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ON THE DISTRIBUTION OF INTEGRAL AND PRIME DIVISORS WITH EQUAL NORMS

by B. Z. MOROZ (*)

This is an exposition of the material presented in my lectures given at Orsay in March 1983.

1.

Consider r finite extensions k_1, \dots, k_r of an algebraic number field k , a finite extension of \mathbf{Q} , and fix an ideal class A_j in k_j , $1 \leq j \leq r$. Let

$$V(A) = \{a \mid a_j \in A_j, N_{k_1/k} a_1 = \dots = N_{k_r/k} a_r\}$$

be the set of r -tuples of divisors having equal norms. Following E. Hecke, [1], one associates to a divisor of a number field a point in Minkowski space, the real vector space corresponding to this field; we study the distribution of integral and prime divisors in $V(A)$ regarded as points of a real manifold, in the spirit of [1]. For technical reasons we consider here only the case $k = \mathbf{Q}$ (compare [2] and the appendix to this paper).

We use the following notations: $\text{card } S$, or simply $|S|$, denotes the cardinality of a finite set S . Let L be an algebraic number field of degree n over \mathbf{Q} :

\mathfrak{o} is the ring of integers of L ,
 \mathfrak{o}^* is its group of units,
 I is the group of fractional divisors of L ,
 I_0 is the monoid of integral divisors,

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\mathcal{P} is the set of prime divisors,

S_2 and S_1 are the sets of complex and real places of L ,

$S = S_1 \cup S_2$, $|S_j| = r_j$ ($j=1,2$), $n = r_1 + 2r_2$,

$L_w = \begin{cases} \mathbf{R}, & w \in S_1 \\ \mathbf{C}, & w \in S_2 \end{cases}$ denotes the completion of L at $w \in S$,

$\|x\| = \begin{cases} |x|, & w \in S_1 \\ |x|^2, & w \in S_2 \end{cases}$ for $x \in L_w$.

Let us introduce the algebra $X = \prod_{w \in S} L_w$ of dimension n over \mathbf{R} , referred to as Minkowski space associated with L . Let $\psi: L \rightarrow X$ be the componentwise embedding of L in X . The group v^* of units acts freely as a discrete group of transformations on the multiplicative group $X^* = \prod_{w \in S} L_w^*$ of non-zero elements of X ; let $Y = X^*/\psi(v^*)$ be the group of its orbits. E. Hecke, [1], introduces « ideal numbers » (compare also, [3]-[6]) and defines Größencharaktere to be able to study the distribution of integral and prime divisors among the areas of Y . We recall this construction, as well as the results of [3]-[5] to be generalized here. Let $N: X \rightarrow \mathbf{R}_+$ and $N^{-1}: \mathbf{R}_+ \rightarrow X$ denote the norm map $N: x \rightarrow \prod_{w \in S} \|x_w\|$ and its right inverse $N^{-1}: t \rightarrow (t^{1/n}, \dots, t^{1/n})$. Since N is trivial on $\psi(v^*)$, one obtains $Y = \mathbf{R}_+ \times Y_0$, where

$$Y_0 := X_0/\psi(v^*), \quad X_0 := \{x \mid x \in X, N(x) = 1\}.$$

Let \hat{Y}_0 be the group of characters of Y_0 and $\lambda \in \hat{Y}_0$; one can regard λ as a character of X^* trivial on $\psi(v^*)$ and on $N^{-1}\mathbf{R}_+$. Thus

$$(1) \quad \lambda(x) = \prod_{w \in S} \|x_w\|^{t_w} \left(\frac{x_w}{|x_w|} \right)^{a_w},$$

where $a_w \in \mathbf{Z}$, $t_w \in \mathbf{R}$, x_w denotes the projection of x on L_w , and, moreover, $\lambda(\varepsilon x) = \lambda(x)$ for $\varepsilon \in \psi(v^*)$,

$$\sum_{w \in S_1} t_w + 2 \sum_{w \in S_2} t_w = 0, \quad a_w \in \{0, 1\} \text{ for } w \in S_1.$$

It follows from the Dirichlet theorem on units (compare [1], [6]) that $Y = \mathbf{R}_+ \times \mathfrak{I}_L \times (Z/2Z)^{r_0}$, where \mathfrak{I}_L is a torus of dimension $n - 1$, and $r_0 \leq r_1$. Therefore, $\hat{Y}_0 \cong Z^{n-1} \times (Z/2Z)^{r_0}$, and there exist characters $\lambda_1, \dots, \lambda_{n-1}$ multiplicatively independent over Z and such

that any $\lambda \in \hat{Y}_0$ has the form

$$(2) \quad \lambda = \prod_{v=1}^{n-1} \lambda_v^{m_v} \lambda', \quad m_v \in \mathbb{Z},$$

where $\lambda'(x) = \prod_{w \in S_1} \left(\frac{x_w}{|x_w|} \right)^{a_w}$, $a_w \in \{0,1\}$. The map ψ induces an embedding

$$\varphi : L^*/\mathfrak{o}^* \rightarrow Y$$

of the group of principal divisors L^*/\mathfrak{o}^* of L in Y . Composing φ with the projection of Y on $\mathbb{R}_+ \times \mathfrak{I}_L$ one obtains an embedding

$$\varphi_0 : L^*/\mathfrak{o}^* \rightarrow \mathbb{R}_+ \times \mathfrak{I}_L.$$

Since the group $H := I/L^*$ of ideal classes is finite, one can define an embedding

$$(3) \quad f : I \rightarrow \mathbb{R}_+ \times \mathfrak{I}_L$$

which coincides with φ_0 on L^*/\mathfrak{o}^* . It follows from the work cited above (see, in particular, [1] and [3]-[5]) that both integral and prime divisors are asymptotically equidistributed when identified by means of (3) with points of the real manifold $\mathbb{R}_+ \times \mathfrak{I}_L$. To be more precise, let us introduce a parametrisation of \mathfrak{I}_L induced by the basic characters $\lambda_j(x) = \exp(2\pi i \varphi_j(x))$, $1 \leq j \leq n-1$, $0 \leq \varphi_j(x) < 1$, and identify a point $x \in \mathfrak{I}_L$ with its image $(\lambda_1(x), \dots, \lambda_{n-1}(x)) \in T^{n-1}$, where T denotes the unit circle in \mathbb{C}^* . We call a subset

$$\tau = \{x \mid \lambda_j \leq \varphi_j(x) < \lambda_j + \delta_j, 1 \leq j \leq n-1\}$$

of \mathfrak{I}_L *elementary* whenever $0 \leq \lambda_j < \lambda_j + \delta_j \leq 1$. A set $\tau \subseteq \mathfrak{I}_L$ is called *smooth* if there exists a constant $C(\tau) > 0$ such that for every $\Delta > 0$ one can find a system $t = \{\tau_v\}$ of elementary sets with the following properties: $\text{card}(t) < \Delta^{-(n-1)}$,

$$\tau_v \cap \tau_{v'} = \emptyset \quad \text{for } v \neq v', \quad \tau \subseteq \bigcup_{\tau_v \in t} \tau_v, \quad \text{mes} \left(\bigcup_{\tau_v \cap \partial \tau \neq \emptyset} \tau_v \right) < C(\tau) \Delta,$$

where mes is the normalized Haar measure on \mathfrak{I}_L (so that $\text{mes}(\mathfrak{I}_L) = 1$) and $\partial \tau$ denotes the boundary of τ . The following theorem has been proved by J. P. Kubilius, [4], and, a few years later, by T. Mitsui, [5].

THEOREM 1. — For any smooth set $\tau \subseteq \mathfrak{I}_L$ and any ideal class $A \in H$

$$\begin{aligned} \text{card} \{a \mid a \in I_0, f(a) \in (0, x) \times \tau, a \in A\} &= \frac{\omega_L \text{mes}(\tau)}{h} x + O(x^{1-c_1}) \\ \text{card} \{p \mid p \in \mathcal{P}, f(p) \in (0, x) \times \tau, p \in A\} \\ &= \frac{\text{mes}(\tau)}{h} \int_2^x \frac{dx}{\log x} + O(\exp(-c_2 \sqrt{\log x})x), \end{aligned}$$

where the constants $c_1, c_2 > 0$ depend on L , but not on $x \rightarrow \infty$, and ω_L denotes the residue of the zeta-function of L at $s = 1$, $h := |H|$ is the class number of L .

The characters $\mu_j = \lambda_j \circ f$ are called basic Größencharaktere; the group

$$\hat{I} = \left\{ \mu \mid \mu = \chi \prod_{j=1}^{n-1} \mu_j^{m_j}, m_j \in \mathbb{Z}, \chi \in \hat{H} \right\},$$

where \hat{H} is the group of ideal class characters, can be identified (see, e.g., [6]) with the set of unramified idele-class characters trivial on \mathbb{R}_+ . The map

$$(3') \quad g' : I \rightarrow \mathbb{R}_+ \times T^{n-1}$$

given by

$$g' : a \mapsto (N_{L/Q} a, \mu_1(a), \dots, \mu_{n-1}(a))$$

is compatible with (3) under the above identification of \mathfrak{I}_L and T^{n-1} . Theorem 1 may be viewed as a multidimensional equidistribution principle, in the spirit of the classic memoir of Hecke's, [1]. We should like to refer to [8], [9], [10] for some applications of this principle. One can improve the error term in the second formula using the method of trigonometric sums (see, [3], chapter 2, and [7]). About thirty years ago Yu. V. Linnik suggested (and communicated to his colleagues and students, [11]) that one could generalize Theorem 1 to treat the integral and prime divisors in $V(A)$. As an example of this programme (compare [2] and references therein), we prove here the following result. Let $I_0^j, \mathcal{P}_j, \mathfrak{I}_j$ and h_j denote the monoid of integral divisors, the set of prime divisors, the torus \mathfrak{I}_{k_j} and the class number of k_j respectively; let $h = \prod_{j=1}^r h_j$ and $\mathfrak{I} = \mathfrak{I}_1 \times \dots \times \mathfrak{I}_r$, moreover, let $\mathcal{P} = \{p \mid p_j \in \mathcal{P}_j\}$ and $I_0 = \{a \mid a_j \in I_0^j\}$ be the sets of r -tuples of prime and integral divisors

respectively; let $K = k_1 \dots k_r$ be the composite of the fields k_1, \dots, k_r , let n_j and D_j be the degree $[k_j : \mathbf{Q}]$ and the discriminant of k_j and n be the degree $[K : \mathbf{Q}]$ of K . Consider the map

$$g_j: I_0^j \rightarrow \mathfrak{I}_j$$

induced by the embedding (3'), so that, when \mathfrak{I}_j is identified with T^{n_j-1} ,

$$g_j: \mathfrak{a}_j \mapsto (\mu_{j1}(\mathfrak{a}_j), \dots, \mu_{jn_j-1}(\mathfrak{a}_j)), \quad \mathfrak{a}_j \in I_0^j,$$

where $\{\mu_{j\ell} | 1 \leq \ell \leq n_j - 1\}$ is the set of basic Größencharaktere of k_j , $j = 1, \dots, r$, and introduce a zeta-function

$$(4) \quad Z(k_1, \dots, k_r; s) = \sum_{m=1}^{\infty} a_m^{(1)} \dots a_m^{(r)} m^{-s},$$

where $a_m^{(j)} = \text{card} \{\mathfrak{a}_j | \mathfrak{a}_j \in I_0^j, N_{k_j/\mathbf{Q}} \mathfrak{a}_j = m\}$ is the number of integral divisors of k_j whose norm is equal to m . One can show (see [12], [13])

that if $n = \prod_{j=1}^r n_j$, then

$$(5) \quad Z(k_1, \dots, k_r; s) = \frac{Z_K(s)}{L(s, \Phi)},$$

where $L(s, \Phi) = \prod_p \Phi^{(p)}(p^{-s})^{-1}$, $\Phi^{(p)}(t)$ is a rational function of t , p varies over rational primes, and, moreover, $\Phi^{(p)}(p^{-s}) \neq 0, \infty$ for $\text{Re } s > \frac{1}{2}$; for almost all p the function $\Phi^{(p)}(t)$ is a polynomial of degree not larger than $n - 1$ and such that $\Phi^{(p)}(0) = 1, \frac{d}{dt} \Phi^{(p)}|_{t=0} = 0$. In particular, the Euler product

$$L(s, \Phi) = \prod_p \Phi^{(p)}(p^{-s})^{-1}$$

converges absolutely for $\text{Re } s > \frac{1}{2}$.

THEOREM 2. — *If k_j is Galois over \mathbf{Q} for every j , $n = \prod_{j=1}^r n_j$ and $(D_j, D_\ell) = 1$ for $j \neq \ell$ (the discriminants are pairwise coprime), then for*

any smooth set $\tau \subseteq \mathfrak{I}$ one has

$$\text{card} \{a \mid a \in V(A) \cap I_0, |a| < x, g(a) \in \tau\} = \frac{\omega_k \text{mes}(\tau)}{hL(1, \Phi)} x + O(x^{1-c_1}),$$

$$\begin{aligned} \text{card} \{p \mid p \in V(A) \cap \mathcal{P}, |p| = x, g(p) \in \tau\} \\ = \frac{\text{mes}(\tau)}{h} \text{li}(x) + O(x \exp(-c_2 \sqrt{\log x})) \end{aligned}$$

for some $c_1, c_2 > 0$ depending on k_1, \dots, k_r , but not on $x \rightarrow \infty$, where

$$|a| := \left(\sum_{j=1}^r N_{k_j/Q} a_j \right) \frac{1}{r} \text{ for } a = \{a_1, \dots, a_r \mid a_j \in I_0^j\},$$

and

$$\text{li}(x) := \int_2^x \frac{du}{\log u}; \quad g = (g_1, \dots, g_r).$$

One can view Theorem 2 as a statement about statistical independence of the fields k_1, \dots, k_r . To be more precise, let

$$\tau = \tau_1 \times \dots \times \tau_r, \quad \tau_j \subseteq \mathfrak{I}_j,$$

then (under the above assumptions) the probability to find $a \in V(A)$ with $g(a) \in \tau$ is equal to the product of the probabilities that $a_j \in A_j$ and $g_j(a_j) \in \tau_j$, $j = 1, \dots, r$. Thus the condition

$$(6) \quad N_{k_1/Q} a_1 = \dots = N_{k_r/Q} a_r$$

affects the probability of the event:

$$\ll a_1 \in A_1, \dots, a_r \in A_r, g_1(a_1) \in \tau_1, \dots, g_r(a_r) \in \tau_r \gg$$

neither for r -tuples of integral, nor of prime divisors. On the other hand, Theorem 2 may be regarded as an assertion on representation of integers by decomposable forms. As a special case of this theorem ($n_1 = \dots = n_r = 2$), one obtains the following result.

PROPOSITION 3. — *Let f_1, \dots, f_r be binary positive definite primitive quadratic forms with pairwise co-prime fundamental discriminants. Then the number of integral solutions*

$$(x_1, x_2, \dots, x_{2r-1}, x_{2r})$$

of the system of equations

$$f_1(x_1, x_2) = \dots = f_r(x_{2r-1}, x_{2r})$$

subject to the condition $f_1(x_1, x_2) \leq N$ is equal to

$$AN + O(N^{1-c})$$

for some $A > 0$, $c > 0$ independent on N .

It turns out that for two quadratic fields ($n_1 = n_2 = r = 2$)

$$L(s, \Phi) = L(2s, \chi_0),$$

where $\chi_0(n) = \left(\frac{D_1 D_2}{n}\right)$ (see, e.g., [13], § 5). Therefore we obtain the following result.

PROPOSITION 4. — Let $k_j = \mathbf{Q}(\sqrt{D_j})$, $j = 1, 2$, $(D_1, D_2) = 1$. Then

$$\text{card} \{ \mathfrak{a} \mid \mathfrak{a} \in V(\mathbf{A}) \cap I_0, |\mathfrak{a}| < x, g(\mathfrak{a}) \in \tau \} = \frac{\omega_K \text{mes}(\tau)}{hL(2, \chi_0)} x + O(x^{1-c_1})$$

with $c_1 > 0$ independent on x .

We remark finally that the O-constants depend on τ only through the « constant of smoothness » $C(\tau)$, as can be readily observed from the proof of Theorem 2 given below.

2.

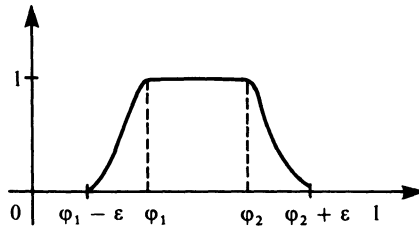
Further on we write $I_0(\mathbf{K})$, $\mathcal{P}(\mathbf{K})$, $H(\mathbf{K})$, $\mu(\mathbf{K})$ for the monoid of the integral divisors, set of prime divisors, class group and the set of basic Größencharaktere of \mathbf{K} . Theorem 2 will be deduced from the following four lemmas.

LEMMA 1. — Let $\varphi_1, \varphi_2, \varepsilon$ satisfy the inequalities

$$0 \leq \varphi_1 - \varepsilon < \varphi_1 < \varphi_2 < \varphi_2 + \varepsilon \leq 1.$$

There exists a real valued function $f \in C^\infty[0,1]$ such that $0 \leq f(t) \leq 1$ for $t \in [0,1]$, $f(t) = 1$ for $t \in [\varphi_1, \varphi_2]$, $f(t) = 0$ for $t \notin [\varphi_1 - \varepsilon, \varphi_2 + \varepsilon]$,

$f'(t) \neq 0$ for $\varphi_1 - \varepsilon < t < \varphi_1$ and $\varphi_2 < t < \varphi_2 + \varepsilon$:



This is a well-known lemma of elementary calculus; we choose one of such functions to be denoted by $f(\varphi_1, \varphi_2, \varepsilon; \cdot)$.

Let C_j, C_K be the idele class groups of k_j, K , and χ_j be an idele class character of k_j trivial on \mathbf{R}_+ ; we define an idele class character

$$(7) \quad \chi := \prod_{j=1}^r \chi_j \circ N_{K/k_j}$$

in K , and an L-function

$$L(\chi_1, \dots, \chi_r; s) := \sum_{\mathfrak{a} \in V} \chi_1(\mathfrak{a}_1) \dots \chi_r(\mathfrak{a}_r) |\mathfrak{a}|^{-s},$$

where $V = \{\mathfrak{a} | \mathfrak{a}_j \in I_0^j, N_{k_j/Q} \mathfrak{a}_1 = \dots = N_{k_r/Q} \mathfrak{a}_r\}$.

LEMMA 2. — If $n = \prod_{j=1}^r n_j$, then $L(\chi_1, \dots, \chi_r; s) = L(s, \chi) L(s, \Phi)^{-1}$, where $L(s, \chi) = \sum_{\mathfrak{a} \in I_0(K)} \chi(\mathfrak{a}) N_{K/Q} \mathfrak{a}^{-s}$ for $\operatorname{Re} s > 1$, and $L(s, \Phi)$ as defined in (5) with $\Phi^{(p)}$ depending on χ_1, \dots, χ_r and having the properties similar to those of the polynomials in (5).

This follows from the results cited before, [12] (or [13]).

LEMMA 3. — Let $n = \prod_{j=1}^r n_j$, then

$$(8) \quad \sum_{\mathfrak{a} \in V, |\mathfrak{a}| < x} \chi_1(\mathfrak{a}_1) \dots \chi_r(\mathfrak{a}_r) = g(\chi) \frac{\omega_K x}{L(1, \Phi)} + O(a(\chi) \frac{3n+1}{2} x^{1-c_1}),$$

$$(9) \quad \sum_{\mathfrak{a} \in V \cap \mathcal{P}, |\mathfrak{a}| < x} \chi_1(\mathfrak{a}_1) \dots \chi_r(\mathfrak{a}_r) \\ = g(\chi) \int_2^x \frac{dx}{\log x} + O\left(x \exp\left(-c_2 \frac{\log x}{\log a(\chi) + \sqrt{\log x}}\right)\right)$$

where $c_1, c_2 > 0$, $g(\chi) = \begin{cases} 0, & \chi \neq 1 \\ 1, & \chi = 1 \end{cases}$, the O -constants and c_1, c_2 depend on k_1, \dots, k_r , but not on χ_1, \dots, χ_r unless $\chi^2 = 1$, nor on x ;
 $\sum_{w \in S} (|a_w| + |b_w|) =: a(\chi)$, when χ is given by

$$(10) \quad \chi(\alpha) = \prod_{w \in S} \left(\frac{\alpha_w}{|\alpha_w|} \right)^{a_w} \cdot |\alpha_w|^{ib_w}$$

for $\alpha \equiv 1 \pmod{f(\chi)}$, $\alpha \in K^*$, $a_w \in \mathbb{Z}$, $b_w \in \mathbb{R}$; α_w denotes the image of α in K_w for $w \in S$ and $f(\chi)$ is the conductor of χ .

Proof. — To prove (9) one remarks (see, e.g., [14], Lemma 1) that for any $a \in V \cap \mathcal{P}$ satisfying the condition « $|a| = q$ is a rational prime» there exists one and only one prime $\mathfrak{p} \in \mathcal{P}(K)$ such that $N_{K/k, \mathfrak{p}} = a_j$. Therefore,

$$\begin{aligned} \sum_{a \in V \cap \mathcal{P}, |a| < x} \chi_1(a_1) \dots \chi_r(a_r) &= \sum_{\substack{a \in V \cap \mathcal{P}, |a| = q \\ q < x}} \chi_1(a_1) \dots \chi_r(a_r) + O(x^{1/2}) \\ &= \sum_{\mathfrak{p} \in \mathcal{P}(K), N_{K/Q, \mathfrak{p}} < x} \chi(\mathfrak{p}) + O(x^{1/2}) \end{aligned}$$

and (9) follows from estimates obtained in the work cited above (see [4], ch. I, § 8, lemma 4, or [5], § 2, lemma 6) (*). By a standard argument one obtains (see, e.g., [15], lemma 3.12)

$$\begin{aligned} A(x) &:= \sum_{a \in V, |a| < x} \chi_1(a_1) \dots \chi_r(a_r) \\ &= \frac{1}{2\pi i} \int_{c-i\pi}^{c+i\pi} \frac{x^s}{s} L(\chi_1, \dots, \chi_r; s) ds + O_\varepsilon\left(\frac{x^{1+\varepsilon}}{T}\right), \end{aligned}$$

where $c = 1 + (\log x)^{-1}$, $T > 0$. It follows from lemma 2 that

$$\begin{aligned} A(x) &= \frac{1}{2\pi i} \int_{1/2+\varepsilon-i\pi}^{1/2+\varepsilon+i\pi} \frac{x^s}{s} L(s, \chi) L(s, \Phi)^{-1} ds + g(\chi) \frac{\omega_K x}{L(1, \Phi)} \\ &\quad + O_\varepsilon\left(\frac{x^{1+\varepsilon}}{T}\right) + O_\varepsilon\left(\int_{1/2+\varepsilon}^c (|L(\sigma+i\pi, \chi)| + |L(\sigma-it, \chi)|) \frac{x^\sigma}{T} d\sigma\right) \end{aligned}$$

because $L(s, \Phi)^{-1} = O_\varepsilon(1)$ for $\operatorname{Re} s > \frac{1}{2} + \varepsilon$.

(*) Alternatively one can deduce (9) from lemma 2.

By a Phragmén-Lindelöf type of argument (compare, [6], pp. 92-93 and [5], pp. 14-15) one deduces from the functional equation for $L(s, \chi)$ and Stirling's formula for the Γ -function an estimate

$$(11) \quad L(\sigma + it, \chi) = O_\varepsilon \left((1 + |t|)^{\frac{3n}{2}(1 - \sigma + \varepsilon)} a(\chi)^{\frac{3n}{2} + \varepsilon} \right)$$

in the region $0 \leq \sigma \leq c$. Substitution of (11) into the estimate for $A(x)$ we have just written out leads to (8).

LEMMA 4. — Let k_j be Galois over Q for each j , $n = \prod_{j=1}^r n_j$, $(D_j, D_\ell) = 1$ for $j \neq \ell$, $\chi = 1$, and χ_j be unramified for each j . Then $\chi_j = 1$ for every j .

Proof. — Let us assume first that χ_j is of finite order for every j ; then, being unramified, it is an ideal class character. One can deduce from class field theory, [17], that (under the above conditions)

$$\{(N_{K/k_1}A, \dots, N_{K/k_r}A) | A \in H_K\} = H_1 \times \dots \times H_r,$$

where H_j is the ideal class group of k_j ; in particular, for any $A_j \in H_j$ there exists $A \in H_K$ such that $N_{K/k_j}A = A_j$; $N_{K/k_\ell}A = 1$ for $\ell \neq j$. If $\chi = 1$, then

$$1 = \prod_{\ell=1}^r (\chi_\ell \circ N_{K/k_\ell})(A) = \chi_j(A_j);$$

and we see that $\chi_j = 1$. Assuming $\chi = 1$ we deduce now that χ_j is of finite order for any j . Let G_j be the Galois group of k_j and G be the Galois group of K ; since $n = \prod_{j=1}^r n_j$, we have $G \cong G_1 \times \dots \times G_r$.

The character

$$(\chi_j \circ N_{K/k_j})^{-1} = \prod_{\ell \neq j} \chi_\ell \circ N_{K/k_\ell}$$

is, therefore, G_j -invariant; since $[C_j : N_{K/k_j}C_K] = d_j$ is finite, we see that $\chi_j^{d_j}$ is G_j -invariant. Take $p \in \mathcal{P}_j$; since $\chi_j^{d_j}(p) = \chi_j^{d_j}(p^\gamma)$ for $\gamma \in G_j$, we see that $(\chi_j(p))^{n_j d_j} = (\chi_j(p))^{d_j}$, where $N_{k_j/Q}p = p^{n_j}$. But any idèle class character in Q is of finite order, and it follows, therefore, that $\chi_j' = 1$ for some ℓ .

3.

Theorem 2 can be deduced from lemma 3 and lemma 4 on purely formal lines. It is an easy consequence of these lemmas and the following form of the Weyl's equidistribution principle (compare [1], p. 37, and [18], Satz 3). To state it we appeal to lemma 1 and write

$$f(\varphi_1, \varphi_2, \varepsilon; t) = \sum_{n=-\infty}^{\infty} c_n \exp(2\pi i n t),$$

so that

$$(12) \quad c_0 = (\varphi_2 - \varphi_1) + O(\varepsilon), \quad c_n = O\left(\frac{1}{|n|^k \varepsilon^{k-1}}\right)$$

for any fixed integral $k \geq 1$.

PROPOSITION 5. — *Let*

$$\mathfrak{I} = \{\exp(2\pi i \varphi_1), \dots, \exp(2\pi i \varphi_m) \mid 0 \leq \varphi_j < 1, j=1, \dots, m\}$$

be a torus of dimension m , τ be a smooth subset of \mathfrak{I} , G be a finite Abelian group with the group of characters \hat{G} and

$$\hat{\mathfrak{I}} = \{\lambda_1^{\ell_1} \dots \lambda_m^{\ell_m} \mid \ell_j \in \mathbb{Z}, \lambda_j: x \mapsto x_j\}$$

be the group of characters of \mathfrak{I} , $x = (\dots, \exp(2\pi i \varphi_j) = x_j, \dots) \in \mathfrak{I}$. Consider a set W and three maps:

$$g_1: W \rightarrow \mathfrak{I}, \quad g_2: W \rightarrow G, \quad N: W \rightarrow \mathbb{R}_+;$$

we denote by \hat{W} the set of functions on W defined by

$$\hat{W} = \{\mu \mid \mu(a) = (\lambda \circ g_1)(a) (\lambda' \circ g_2)(a), \lambda \in \hat{\mathfrak{I}}, \lambda' \in \hat{G}\},$$

where a varies over the elements of W . If

$$(13) \quad \sum_{Na < x} \chi(a) = g(\chi) A(x) + O(x B(x, a(\chi))^{-1})$$

for $\chi \in \hat{W}$, where

$$g(\chi) = \begin{cases} 1, & \lambda=1 \text{ and } \lambda'=1 \\ 0, & \text{otherwise} \end{cases}; \quad A(x) = O(x), \quad a(\chi) := \sum_{j=1}^m |\ell_j|$$

for

$$\chi = (\lambda \circ g_1)(\lambda' \circ g_2), \quad \lambda' \in \hat{G}, \quad \lambda = \prod_{j=1}^m \lambda_j^{\ell_j},$$

then for any smooth subset τ of \mathfrak{X} and any $\gamma \in G$ we have

$$(14) \quad \text{card} \{a \mid a \in W, g_2(a) = \gamma, g_1(a) \in \tau, Na < x\} \\ = A(x) \frac{\text{mes}(\tau)}{|G|} + O\left(\frac{x}{b(x)}\right),$$

where $b(x)$ can be chosen to be equal to $b_1(x)^\nu$ with $\nu > 0$, and $b_1(x)$ is determined by

$$\sum_{\ell_1, \dots, \ell_m = -\infty}^{\infty} \frac{1}{B(x, a(\ell))} \alpha(\ell) = b_1(x)^{-1}, \quad a(\ell) = \sum_{j=1}^m |\ell_j|$$

with $\alpha(\ell) = \prod_{j=1}^m \alpha_j(\ell_j)$, $\alpha_j(\ell_j) = \begin{cases} 1, & \ell_j = 0 \\ \ell_j^{-k}, & \ell_j \neq 0 \end{cases}$, k can be chosen to be any positive integer.

Proof. — We deduce (14) from (13) for rectangular τ by means of lemma 1 and then prove (14) for any smooth $\tau \subseteq \mathfrak{X}$. Let

$$\tau = \{\varphi \mid \psi_j \leq \varphi_j < \psi_j + \delta_j, j = 1, \dots, m\}.$$

Choose $\varepsilon > 0$ and set (using notations of lemma 1)

$$f_j^+(\varphi_j) = f(\psi_j, \psi_j + \delta_j, \varepsilon; \varphi_j), \\ f_j^-(\varphi_j) = f(\psi_j - \varepsilon, \psi_j - \varepsilon + \delta_j, \varepsilon; \varphi_j), \\ F^\pm = \prod_{j=1}^m f_j^\pm.$$

Let \mathcal{N} denote the left hand side in (14). Obviously,

$$\sum_{\substack{Na < x \\ g_2(a) = \gamma}} F^-(g_1(a)) \leq \mathcal{N} \leq \sum_{\substack{Na < x \\ g_2(a) = \gamma}} F^+(g_1(a)).$$

On the other hand,

$$(16) \quad \sum_{\substack{Na < x \\ g_2(a) = \gamma}} F^\pm(g_1(a)) = \frac{1}{|G|} \sum_{Na < x} \sum_{\chi \in \hat{G}} \overline{\chi(\gamma)} F^\pm(g_1(a)) \chi(g_2(a)).$$

Write $f_j^\pm(t) = \sum_{n=-\infty}^{\infty} c_{nj}^\pm \exp(2\pi int)$ and denote the left hand side in (16) by \mathcal{N}^\pm . It follows from (16) that

$$\mathcal{N}^\pm = \sum_{\mu \in \mathbb{W}} c^\pm(\mu) \sum_{Na < x} \mu(a),$$

where

$$c^\pm(\mu) = \frac{1}{|G|} \bar{\chi}(\gamma) \prod_{j=1}^m c_{j_j}^\pm \quad \text{for } \mu = ((\lambda_1^{\prime 1} \dots \lambda_m^{\prime m}) \circ g_1)(\chi \circ g_2).$$

Equation (13) and estimate (12) give

$$\begin{aligned} \mathcal{N}^\pm &= \frac{1}{|G|} \left(\prod_{j=1}^m \delta_j \right) A(x) + O(x\varepsilon) + \sum_{\substack{\mu \in \mathbb{W} \\ \mu \neq 1}} |c^\pm(\mu)| \left| \sum_{Na < x} \mu(a) \right| \\ &= A(x) \frac{\text{mes}(\tau)}{|G|} + O(x\varepsilon) + O\left(\sum_{\mu \in \mathbb{W}} |c^\pm(\mu)| B(x, a(\mu))^{-1} x \right). \end{aligned}$$

Thus

$$\mathcal{N}^\pm = A(x) \frac{\text{mes}(\tau)}{|G|} + O(x\varepsilon) + O(\varepsilon^{-km} x b_1(x)^{-1}).$$

By choosing $\varepsilon^{km+1} = b_1(x)^{-1}$ one obtains (14) with $b(x) = b_1(x)^{1/km+1}$. Now let $\tau \subseteq \mathfrak{I}$ be a smooth set and $t = \{\tau_v\}$ a system of elementary sets with the properties

$$\text{card}(t) < \Delta^{-m}, \quad \tau_v \cap \tau_{v'} = \emptyset \quad \text{for } v \neq v',$$

$$\tau \subseteq \bigcup_{\tau_v \in t} \tau_v, \quad \text{mes} \left(\bigcup_{\tau_v \in t} \tau_v \right) < C(\tau) \cdot \Delta$$

for some $\Delta > 0$. Applying (14) to every $\tau_v \in t$ one obtains

$$\mathcal{N} = A(x) \frac{\text{mes}(\tau)}{|G|} + O(C(\tau) \Delta x) + O\left(\frac{x}{\Delta^m b(x)} \right),$$

and it is enough to choose $\Delta^{m+1} = \frac{1}{b(x)}$ to finish the proof.

To deduce Theorem 2 from Proposition 5 we take $G = H_1 \times \cdots \times H_r$, where H_j denotes the ideal class group of k_j , and define W to be either $V(A) \cap I_0$, or $V(A) \cap \mathcal{P}$. By lemma 3, one can take

$$A(x) = \frac{\omega_K}{L(1, \Phi)} x, \quad B(x, a(\chi)) = \frac{x^{c_1}}{a(\chi)^{\frac{3n+1}{2}}}$$

in the former case, and

$$A(x) = \int_2^x \frac{dx}{\log x}, \quad B(x, a(\chi)) = \exp\left(\frac{c_2 \log x}{\log a(\chi) + \sqrt{\log x}}\right)$$

in the latter case. Lemma 4 assures that $g(\chi) = 0$ for a non-trivial character (χ_1, \dots, χ_r) of H ; it can be checked easily that $a(\chi) \leq c_3 \sum_{j=1}^r a(\chi_j)$ for some constant c_3 depending only on the fields k_1, \dots, k_r , and that in both cases $b(x)$ has the required form to assure the right error terms in theorem 2.

4.

The condition $(D_j, D_\ell) = 1$ for $j \neq \ell$ in theorem 2 and in lemma 4 can be replaced by a weaker one: for every rational prime p one has $(e_j(p), e_i(p)) = 1$ for $j \neq i$, where $e_j(p)$ denotes the ramification degree of p in k_j (compare [17]). Following the interpretation given to the scalar product of L-functions in [19] one may try to interpret theorem 2 as a statement about distribution of integral points on algebraic tori. Finally we should like to refer to [20]-[24], where the problem discussed here or similar questions were studied.

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Appendix.

Following [2] we discuss here the general situation making no a priori assumptions on k_j , $1 \leq j \leq r$, and k . As before, K denotes the

composite field of k_1, \dots, k_r . Given any idele-class character $\chi_j: C_j \rightarrow C^*$ normalized by the conditions $\chi_j \circ N^{-1} = 1$ and $|\chi_j(\alpha)| = 1$, put

$$b_n(\chi_j) = \sum_{N_{k_j/k} a = n} \chi_j(a),$$

and define

$$L(s; \chi_1, \dots, \chi_r) = \sum_n b_n(\chi_1) \dots b_n(\chi_r) |n|^{-s},$$

where n , a vary over integral divisors of k, k_j . It follows then from the results cited above (see [12], [13]) that

$$(A.0) \quad L(s; \chi_1, \dots, \chi_r) = \prod_{j=1}^v L(s, \psi_j) L(s, \Phi)^{-1},$$

where $L(s, \psi_j)$ are Hecke L-functions,

$$(A.1) \quad L(s, \Phi) = \prod_p \Phi^{(p)}(|p|^{-s})^{-1},$$

$\Phi^{(p)}(t)$ is a rational function such that $\Phi^{(p)}(t) = 1 + t^2 g^{(p)}(t)$, $g^{(p)} \in C[t]$ for almost all p (here p varies over the prime divisors of k). Moreover, both ψ_1, \dots, ψ_v and $\Phi^{(p)}$ are exactly computable as soon as χ_1, \dots, χ_r are given. In particular, the product (A.1) converges absolutely for $\text{Re } s > \frac{1}{2}$ and

$$L(s, \Phi) \neq 0, \infty$$

in this half-plane. If k_1, \dots, k_r are linearly disjoint over k , then $v = 1$ and $\psi_1 = \prod_{j=1}^r \chi_j \circ N_{K/k_j}$ is an idele-class character in K ; if $r = 2$ and k_1, k_2 are quadratic extensions of k with co-prime discriminants, then $L(s, \Phi) = L(2s, \chi_0)$ for some idele class character χ_0 of k (depending on χ_1, χ_2). We now apply these results to obtain estimates for the sums

$$S = \sum_{\substack{a \in V_0 \\ |a| < x}} \chi_1(a_1) \dots \chi_r(a_r),$$

$$S_{pr} = \sum_{\substack{p \in V_{pr} \\ |p| < x}} \chi_1(p_1) \dots \chi_r(p_r),$$

where $V_0 = \{\mathfrak{a} \mid N_{k_1/k} \mathfrak{a}_1 = \cdots = N_{k_r/k} \mathfrak{a}_r, \mathfrak{a}_j \in I_0^j\}$,

$$V_{pr} = \{\mathfrak{p} \mid \mathfrak{p} \in V_0, \mathfrak{p}_j \in \mathcal{P}\}.$$

The implied constants in O-symbols depend on χ_1, \dots, χ_r ; this dependence can be expressed in terms of $a(\chi_1), \dots, a(\chi_r)$ but we shall not do it here. Let v_0 be the number of trivial ψ_j :

$$v_0 = |\{j \mid \psi_j = 1\}|,$$

then

$$(A.2) \quad S = \sum_{k=1}^{v_0} (\log x)^{k-1} c_k x + O(x^{1-r}),$$

$$(A.3) \quad S_{pr} = v_0 \int_2^x \frac{dx}{\log x} + O(x \exp(-\gamma' \sqrt{\log x}))$$

for some exactly computable constants c_1, \dots, c_{v_0} and $\gamma > 0$, $\gamma' > 0$.

The estimates (A.2) and (A.3) follow from the properties of the L-functions (A.0) and (A.1) along the same lines as the corresponding estimates in the text.

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