug}})$ the capacitated flow polytope of the graph $\mathrm{Gr}^{\mathrm{aug}}(G)$ with netflow $\boldsymbol{a}_G^{\mathrm{aug}}$ and with the capacity constraints $0\leqslant y_{ij}\leqslant 1$ for all $1\leqslant j<i\leqslant n$. By construction, the points in $\mathcal{F}^{c}_{\mathrm{Gr}^{\mathrm{aug}}(G)}(\boldsymbol{a}_G^{\mathrm{aug}})$ are exactly the solutions to the augmented constraint array of G.

Definition 4.6. *Similar to Theorem* 3.15, *contracting the edges* $\{a_{ij}\mid 1\leqslant j\leqslant i<n\}$ *of* $\mathrm{Gr}^{\mathrm{aug}}(G)$ *and relabeling the representative vertices* v_{nj} *by* j *and* $v_{n+1,n+1}$ *by* t, *we obtain a graph called the* augmented graph of G. *This graph is denoted* G^{aug} *and is defined on vertices* $[n]\cup\{t\}$ *with labeled edges* $E_a\cup E_z\cup E_y$ *where*

E_a *consists of edges* $a_{nj}: j\to t$ *for* $j\in[n]$;

E_z *consists of edges* $z_{ij}: j\to i$ *for* $(j,i)\in E(G\backslash 0)$;

E_y *consists of edges* $y_{ij}: j\to t$ *for* $(j,i)\in E(G\backslash 0)$.

Example 4.7. For $G=K_4$, the graphs $\mathrm{Gr}^{\mathrm{aug}}(G)$ and G^{aug} are shown in Figure 8.

Before proceeding, recall the netflow vector

$$\boldsymbol{b}_G^F=(\mathrm{indeg}_G(1)-\mathrm{outdeg}_F(1),\ldots,\mathrm{indeg}_G(n)-\mathrm{outdeg}_F(n),-\#E(G\backslash F))$$

for any $F\subseteq E(G\backslash 0)$. Denote by $\mathcal{F}^{c}_{G^{\mathrm{aug}}}\left(\boldsymbol{b}_G^{\varnothing}\right)$ the capacitated flow polytope of the graph G^{aug} with netflow $\boldsymbol{b}_G^{\varnothing}$ and the capacity constraints $0\leqslant y_{ij}\leqslant 1$ for all $1\leqslant j<i\leqslant n$.

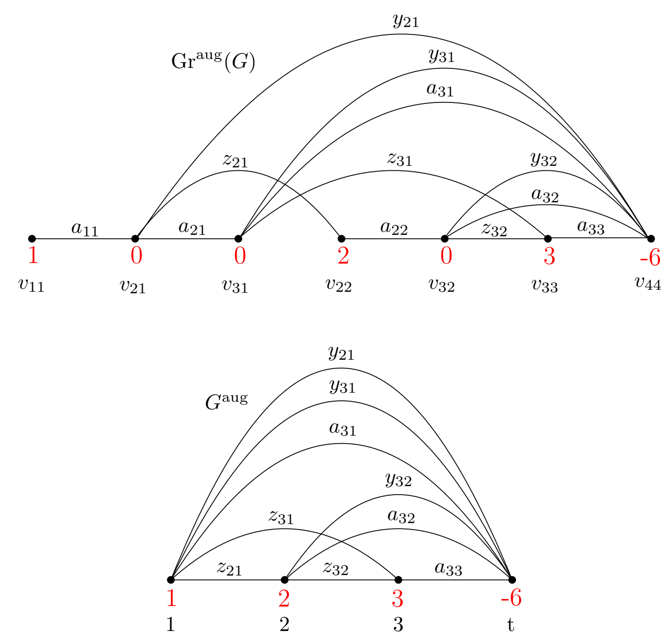


Figure 8. The graphs $\mathrm{Gr}^{\mathrm{aug}}(G)$ and G^{aug} for G a complete graph on $[0,3]$.

Theorem 4.8. *Let A denote the augmented constraint array of G and* $\mathrm{Poly}(A)$ *the polytope defined by the real valued solutions to A with the additional constraints $0 \leqslant y_{ij} \leqslant 1$ for all i and j with $1 \leqslant j < i \leqslant n$. Then, the capacitated flow polytopes*

$$\mathrm{Poly}(A), \quad \mathcal{F}^{c}_{\mathrm{Gr}^{\mathrm{aug}}(G)}\left(\boldsymbol{a}_G^{\mathrm{aug}}\right), \ \textit{and} \quad \mathcal{F}^{c}_{G^{\mathrm{aug}}}\left(\boldsymbol{b}_G^{\varnothing}\right)$$

are all integrally equivalent.

Proof. Follows immediately from the constructions of Definitions 4.5 and 4.6. □

Theorem 4.9. *For G a graph on $[0,n]$, the Newton polytope of the left-degree polynomial $L_G(\boldsymbol{t})$ and the capacitated flow polytope $\mathcal{F}^{c}_{G^{\mathrm{aug}}}\left(\boldsymbol{b}_G^{\varnothing}\right)$ satisfy*

$$\mathrm{Newton}(L_G(\boldsymbol{t})) = \psi\left(\mathcal{F}^{c}_{G^{\mathrm{aug}}}\left(\boldsymbol{b}_G^{\varnothing}\right)\right),$$

where where ψ is the projection that takes a flow on $\mathcal{F}^{c}_{G^{\mathrm{aug}}}\left(\boldsymbol{b}_G^{\varnothing}\right)$ to its values on the edges labeled $\{a_{nj} \mid j \in [n]\}$.

Proof. Let $\alpha \in \mathrm{LD}(G,F)$ for $F \subseteq E(G\backslash 0)$. Consider the set of integer flows on G^{aug} such that each edge y_{ij} has flow 1 if $(j,i) \in F$ and zero otherwise. By the construction of G^{aug}, these are in bijection with the integer flows on $\widetilde{G}\backslash\{s,0\}$ with netflow vector $\boldsymbol{b}_G^{F}$, which in turn are in bijection to $\mathrm{LD}(G,F)$ (Corollary 3.16). Thus α is the projection of a capacitated flow on G^{aug} with netflow $\boldsymbol{b}_G^{\varnothing}$.

Conversely, let $\alpha = (\alpha_1, \dots, \alpha_n) \in \psi\left(\mathcal{F}^{c}_{G^{\mathrm{aug}}}\left(\boldsymbol{b}_G^{\varnothing}\right)\right)$ be an integer point. Then, there exists some flow f (not necessarily integral) on G^{aug} with netflow $\boldsymbol{b}_G^{\varnothing}$ having the integer values α_j on the a-edges (j,t). If we remove these edges and modify the netflow vector accordingly, the new flow polytope we get is the (integrally capacitated) flow polytope of a graph with an integral netflow vector. Any such polytope has integral vertices [17, Theorem 13.1]. Thus, we can choose f to be an integral flow.

Since the edges labeled y_{ij} are constrained between 0 and 1, f takes value 0 or 1 on these edges. If we let $F=\{(j,i)\in E(G\backslash 0) \mid f$ takes value 1 on the edge labeled by $y_{ij}\}$, then f induces a flow on $\widetilde{G}\backslash\{s,0\}$ with netflow vector $\boldsymbol{b}_G^F$, so $\alpha \in \mathrm{LD}(G,F)$. □

Corollary 4.10. *For any graph G on $[0,n]$, $L_G(\boldsymbol{t})$ has SNP.*

Proof. The second half of the proof of Theorem 4.9 demonstrated that any integer point $\alpha \in \psi\left(\mathcal{F}^c_{G^{\mathrm{aug}}}\left(\boldsymbol{b}_G^{\varnothing}\right)\right)$ satisfied $\alpha \in \mathrm{LD}(G,F)$ for some F. Thus $\alpha \in \mathrm{LD}(G)$. □

We now analyze the component polytopes $\mathrm{Newton}(L_{G,F}(\boldsymbol{t}))$ and show that they are generalized permutahedra. We first briefly recall the relevant definitions from [16].

A *generalized permutahedron* is a deformation of the usual permutahedron obtained by parallel translation of the facets. Generalized permutahedra are parameterized by real numbers $\{z_I\}_{I\subseteq[n]}$ with $z_\varnothing = 0$ and satisfying the supermodularity condition

$$z_{I\cup J} + z_{I\cap J} \geqslant z_I + z_J \text{ for any } I, J \subseteq [n].$$

For a choice of parameters $\{z_I\}_{I\subseteq[n]}$, the associated generalized permutahedron $P_n^z(\{z_I\})$ is defined by

$$P_n^z(\{z_I\}) = \left\{\boldsymbol{t}\in\mathbb{R}^n \mid \sum_{i\in I} t_i \geqslant z_I \text{ for } I\neq[n], \text{ and } \sum_{i=1}^n t_i = z_{[n]}\right\}.$$

There is a subclass of generalized permutahedra given by Minkowski sums of dilations of the faces of the standard $(n-1)$-simplex. For $I\subseteq[n]$, let $\Delta_I = \mathrm{Conv}(\{e_i \mid i\in I\})$, where e_i is the *ith* standard basis vector in $\mathbb{R}^n$ and $\Delta_\varnothing$ is the origin. Given a set $\{y_I\}_{I\subseteq[n]}$ of nonnegative real numbers with $y_\varnothing = 0$, consider the polytope $\sum_{I\subseteq[n]} y_I\Delta_I$.

Proposition 4.11 ([16], Proposition 6.3). *Given nonnegative real numbers $\{y_I\}_{I\subseteq[n]}$, set $z_I = \sum_{J\subseteq I} y_J$. Then*

$$P_n^z(\{z_I\}) = \sum_{I\subseteq[n]} y_I\Delta_I.$$

We now return to left-degree polynomials. Our goal is to show that

$$\mathrm{Newton}(L_{G,F}(\boldsymbol{t})) = P_n^z\left(\{z_I^F\}_{I\subseteq[n]}\right)$$

for some parameters $\{z_I^F\}_{I\subseteq[n]}$. The proof relies on the following fact about flow polytopes, which readily follows from the max-flow min-cut theorem.

Lemma 4.12. *Let G be a graph on $[0,n]$ and $\boldsymbol{\alpha} = (\alpha_0,\dots,\alpha_n)\in\mathbb{R}^{n+1}$. Then $\mathcal{F}_G(\boldsymbol{\alpha})$ is nonempty if and only if*

$$\sum_{i=0}^n \alpha_i = 0 \text{ and } \sum_{i\in S}\alpha_i \leqslant 0 \text{ for all } S\subseteq[0,n] \text{ with } \mathrm{outdeg}_G(S) = 0. \tag{7}$$

Proof. Observe that the conditions (7) are necessary in order for $\mathcal{F}_G(\boldsymbol{\alpha})$ to be nonempty. We now show they are also sufficient. For this, we rephrase the problem as a max-flow problem on another graph. Let

$$G' = (V(G)\cup\{s,t\}, E(G)\cup\{(s,i)\mid i\in[0,n],\alpha_i>0\}\cup\{(i,t)\mid i\in[0,n],\alpha_i<0\}).$$

Direct edges of G' from smaller to larger vertices, where s is the smallest and t is the largest.

Let the edges $\{(s,i)\mid i\in[0,n],\, \alpha_i>0\}$ have upper capacity α_i, and the edges $\{(i,t)\mid i\in[0,n],\, \alpha_i<0\}$, have upper capacity $-\alpha_i$. Let the edges belonging to both G and G' have the upper capacity $\sum_{i\in[0,n],\alpha_i>0}\alpha_i$. Assign all edges of G' the lower capacity of 0.

If the maximum flow on G' saturates the edges incident to s (equivalently, to t), then $\mathcal{F}_G(\boldsymbol{\alpha})$ is nonempty. We thus proceed to show that if $\boldsymbol{\alpha}$ satisfies (7) with the given G, then the maximum flow on G' saturates the edges incident to s. In other words, if $\boldsymbol{\alpha}$ satisfies (7) with the given G, then the value of the maximum flow on G' is $\sum_{i\in[0,n],\alpha_i>0}\alpha_i$.

Recall that by the max-flow min-cut theorem [17, Theorem 10.3] the maximum value of an $s-t$ flow on G' subject to the above capacity constraints equals the minimum capacity of an $s-t$ cut in G'. For the cut $(\{s\}, V(G)\backslash\{s\})$ the capacity is $\sum_{i\in[0,n],\alpha_i>0}\alpha_i$, and we show that this is the minimum capacity of an $s-t$ cut in G'. If the cut contains any edge not incident to s or t, then the capacity of that edge is already $\sum_{i\in[0,n],\alpha_i>0}\alpha_i$.

On the other hand, if the cut does not contain any edge not incident to s or t, the partition of vertices is of the form $(\{s\}\cup S, S^c\cup\{t\})$, where $S\subseteq[0,n]$ with $\mathrm{outdeg}_G(S)=0$ and $S^c=[0,n]\backslash S$. Thus, by (7) we have $\sum_{i\in S}\alpha_i\leqslant 0$. The capacity of the cut $(\{s\}\cup S, S^c\cup\{t\})$ is

$$\sum_{i\in S^c,(s,i)\in G'}\alpha_i-\sum_{i\in S,(i,t)\in G'}\alpha_i.$$

Note that

$$0\geqslant\sum_{i\in S}\alpha_i=\sum_{i\in S,\alpha_i>0}\alpha_i+\sum_{i\in S,(i,t)\in G'}\alpha_i.$$

Consequently,

$$\begin{aligned}\sum_{i\in S^c,(s,i)\in G'}\alpha_i-\sum_{i\in S,(i,t)\in G'}\alpha_i &\geqslant \sum_{i\in S^c,(s,i)\in G'}\alpha_i+\sum_{i\in S,\alpha_i>0}\alpha_i\\ &=\sum_{i\in[0,n],\alpha_i>0}\alpha_i.\end{aligned}$$

In other words, the capacity of any cut is at least $\sum_{i\in[0,n],\alpha_i>0}\alpha_i$, and we saw that this is achieved. Thus, the value of the maximum flow on G' is $\sum_{i\in[0,n],\alpha_i>0}\alpha_i$, as desired. ☐

For $F\subseteq E(G\backslash 0)$, recall the numbers f_{ij} given by

$$f_{ij}=\#\{(j,k)\in F\mid k\leqslant i\}.$$

By Corollary 3.16 (Theorem 6.3 for the general case), $\mathrm{LD}(G,F)$ is in bijection with integral flows on the graph $\widetilde{G}\backslash\{s,0\}$ with the netflow vector $\boldsymbol{b}_G^F$ defined by

$$\boldsymbol{b}_G^F=(\mathrm{indeg}_G(1)-\mathrm{outdeg}_F(1),\dots,\mathrm{indeg}_G(n)-\mathrm{outdeg}_F(n),-\#E(G\backslash F))$$

via projection onto the edges (i,t).

Definition 4.13. *For a collection of vertices I of a graph G, define the outdegree $\mathrm{outdeg}_G(I)$ to be the number of edges from vertices in I to vertices not in I.*

To each $I\subseteq[n]$, associate the integer z_I^F given by

$$z_I^F=\sum_{i\in S}\mathrm{indeg}_G(i)-\mathrm{outdeg}_F(i) \tag{8}$$

where $S\subseteq I$ is the maximal subset with $\mathrm{outdeg}_G(S)=0$.

Theorem 4.14. *For a simple graph G, $F\subseteq E(G\backslash 0)$, and $\{z_I^F\}$ the parameters defined by* (8)*, $\mathrm{Newton}(L_{G,F}(\boldsymbol{t}))$ is the generalized permutahedron*

$$\mathrm{Newton}(L_{G,F}(\boldsymbol{t}))=\mathrm{Conv}(\mathrm{LD}(G,F))=P_n^z\left(\{z_I^F\}_{I\subseteq[n]}\right).$$

Furthermore, each integer point of $P_n^z\left(\{z_I^F\}\right)$ is in $\mathrm{LD}(G,F)$, so $L_{G,F}(\boldsymbol{t})$ has SNP.

Proof. First, it is easy to check that the parameters z_I^F satisfy the supermodularity condition. Thus, $P_n^z\left(\{z_I^F\}_{I\subseteq[n]}\right)$ is a generalized permutahedron. To observe that $\mathrm{Conv}(\mathrm{LD}(G,F))\subseteq P_n^z\left(\{z_I^F\}\right)$, simply recall that $\mathrm{LD}(G,F)$ equals the projection of integral flows on $\widetilde{G}\backslash\{s,0\}$ with netflow $\boldsymbol{b}_G^F$ onto the edges $\{(j,t)\}_{j\in[n]}$.

For the reverse direction, let $\boldsymbol{d}$ denote the truncation of $\boldsymbol{b}_G^F$ by its last entry, that is let $\boldsymbol{d}=(d_1,\dots,d_n)$ where

$$d_i=\mathrm{indeg}_G(i)-\mathrm{outdeg}_F(i).$$

We must show that each point $\boldsymbol{x}=(x_1,\dots,x_n)\in P_n^z\left(\{z_I^F\}\right)$, the assignment $a_{nj}=x_j$ in $\widetilde{G}\backslash\{s,0\}$ can be extended to a flow on $\widetilde{G}\backslash\{s,0\}$. This is equivalent to showing

$$\mathcal{F}_{G\backslash 0}(\boldsymbol{d}-\boldsymbol{x})\neq\varnothing.$$

By Lemma 4.12, it suffices to note that

$$\sum_{i\in S} d_i-x_i\leqslant 0 \text{ for all } S\subseteq[n] \text{ with } \mathrm{outdeg}_G(S)=0.$$

However, since $\mathrm{outdeg}_G(S)=0$, we have

$$\sum_{i\in S}x_i\geqslant z_S=\sum_{i\in S}d_i.$$ □

We further show that $\mathrm{Newton}(L_{G,F}(\boldsymbol{t}))$ can be written as $\sum_{I\subseteq[n]}y_I\Delta_I$ for some parameters y_I. Let $L=\{J\subseteq[n]\mid \mathrm{outdeg}_G(J)=0\}$. L is a lattice under union and intersection, so consider the set Q of join-irreducible elements of L (elements that cannot be written as the union of other elements).

We explicitly describe the members of Q. Let $\delta(i)$ denote all the vertices of G that can be reached from i by a directed path (including i itself).

Lemma 4.15. *An element $J\in L$ is join-irreducible if and only if $J=\delta(i)$ for some $i\in[n]$.*

For $J\subseteq[n]$, define

$$y_J^F=\begin{cases}\mathrm{indeg}_G(k)-\mathrm{outdeg}_F(k) & \text{if } J\in Q,\ J \text{ covers } J' \text{ in } L,\ J\backslash J'=\{k\},\\ 0 & \text{if } J\notin Q.\end{cases} \tag{9}$$

Proposition 4.16. *For any simple graph G and $F\subseteq E(G\backslash 0)$,*

$$P_n^z\left(\{z_I^F\}\right)=\sum_{I\subseteq[n]}y_J^F\Delta_I.$$

Proof. Note that $z_I^F=z_{I_1}^F$ where I_1 is the largest element of L contained in I. Thus,

$$z_I^F=z_{I_1}^F=\sum_{k\in I_1}b_k=\sum_{\substack{J\in Q\\ J\subseteq I_1}}y_J^F=\sum_{J\subseteq I}y_J^F.$$

Apply Proposition 4.11. □

From (9), we can read off the $\{y_I^F\}$ decomposition of $\mathrm{Newton}(L_{G,F}(\boldsymbol{t}))$. Then,

$$\mathrm{Newton}(L_{G,F}(\boldsymbol{t}))=\sum_{i=1}^n(\mathrm{indeg}_G(i)-\mathrm{outdeg}_F(i))\Delta_{\delta(i)}. \tag{10}$$

Example 4.17. For a simple graph G, recall that the transitive closure of G is the simple graph formed by adding edges (i,j) to $E(G)$ whenever the vertices $i \neq j$ are connected by a directed path in G. If G is a simple graph on $[0,n]$ such that the transitive closure of $G\backslash\{0\}$ is complete, then for each $F \subseteq E(G\backslash 0)$,

$$\mathrm{Newton}(L_{G,F}(\boldsymbol{t})) = \Pi_n\left(\mathrm{indeg}_G(1) - \mathrm{outdeg}_F(1), \dots, \mathrm{indeg}_G(n) - \mathrm{outdeg}_F(n)\right)$$

where $\Pi_n(\boldsymbol{x})$ is the Pitman–Stanley polytope as defined in [18], but shifted up one dimension in affine space, that is

$$\begin{aligned}\Pi_n(\boldsymbol{x}) &= \left\{\boldsymbol{t} \in \mathbb{R}^n_{\geqslant 0} \mid \sum_{p=1}^{k} t_p \leqslant \sum_{p=1}^{k} x_p \text{ for } k \in [n-1], \text{ and } \sum_{p=1}^{n} t_p = \sum_{p=1}^{n} x_p\right\} \\ &= x_n\Delta_{\{n\}} + x_{n-1}\Delta_{\{n-1,n\}} + \cdots + x_1\Delta_{[n]}.\end{aligned}$$

Proposition 4.18. *If T is a tree on $[0,n]$, then $\mathrm{Newton}(L_{T,F}(\boldsymbol{t}))$ is a simple polytope.*

Proof. By the Cone-Preposet Dictionary for generalized permutahedra, ([15], Proposition 3.5) it is enough to show that each vertex poset Q_v is a tree-poset, that is, its Hasse diagram has no cycles. To show this, let $I \subseteq [n]$ and consider the normal fan $N(\Delta_I)$ of the simplex Δ_I. By (10), the normal fan of $\mathrm{Newton}(L_{G,F}(\boldsymbol{t}))$ is the refinement of normal fans $N(\Delta_I)$.

Thus, a maximal cone of the normal fan of $\mathrm{Newton}(L_{G,F}(\boldsymbol{t}))$ is given by an intersection of maximal cones in each $N(\Delta_I)$ for $I = \delta(j)$, $j \in [n]$, $\mathrm{indeg}_T(j) > 0$. A maximal cone in $N(\Delta_I)$ gives the vertex poset relations $x_i > x_j$ for all $j \in I$ and any chosen $i \in I$. Thus, relations in the Hasse diagram of a vertex poset lift to undirected paths in T.

If some Q_v has a cycle C, then we can lift the relations to get two different paths in T between two vertices. This subgraph will contain a cycle, contradicting that T is a tree. ☐

The Newton polytopes of the homogeneous components of $L_G(\boldsymbol{t})$ are also generalized permutahedra.

Definition 4.19. *For each $k \geqslant 0$ let $L^k_G(\boldsymbol{t})$ denote the degree $\#E(G)-k$ homogeneous component of $L_G(\boldsymbol{t})$, that is*

$$L^k_G(\boldsymbol{t}) = \sum_{\substack{F\subseteq E(G\backslash 0)\\ \#F=k}} L_{G,F}(\boldsymbol{t}).$$

For a simple graph G on $[0,n]$, Theorem 4.9 showed that the augmented graph G^{aug} of Definition 4.6 has the property that the projection of integral flows on G^{aug} with netflow

$$\boldsymbol{b}^{\varnothing}_G = (\mathrm{indeg}_G(1), \dots, \mathrm{indeg}_G(n), -\#E(G))$$

and capacitance $0 \leqslant y_{ij} \leqslant 1$ for all $1 \leqslant j < i \leqslant n$ onto the edges labeled a_{nj} for $j \in [n]$ is exactly $\mathrm{LD}(G)$. The following construction is a variation on this theme designed so its integral flows will only project to left-degree sequences whose entries have a particular sum.

Definition 4.20. *Given a simple graph G on $[0,n]$ and $k \geqslant 0$, let $G^{(k)}$ be the graph on $[1,n+1] \cup \{t\}$ with labeled edges $E_a \cup E_z \cup E_y$ where*

E_a *consists of edges* $a_{nj} : j \to t$ *for* $j \in [n]$;

E_z *consists of edges* $z_{ij} : j \to i$ *for* $(j,i) \in E(G\backslash 0)$;

E_y *consists of edges* $y_{ij} : j \to n+1$ *for* $(j,i) \in E(G\backslash 0)$.

The flow polytope $\mathcal{F}^c_{G^{(k)}}(\boldsymbol{b}_G^{(k)})$ *is the flow polytope of* $G^{(k)}$ *with netflow vector* $\boldsymbol{b}_G^{(k)} = (\mathrm{indeg}_G(1), \dots, \mathrm{indeg}_G(n), -k, k - \#E(G))$ *and capacities* 1 *on the edges* y_{ij}*.*

Example 4.21. For G the complete graph on $[0,3]$, $G^{(k)}$ is shown in Figure 9 alongside G^{aug} for comparison.

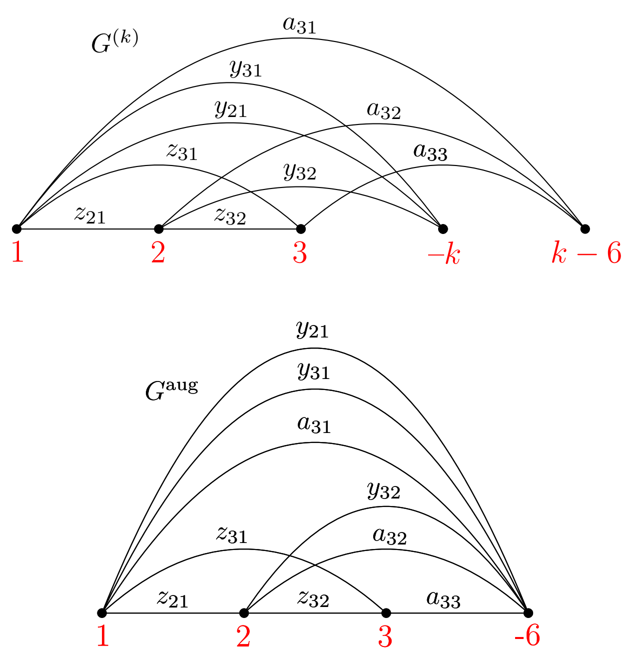


Figure 9. The graphs $G^{(k)}$ and G^{aug} for G a complete graph on $[0,3]$.

Note that capacitated integral flows on $G^{(k)}$ with netflow $\boldsymbol{b}_G^{(k)}$ are in bijection with capacitated integral flows on G^{aug} with netflow $\boldsymbol{b}_G^{\varnothing}$ where exactly k edges y_{ij} have flow 1, and the bijection preserves the values on the edges $\{a_{nj} \mid j \in [n]\}$.

Theorem 4.22. *For* $k \geqslant 0$*, if* ψ *is the projection that takes a flow on* $\mathcal{F}^c_{G^{(k)}}\left(\boldsymbol{b}_G^{(k)}\right)$ *to the tuple of its values on the edges labeled* a_{nj} *for* j *in* $[n]$*, then*

$$\mathrm{Newton}\left(L_G^k(\boldsymbol{t})\right) = \psi\left(\mathcal{F}^c_{G^{(k)}}\left(\boldsymbol{b}_G^{(k)}\right)\right).$$

Proof. Let α be an integer point in $\mathrm{Newton}\left(L_G^k(\boldsymbol{t})\right)$, so $\alpha \in \mathrm{LD}(G,F)$ for $F \subseteq E(G\backslash 0)$ with $\#F = k$. Then, α corresponds to a capacitated integral flow on G^{aug} with netflow $\boldsymbol{b}_G^{\varnothing}$, which in turn corresponds to a capacitated integral flow on $G^{(k)}$ with netflow $\boldsymbol{b}_G^{(k)}$ that ψ takes to α.

Conversely, let α be an integer point in $\psi\left(\mathcal{F}^c_{G^{(k)}}\left(\boldsymbol{b}_G^{(k)}\right)\right)$. Lift α to an integral flow f on $G^{(k)}$. The flow f corresponds to an integral flow on G^{aug}, so if $F = \{(j,i) \mid y_{ij} = 1 \text{ in } f\}$, then $\#F = k$ and $\alpha \in \mathrm{LD}(G,F)$. □

Similar to the proof of Theorem 4.14, for $k \geqslant 0$ and $I \subseteq [n]$, define parameters $z_I^{(k)}$ by

$$z_I^{(k)} = \min\left\{\sum_{i \in I} f(i,t) \mid f \text{ is a flow on } G^{(k)} \text{ with netflow vector } \boldsymbol{b}_G^{(k)}\right\}. \tag{11}$$

Theorem 4.23. *For $k \geqslant 0$ and $\{z_I^{(k)}\}$ the parameters defined by* (11), $\mathrm{Newton}(L_G^k(\boldsymbol{t}))$ *is the generalized permutahedron*

$$\mathrm{Newton}(L_G^k(\boldsymbol{t})) = P_n^z\left(\{z_I^{(k)}\}_{I\subseteq[n]}\right).$$

Furthermore, each integer point of $P_n^z\left(\{z_I^{(k)}\}\right)$ is a left-degree sequence, so $L_G^k(\boldsymbol{t})$ has SNP. Additionally, if G is a tree, then $L_G^0(\boldsymbol{t})$ is the integer point transform of its Newton polytope.

Proof. The proof of the first two statements is analogous to that of Theorem 4.14. Alternatively, SNP follows from the fact that the $\mathrm{Newton}(L_G^k)$ is the intersection of $\mathrm{Newton}(L_G)$ by a hyperplane.

Recall that the integer point transform of a polytope $P \subseteq \mathbb{R}^m$ is the polynomial

$$L_P(x_1,\dots,x_m) = \sum_{p\in P\cap\mathbb{Z}^m} \boldsymbol{x}^p.$$

To prove the third statement we must show that if G is a tree, all nonzero coefficients of L_G^0 are 1. It follows from Corollary 3.16 (Theorem 6.3) that $\mathrm{LD}(G,\varnothing)$ equals the multiset of projections of integral flows on $\widetilde{G}\backslash\{s,0\}$ with the netflow vector $\boldsymbol{b}_G^{\varnothing}$. Then, the multiplicity of any particular $\alpha \in \mathrm{LD}(T,\varnothing)$ is the number of flows on $G\backslash 0$ with netflow $\boldsymbol{b}_G^{\varnothing} - \alpha$. However, trees admit at most one flow for any given netflow vector, so every element of $\mathrm{LD}(G,\varnothing)$ has multiplicity 1. This implies all coefficients in L_G^0 are 0 or 1. □

Theorems 4.9 and 4.23 imply the following.

Corollary 4.24. *Given a graph G on the vertex set $[0,n]$ with m edges, we have that*

$$\mathrm{Newton}(L_G(\boldsymbol{t})) \cap \left\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid \sum_{i=1}^n x_i = m-k\right\} = P_n^z\left\{z_I^{(k)}\right\}_{I\subseteq[n]},$$

for the parameters $\{z_I^{(k)}\}$ given in (11).

Proof. We have that $\mathrm{Newton}(L_G(\boldsymbol{t})) \cap \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n = m-k\} = \mathrm{Newton}(L_G^k(\boldsymbol{t}))$, which by Theorem 4.23 equals $P_n^z\left(\{z_I^{(k)}\}_{I\subseteq[n]}\right)$. □

Theorems 3.17 and 4.23 imply:

Corollary 4.25. *If G is a tree on $[0,n]$, then the normalized volume of the flow polytope of $\widetilde{G}$ is*

$$\mathrm{Vol}\,\mathcal{F}_{\widetilde{G}} = \mathrm{Ehr}(P_G^0, 1),$$

where $P_G^0 = \mathrm{Newton}(L_G^0(\boldsymbol{t}))$ is the generalized permutahedron specified in Theorem 4.23.

Corollary 4.25 is of the same flavor as Postnikov's following beautiful result; for the details of the terminology used in this theorem refer to [16].

Theorem 4.26. [16, Theorem 12.9] *For a bipartite graph G, the normalized volume of the root polytope Q_G is*

$$\mathrm{Vol}\,Q_G = \mathrm{Ehr}(P_G^-, 1),$$

where P_G^- is the trimmed generalized permutahedron.

Root polytopes and flow polytopes are closely related, as can be seen by contrasting the techniques and results in the papers [9, 10, 11, 12, 16]. It is thus reasonable to expect that Corollary 4.25 and Theorem 4.26 are related mathematically. We invite the interested reader to investigate their relationship.

5. Newton polytopes of Schubert and Grothendieck polynomials

In this section, we discuss the connection between left-degree sequences, Schubert polynomials, and Grothendieck polynomials discovered in [5] and relate it to their Newton polytopes. Our main theorem is the following.

Theorem C. *Let $\pi \in S_{n+1}$ be of the form $\pi = 1\pi'$ where π' is a dominant permutation of $\{2, 3, \dots n+1\}$. Then the Grothendieck polynomial $\mathfrak{G}_\pi$ has SNP and the Newton polytope of each homogeneous component of $\mathfrak{G}_\pi$ is a generalized permutahedron. In particular, the Schubert polynomial $\mathfrak{S}_\pi$ has SNP and* $\mathrm{Newton}(\mathfrak{S}_\pi)$ *is a generalized permutahedron. Moreover, $\mathfrak{S}_\pi$ is the integer point transform of its Newton polytope.*

Theorem C implies that the recent conjectures of Monical, Tokcan, and Yong [14, Conjecture 5.1 & 5.5] are true in the special case of permutations $1\pi'$, where π' is a dominant permutation. The authors and Alex Fink prove [14, Conjecture 5.1] in its full generality in [6]. The following conjecture, discovered jointly with Alex Fink, is a strengthening of [14, Conjecture 5.5] based on the results of this paper. We have tested it for all $\pi \in S_n$, for $n \leqslant 8$.

Conjecture 5.1. *The Grothendieck polynomial $\mathfrak{G}_\pi$ has SNP and the Newton polytope of each homogeneous component of $\mathfrak{G}_\pi$ is a generalized permutahedron.*

Since [5] uses right-degree sequences and right-degree polynomials instead of their left-degree counterparts, we will adopt this convention throughout this section. To simplify notation, all graphs in this section will be on the vertex set $[n+1]$. Note the following easy relation between right-degree and left-degree.

Given a graph G on vertex set $[n+1]$, let G^* be the mirror image of the graph G with vertex set shifted to $[0, n]$. More formally, let G^* be the graph on vertices $[0, n]$ with edges

$$E(G^*) = \{(n+1-j, n+1-i) \mid (i,j) \in E(G)\}.$$

The right-degree sequences of G are exactly the left-degree sequences of G^* read backwards. Via Theorem A of Section 6 in hand, we define the *right-degree multiset* $\mathrm{RD}(G)$ as the multiset of right-degree sequences of leaves in any reduction tree of G, and $\mathrm{RD}(G, \varnothing)$ the submultiset of sequences whose components sum to $\#E(G)$ (notation consistent with $\mathrm{LD}(G, F)$ in Definition 3.9).

Definition 5.2. *For any graph G on $[n+1]$, define the* right-degree polynomial *R_G by*

$$R_G(t_1, t_2, \dots t_n) = L_{G^*}(t_n, t_{n-1}, \dots, t_1) = \sum_{\alpha \in \mathrm{RD}(G)} (-1)^{\mathrm{codim}(\alpha)} t_1^{\alpha_1} t_2^{\alpha_2} \dots t_n^{\alpha_n}$$

where $\mathrm{codim}(\alpha) = \#E(G) - \sum_{i=1}^n \alpha_i$. *For $k \geqslant 0$, let $R_G^k(\boldsymbol{t})$ denote the degree $\#E(G) - k$ homogeneous component of $R_G(\boldsymbol{t})$.*

Define the reduced right-degree polynomial $\widetilde{R}_G$ as follows: If $\{v_{i_1}, \dots v_{i_k}\}$ are the vertices of G with positive outdegree, then R_G is a polynomial in $t_{i_1}, \dots, t_{i_k}$. Obtain $\widetilde{R}_G$ by relabeling the variables t_{i_m} by t_m for each m. Note that R_G^0 (resp. $\widetilde{R_G^0}$) is the top homogeneous component of R_G (resp. $\widetilde{R}_G$), and is given by

$$R_G^0(t_1, \dots, t_n) = \sum_{\alpha \in \mathrm{RD}(G, \varnothing)} t_1^{\alpha_1} t_2^{\alpha_2} \dots t_n^{\alpha_n}.$$

The following statement collects the right-degree analogues of Corollary 4.10 and Theorem 4.23 from the previous section.

THEOREM 5.3. *Let G be a graph on $[n+1]$. Then, $R_G(\boldsymbol{t})$ has SNP, and the Newton polytope of each homogeneous component R_G^k is a generalized permutahedron. Additionally, if G is a tree, then $R_G^0(\boldsymbol{t})$ equals the integer point transform of its Newton polytope.*

We now recall the definition of pipe dreams of a permutation and the characterization of Schubert and Grothendieck polynomials in terms of pipe dreams.

DEFINITION 5.4. *A* pipe dream *for $\pi \in S_{n+1}$ is a tiling of an $(n+1)\times(n+1)$ matrix with two tiles, crosses $+$ and elbows ⌟⌜, such that*

- *all tiles in the weak south-east triangle are elbows, and*
- *if we write $1, 2, \ldots, n+1$ on the top and follow the strands (ignoring second crossings among the same strands), they come out on the left and read π from top to bottom.*

A pipe dream is reduced *if no two strands cross twice.*

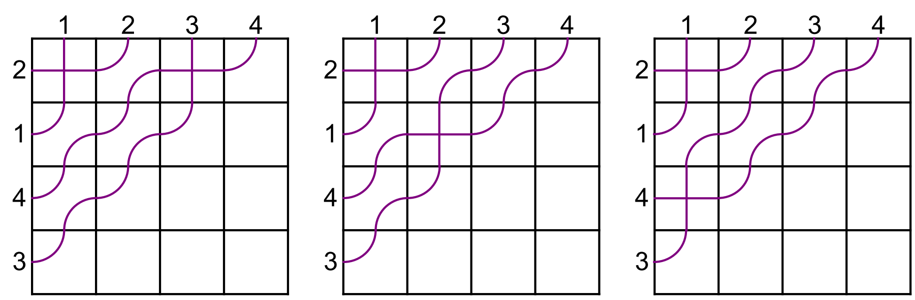


FIGURE 10. The reduced pipe dreams for $\pi = 2143$. All tiles not shown are elbows.

For $\pi \in S_{n+1}$ let $\mathrm{PD}(\pi)$ denote the collection of all pipe dreams of π and $\mathrm{RPD}(\pi)$ the collection of all reduced pipe dreams of π. For $P \in \mathrm{PD}(\pi)$, define the weight of P by

$$wt(P) = \prod_{(i,j)\in \mathrm{cross}(P)} t_i$$

where $\mathrm{cross}(P)$ denotes the set of indices of all crosses in P.

Recall that for any $\pi \in S_{n+1}$, the Grothendieck polynomial $\mathfrak{G}_\pi$ can be represented in terms of pipe dreams of π by

$$\mathfrak{G}_\pi(t_1, \ldots, t_n) = \sum_{P \in \mathrm{PD}(\pi)} wt(P),$$

and the Schubert polynomial $\mathfrak{S}_\pi$ is the lowest degree homogeneous component of the Grothendieck polynomial:

$$\mathfrak{S}_\pi(t_1, \ldots, t_n) = \sum_{P \in \mathrm{RPD}(\pi)} wt(P).$$

In [5, Theorem 5.1], it is proved that for any noncrossing tree T, the right-degree sequences $\mathrm{RD}(T)$ (see paragraph preceding Definition 5.2) are independent of the choice of reduction tree for T, and the following connection to Grothendieck polynomials is shown.

Theorem 5.5 ([5, Theorem 5.3]). *Let $\pi \in S_{n+1}$ be of the form $\pi = 1\pi'$ where π' is a dominant permutation of $\{2, 3, \ldots n+1\}$. Then, there is a tree $T(\pi)$ and nonnegative integers $g_i = g_i(\pi)$ such that*

$$\widetilde{R}_{T(\pi)}(\boldsymbol{t}) = \left(\prod_{i=1}^{n} t_i^{g_i}\right) \mathfrak{G}_\pi(t_1^{-1}, \ldots, t_n^{-1}).$$

Explicitly, if $C(\pi)$ denotes the set $\operatorname{core}(\pi) \cup \{(1,1)\}$, *then $g_i(\pi)$ is the number of boxes in column i of $C(\pi)$.*

In terms of Newton polytopes, Theorem 5.5 implies

$$\text{Newton}\left(\mathfrak{G}_\pi\right) = \varphi\left(\text{Newton}\left(\widetilde{R}_{T(\pi)}\left(\boldsymbol{t}\right)\right)\right)$$

and

$$\text{Newton}\left(\mathfrak{S}_\pi\right) = \varphi\left(\text{Newton}\left(\widetilde{R}^0_{T(\pi)}\left(\boldsymbol{t}\right)\right)\right)$$

where φ is the integral equivalence

$$(x_1, \ldots, x_n) \mapsto (g_1 - x_1, \ldots, g_n - x_n).$$

Proof of Theorem C. By Theorem 5.3, right-degree polynomials $R_G(\boldsymbol{t})$ have SNP. Since Newton $\left(\widetilde{R}_{T(\pi)}\right)$ is the image of Newton $\left(R_{T(\pi)}\right)$ by a projection forgetting coordinates that are always zero, it follows from Theorem 5.5 that $\mathfrak{G}_\pi$ has SNP.

Theorem 5.3 and Theorem 5.5 also yield that each homogeneous component of $\mathfrak{G}_\pi$ has SNP and that their Newton polytopes are generalized permutahedra. In particular, this holds for the Schubert polynomial. Since by [5] the Schubert polynomial of $\pi = 1\pi'$, where π' is a dominant permutation, has $0, 1$ coefficients, the last statement also follows. □

From the proof of Theorem 5.5 in [5], one can infer the following new transition rule for Schubert polynomials of permutations of the form $1\pi'$ with π' dominant.

Lemma 5.6 (Transition rule for $1\pi'$ Schubert polynomials). *Let $\pi \in S_{n+1}$ be of the form $\pi = 1\pi'$ with π' a dominant permutation of $\{2, \ldots, n+1\}$. Let π' have diagram given by the partition $\lambda(\pi') = (\lambda_1, \ldots, \lambda_z)$ with $\lambda_z = k$. For $0 \leqslant l \leqslant k$, let w_l be the permutation on $\{2, \ldots, n+1\}$ whose diagram is the partition $(\lambda_1 - (k-l), \ldots, \lambda_{z-1} - (k-l))$. Then*

$$\mathfrak{S}_\pi(\boldsymbol{x}) = \sum_{l=0}^{k} \left(\prod_{m=1}^{l} x_m\right) \left(\prod_{p=l+2}^{k+1} x_p^z\right) \mathfrak{S}_{1w_l}(\boldsymbol{x}_{\phi_l})$$

where $\boldsymbol{x} = (x_1, x_2, \ldots)$, $\boldsymbol{x}_{\phi_l} = (x_{\phi_l(1)}, x_{\phi_l(2)}, \ldots)$, and $\phi_l(i) = \begin{cases} i & \text{if } i \leqslant l+1, \\ i+k-l & \text{if } i \geqslant l+2. \end{cases}$

Example 5.7. Let $\pi = 14523$. Then, $\pi' = 4523$, so $\lambda(\pi') = (2,2)$. For $0 \leqslant l \leqslant 2$, the permutation w_l will have diagram given by the partition (l). These permutations are $w_0 = 2345$, $w_1 = 3245$, and $w_2 = 3425$. Hence, the terms in the transition rule are

$$\begin{aligned}
&(1)(x_2^2 x_3^2)\mathfrak{S}_{1w_0}(x_1, x_4, x_5, x_6) = x_2^2 x_3^2\\
&(x_1)(x_3^2)\mathfrak{S}_{1w_1}(x_1, x_2, x_4, x_5) = x_1^2 x_3^2 + x_1 x_2 x_3^2\\
&(x_1 x_2)(1)\mathfrak{S}_{1w_2}(x_1, x_2, x_3, x_4) = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3.
\end{aligned}$$

Adding these terms together gives the expected polynomial

$$\mathfrak{S}_\pi(x_1, x_2, x_3, x_4) = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_1 x_2 x_3^2 + x_2^2 x_3^2.$$

6. Left-degree sequences as invariants

In this section we prove the results of Section 3 without the assumption that G is simple. Similar adjustments can be made to generalize Sections 4 and 5 away from simple graphs. In this setting, we also prove Theorem A, which was first proved independently by Grinberg [7].

To deal with multiple edges in $E(G)$, we view each element of $E(G)$ as being distinct. Formally, we may think of assigning a distinguishing number to each copy of a multiple edge. In this way, we may speak of subsets $F \subseteq E(G\backslash 0)$ in the usual sense.

For G any graph on the vertex set $[0,n]$, we can still construct the reduction tree $\mathcal{T}(G)$ using the same algorithm as before in Definition 3.2. As in the case of simple graphs, the leaves of this specific reduction tree can be encoded as solutions to some constraint arrays. The key is using a generalized version of Lemma 3.1 with multiple incoming and outgoing edges at vertex v. This generalization is derived the same way and is not harder, but far more technical. The arrays we obtain are no longer necessarily triangular, but rather they may be staggered. This is explained below and demonstrated in Examples 6.1 and 6.2. We leave the proofs to the interested reader; they are straightforward generalizations of those in the previous section. With $\mathcal{T}(G)$ in hand, $\mathrm{LD}(G)$ is defined exactly as before.

We now describe how to define the arrays $\mathrm{Tri}_G(\varnothing)$. Start with the array constructed for simple graphs in Definition 3.3. Replace each a_{ij} by $a_{ij}^{(1)}$ in Definition 3.3 and Theorem 3.4. Add variables $a_{ij}^{(k)}$ with $k>1$ for each additional copy of the edge (j,i) appearing in G. When there are $k>1$ copies of the edge $(j,i)\in E(G)$, also replace $a_{ij}^{(1)} \leqslant a_{i-1,j}^{(1)}$ in the constraint array by $a_{ij}^{(1)} \leqslant a_{ij}^{(2)} \leqslant \cdots \leqslant a_{ij}^{(k)} \leqslant a_{i-1,j}^{(1)}$. The following example demonstrates these changes.

Example 6.1. Following Example 3.5, if G is the graph on vertex set $[0,4]$ with

$$E(G) = \{(0,1),(0,1),(0,2),(1,2),(1,2),(2,3),(2,4),(3,4),(3,4)\},$$

we obtain the constraints

$$\begin{aligned}
&0 \leqslant a_{41}^{(1)} = a_{31}^{(1)} = a_{21}^{(1)} \leqslant a_{21}^{(2)} \leqslant a_{11}^{(1)} = 2\\
&0 \leqslant a_{42}^{(1)} \leqslant a_{32}^{(1)} \leqslant a_{22}^{(1)} = 5 - a_{21}^{(1)}\\
&0 \leqslant a_{43}^{(1)} \leqslant a_{43}^{(2)} \leqslant a_{33}^{(1)} = 6 - a_{31}^{(1)} - a_{32}^{(1)}\\
&0 \leqslant a_{44}^{(1)} = 9 - a_{41}^{(1)} - a_{42}^{(1)} - a_{43}^{(1)}.
\end{aligned}$$

Defining $\mathrm{Tri}_G(F)$ for arbitrary G requires analogous modifications. View $E(G)$ as a multiset, so we formally view each copy of a multiple edge (j,i) as a distinct element. Let F vary over subsets of $E(G\backslash 0)$, and define $\mathrm{Tri}_G(F)$ from (the general version of) $\mathrm{Tri}_G(\varnothing)$ as before using the numbers f_{ij} of (3) and treating each $a_{ij}^{(m)}$ identically for different m.

Example 6.2. With G as in Example 6.1 and $F=\{(1,2),(1,2),(2,3)\}$, the array $\mathrm{Tri}_G(F)$ is given by

$$\begin{aligned}
&2 \leqslant a_{41}^{(1)} + 2 = a_{31}^{(1)} + 2 = a_{21}^{(1)} + 2 \leqslant a_{21}^{(2)} + 2 \leqslant a_{11}^{(1)} = 2\\
&1 \leqslant a_{42}^{(1)} + 1 \leqslant a_{32}^{(1)} + 1 \leqslant a_{22}^{(1)} = 3 - a_{21}^{(1)}\\
&0 \leqslant a_{43}^{(1)} \leqslant a_{43}^{(2)} \leqslant a_{33}^{(1)} = 3 - a_{31}^{(1)} - a_{32}^{(1)}\\
&0 \leqslant a_{44}^{(1)} = 6 - a_{41}^{(1)} - a_{42}^{(1)} - a_{43}^{(1)}.
\end{aligned}$$

Using the definition of $\mathrm{Tri}_G(F)$ for arbitrary graphs G, we can extend the definitions of $\mathrm{Sol}_G(F)$ and $\mathrm{LD}(G,F)$ from simple graphs to arbitrary graphs G. As in Proposition 3.13, for each $F \subseteq E(G\backslash 0)$ the polytope $\mathrm{Poly}(\mathrm{Tri}_G(F))$ is integrally equivalent to the flow polytope of a graph $\mathrm{Gr}(G)$, a straightforward generalization of Definition 3.12. The proofs of Theorem 3.15 and Corollary 4.10 then go through with minor changes. In particular, we have the following summary result.

Theorem 6.3. *Let G be a graph on $[0,n]$, ρ be the map that takes a triangular array in any $\mathrm{Sol}_G(F)$ to its first column $\left(a^{(1)}_{n1},\dots,a^{(1)}_{nn}\right)$, and ψ be the map that takes a flow on $\widetilde{G}\backslash\{s,0\}$ to the tuple of its values on the edges $\{(j,t)\mid j\in[n]\}$. For $F\subseteq E(G\backslash 0)$, recall the netflow vector*

$$\boldsymbol{b}_G^F = (\mathrm{indeg}_G(1)-\mathrm{outdeg}_F(1),\dots,\,\mathrm{indeg}_G(n)-\mathrm{outdeg}_F(n),-\#E(G\backslash F)).$$

Then for each $F\subseteq E(G\backslash 0)$,

$$\mathrm{LD}(G,F)=\rho\left(\mathrm{Sol}_G(F)\right)=\psi\left(\mathcal{F}_{\widetilde{G}\backslash\{s,0\}}\left(\boldsymbol{b}_G^F\right)\cap\mathbb{Z}^{\#E(\widetilde{G}\backslash\{s,0\})}\right),\ \textit{so}$$

$$\begin{aligned}\mathrm{LD}(G) &= \bigcup_{F\subseteq E(G\backslash 0)}\mathrm{LD}(G,F)\\ &= \bigcup_{F\subseteq E(G\backslash 0)}\rho\left(\mathrm{Sol}_G(F)\right)\\ &= \bigcup_{F\subseteq E(G\backslash 0)}\psi\left(\mathcal{F}_{\widetilde{G}\backslash\{s,0\}}\left(\boldsymbol{b}_G^F\right)\cap\mathbb{Z}^{\#E(\widetilde{G}\backslash\{s,0\})}\right).\end{aligned}$$

In the proof of Theorem A below, it will be more convenient to use an equivalent formulation of Theorem 6.3. Instead of considering flows on $\widetilde{G}\backslash\{s,0\}$ with netflow vector $\boldsymbol{b}_G^F$, consider flows on $\widetilde{G}\backslash\{s\}$ with netflow vector $(0,\boldsymbol{b}_G^F)$, where

$$(0,\boldsymbol{b}_G^F)=(0,\mathrm{indeg}_G(1)-\mathrm{outdeg}_F(1),\dots,\,\mathrm{indeg}_G(n)-\mathrm{outdeg}_F(n),-\#E(G\backslash F)).$$

Now, we use Theorem 6.3 to prove Theorem A. Before proceeding with the proof, we first recall the relevant notation introduced previously. For a graph G on $[0,n]$, let $\mathcal{R}$ be any reduction tree of G and $\mathcal{T}(G)$ the specific reduction tree whose leaves are encoded by the arrays $\mathrm{Sol}_G(F)$ (constructed in Definition 3.2). Recall that $\mathrm{InSeq}(\mathcal{R})$ denotes the multiset of left-degree sequences of the leaves of $\mathcal{R}$, and $\mathrm{LD}(G)=\mathrm{InSeq}(\mathcal{T}(G))$.

Proof of Theorem A. We proceed by induction on the maximal depth of a reduction tree of G. For the base case, the only reduction tree possible is the single leaf G. For the induction, perform a single reduction on G using fixed edges $r_1=(i,j)$ and $r_2=(j,k)$ with $i<j<k$ to get graphs G_1, G_2, and G_3, with notation as in (2). Note that we are selecting particular edges r_1 and r_2 even if there are multiple edges (i,j) or (j,k). Let r_3 denote the new edge (i,k) in G_m for each $m\in[3]$. Let $\mathcal{R}(G_m)$ be the reduction tree of G_m, $m\in[3]$, induced from $\mathcal{R}$ by restriction to the node labeled by G_m and all of its descendants.

By the induction assumption, $\mathrm{InSeq}(\mathcal{R}(G_m))$ is exactly $\mathrm{LD}(G_m)$, so

$$\mathrm{InSeq}(\mathcal{R})=\bigcup_{m\in[3]}\mathrm{InSeq}(\mathcal{R}(G_m))=\bigcup_{m\in[3]}\mathrm{LD}(G_m).$$

Thus, we need to show that

$$\mathrm{LD}(G)=\bigcup_{m\in[3]}\mathrm{LD}(G_m) \tag{12}$$

regardless of the choice of r_1 and r_2. However, if ρ is the map that takes an array to its first column, then Theorem 6.3 yields the disjoint union decomposition

$$\mathrm{LD}(G) = \bigcup_{F \subseteq E(G\backslash 0)} \rho\left(\mathrm{Sol}_G(F)\right).$$

Similarly, for each $m \in [3]$,

$$\mathrm{LD}(G_m) = \bigcup_{F \subseteq E(G_m\backslash 0)} \rho\left(\mathrm{Sol}_{G_m}(F)\right).$$

Thus, to prove (12), it suffices to show

$$\bigcup_{F \subseteq E(G\backslash 0)} \rho\left(\mathrm{Sol}_G(F)\right) = \bigcup_{m\in[3]} \bigcup_{F \subseteq E(G_m\backslash 0)} \rho\left(\mathrm{Sol}_{G_m}(F)\right). \tag{13}$$

To show (13), to each $F \subseteq E(G\backslash 0)$, we associate a tuple $(F_m)_{m\in I(F,r_1,r_2)}$ with $I(F,r_1,r_2) \subseteq [3]$ and $F_m \subseteq E(G_m\backslash 0)$, $m \in [3]$, such that each subset of any $E(G_m\backslash 0)$ is in exactly one tuple and for each $F \subseteq E(G\backslash 0)$,

$$\rho\left(\mathrm{Sol}_G(F)\right) = \bigcup_{m\in I(F,r_1,r_2)} \rho\left(\mathrm{Sol}_{G_m}(F_m)\right).$$

By Theorem 6.3, we verify the equivalent condition

$$\psi\left(\mathcal{F}_{\widetilde{G}\backslash\{s\}}\left(0, \boldsymbol{b}_G^F\right) \cap \mathbb{Z}^{\#E(\widetilde{G}\backslash\{s\})}\right) = \bigcup_{m\in I(F,r_1,r_2)} \psi\left(\mathcal{F}_{\widetilde{G_m}\backslash\{s\}}\left(0, \boldsymbol{b}_{G_m}^F\right) \cap \mathbb{Z}^{\#E(\widetilde{G_m}\backslash\{s\})}\right).$$

To make the notation more compact, let $H = \widetilde{G}\backslash\{s\}$ and $H_m = \widetilde{G_m}\backslash\{s\}$ for $m \in [3]$. We proceed in several cases depending on F, r_1, r_2. In each case, the argument is very similar to the proof of Proposition 2.3.

I. Suppose that r_1 is not incident to vertex 0. The following four cases deal with this scenario.

Case 1. $r_1, r_2 \notin F$: Associate to F the tuple (F_1, F_2) with

$$F_1 = F \text{ and } F_2 = F.$$

Let h be an integral flow on H with netflow vector $(0, \boldsymbol{b}_G^F)$. For $m \in [3]$, we define integral flows on H_m with netflow $(0, \boldsymbol{b}_{G_m}^F)$ having the same image under ψ.

- If $h(r_1) \geqslant h(r_2)$, define h_1 on H_1 with netflow $\boldsymbol{b}_{G_1}^{F_1}$ by

$$h_1(e) = \begin{cases} h(r_2) & \text{if } e = r_3, \\ h(r_1) - h(r_2) & \text{if } e = r_1, \\ h(e) & \text{otherwise.} \end{cases}$$

- If $h(r_1) < h(r_2)$, define h_2 on H_2 with netflow $\boldsymbol{b}_{G_2}^{F_2}$ by

$$h_2(e) = \begin{cases} h(r_1) & \text{if } e = r_3, \\ h(r_2) - h(r_1) - 1 & \text{if } e = r_2, \\ h(e) & \text{otherwise.} \end{cases}$$

For the inverse map, given integral flows h_m on H_m with netflow $\boldsymbol{b}_{G_m}^{F_m}$ for $m \in [2]$, define flows $h^{(m)}$ on H by

$$h^{(1)}(e) = \begin{cases} h_1(r_1) + h_1(r_3) & \text{if } e = r_1, \\ h_1(r_3) & \text{if } e = r_2, \\ h_1(e) & \text{otherwise,} \end{cases}$$

and

$$h^{(2)}(e) = \begin{cases} h_2(r_3) & \text{if } e = r_1, \\ h_2(r_2) + h_2(r_3) + 1 & \text{if } e = r_2, \\ h_2(e) & \text{otherwise.} \end{cases}$$

Case 2. $r_1 \in F$, $r_2 \notin F$: Associate to F the tuple (F_1, F_2) with

$$F_1 = F\backslash\{r_1\} \cup \{r_3\} \text{ and } F_2 = F\backslash\{r_1\} \cup \{r_3\}.$$

Use the same maps on flows given in Case 1.

Case 3. $r_1 \notin F$, $r_2 \in F$: Associate to F the tuple (F_1, F_2, F_3) with

$$F_1 = F\backslash\{r_2\} \cup \{r_1\},\ F_2 = F,\ \text{and } F_3 = F\backslash\{r_2\}.$$

Let h be an integral flow on H with netflow vector $(0, \boldsymbol{b}_G^F)$. For $m \in [3]$, we define integral flows on H_m with netflow $(0, \boldsymbol{b}_{G_m}^{F_m})$ having the same image under ψ.

- If $h(r_1) > h(r_2)$, define h_1 on H_1 with netflow $\boldsymbol{b}_{G_1}^{F_1}$ by

$$h_1(e) = \begin{cases} h(r_2) & \text{if } e = r_3, \\ h(r_1) - h(r_2) - 1 & \text{if } e = r_1, \\ h(e) & \text{otherwise.} \end{cases}$$

- If $h(r_1) < h(r_2)$, define h_2 on H_2 with netflow $\boldsymbol{b}_{G_2}^{F_2}$ by

$$h_2(e) = \begin{cases} h(r_1) & \text{if } e = r_3, \\ h(r_2) - h(r_1) - 1 & \text{if } e = r_2, \\ h(e) & \text{otherwise.} \end{cases}$$

- If $h(r_1) = h(r_2)$, define h_3 on H_3 with netflow $\boldsymbol{b}_{G_3}^{F_3}$ by

$$h_3(e) = \begin{cases} h(r_1) & \text{if } e = r_3, \\ h(e) & \text{otherwise.} \end{cases}$$

Given integral flows h_m on H_m with netflows $\boldsymbol{b}_{G_m}^{F_m}$ for $m \in [3]$, construct the inverse map by defining flows $h^{(m)}$ on H for $m \in [3]$. Let $h^{(2)}$ be the same as in Case 1, and define

$$h^{(1)}(e) = \begin{cases} h_1(r_1) + h_1(r_3) + 1 & \text{if } e = r_1, \\ h_1(r_3) & \text{if } e = r_2, \\ h_1(e) & \text{otherwise,} \end{cases} \quad \text{and} \quad h^{(3)}(e) = \begin{cases} h_3(r_3) & \text{if } e = r_1, \\ h_3(r_3) & \text{if } e = r_2, \\ h_3(e) & \text{otherwise.} \end{cases}$$

Case 4. $r_1, r_2 \in F$: Associate to F the tuple (F_1, F_2, F_3) with

$$F_1 = F\backslash\{r_2\} \cup \{r_3\},\ F_2 = F\backslash\{r_1\} \cup \{r_3\},\ \text{and } F_3 = F\backslash\{r_1, r_2\} \cup \{r_3\}.$$

Use the maps on flows given in Case 3.

A straightforward check shows that every $F \subseteq E(G_m\backslash 0)$ for $m \in [3]$ is reached exactly once by Cases 1–4.

II. Suppose that r_1 is incident to vertex 0. The following two cases deal with this scenario.

Case 1′. $r_2 \notin F$: Associate to F the tuple (F_1, F_2) with

$$F_1 = F \text{ and } F_2 = F.$$

Use the maps on flows given in Case 1.

Case 2′. $r_2 \in F$: Associate to F the tuple (F_2, F_3) with

$$F_2 = F \text{ and } F_3 = F \backslash \{r_2\}.$$

Use the maps on flows for H_2 and H_3 given in Case 3.

A straightforward check shows that every $F \subseteq E(G_m \backslash 0)$ for $m \in [3]$ is reached exactly once by cases 1′–2′. □

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