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# BROWNIAN MOTION AND RANDOM WALKS ON MANIFOLDS

by Nicolas Th. VAROPOULOS

## 0. Introduction.

In this paper I shall examine, from a general point of view, some of the questions that were raised in [1]. Let M be a complete, connected, non-compact Riemannian manifold, and let $\Gamma = \{m_i \in M,\ i = 1, 2, \dots\}$ be a discrete grid of M i.e. a subset such that $d(m_i, m_j) > 1/A$ $(i \neq j)$ and $d(m, \Gamma) < A$ $(m \in M)$ for the Riemannian distance $d$ and some $A > 0$. The distance $d$ induces then a distance on the countable space $\Gamma$, and we shall say that a random walk on $\Gamma$ is an admissible walk if its leaps are neither too long nor too short for that distance (cf. § 4 for the exact definition).

The main aim of this paper is to compare the canonical Brownian motion on M (cf. [2], [3]) with the admissible random walks on $\Gamma$. We prove, for instance, that Brownian motion on M is recurrent if and only if the above random walks are recurrent. As a corollary we recover two of the main results of [1] (cf. [15] for a special case) :

THEOREM. — *Let* M *be a compact Riemannian manifold and let* $\tilde{M} \to M$ *be a regular connected covering of* M *with* G *as a deck transformation group* [*i.e.* $G = \pi_1(M)/N$ $N \lhd \pi_1(M)$]. *Then Brownian motion is transient on* $\tilde{M}$ *if and only if* G *is a transient group.*

The other result deals with a regular covering of the $k$-punctured sphere, i.e. $\Sigma \to S \backslash \{z_1, z_2, \dots, z_k\} = S_k$ where S is the Riemannian sphere $z_1, z_1, \dots, z_k \in S$ and $\Sigma$ is some regular covering with G as a deck transformation group. We shall construct then a random walk on $D \times G$ (where $D$ is the disjoint union of $k$ copies of the non-negative integers with their $k$ 0-points identified) and we shall show that the Riemann surface $\Sigma$ is hyperbolic if and only if that walk is transient. The case $G = H^1(S_k) = \pi_1(S_k)/[\pi_1, \pi_1]$ is the Lyons-McKean case [4] (for $k = 3$) and then the walk is transient.

One way of seeing the above general considerations, is to say that one « discretises the potential theory of $M$ » to the potential theory of a random walk on $\Gamma$. Another application of the above is an estimate on the Heat diffusion kernel $p_t(x,y)$ defined by

$$p_t(x,y)\, dV(y) = \mathbf{P}_x[z(t) \in dV(y)]; \qquad x, y \in M, \quad t > 0$$

where $\{z(t) \in M; t>0\}$ is the Brownian motion of $M$. (I assume that there is no explosion cf. [2], [3].) Indeed we have :

THEOREM. — *Let* $M$ *be a complete, connected, non-compact manifold and let us assume that* $K(V)$, *the sectional curvature of* $M$, *is uniformly bounded on* $M$ *and that* $i_x(M)$ $(x \in M)$, *the injectivity radius of* $M$, *is bounded from below* (cf. [5]). *There exist then* $C_1^{(\varepsilon)}$, $C_2^{(\varepsilon)} \ldots$ *constants depending only on* $M$ *and* $\varepsilon > 0$ *such that*

$$\sup_{x,y \in M} |p_t(x,y)| \leqslant C_1^{(\varepsilon)} t^{-1/2+\varepsilon}; \qquad t > 1$$

$$\sup_{x,y \in M} |p_t(x,y)| \leqslant C_2^{(\varepsilon)} t^{-1/2} (\log t)^{1+\varepsilon}; \quad t > 1$$

etc.

The correct bound is no doubt $Ct^{-1/2}$, but I cannot prove that.

Analogously, for $\mu \in \mathbf{P}(G)$ any symmetric probability measure, on a discrete, finitely generated, infinite group, $G$, we have estimates :

$$\sup_{g \in G} |\mu^{*n}(g)| \leqslant \tilde{C}_1^{(\varepsilon)} n^{-1/2+\varepsilon} \qquad (n \geqslant 1)$$

$$\sup_{g \in G} |\mu^{*n}(g)| \leqslant \tilde{C}_2^{(\varepsilon)} n^{-1/2} (\log n)^{1+\varepsilon} \quad (n \geqslant 1) \quad \text{etc.}$$

(I assume here that $\operatorname{supp} \mu$ generates $G$.)

## 1. Metrics and graphs on discrete spaces.

We shall use throughout $\mathbf{N} = \{1,\dots\}$ as a model of a discrete space. Let $d$ be a distance on $\mathbf{N}$ so that $(\mathbf{N},d)$ is a metric space, we shall say that $d$ is *discrete* if

$$\sup_i \operatorname{Card}(\mathrm{B}_a(i)) < +\infty \quad (a>0) \tag{1.1}$$

where

$$\mathrm{B}_a(i) = \{j \in \mathbf{N} \mid d(i,j) \leqslant a\}.$$

We shall say that $d$ is connected if there exists some $a > 0$ such that for every choice of $i, j \in \mathbf{N}$ we can find $i = i_0,\ i_1, \dots, i_k = j$ such that:

$$d(i_p, i_{p+1}) \leqslant a, \quad p = 0, 1, \dots, k-1; \qquad k \leqslant ad\,(i,j). \tag{1.2}$$

We shall say that $\Gamma$ is a graph on $\mathbf{N}$ if $\mathscr{E}_\Gamma = \mathscr{E} \subset \mathbf{N} \times \mathbf{N}$ the set of edges is a symmetric subset that contains the diagonal. It is clear then that a connected (in the ordinary sense of the word) graph on $\mathbf{N}$ defines a connected metric on $\mathbf{N}$ by:

$$d_\Gamma(i,j) = \inf\{n \mid i, j \text{ can be connected by } n \text{ edges}\} \text{ for } i \neq j. \tag{1.3}$$

Conversely an arbitrary metric $d$ on $\mathbf{N}$ defines the graph

$$\mathscr{E}_d = \{(i,j) \in \mathbf{N} \times \mathbf{N} \mid d(i,j) \leqslant a\} \tag{1.4}$$

where $a > 0$ is some fixed $a > 0$.

If the $a$ in (1.4) verifies (1.2) then the above graph is connected. If the metric $d$ is discrete then the graph defined by (1.3) satisfies

$$\sup_i \operatorname{Card} \Sigma_\Gamma(i) < +\infty \tag{1.5}$$

where $\Sigma_\Gamma(i)$ denotes the star of $i$ in $\Gamma$ i.e. the elements of $\mathbf{N}$ that can be connected to $i$ by one edge of $\Gamma$. Conversely a connected graph that verifies (1.5) gives rise in (1.3) to a discrete metric.

The above correspondance between metrics and graphs on $\mathbf{N}$ is essentially biunique and will be used systematically.

We have:

PROPOSITION. — *Let* $\Gamma$ *be a connected graph on* $\mathbf{N}$ *that verifies* (1.5). *For all* $i \in \mathbf{N}$ *there exists then a sequence of distinct points* $i = i_0, i_1, \ldots$ *such that* $d(i_p, i_{p+q}) = q$ *for* $p, q = 0, 1, \ldots$.

The proof is an easy application of a diagonal process. [i.e. Tychonov's Theorem on products of compact spaces !]

To illustrate the above ideas let G be a discrete group generated by the finite set $g_1, g_2, \ldots, g_s \in G$. Let us denote by:

$$|g| = \inf\{n \mid g = g_{i_1}^{\varepsilon_1} \ldots g_{i_n}^{\varepsilon_n};\ 1 \leqslant i_k \leqslant s,\ \varepsilon_k = \pm 1,\ k = 1, \ldots, n\}.$$

Different sets of generators give different $|\ |$ but we have:

$$A^{-1}|g|^{\text{old}} \leqslant |g|^{\text{new}} \leqslant A|g|^{\text{old}}$$

for some $A > 0$. Two connected discrete metrics can then be defined on G by $d_\ell(g,h) = |g^{-1}h|$ and $d_r(g,h) = |gh^{-1}|$. In general the above two metrics are not equivalent and the antiisomorphism $x \to x^{-1}$ on G is an isometry between $d_\ell$ and $d_r$. $d_\ell$ (resp. $d_r$) is left (resp. right) invariant.

The above metrics arise naturally in the theory of the Riemannian covering spaces. Let M be a compact Riemannian manifold and let $\tilde{M} \xrightarrow[p]{} M$ be a regular connected covering (this means that the deck transformation group is transitive on each fiber $p^{-1}(x_0)$ $(x_0 \in M)$ and that $\tilde{M}$ is obtained from the universal covering space by identifying the points in a normal subgroup $N \lhd \pi_1(M)$). $\tilde{M}$ is then endowed with a natural Riemannian structure.

Let us fix a base point $x \in \tilde{M}$ let $x_0 = p(x)$ and let us define the deck transformation group $G = \pi_1/N$ with respect to that point $x$.

We can then identify $p^{-1}(x_0)$ with the group G since

$$p^{-1}(x_0) = \{gx \mid g \in G\}$$

and the metric on $\tilde{M}$ induces thus a distance $d^*$ on G which is left invariant (and depends, but only in a very inessential way, on the choice of the base point $x \in \tilde{M}$). The above distance is equivalent to $d_\ell$ on G (observe that $\pi_1$ and therefore G is *finitely generated*) cf. [14]. Indeed the above two distances on G are both *left* invariant and the assertion is a consequence of the following

LEMMA (Milnor [6]). — *Let* $\tilde{M}$, M *and* $x_0 = p(x)$ *be as above and let* $\gamma_1, \ldots, \gamma_s \in \pi_1(M;x_0)$ *be a (prefered) set of generators of the fundamental group. There exists then* $A > 0$ *such that*

$$A^{-1}|g| \leqslant d(x,gx) \leqslant A|g|, \quad g \in G$$

where $d$ *denotes the Riemannian distance on* $\tilde{M}$.

*Proof (I give it for completeness).* — Let us use $\dot\gamma_1, \ldots, \dot\gamma_s \in G = \pi_1/N$ (the canonical images of $\gamma_i$ in G) as generators of G, and let $g \in G$. Let $g = \dot\gamma_{i_1}^{\varepsilon_1} \ldots \dot\gamma_{i_n}^{\varepsilon_n}$ we can join then $x$ with $gx$ by a curve of length $\leqslant n.\max_{1 \leqslant j \leqslant s}$ [length in M of a loop representing $\gamma_j$]. This proves that $f(x,gx) \leqslant A|g|$.

Let us now fix $D \subset \tilde{M}$ some relatively compact fundamental domain such that $x_0 \in \mathring{D} \subset D \subset \overline{\mathring{D}}$. [e.g. we can construct D by the exponential map at $x_0$ so that $\mathring{D}$ can be identified to M\Cut locus of $x_0$]. Let $D_1$ be the ε-N*hd* of D [i.e. points at a distance $\leqslant \varepsilon$] and let $D_3$ be the 3ε-N*hd* of D (for some $\varepsilon > 0$).

$\{g\mathring{D}_1 \mid g \in G\}$ is then an open covering of $\tilde{M}$ and the family $\{g\mathring{D}_3 \mid g \in G\}$ is locally finite [for ε sufficiently small]. Indeed the $g\mathring{D}_3$ contain disjoined open balls [if ε is sufficiently small].

Let us fix $g \in G$ and let us join $x$ with $gx$ by a [lengh parameter] minimizing geodesic $\ell$ in $\tilde{M}$:

$$\ell = \{\ell(t)/0 \leqslant t \leqslant d = d(x,gx)\}.$$

Let $x_j = \ell(t_j)$ with $t_j = j\dfrac{d}{L}$ $(j = 0,1,\ldots,L)$ where $L \geqslant \varepsilon^{-1} d$ and let $x_j \in h_j\mathring{D}_1$ $(h_j \in G \mid j = 0,1,\ldots,L)$ with $h_0$ = identity of G and $h_L = g$. It follows that $g = f_0 f_1 \ldots f_{L-1}$ where $f_k = h_k^{-1} h_{k+1}$ $(k = 0,1,\ldots,L-1)$ and $f_k\mathring{D}_3 \cap \mathring{D}_3 \neq \emptyset$ $(k = 0,1,\ldots,L-1)$. But this last relation and the local finiteness of the covering $\{g\mathring{D}_3 \mid g \in G\}$ force all the $f_k$ 's to be in some fixed finite set $F \subset G$. If we use $\{\dot\gamma_1,\ldots,\dot\gamma_s\} \cup F$ as a set of generators in G we conclude therefore that $|g| \leqslant L$ that can be chosen to be less than $[\varepsilon^{-1} d] + 1$. This proves that $|g| \leqslant A\, d(x,gx)$.

## 2. — S-operators : Asymptotic estimates.

Let H be a *real* Hilbert space and let $T_1, T_2, T_3$ be three contractions on H (i.e. $\|T_i\| \leqslant 1$, $i=1,2,3$) that satisfy

$$T_1 = \alpha T_2 + (1-\alpha) T_3$$

for some $0 < \alpha \leqslant 1$ and such that $T_2$ is symmetric (i.e. $\langle T_2 x, y\rangle = \langle x, T_2 y\rangle$, $x,y \in H$). We then have :

$$\sum_{n=0}^{\infty} \lambda^n \langle T_1^n f, f\rangle \leqslant \frac{1}{\alpha} \sum_{n=0}^{\infty} \lambda^n \langle T_2^n f, f\rangle ; \qquad f \in H, \quad 0 \leqslant \lambda < 1$$

(cf. [7], [1]).

From this it follows that for all $\mu$ probability measure on $[0, 1)$ and

(2.1) $$\xi_n = \int_0^1 \lambda^n \, d\mu(\lambda) \quad (n \geqslant 0)$$

we have :

(2.2) $$\sum_{n=0}^{\infty} \xi_n \langle T_1^n f, f\rangle \leqslant \frac{1}{\alpha} \sum_{n=0}^{\infty} \xi_n \langle T_2^n f, f\rangle .$$

If we suppose that :

(2.3) $$\langle T_1^n f, f\rangle , \quad \langle T_2^n f, f\rangle \geqslant 0 \quad (n \geqslant 0)$$

then we can suppose that $\mu$ is a measure on $[0, 1]$ in particular it can be chosen to be $\delta_1$ and we have :

(2.4) $$\sum_{n=0}^{\infty} \langle T_1^n f, f\rangle \leqslant \frac{1}{\alpha} \sum_{n=0}^{\infty} \langle T_2^n f, f\rangle .$$

If (2.3) holds and $\mu$ is such that :

$$\xi_n \langle T_1^n f, f\rangle \text{ is decreasing in } n; \qquad \sum_{n=0}^{\infty} \xi_n \langle T_2^n f, f\rangle < +\infty$$

then it follows that :

(2.5) $$\langle T_1^n f, f\rangle = O\left(\frac{1}{n\xi_n}\right) \cdot$$

Let now $(X;dx)$ be a measure space and let $\{K(x,y);x,y\geqslant 0\}$ be a positive kernel on $X$ we shall say that $K$ is an S-operator or an S-kernel if

$$\int K(x,y_0)\,dx = \int K(x_0,y)\,dy = 1\,; \qquad x_0, y_0 \in X.$$

We shall say that $K$ is symmetric if $K(x,y) = K(y,x)$ $(\forall x,y \in X)$. We have then the following order relation on S-kernels:

$$K_1 \gg K_2 \Leftrightarrow \exists \alpha > 0 \text{ s.t. } K_1(x,y) \geqslant \alpha K_2(x,y)\,; \quad x, y \in X.$$

We have:

PROPOSITION. – *Let* $K_1$, $K_2$ *be two* S*-operators on* $(X,dx)$ *s.t.* $K_2$ *is symmetric and* $K_1 \gg K_2$. *Then there exists some* $A > 0$ *s.t.*

$$(2.6) \qquad \sum_{n=0}^{\infty} \xi_n \langle K_1^n f,f\rangle \leqslant A \sum_{n=0}^{\infty} \xi_n \langle K_2^n f,f\rangle$$

*for every* $\xi_n$ *as in* (2.1) *and* $f \in L^2(X;dx)$.

*Proof.* – Indeed if $\alpha > 0$ is small enough we have

$$K_1 = \alpha K_2 + (1-\alpha)K_3$$

with $K_3$ an S-operator. It remains to observe that S-operators are contractions on the Hilbert space $L^2(X;dx)$ and to apply (2.2).

If $f \in L^2(X;dx)$ is *non-negative* then the analogue of (2.4) holds and similarly for (2.5).

Observe finally that for an appropriate choice of $\mu$ we can have:

$$\xi_n \sim n^{-\beta}$$

for any $\beta \geqslant 0$, or we can have:

$$\xi_n \sim n^{-\beta} (\log n)^A$$

for any $\beta > 0$ and $A \in \mathbf{R}$; and so on.

Let $\{\xi_n\}$ be as in (2.1) let $0 \leqslant f \in L^2(X;dx)$ and let $K_1 \gg K_2$ be two S-operators such that $K_2$ is symmetric and such that:

$$\sum_{n=0}^{\infty} \xi_n \langle K_2^n f,f\rangle < +\infty ; \qquad \xi_n \langle K_1^n f,f\rangle \ \text{decreasing in } n .$$

It follows then from (2.5) that

$$\langle K_1^n f,f\rangle = O\left(\frac{1}{n\xi_n}\right) \cdot \tag{2.7}$$

## 3. Discretising S-operators.

Let $(X;dx)$ be a measure space and let $X = \bigcup_{j=1}^{\infty} X_j$ be a disjoined partition into Borel subsets of positive finite measure. The above partition induces two positive norm decreasing mappings:

$$T: \ L^p(X;dx) \to \ell^p(d\lambda); \quad T^*: \ell^q(d\lambda) \to L^q(X;dx)$$

for $1 \leqslant p, q \leqslant +\infty$ where $d\lambda$ is the measure on $\mathbf{N}$ given by $\lambda_j = \lambda(\{j\}) = dx$ measure of $X_j$.

We define the two mappings by:

$$(Tf)(j) = \frac{1}{\lambda_j}\int_{X_j} f\,dx; \qquad j = 1, \ldots, f \in L^p(X;dx)$$
$$(T^*f)(x) = f(j) \ \forall x \in X_j; \qquad j = 1, \ldots, f \in \ell^p(d\lambda).$$

The two mappings are adjoint of each other and satisfy $T \circ T^* = \text{Identity}$.

Using the above two mappings we can establish a correspondance between the S-operators on $(X;dx)$ and the S-operators on $(\mathbf{N};d\lambda)$. Indeed if $K$ is a (symmetric) S-operator on $(X;dx)$ then $TKT^*$ is a (symmetric) S-operator on $(\mathbf{N};d\lambda)$ and conversely if $L$ is a (symmetric) S-operator on $(\mathbf{N};d\lambda)$ then $T^*LT$ is a (symmetric) S-operator on $(X;dx)$. The above facts can be verified directly or one can observe that

S-operators are *characterized* by the properties that together with their adjoints they are positive, norm decreasing on all the $L^p$ spaces $(1 \leqslant p \leqslant +\infty)$ and that they preserve the identity (or that they preserve total mass of positive functions of $L^1$).

In the applications that we have in mind $(X;dx)$ will be a complete, connected, non-compact Riemannian manifold assigned with its canonical volume which we shall denote by $(M;dV)$. We shall consider then decompositions of $M$ into disjoint relatively compact subsets $M = \bigcup_{j=1}^{\infty} M_j$; we shall fix points $m_j \in M_j$ $(j=1,\ldots)$ and we shall suppose throughout that for some $A > 0$ the following condition is verified:

$$(3.1) \quad A^{-1}\, d(x,y) \leqslant d(m_i,m_j) \leqslant A(d(x,y)+1); \quad \forall x \in M_i, \quad y \in M_j, \quad i,j = 1, \ldots, \quad i \neq j$$

($d$ denotes of course the Riemannian distance on $M$).

Let us then denote by $p_t(x,y)$ the Heat diffusion kernel on $M$ which in terms of the canonical Brownian motion $\{z(t);0<t<e\}$ on $M$ (cf. [3], [4]) can be defined by:

$$p_t(x,y)\, dV(y) = \mathbf{P}_x[z(t) \in dV(y); t<e].$$

For convenience I shall assume that $e = +\infty$ i.e. I shall assume that diffusion is conservative on $M$:

$$(3.2) \qquad \int p_t(x,y)\, dV(y) = 1; \qquad t > 0, \quad x \in M.$$

This holds under very general conditions on $M$ (cf. [8], [3]) and certainly under condition (3.6) below.

I shall make a number of further assumptions on $p_t$. To avoid repetition in expressing the conditions below I shall resort throughout to standard practice and denote the dependance of a constant that appears in the text by brackets (e.g. $C = C(\lambda_1,\lambda_2,\ldots)$ means that the constant $C$ depends only on the parameters $\lambda_1, \lambda_2, \ldots$).

Let us then fix $M$ and let us assume that (3.2) holds. The positive constants below $A, C_1, K_1, \ldots$ all depend on $M$ and whatever parameter appears in their bracket.

Here are the conditions that we shall need on $p_t$:

$$(3.3)\quad p_t(x,y) \geqslant A(d_0, t_1, t_2)\,; \quad x, y \in M, \ d(x,y) \leqslant d_0, \ t \in [t_1, t_2]$$

$$(3.4)\quad p_t(x,y) \geqslant C_1(t_1, t_2) e^{-K_1 \frac{d^2(x,y)}{t}}\,; \quad x, y \in M, \ t \in [t_1, t_2]$$

$$(3.5)\quad p_t(x,y) \leqslant C_2(t_1, t_2) e^{-K_2 \frac{d^2(x,y)}{t}}\,; \quad x, y \in M, \ t \in [t_1, t_2]$$

where $0 < t_1 < t_2$.

The condition (3.4) is stronger than (3.3). Condition (3.4) is a consequence of the following geometric condition

$$(3.6)\qquad \mathrm{Ric}_x(X,X) \geqslant -K|X|^2\,; \qquad X \in T_x(M), \ x \in M$$

where K is some fixed constant. This fact is contained in [9] and is implicit in [10]. The advantage of (3.3) over (3.4) is that it is much « cheaper » to prove and it suffices for most of our purposes (all in fact). (3.3) is automatic if we assume for instance that a discrete group G acts uniformly on M (i.e. M/G is compact).

The condition (3.5) is a consequence of the following hypothesis on M :

$$(3.7)\qquad \sup_{x \in M} |K_x(V)| < +\infty\,; \qquad \inf_{x \in M} i_x(M) > 0,$$

$K_x(V)$ denotes the sectional curvature at $x$ $(V \in T_x^2(M))$ and $i_x(M)$ denotes the injectivity radius at $x \in M$. The fact that (3.7) implies (3.5) is proved in [11], [16] (cf. also [17]).

I shall now apply the discretising procedure described above to the symmetric S-operators $p_t(x,y)$ $(t>0)$ on $(M; dV)$ relative to the decomposition (3.1).

I shall denote by :

$$(3.8)\qquad k_t = Tp_tT^*\,; \qquad K_t = T^*Tp_tT^*T, \quad (t>0)$$

which are clearly symmetric S-operators on $(N; d\lambda)$ and $(M; dV)$ provided that (3.2) is verified.

The point now is that under conditions (3.1), (3.2) and (3.4) we have :

$$(3.9)\quad k_t(i,j) \geqslant C_1'(t_1, t_2) e^{-K_1' \frac{d^2(m_i, m_j)}{t}}\,; \quad i, j = 1, 2, \ldots, t \in [t_1, t_2]$$

and under conditions (3.1), (3.2) and (3.5) we have

$$(3.10)\quad k_t(i,j) \leqslant C_2'(t_1,t_2)e^{-K_2'\frac{d^2(m_i,m_j)}{t}}\ ;\quad i,j = 1, 2, \ldots, t \in [t_1,t_2]$$

(with $0 < t_1 < t_2$). We conclude:

PROPOSITION. — *Let* M *be a complete, connected, non-compact Riemannian manifold that satisfies* (3.7) *and let* $M = \bigcup_{j=1}^{\infty} M_j$ *be a decomposition that satisfies* (3.1).

*For every time interval* $[t_1,t_2]$ $(0<t_1<t_2)$ *there exists then* $t_0 > 0$ *and constants* $A > 0$ *such that:*

$$p_t(x,y) \geqslant AK_{t_0}(x,y);\qquad K_t(x,y) \geqslant Ap_{t_0}(x,y)$$
$$t \in [t_1,t_2],\qquad x, y \in M$$

*where* $K_t = Tk_tT^*$ *is the kernel defined in* (3.8). *Furthermore* $k_t$ *satisfies* (3.9) *and* (3.10).

The full strenght of (3.4) was used in the proof of the above proposition.

The weaker condition would yield, however, estimates that are almost as good and at any rate sufficient to push the rest of the theory through. The interested reader will convince himself of this fact without any difficulty, I am sure. Observe also that the above proposition will only be used for arbitrarily small values of $t_1$, $t_2$. The estimate (3.5) is somewhat easier to obtain then.

## 4. Discrete random walks.

A time homogeneous random walk $\{x(n);\, n=1,2,\ldots\}$ with values in the discrete countable space $N = \{1,2,\ldots\}$ will be identified in what follows with its Markovian transition matrix $P = \{P(i,j);\, i,j=1,2,\ldots\}$ where

$$P(i,j) = \text{Prob}\,[x(n+1)=j\,|\,x(n)=i].$$

Let P be as above and let:

$$Q = e^{-1}\,\mathrm{Exp}\,(P) = e^{-1}\left[I + P + \frac{1}{2!}P^2 + \cdots\right]$$

$$R_\alpha = \alpha I + (1-\alpha)P, \qquad 0 \leqslant \alpha \leqslant 1$$

which are also Markovian matrices.

Direct computation shows then that:

$$\text{(4.1)} \qquad \sum_{n=0}^{\infty} Q^n = \sum_{n=0}^{\infty} \lambda_n P^n, \qquad C^{-1} \leqslant \lambda_n \leqslant C \qquad n \geqslant 1$$

$$\text{(4.2)} \qquad \sum_{n=0}^{\infty} R_\alpha^n = \frac{1}{1-\alpha}\sum_{n=0}^{\infty} P^n, \qquad 0 \leqslant \alpha < 1$$

where $C > 0$ is numerical ($P^n, Q^n$ etc. is the ordinary matrix multiplication). Indeed we have:

$$\lambda_p = \sum_{n=0}^{\infty} e^{-n}\frac{n^p}{p!} \approx \frac{1}{p!}\int_0^\infty e^{-t}t^p\,dt = 1\,.$$

[To convince yourself that the $\approx$ is uniform with respect to $p$ observe that the function $e^{-t}t^p$ has only one maximum at $t = p$ and that that maximum is of the order $\left.\frac{p!}{\sqrt{p}}\right]$.

A Markovian matrix P induces the usual $\tilde{P}: \ell^\infty \to \ell^\infty$ mapping by:

$$(\tilde{P})_i = \sum_{j=1}^{\infty} P(i,j)f_j, \qquad i = 1, 2, \ldots.$$

Let then $\lambda$ be a positive measure on $\mathbf{N}$ such that $\lambda(\{j\}) = \lambda_j > 0$ ($j=1,2,\ldots$), we then say that $\lambda$ is a symmetrising measure for the random walk P if the matrix $\{\lambda_i P(i,j) \mid i,j=1,\ldots\}$ is symmetric i.e. if for the scalar product $\langle\ \ \rangle$ on $\ell^2(d\lambda)$ we have $\langle \tilde{P}f,g\rangle = \langle f,\tilde{P}g\rangle$ for $f, g$ compactly supported elements of $\ell^\infty$. Walks that admit a symmetrising measure will be called symmetrisable.

We then have

$$(\tilde{P}f)_i = \sum_{j=1}^{\infty} K(i,j)f_j\lambda_j$$

where

$$\{K(i,j) = P(i,j)\lambda_j^{-1} \mid i,j=1,2,\ldots\}$$

is a symmetric S-operator on $(\mathbf{N};d\lambda)$. Conversely of course if

$$\{\tilde{K}(i,j)\,|\,i,j=1,\ldots\}$$

is a symmetric S-operator on $(\mathbf{N};d\tilde{\lambda})$ for some $\tilde{\lambda}$ such that

$$\tilde{\lambda}(\{j\}) > 0 \quad (j \geqslant 1)$$

then $\tilde{P}(i,j) = \tilde{K}(i,j)\tilde{\lambda}(\{j\})$ is a symmetrisable random walk.

For two random walks we shall say that $P \gg Q$ if for some $\alpha > 0$ we have $P(i,j) \geqslant \alpha Q(i,j)$ $(i,j=1,2,\ldots)$. If P and Q can both be symmetrised by the same measure $\lambda$ on $\mathbf{N}$ then $P \gg Q$ if and only if the corresponding S-operators $K_P$ and $K_Q$ satisfy $K_P \gg K_Q$.

PROPOSITION 4.1. – *Let $d$ be a connected metric on $\mathbf{N}$ and let $P, Q$ be two random walks on $\mathbf{N}$ that satisfy :*

(i) $P(i,j) \leqslant C d(i,j)^{-bd(i,j)}$ $(i \neq j)$,

(ii) $\inf\{Q(i,j)\,|\,d(i,j) \leqslant a,\, i \neq j\} = \lambda > 0$,

*where $a > 0$ satisfies* (1.1), *C is a positive constant and $b > a$.*

*Then we have $e^{-1}e^{Q} \gg P$.*

*Proof.* – Let $i \neq j$ and let $i = i_0, i_1, \ldots, i_{n-1}, i_n = j$, $n+1$ *distinct* points such that $d(i_k, i_{k+1}) \leqslant a$, $n \leqslant ad(i,j)$ it follows from (ii) that $Q^n(i,j) \geqslant \lambda^n$ so that :

$$e^{-1}e^{Q}(i,j) \geqslant e^{-1}\frac{\lambda^n}{n!} \geqslant \alpha\left(\frac{\lambda}{ea}\right)^{ad(i,j)} d(i,j)^{-ad(i,j)}$$

for some $\alpha > 0$.

On the other hand it is also clear that $e^{-1}e^{Q}(i,i) \geqslant e^{-1}$. From this and (i) our proposition follows.

DEFINITION. – *Let $d$ be a connected metric on $\mathbf{N}$ we then say that a random walk on $\mathbf{N}$ is admissible with respect to $d$ if it satisfies conditions* (i) *and* (ii) *of the Proposition* 4.1.

Observe that no admissible random walks can exist unless the metric is discrete. Conversely if the metric is discrete plenty of admissible random walks can be constructed.

To see that, let more generally $\Gamma$ be a graph on $\mathbf{N}$ that is not necessarily connected but such that (1.5) is satisfied. If we set then

$$P(i,j) = L(\lambda_i \delta_{i,j} + I_{\mathscr{E}}(i,j))$$

where $I_{\mathscr{E}}$ is the characteristic function of $\mathscr{E} \subset \mathbf{N} \times \mathbf{N}$ we see that for an appropriate choice of $\lambda_i \geqslant -1$ and of $L > 0$ we have a Markovian matrix that is subordinated to $\Gamma$ in the sense that $P(i,j) = 0$ if $(i,j) \notin \mathscr{E}_\Gamma$. The above matrix is clearly symmetric and if $\Gamma$ is a connected graph the random walk induced by P is admissible with respect to metric $d_\Gamma$ induced by $\Gamma$ (as explained in § 1).

From Proposition 4.1 we deduce the following key:

COROLLARY. — *Let $d$ be a connected discrete metric on $\mathbf{N}$ and let $P_1, P_2$ be two admissible random walks (with respect to $d$) that admit a common symmetrising measure. Then there exists a constant $C > 0$ such that*

$$C^{-1} \sum_{n=0}^{\infty} P_2^n(i,i) \leqslant \sum_{n=0}^{\infty} P_1^n(i,i) \leqslant C \sum_{n=0}^{\infty} P_2^n(i,i), \quad i \in \mathbf{N}.$$

*Proof.* — Indeed let $\lambda (= \{\lambda_j\}_{j=1}^{\infty})$ be the symmetrising measure of these two walks and let $K_\alpha(i,j) = P_\alpha(i,j)\lambda_j^{-1}$ $(\alpha = 1,2)$ be the corresponding symmetric S-kernels. The same measure $\lambda$ also symmetrises the two walks $Q_\alpha = e^{-1}e^{P_\alpha}$ $(\alpha = 1,2)$ and if we denote by $L_\alpha(i,j) = Q_\alpha(i,j)\lambda_j^{-1}$, the corresponding S-kernel, we clearly have

$$Q_1 \gg P_2; \qquad Q_2 \gg P_1$$

by Proposition 4.1 and also

$$L_1 \gg K_2; \qquad L_2 \gg K_1.$$

By the proposition of § 2 it follows therefore that

$$\sum_{n=0}^{\infty} L_1^n(i,i) \leqslant C \sum_{n=0}^{\infty} K_2^n(i,i), \quad i \in \mathbf{N}$$

for some $C > 0$ where the $n^{\text{th}}$ power of $K_2$ and $L_1$ are now the $n^{\text{th}}$ power of the corresponding operators in $\ell^2(d\lambda)$ and we have

$$K_2^n(i,i) = P_2^n(i,i)\lambda_i^{-1}$$
$$L_1^n(i,i) = Q_1^n(i,i)\lambda_i^{-1}.$$

The conclusion is that:

$$\sum_{n=0}^{\infty} Q_1^n(i,i) \leqslant C \sum_{n=0}^{\infty} P_2^n(i,i)$$

which together with (4.1) gives one of the two inequalities of the corollary. The other inequality follows by symmetry.

The upshot of the corollary is of course that the transience of symmetrisable admissible walks on $(\mathbf{N},d)$ is an invariant of the metric $d$ and of the symmetrising measure. More often than not of course the symmetrising measure will simply be the « standard » measure that gives mass 1 to each point $(\lambda_j = 1, j = 1, \ldots)$.

Interesting examples of random walks that satisfy the above conditions are supplied by finitely generated groups. Let G be such a group and let us assign it with its left (resp. right) metric associated to some fixed set of generators $\{g_i; i = 1, \ldots s\}$ as in § 1.

Let $\mu \in \mathbf{P}(G)$ be a probability measure and let us set:

$$\text{(4.3)} \qquad P(g,h) = \mu(\{g^{-1}h\}) \quad (\text{resp.}: \mu(\{gh^{-1}\}).$$

This gives us a random walk on G. This walk is admissible with respect to the above left (resp.: right) metric of G if (and only if):

$$\text{(i)} \qquad \mu(\{g\}) \leqslant C|g|^{-b|g|}; \qquad g \in G$$

$$\text{(ii)} \qquad \mu(\{g_i^{\varepsilon}\}) > 0, \qquad i = 1, \ldots, s, \quad \varepsilon = \pm 1$$

for some $b > 1$

We obtain thus left (resp.: right) invariant random walks on G. The mapping $x \to x^{-1}$ identifies the left invariant walks to the right invariant walks with the same $\mu \in \mathbf{P}(G)$.

The walk defined by (4.3) is symmetric (with respect to the Haar measure of G) if and only if $\mu$ is a symmetric measure i.e. if

$$\text{(iii)} \qquad \mu(\{g\}) = \mu(\{g^{-1}\}); \qquad g \in G.$$

As a consequence of the corollary we see that for a given discrete finitely generated group all the admissible symmetric walks (right or left) are transient or recurrent at the same time.

We shall say that G is a transient group if these walks are transient and we say that G is recurrent in the opposite case.

PROPOSITION 4.2. — *Let* G *be an infinite group generated by* $\{g_i; i=1,\ldots,s\}$ *and let* $\mu \in \mathrm{P}(\mathrm{G})$ *be a symmetric probability measure such that* $\mu(\{g_i\}) > 0$ $(i=1,\ldots,s)$. *For all* $0 < \varepsilon < 0.1$ *there exist then constants* $\mathrm{C}_1 = \mathrm{C}_1(\varepsilon)$, $\mathrm{C}_2 = \mathrm{C}_2(\varepsilon)$ *etc. such that :*

$$\sup_{g \in \mathrm{G}} |\mu^{*n}(g)| \leqslant \min \{\mathrm{C}_1 n^{-1/2+\varepsilon}, \mathrm{C}_2 n^{-1/2}(\log n)^{1+\varepsilon} \text{ etc...}\}.$$

Observe first of all that we can replace $\mu$ by $\mu^2$ [for convenience I shall replace in my notations convolution by ordinary multiplication] and that we can therefore assume that our measure charges $e$ the identity of G and that $\|\mu^n\|_\infty = \mu^n(\{e\})$ which is then decreasing in $n$.

Let then $\Gamma = \{g_n \in \mathrm{G}; n \in \mathbf{Z}\}$ be a sequence of distinct points such that $g_0 = e$ and $d_r(g_k, g_{k+p}) = |p|$ $(k,p \in \mathbf{Z})$. This can be constructed by first constructing *finite* sequences $\Gamma_{\mathrm{N}} = \{g_n^{(\mathrm{N})} \in \mathrm{G};\ -\mathrm{N} \leqslant n \leqslant \mathrm{N}\}$ that have the required property and then using a diagonal process [i.e. Tychonov's theorem].

Let then K be the random walk defined on G by :

$$\mathrm{K}(x,y) = \begin{cases} 1/2; & x = g_k,\ y = g_{k\pm 1},\ k \in \mathbf{Z} \\ 1\ ; & \text{if } x = y \notin \Gamma \end{cases}$$

$\mathrm{K} \equiv 0$ in all the other cases.

Clearly K is symmetric with respect to the Haar measure of G and if we denote by $\mathrm{M}(x,y) = \mu(\{x^{-1}y\})$ we have $\mathrm{M} \gg \mathrm{K}$. Our proposition then follows from (2.7).

Using the final remark of § 1 and the above method we can prove the more general.

PROPOSITION 4.3. — *Let* P *be a (not necessarily symmetric) random walk on* N *and let us assume that* P *satisfies the condition* (ii) *of Proposition* 4.1 *(for some connected metric* $d$*). Then for every* $i \in \mathrm{N}$ *and* $\varepsilon > 0$ *we have :*

$$(4.4) \qquad \sum_{n=1}^{\infty} \frac{\mathrm{P}^n(i,i)}{n^{1/2+\varepsilon}} < +\infty\,; \qquad \sum_{n=1}^{\infty} \frac{\mathrm{P}^n(i,i)}{n^{1/2}(\log n)^{1+\varepsilon}} < +\infty$$

*etc.*

To make the above proof work we have to assume that $\inf_i P(i,i) > 0$ and this, in general, is of course not true. It is here that (4.2) comes to our rescue. Indeed let us replace P by $R_\alpha = \alpha I + (1-\alpha)P$ for some small $\alpha$ $(0<\alpha<1)$, our method then applies and (4.4) holds with P replaced by $R_\alpha$. But it is clear that:

$$(4.5) \qquad \sum_{n=1}^{\infty} \frac{P^n(i,i)}{n^{1/2+\varepsilon}} \leqslant \frac{C}{(1-\alpha)} \sum_{n=1}^{\infty} \frac{R_\alpha^n(i,i)}{n^{1/2+\varepsilon}}$$

where C only depends on $\varepsilon$ and the result follows. The details will be left to the reader. ((4.5) holds for the same reason that (4.2) holds.)

## 5. Riemannian manifolds.

Let M be a complete, connected, non-compact Riemannian manifold and let $\Gamma = \{m_i; i \in \mathrm{N}\} \subset \mathrm{M}$ be a discrete subset of M that verifies

$$d(m_i,m_j) > 1/\mathrm{A} \ (i \neq j) \qquad \text{and} \qquad d(m,\Gamma) < \mathrm{A} \ (m \in \mathrm{M})$$

for some $\mathrm{A} > 0$. We shall say that $\Gamma$ is a grid in M. One way to construct a grid is to construct a subset $\{m_i^*; i \in \mathrm{N}\}$ that is maximal under the condition $d(m_i^*,m_j^*) \geqslant \mathrm{A} > 0$. Given a grid it is easy then to decompose $\mathrm{M} = \bigcup_{j=1}^{\infty} \mathrm{M}_j$ into disjoint Borel subsets that satisfy (3.1) and satisfy also the more restrictive condition

$$(5.1) \qquad \mathrm{B}_{\alpha_1}(m_i) \subset \mathrm{M}_i \subset \mathrm{B}_{\alpha_2}(m_i); \qquad i = 1, 2, \ldots$$

with $\alpha_1 = \dfrac{1}{10\,\mathrm{A}}$ and $\alpha_2 = 10\,\mathrm{A}$ say. We can for instance define:

$$\mathrm{M}_i = \{m \in \mathrm{M}/d(m, m_i) \leqslant d(m,m_j), \ \forall j \neq i\}$$

and make the appropriate modification to make them disjoint.

An interesting way of constructing a grid exists when $\tilde{\mathrm{M}} \xrightarrow[p]{} \mathrm{M}$ is a regular covering over a compact Riemannian manifold M (as in § 1) with G as deck transformation group. Let D be a relatively compact fundamental domain such that $\overline{\mathring{\mathrm{D}}} = \bar{\mathrm{D}}$ and let us fix $x \in \mathring{\mathrm{D}}$, we clearly obtain a grid by setting $\Gamma = \{gx/g \in \mathrm{G}\}$ [where we use $x \in \tilde{\mathrm{M}}$ as a *base* point for the action of G, i.e. we identify G with the quotient $\pi_1(\mathrm{M},p(x))/\pi_1(\tilde{\mathrm{M}},x)$].

We also obtain the decomposition $\tilde{\mathrm{M}} = \bigcup_{g \in \mathrm{G}} g\mathrm{D}$ that clearly satisfies (3.1) and (5.1). The above grid can, and will, in what follows be identified with G itself.

Let us go back now to a general connected complete non-compact manifold M and let $\Gamma = \{m_i ; i \in \mathrm{N}\} \subset \mathrm{M}$ be a grid in M and $\mathrm{M} = \cup \mathrm{M}_i$ a decomposition into Borel subsets that satisfies (3.1) and (5.1). Let us further assume that M satisfies (3.7), and let $k_t$ and $\mathrm{K}_t$ be as in (3.8) for the above decomposition.

With the usual notation $\lambda_j = \mathrm{Vol}\,(\mathrm{M}_j)$ $(j=1,2,\ldots)$ let then :

$$\mathrm{P}_t(i,j) = k_t(i,j)\lambda_j$$

which gives then for every fixed $t > 0$ a random walk on $\Gamma$ which is clearly admissible with respect to the metric $d^*$ induced on $\Gamma$ by M.

Indeed the conditions (5.1) and (3.7) imply that

$$0 < \inf_j \lambda_j \leqslant \sup_j \lambda_j < +\infty ,$$

and that $d^*$ is a connected discrete metric [to see that $d^*$ is connected join two points with a minimal geodesic and to see that it is discrete use the previous uniform bound on the volumes]. The fact that $\mathrm{P}_t$ is an admissible walk follows then from (3.9) and (3.10).

Furthermore $\{\lambda_j\}$ gives a symmetrising measure for all these walks. We have then :

THEOREM. — *Let* M *be a complete, connected, non-compact Riemannian manifold such that its sectional curvature is uniformly bounded (both from above and below) and its injectivity radius is bounded from below.*

*Let* $\Gamma \subset \mathrm{M} = \cup \mathrm{M}_j$ *be a grid and a decomposition that satisfy* (3.1) *and* (5.1). *Then Brownian motion is transient on* M *if and only if the admissible (with respect to the induced metric ) random walks on* $\Gamma$ *that are symmetric with respect to* $\{\lambda_j = \mathrm{Vol}\,(\mathrm{M}_j); j=1,\ldots\}$ *are transient* (cf. Appendix).

Indeed the above walks are transient if and only if $\sum_{n=0}^{\infty} \mathrm{P}_t^n(i,i) < +\infty\ (t>0)$ i.e. if and only if $\sum_{n=0}^{\infty} k_t^n(i,i) < +\infty$ which by the Proposition in § 3 happens if and only if $\sum_{n=0}^{\infty} \langle p_{nt}f,f\rangle < +\infty$, $0 \leqslant f \in \mathrm{L}^2(\mathrm{M};d\mathrm{V})$. To conclude that this is equivalent to the existence of Green's function on M some uniformity of the above convergence is needed for $t \in [t_1, t_2]$, $0 < t_1 < t_2$. To avoid repetition I shall refer the reader to [1], § 5 and 6.

If we apply the above theorem to $\tilde{\mathrm{M}}$ where $\tilde{\mathrm{M}} \underset{p}{\rightarrow} \mathrm{M}$ is a normal covering of a compact manifold and if we use the grid and the decomposition explained at the beginning of this section we obtain the result stated at the introduction about the transience of Brownian motion on $\tilde{\mathrm{M}}$.

Let now M be as in the previous theorem we can then find a *special* grid $\Gamma \subset \mathrm{M} = \cup \mathrm{M}_j$ that satisfies (3.1), (5.1) and which in addition satisfies the following condition:

$$\begin{aligned} d(m_{2j}, m_{2j+2}) &\leqslant \mathrm{K};\quad j = 1,2,\ldots \\ \mathrm{Vol}\,(\mathrm{M}_{2j}) &= \mathrm{L};\quad j = 1,2,\ldots \end{aligned} \tag{5.2}$$

for some K, L > 0 (by renormalising the metric I could even assume that L = 1). We can also assume that $m_2$ is a preassigned point $m_2 = \tilde{m} \in \mathrm{M}$. I shall postpone the construction of this special grid until Appendix II.

We can then prove:

THEOREM. — *Let* M *be a complete, connected, non-compact Riemannian manifold such that its sectional curvature is bounded both from above and below and its injectivity radius is bounded from below. Then for all* $0 < \varepsilon < 0.1$ *there exist constants* $\mathrm{C}_1 = \mathrm{C}_1(\varepsilon)$, $\mathrm{C}_2 = \mathrm{C}_2(\varepsilon)$ *etc. for which the Heat kernel of* M *satisfies*

$$\sup_{m,n\in \mathrm{M}} |p_t(m,n)| \leqslant \mathrm{Min}\,[\mathrm{C}_1 t^{-1/2+\varepsilon}; \mathrm{C}_2 t^{-1/2}(\log t)^{1+\varepsilon}; \text{etc.}],$$

*for all* $t > 1$.

*Proof.* – Let us fix $\tilde{m} \in \mathrm{M}$ some point of M and let $\Gamma \subset \mathrm{M} = \cup \mathrm{M}_j$ be a special grid and a decomposition satisfying (3.1), (5.1) and (5.2) with $m_2 = \tilde{m}$.

Let then P be the random walk on $\Gamma$ defined as follows:

$$\begin{aligned} \mathrm{P}(m_{2i+1}, m_{2i+1}) &= 1; \quad i = 0, 1, \ldots \\ \mathrm{P}(m_2, m_2) &= 1/2; \quad \mathrm{P}(m_2, m_4) = 1/2 \\ \mathrm{P}(m_{2k}, m_{2(k\pm 1)}) &= 1/2; \quad k = 2,3 \ldots \end{aligned}$$

and $\mathrm{P}(m_i, m_j) = 0$ in all the other choices of $m_i$ and $m_j$.

By our hypothesis on $\cup \mathrm{M}_j$ the measure $\{\lambda_j = \mathrm{Vol}(\mathrm{M}_j); j = 1,2,\ldots\}$ is a symmetrising measure for the above walk and the symmetric S-operator $\mathrm{K}(i,j) = \mathrm{P}(m_i, m_j)\lambda_j^{-1}$ satisfies $k_t \gg \mathrm{K}$ for all $(t>0)$ [cf. (3.8)]. Now it is an easy matter to prove that $\mathrm{P}^n(m_{2j}; m_{2j}) = \mathrm{O}(n^{-1/2})$ (uniformly on $j$). Indeed on the even integers P reduces essentially to a reflecting standard coin tossing game on the non-negative integers. From this and § 2 it follows that

$$\sum_{n=0}^{\infty} \xi_n k_t^n(2,2) < +\infty \quad (t>0)$$

for

$$\xi_n = n^{-1/2-\varepsilon}, \qquad \xi_n = n^{-1/2} \log n^{-1-\varepsilon}$$

etc.

But from the proposition of § 3 it then follows that if $0 \leqslant f \in \mathrm{C}^{\infty}(\mathrm{M})$ has its support in $\mathrm{M}_2$ then:

$$\sum_{n=0}^{\infty} \xi_n \langle p_{nt} f, f \rangle < +\infty \quad (t>0) \tag{5.3}$$

and the above sum stays bounded for $t \in [t_1, t_2]$ for fixed $0 < t_1 < t_2$. From this we can deduce that:

$$\sum_{n=0}^{\infty} \xi_n p_{nt}(\tilde{m}, \tilde{m}) < +\infty \tag{5.4}$$

and that the sum stays bounded for $t \in [t_1, t_2]$. Indeed (5.4) is an immediate consequence of (5.3) and of the fact $|d_x p_t(x, y)|$ stays bounded as $t \in [t_1, t_2]$ uniformly in $x, y \in \mathrm{M}$ (cf. [11]).

To finish the proof up we just have to observe that:

$$p_{2t}(\tilde{m},\tilde{m}) = \int_M p_t^2(\tilde{m},x)\, dV(x) = \|p_t(\tilde{m},.)\|_2^2$$

which therefore by the semi-group property of $p_t$ is a decreasing function of $t$.

(5.4), (2.7) and the above give us that:

$$p_t(\tilde{m},\tilde{m}) = O\left(\frac{1}{n\xi_n}\right) \quad \text{as } t \to \infty \quad (n=[t])$$

and of course the 0 is uniform with respect to $\tilde{m}$ since all the above constructions (cf. Appendix II) are. The Theorem follows.

## 6. The Riemann surface.

Let S be the Riemannian sphere with its standard conformal structure ($S \subset \mathbf{R}^3$) and let $\Sigma = S \setminus \{z_1, \ldots, z_m\}$ the Riemann surface that we obtain by deleting $m$ of its points. We shall consider $\Sigma_G \to \Sigma$ a Galois [regular, in the terminology of [12] Ch. 9] covering with $G = \pi_1(\Sigma)/N$ as a deck transformation group ($N \lhd \pi_1(\Sigma)$). Let us fix a base point $z_0 \in \Sigma$ for the fundamental group and let $\gamma_1, \ldots, \gamma_k$ be the generators of $\pi_1(\Sigma; z_0)$ that we obtain by the loops that (based at $z_0$) go round the points $z_1, \ldots, z_k$ once, anticlockwise.

I shall denote by $g_i = \dot{\gamma}_i$, $i = 1, \ldots, k$ the images of the $\gamma_i$ 's on $G = \pi_1/N$ which are then a set of generators for G.

Let D be « the infinite star with $k$ spikes ». More exactly D is the disjoint union of $k$ copies of the non-negative integers

$$\mathbf{N} \cup 0 = \{0,1,2,\ldots\}$$

with their 0 points all identified. I shall denote the points of D by $x_i^j$, $j = 1, \ldots, k$; $i = 0, 1, \ldots$ and the above identification means that $x_0^1 = x_0^2 = \cdots = x_0^k$.

On the discrete space $D \times G = \Theta$ I shall now define the random walk :

$$(6.1) \qquad P[(x_0^j, g), (x_1^j, h)] = \frac{1}{n_j k}; \qquad j = 1, \ldots, k;$$
$$g, h \in G, \quad g^{-1}h = g_j^\varepsilon, \quad \varepsilon = \pm 1, 0$$

where

$$n_j = \text{Card}\,\{g_j^\varepsilon; \varepsilon = 0, \pm 1\}$$

$$(6.2) \qquad P[(x_i^j, g), (x_{i\pm 1}^j, h)] = \frac{1}{2n_j}; \quad j = 1, \ldots, k;$$
$$g, h \in G, \quad g^{-1}h = g_j^\varepsilon, \quad \varepsilon = \pm 1, 0, \quad i = 1, 2, \ldots.$$

We also set $P(\theta, \theta') = 0$ for all other choices of $\theta, \theta' \in \Theta$.
It is clear then that the subset

$$\{(\theta, \theta') \in \Theta \times \Theta \mid P(\theta, \theta') \neq 0\}$$

is symmetric about the diagonal. It follows that we can find a symmetrising measure $\lambda$ for $P$ on $\Theta$. Indeed it is enough to set

$$(6.3) \qquad \lambda[\{(x_0^j, g)\}] = \alpha, \qquad \lambda[\{(x_i^j, g)\}] = \beta$$

for $j = 1, 2, \ldots, k$, $i = 1, 2, \ldots$ and $g \in G$ and a choice of $\alpha, \beta > 0$ that satisfies

$$(6.4) \qquad \frac{\alpha}{k} = \frac{\beta}{2}.$$

To see this observe that $P((x_i^j, g), (x_r^k, h))$ is zero unless $j = k$ and then we can verify that $\lambda(\{\theta\})$, $P(\theta, \theta')$ is symmetric in $\theta, \theta'$.

We have then :

THEOREM. — *The Riemann surface $\Sigma_G$ is hyperbolic (i.e. the Brownian motion is transient on $\Sigma_G$) if and only if the above random walk is transient.*

One half of the above theorem was proved in [1]. I shall show that in fact the above theorem follows from the main Theorem of § 5. I shall be brief and I shall use all the notations and ideas of [1] § 7, 8 (where $k$ was taken to be $k = 3$).

In [1] § 8 I assigned $\Sigma$ with a conformal metric that makes $\Sigma$ a complete manifold and that is flat outside some compact subset. That metric lifts to $\Sigma_G$.

I then proceed to construct a decomposition of $\Sigma_G$ and a grid $\Theta^*$ in $\Sigma_G$ such that $\Theta^*$ can be identified with $\Theta$. Furthermore in the discretising procedure of § 3 the volumes of the subsets of the decomposition are compatible with (6.3), (6.4).

It is an easy matter to verify that some power $P^n$ of our random walk is admissible with respect to the induced distance $d$ on $\Theta$. The Theorem follows.

In general it is not clear how one decides whether the above random walk on $\Theta$ is transient or not. Here are some particular cases (*):

(i) G is a transient group. Then the above random walk is transient; this was proved in [1], § 7.

(ii) $G \cong \mathbf{Z}$ (or a finite extension of $\mathbf{Z}$). It is easy to show that the above walk is then recurrent. (Implicit in [1], § 7.)

(iii) $G \cong \mathbf{Z}^2$ (or a finite extension of $\mathbf{Z}^2$). Then the above random walk is transient. This is implicit in [1], § 7.

The special case $k = 3$, $G = H^1(\Sigma) = \pi_1/[\pi_1, \pi_2]$ is the case that was treated by Lyons and McKean [4].

The above three cases might well be *exhaustive* for G and then the problem of deciding whether $\Sigma_G$ is hyperbolic would be completely solved. But this is an open question (cf. [13]).

(iv) If the canonical generators $g_i$, $i = 1, \ldots, k$ of G satisfy $g_i^N = e$ (= the neutral element of G) for some $N \geqslant 1$ then our random walk on $\Theta$ is transient if and only if G is a transient group.

Indeed at the end of § 7 [1] I have constructed $\mu$ a symmetric probability measure on G that charges the generators $g_i$ $(i=1,\ldots,k)$ and is such that our walk on $\Theta$ is transient if and only if $\mu$ is a transient measure on G $\left(\text{i.e. } \sum_{n=0}^{\infty} \mu^n(e) < +\infty\right)\cdot$ $\mu$ is not in general finitely supported and so we can very well have $\sum_{n=0}^{\infty} \mu^n(e) < +\infty$ for a recurrent group G (but not the other way round).

The point now is that under the condition $g_i^N = e$ $(i=1,\ldots,k)$ $\mu$ is finitely supported and this proves our assertion.

(*) The problem was completely solved in [18].

## 7. The non regular coverings.

Let G be a finitely generated group and let $H \subset G$ a (not necessarily normal) subgroup. We shall denote by

$$G\,|\,H = \{Hg;\, g \in G\}$$

the left coset space of $[G:H]$.

The metric $d_l$ on G (cf. § 1) induces then a quotient metric $\mathring{d}$ on $G\,|\,H$ which is clearly also connected and discrete. Let now $P(g,h) = \mu(\{g^{-1}h\})\,(g,h \in G)$ be a random walk on G induced by some probability measure $\mu \in \mathbf{P}(G)$. It is clear then that

$$\mathring{P}(\mathring{g},\mathring{h}) = \sum_{h \in \mathring{h}} P(g,h);\quad g \in \mathring{g},\, \mathring{h} \in G\,|\,H \tag{7.1}$$

is the transition matrix of a random walk on $G\,|\,H$ [the above definition clearly does not depend on the particular choice of $g \in \mathring{g}$].

The point is that if $\mu$ is chosen to satisfy:

(i) $\mu = \check{\mu}$ (i.e. $\mu(\{g\}) = \mu(\{g^{-1}\})$),

(ii) $Gp\,(\operatorname{supp}\mu) = G$,

(iii) $\mu(\{g\}) \leqslant C|g|^{-b|g|}$ for some $C > 0$ and $b > 1$,

then the walk (7.1) is an admissible (*w.r.t.* $\mathring{d}$) and symmetric (i.e. $\mathring{P}(\mathring{g},\mathring{h}) = \mathring{P}(\mathring{h},\mathring{g})$, $\forall \mathring{g},\ \mathring{h} \in G|H$) walk on $G|H$. The transience or recurrence of the above walk does not therefore depend on the particular choice of $\mu$ [that satisfy (i), (ii), (iii)].

We say that $G|H$ is recurrent if and only if the above walks are recurrent. An intuitive way of saying that $G\,|\,H$ is recurrent is to say that the random particle that performs a (left) invariant walk on G returns infinitely often to the subgroup H.

Let now $\tilde{M} \xrightarrow{p} M$ be a general (i.e. not necessarily regular) covering of a *compact* Riemannian manifold ($\tilde{M}$ is connected).

What we have is $H \subset \pi_1(M) = G$ a subgroup of the fundamental group of M and $\tilde{M}$ is just $\check{M}/H$ where $\check{M}$ is the universal (simply connected) cover of M. We have also covering mappings

$$\check{M} \xrightarrow{\check{p}} \tilde{M} \xrightarrow{p} M.$$

THEOREM. — *Let* $\tilde{M} \xrightarrow{p} M$ *be as above then Brownian motion is recurrent on* $\tilde{M}$ *if and only if the coset space* $G|H$ *is recurrent.*

The proof is immediate. Indeed fix some $\check{m} \in \check{M}$ and let

$$\check{\Gamma} = \{g\check{m} \mid g \in G = \pi_1(M)\}$$

be the canonical grid obtained on $\check{M}$.

The set $\check{p}(\check{\Gamma}) = \Gamma \subset \tilde{M}$ is then a grid on M and can be identified with $G|H$. Further more that identification brings the metric on $\Gamma$ induced by the Riemannian structure to a metric on $G|H$ that is equivalent (in the obvious sense of § 1) to $\mathring{d}$.

The machinery of § 5 therefore applies and we have our Theorem.

We also have cf. [1] as a corollary that coset spaces are transient only if

$$\sum_{n=0}^{\infty} \frac{n}{\gamma(n)} < +\infty$$

$$[\gamma(n) = \operatorname{Card}\{y \in G|H \mid \mathring{d}(x,y) \leqslant n\}].$$

## Appendix I.

In view of a recent theorem of T. Lyons (the Annals of Prob., 1983, vol 11, n° 2, pp. 393-402) it follows that the exact value of $\lambda_j = \operatorname{vol}(M_j)$ in the main theorem of § 5 is irrelevant and we can simply talk of « symmetrisable admissible random walks ».

Indeed Lyons' theorem says that if $P_1(i,j)$, $P_2(i,j)$ are two random walks on N the first symmetrisable by $\{\lambda_j^{(1)}\}$ the second by $\{\lambda_j^{(2)}\}$ then the transience of one implies the transience of the other provided that

$$C^{-1} \leqslant \frac{\lambda_i^{(1)} P_1(i,j)}{\lambda_i^{(2)} P_2(i,j)} \leqslant C$$

for some $C > 0$.

**Appendix II (construction of the special grid of § 5).**

Let M satisfy the conditions of Theorem. Let us start with

$$\Gamma = \{m_j \in \mathrm{M}_j, j=1,2,\ldots\}$$

and $\mathrm{M} = \cup \mathrm{M}_j$, $m_j \in \mathring{\mathrm{M}}_j$ a grid in M that satisfies the conditions (3.1) and (5.1). Let $\tilde{m}$ be a fixed point of M and let $\gamma = \{\gamma(t); t \geqslant 0\}$ be a ray in M going off to infinity and starting at $\tilde{m}$, this means that $\gamma$ is a geodesic such that $\gamma(0) = \tilde{m}$ and such that for any two $0 \leqslant t_1 < t_2$ we have $d(\gamma(t_1), \gamma(t_2)) =$ length of $\{\gamma(t); t_1 < t < t_2\} = t_2 - t_1$.

The existence of such a ray in any non-compact, complete manifold is easy to establish.

Let then $t > 0$ such that the tubular N*hd* $\gamma_\varepsilon = \{m \in \mathrm{M} \mid d(m,\gamma) < \varepsilon\}$ is really nice i.e. looks like a cylinder; this can clearly be achieved as soon as $\varepsilon$ is sufficiently small.

The next thing to do is to slice that semi-infinite cylinder into disjoint portions $p_j$ $(j=1,\ldots)$ along its length, of equal volume. A typical portion could be of the form

$$\gamma_\varepsilon^j = \{m \in \mathrm{M} \mid d(m,\gamma^j) < \varepsilon\}$$

where

$$\gamma^j = \{\gamma(t) \mid t_j \leqslant t \leqslant t_{j+1}\}.$$

A correction has to be made at the two ends if we want these portions to be disjoint. Clearly this can be done by using the normal bundle of $\gamma$.

Let L be the common volume of these portions. By choosing first $\varepsilon > 0$ and then L sufficiently small we can also make sure that $\operatorname{diam}(p_j) \leqslant \delta$ for a preassigned $\delta$.

We shall consider then the new decomposition

$$\mathrm{M} = \Big(\bigcup_j \tilde{\mathrm{M}}_j\Big) \cup \Big(\bigcup_j p_j\Big)$$

where $\tilde{\mathrm{M}}_j = \mathrm{M}_j \backslash \gamma_\varepsilon$.

For $\varepsilon$ sufficiently small we see that this new decomposition satisfies all the required conditions; it simply remains to renumber it!

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## BIBLIOGRAPHY

[1] N. Th. VAROPOULOS, Brownian Motion and Transient Groups, *Ann. Inst. Fourier*, 33-2 (1983), 241-261.
[2] H. P. McKEAN Je., *Stochastic Integrals*, Academic Press, 1969.
[3] N. Th. VAROPOULOS, Potential Theory and Diffusion on Riemannian Manifolds, Conference on Harmonic analysis in honor of Antoni Zygmund. (Wadsworth).
[4] T. J. LYONS and H. P. McKEAN, Winding of the Plane Brownian Motion (preprint).
[5] J. CHEEGER and D. G. EBIN, *Comparison Theorems in Riemannian Geometry*, North-Holland, 1975.
[6] J. MILNOR, A Note on Curvature and Fundamental Group, *J. Diff. Geometry*, 2 (1968), 1-7.
[7] P. BALDI, N. LOHOUÉ et J. PEYRIÈRE, *C.R.A.S.*, Paris, t. 285 (A), 1977, 1103-1104.
[8] S. T. YAU, On the Heat Kernel of a Complete Riemannian Manifold, *J. Math. Pure et Appl.*, 57 (1978), 191-201.
[9] J. CHEEGER and S. T. YAU, A Lower Bound for the Heat Kernel, *Comm. Pure and Appl. Math.*, vol. XXXIV (1981), 465-480.
[10] H. DONNELLY and P. LI, Lower Bounds for the Eigen Values of Negatively Curved Manifolds, *Math. Z.*, 172 (1980), 29-40.
[11] S. Y. CHENG, P. LI, and S. T. YAU, On the Upper Estimate of the Heat Kernel of a Complete Riemannian Manifold, *Amer. J. of Math.*, Vol. 103(5) (1980), 1021-1063.
[12] L. V. AHLFORS, *Conformal Invariants*, New-York, McGraw-Hill.
[13] N. Th. VAROPOULOS, Random Walks on Soluble Groups, *Bull. Sci. Math.*, 2e série, 107 (1983), 337-344.
[14] M. GROMOV, *Structures Métriques pour les variétés Riemanniennes*, Cedic/Fernand Nathan (1981).
[15] Y. GUIVARC'H, *C.R.A.S.*, Paris, t. 292 (I) (1981), 851-853.
[16] J. VAUTHIER, Théorèmes d'annulation, *Bull. Sc. math.*, 2e série, 103 (1979), 129-177.
[17] H. DONNELLY, Spectral geometry, *Math Z.*, 169 (1979), 63-76.
[18] N. Th. VAROPOULOS, *C.R.A.S.*, t. 297 (I), p. 585.

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