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# NON-DEGENERESCENCE OF SOME SPECTRAL SEQUENCES

by K.S. SARKARIA

## 1. Introduction.

Given a Lie algebra $\mathfrak{F}$ of vector fields (smooth sections of the complexified tangent bundle) of a smooth manifold $M^m$ one has a decreasing filtered complex

$$A(M) = A_0(\mathfrak{F}) \supseteq A_1(\mathfrak{F}) \supseteq \ldots \supseteq A_m(\mathfrak{F}) \supseteq A_{m+1}(\mathfrak{F}) = 0 ; \quad (1)$$

here $A(M)$ is the differential graded algebra of all smooth complex valued forms on $M$ (the de Rham complex of $M$) and, for each $i > 0$, $A_i(\mathfrak{F})$ denotes the ith power of the ideal generated by 1-forms which vanish on $\mathfrak{F}$. Like any filtered complex, (1) has an associated spectral sequence which will be denoted by $E_k(\mathfrak{F})$. (We will follow Griffiths and Harris [2] regarding terminology for spectral sequences.) Two special cases of this construction will be important for us. (a) If $M^{2n}$ is a complex manifold and $\mathfrak{F}$ consists of vector fields which in local complex coordinates $z_1, z_2, \ldots, z_n$ are of the form $\sum_j \varphi_j \frac{\partial}{\partial \bar{z}_j}$, then $E_k(\mathfrak{F})$ is the well known *Fröhlicher spectral sequence* [1] of the complex manifold $M$. The usual notation for $E_1^{p,q}(\mathfrak{F})$ is $H^q(M, \Omega^p)$: the qth cohomology of $M$ with coefficients in the sheaf $\Omega^p$ of germs of holomorphic $p$-forms on $M$. (b) If $M$ carries a smooth foliation and $\mathfrak{F}$ consists of all vector fields tangent to the leaves, then $E_k(\mathfrak{F})$ will be called the *spectral sequence of the foliated manifold* $M$. It has been considered e.g. in [3].

THEOREM. – (A) *For all integers* $m \geqslant 1$, $0 \leqslant c \leqslant m$, *the m-torus* $T^m$ *admits a codimension* $c$ *real analytic foliation such that* $E_\mu^{\mu,0}(\mathfrak{F})$ *is infinite dimensional; here* $\mu = \min(c, m-c+1)$. (B) *For all integers* $n \geqslant 1$, $T^{2n-1} \times \mathbf{R}$ *admits a complex structure such that* $E_n^{n,0}(\mathfrak{F})$ *is infinite dimensional.* (C) *For all integers* $m \geqslant 1$, $\mathbf{R}^m$ *admits a compactly supported Lie algebra* $\mathfrak{F}$ *of vector fields such that* $E_m^{m,0}(\mathfrak{F})$ *is infinite dimensional.*

This theorem (which will be proved in § 2) is best possible in the sense that dimension considerations show that $E_k(\mathfrak{F})$ is isomorphic to the de Rham cohomology for $k \geqslant \mu + 1$ (resp. $k \geqslant n+1$, resp. $k \geqslant m+1$) and so is finite dimensional for these values of $k$. We note that the cohomology considered by Schwarz [4] is precisely the $E_2^{*,0}(\mathfrak{F})$ of a foliated manifold. In response to a question posed by Bott, he constructed smooth (non-analytic) foliations of compact manifolds for which $E_2^{*,0}(\mathfrak{F})$ is infinite dimensional. Thus (A) can be considered as an improvement on [4]; besides our construction is different and simpler and leads to foliations which are of a natural and non-pathological kind. Regarding (B) we remark that the degenerescence problem for *compact* complex manifolds is much more delicate; we hope to give some results about it in a subsequent paper.

## 2.

In each of the examples (A)–(C) we will check, for $k = \mu, n$ and $m$ respectively, that $\operatorname{Im}\{H^k(A_k(\mathfrak{F})) \longrightarrow H^k(A_1(\mathfrak{F}))\}$ is infinite dimensional. The infinite dimensionality of $E_k^{k,0}(\mathfrak{F})$ (see [2], p. 441 for definition) would follow as an immediate consequence.

*Proof of* (A) (We ignore the trivial cases $c = 0, m$). – The torus $T^m$ will be considered as the quotient $\mathbf{R}^m/(2\pi\mathbf{Z})^m$. Furthermore we put $\nu = m - (2\mu - 1)$ and identify $\mathbf{R}^m$ with $\mathbf{R}^{\mu-1} \times \mathbf{R}^{\mu-1} \times \mathbf{R} \times \mathbf{R}^\nu$; thus a point of $T^m$ will have coordinates $(\theta_{1,0}, \ldots, \theta_{\mu-1,0}; \theta_{1,1}, \ldots, \theta_{\mu-1,1}; r; t_1, \ldots, t_\nu)$. Note that $\mu = \ell + 1$ (here $\ell = m - c$) or $\mu = c$; correspondingly $\mu - 1 = \ell$ or $\mu - 1 + \nu = \ell$. We define $\mu - 1$ real analytic vector fields on $T^m$ by

$$X_i = \frac{\partial}{\partial \theta_{i,0}} + \sin r \cdot \frac{\partial}{\partial \theta_{i,1}}, \quad 1 \leqslant i \leqslant \mu - 1. \tag{2}$$

In case $\ell > \mu - 1$ we also take the vector fields $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_\nu}$. Taken together these $\ell$ vector fields span an Abelian $\ell$ dimensional Lie algebra $\mathfrak{A}$ of analytic vector fields. At each point of $T^m$ these vector fields are linearly independent; so they determine a real analytic $\ell$ dimensional tangent plane field on $T^m$. By Frobenius theorem this involutive plane field is tangent to a real analytic $\ell$ dimensional foliation of $T^m$. We will prove that $E_\mu^{\mu,0}(\mathfrak{F})$ is infinite dimensional for this foliation. (Note that $\mathfrak{A}$ and $\mathfrak{F}$ have the same filtration (1) and so have the same spectral sequence.)

For each smooth function $\varphi(r)$ of period $2\pi$ we define a closed form $\Omega_\varphi$ of degree $\mu$ on $T^m$ by

$$\Omega_\varphi = \sum_{(\alpha_1, \ldots, \alpha_{\mu-1})} \varphi(r)\,(-\sin r)^{\mu-1-\alpha}\, dr \wedge d\theta_{1,\alpha_1} \wedge \ldots \wedge d\theta_{\mu-1,\alpha_{\mu-1}}\,; \tag{3}$$

here the summation is over all multi-indices $(\alpha_1, \ldots, \alpha_{\mu-1})$ with entries 0 or 1, and $\alpha = \alpha_1 + \ldots + \alpha_{\mu-1}$. Using (2) it follows that the interior products $\sim_{X_i}(\Omega_\varphi)$ and $\sim_{\frac{\partial}{\partial t_k}}(\Omega_\varphi)$ vanish for $1 \leqslant i \leqslant \mu - 1$ and $1 \leqslant k \leqslant \nu$. Thus the forms $\Omega_\varphi$ constitute an infinite dimensional subspace of closed degree $\mu$ forms of $A_\mu(\mathfrak{F})$. We assert that if $\Omega_\varphi = d\omega$ where $\omega \in A_1(\mathfrak{F})$, then the form $\Omega_\varphi$ is the zero form (i.e. $\varphi \equiv 0$). This will suffice to prove the infinite dimensionality of Im $\{H^\mu(A_\mu(\mathfrak{F})) \longrightarrow H^\mu(A_1(\mathfrak{F}))\}$.

Let $T^{2\mu-2}$ denote the subtorus of $T^m$ obtained by putting the last $\nu + 1$ coordinates equal to zero. For each $\theta \in T^{2\mu-2}$ we have the translation $L_\theta : T^m \longrightarrow T^m$ given by $L_\theta(u) = \theta + u$. From (3) we see that $\Omega_\varphi$ is preserved by this action of $T^{2\mu-2}$ on $T^m$. Since the infinitesimal generators of this action are $\frac{\partial}{\partial \theta_{i,\alpha_i}}$ and $\left[\frac{\partial}{\partial \theta_{i,\alpha_i}}, X_k\right] = 0 = \left[\frac{\partial}{\partial \theta_{i,\alpha_i}}, \frac{\partial}{\partial t_k}\right]$, we conclude also that our Lie algebra $\mathfrak{A}$ is preserved by this action. Hence if $\omega \in A_1(\mathfrak{F})$ is such that $d\omega = \Omega_\varphi$ we can assume in addition that $\omega$ is also preserved by this action; for, otherwise, we can replace $\omega$ by the form $\int_{T^{2\mu-2}} L_\theta^*(\omega)\, d\theta$ obtained by averaging with respect to the normalized Haar measure $d\theta$ of $T^{2\mu-2}$. Let us now write the expression of $\omega$ in the chosen coordinate system. We have

$$\omega = \sum_{(\alpha_1, \ldots, \alpha_{\mu-1})} \omega_{\alpha_1 \ldots \alpha_{\mu-1}} \, d\theta_{1,\alpha_1} \wedge \ldots \wedge d\theta_{\mu-1,\alpha_{\mu-1}} + \bar{\omega} \quad (4)$$

where $\bar{\omega}$ denotes all terms other than those written out in the beginning. Since $\omega$ is preserved by the action of $T^{2\mu-2}$ every coefficient of $\omega$ in (4) is a function only of $r, t_1, \ldots, t_\nu$. But $d\omega = \Omega_\varphi$ where $\Omega_\varphi$ is given by (3). This shows that $d\bar{\omega} = 0$ and that the coefficients $\omega_{\alpha_1 \ldots \alpha_{\mu-1}}$ are functions only of $r$.

Since $\omega$ is a degree $\mu - 1$ form in $A_1(\mathfrak{F})$ we should get zero when the operator $\sim_{X_{\mu-1}} \circ \ldots \circ \sim_{X_1}$ is applied to (4). It is clear that on applying this to $\bar{\omega}$ we get zero. On applying it to the first part we get the condition

$$\sum_{(\alpha_1, \ldots, \alpha_{\mu-1})} (\sin r)^\alpha \, \omega_{\alpha_1 \ldots \alpha_{\mu-1}} = 0 . \quad (5)$$

Next we use the conditions $\sim_{X_i}(d\omega) = 0, 1 \leqslant i \leqslant \mu - 1$ on (4). We see that for each multi-index $(\alpha_1, \ldots, \alpha_{\mu-1})_k$ of length $\mu - 2$ obtained by omitting the kth entry we have

$$\omega'_{\alpha_1 \alpha_2 \ldots 0 \ldots \alpha_{\mu-1}} + \sin r \, . \, \omega'_{\alpha_1 \alpha_2 \ldots 1 \ldots \alpha_{\mu-1}} = 0 \quad (6)$$

where primes denote differentiation with respect to $r$ and 0 and 1 are placed at the kth place. This in turn implies the following identities

$$\sum_{\substack{(\alpha_1, \ldots, \alpha_{\mu-1}) \\ s.t. \alpha \geqslant s}} \frac{\alpha!}{(\alpha - s)!} (\sin r)^{\alpha - s} \, \omega'_{\alpha_1 \alpha_2 \ldots \alpha_{\mu-1}} = 0, \quad 0 \leqslant s \leqslant \mu - 2 . \quad (7)_s$$

(To check $(7)_s$ use (6) and the binomial identities

$$\sum_{\alpha = s}^{\mu - 1} \frac{\alpha!}{(\alpha - s)!} \binom{\mu - 1}{\alpha} (-1)^\alpha = 0.)$$

We now differentiate (5) with respect to $r$ and use $(7)_0$ to get

$$\cos r \, . \sum_{\substack{(\alpha_1, \ldots, \alpha_{\mu-1}) \\ s.t. \alpha \geqslant 1}} \alpha (\sin r)^{\alpha - 1} \, . \, \omega_{\alpha_1 \alpha_2 \ldots \alpha_{\mu-1}} = 0$$

which implies –since $\cos r$ is non-zero on an open dense subset of $T^m$ – that

$$\sum_{\substack{(\alpha_1, \ldots, \alpha_{\mu-1}) \\ s.t. \alpha \geqslant 1}} \alpha (\sin r)^{\alpha - 1} \, . \, \omega_{\alpha_1 \alpha_2 \ldots \alpha_{\mu-1}} = 0.$$

Next we differentiate this and use $(7)_1$, etc., etc.; finally we get $\omega_{11\ldots1} = 0$. Now by using (6) it follows that $\omega'_{\alpha_1\alpha_2\ldots\alpha_{\mu-1}} = 0$ for all multi-indices $(\alpha_1, \ldots, \alpha_{\mu-1})$ which means that $d\omega = \Omega_\varphi = 0$.

*Proof of* (B). – We identify $T^{2n-1} \times \mathbf{R}$ with the quotient $(\mathbf{C}^{n-1} \times \mathbf{C})/\Gamma$; here $\Gamma$ is the subgroup of $\mathbf{C}^{n-1} \times \mathbf{C}$ consisting of elements $(w_1, \ldots, w_{n-1}; u)$ with $u$ and each $\operatorname{Re} w_k$, $\operatorname{Im} w_k$ an integral multiple of $2\pi$. Now let $\mathcal{L}$ be the Abelian $n$ dimensional Lie algebra of vector fields of $T^{2n-1} \times \mathbf{R}$ spanned by $\dfrac{\partial}{\partial \bar{u}}$ and

$$X_k = \frac{\partial}{\partial \bar{w}_k} + 2e^{iu}\frac{\partial}{\partial w_k}, \quad 1 \leqslant k \leqslant n-1. \tag{8}$$

We note that at each point $x$ of $T^{2n-1} \times \mathbf{R}$ these $n$ vector fields are linearly independent; so they span an $n$ dimensional subspace $\mathbf{F}(x)$ of the complexified tangent space $\mathbf{T}(x)$. Furthermore $\mathbf{F}(x) + \overline{\mathbf{F}(x)} = \mathbf{T}(x)$ $\left(\text{because } \dfrac{\partial}{\partial \bar{w}_k} + 2e^{iu}\dfrac{\partial}{\partial w_k} \text{ is always}\right.$ $\left.\text{linearly independent from } \dfrac{\partial}{\partial w_k} + 2e^{-i\bar{u}}\dfrac{\partial}{\partial \bar{w}_k}\right)$. By the well known complex Frobenius theorem (of Newlander and Nirenberg) this involutive almost complex structure is integrable i.e. we can choose local complex coordinates $z_1, z_2, \ldots, z_n$ so that the smooth sections of $\mathbf{F}$ are precisely those vector fields which are of the type $\sum_j \varphi_j \dfrac{\partial}{\partial \bar{z}_j}$. We assert that $E_n^{n,0}(\mathfrak{F})$ is infinite dimensional for this complex structure.

For each entire function $\varphi(u)$ of period $2\pi$ we define a closed form of degree $n$ on $T^{2n-1} \times \mathbf{R}$ by

$$\Omega_\varphi = \sum_{(\alpha_1, \ldots, \alpha_{n-1})} \varphi(u)(-2e^{iu})^{n-1-\alpha}\, du \wedge dw_{1,\alpha_1} \wedge \ldots \wedge dw_{n-1,\alpha_{n-1}}. \tag{3$'$}$$

Here the summation is over all multi-indices $(\alpha_1, \ldots, \alpha_{n-1})$ with entries 0, 1 and $\alpha = \alpha_1 + \ldots + \alpha_{n-1}$; furthermore $dw_{k,\alpha_k}$ denotes $d\bar{w}_k$ if $\alpha_k = 0$ and $dw_k$ if $\alpha_k = 1$. We can now prove, by a method entirely analogous to that in part (A), that these $\Omega_\varphi$ constitute an infinite dimensional subspace of closed forms in $A_n(\mathfrak{F})$ and that no non-zero $\Omega_\varphi$ can be the boundary of a form $\omega \in A_1(\mathfrak{F})$.

*Proof of* (C). – Case $m = 1$ is trivial (take $\mathfrak{G} = 0$); so we assume $m \geqslant 2$. Let us take the closed disk D in $\mathbf{R}^2$ with origin as centre and radius 1. In its interior we choose a countably infinite number of disjoint closed disks $D_\beta$. Let us choose polar coordinates $(r_\beta, \phi_\beta)$ for each $D_\beta$ : so $D_\beta$ is given by

$$0 \leqslant \phi_\beta < 2\pi, \ 0 \leqslant r_\beta \leqslant \rho_\beta$$

where $\rho_\beta$ is the radius of $D_\beta$. On the torus $T^{m-2} = \mathbf{R}^{m-2}/\mathbf{Z}^{m-2}$ we choose coordinates $\theta_1, \theta_2, \ldots, \theta_{m-2}$. Now we define $m-1$ smooth vector fields on $T^{m-2} \times D^2$ by

$$\begin{aligned} X_i &= \sum_\beta c_\beta f_\beta(r_\beta) \frac{\partial}{\partial \theta_i} \quad \text{if } 1 \leqslant i < m-1 \\ &= \sum_\beta c_\beta f_\beta(r_\beta) \frac{\partial}{\partial \phi_\beta} \quad \text{if } i = m-1. \end{aligned} \tag{9}$$

Here $f_\beta(r_\beta)$ is a smooth function on $\mathbf{R}^+$, positive on $\left(\frac{\rho_\beta}{2}, \rho_\beta\right)$ and zero outside this interval, while the positive constants $c_\beta$ are so chosen that the sums in (9) converge in the $C^\infty$ topology. One notes that $[X_i, X_j] = 0 \ \forall i, j$ and that each vector field $X_i$ vanishes on the boundary $T^{m-2} \times \partial D^2$ of our solid torus $T^{m-2} \times D^2$. The latter can be smoothly imbedded in $\mathbf{R}^m$ and we can extend these vector fields to all of $\mathbf{R}^m$ by defining them to be zero outside the solid torus. This gives us an Abelian $m-1$ dimensional Lie algebra $\mathfrak{G}$ of compactly supported vector fields on $\mathbf{R}^m$ ; we will prove that $E_m^{m,o}(\mathfrak{G})$ is infinite dimensional.

For each $\beta$, let us choose a smooth $m$-form $\Omega_\beta$ supported inside the region of $T^{m-2} \times D_\beta$ given by $r_\beta < \frac{\rho_\beta}{2}$ and such that

$$\int_{T^{m-2} \times D_\beta} \Omega_\beta \neq 0. \tag{10}$$

Note that each $\Omega_\beta$ lies in $A_m(\mathfrak{G})$. If possible let us suppose that $\Omega$ is a non-trivial finite linear combination of the $\Omega_\beta$ such that $\Omega = d\omega$ where $\omega$ is a smooth $m-1$ form lying in $A_{m-1}(\mathfrak{G})$. This last condition implies that $\omega(X_1, \ldots, X_{m-1}) = 0$ and so in $T^{m-2} \times D_\beta$ we have

$$(c_\beta f_\beta(r_\beta))^{m-1} \omega\left(\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{m-2}}, \frac{\partial}{\partial \phi_\beta}\right) = 0. \tag{11}$$

In the region $\frac{\rho_\beta}{2} < r_\beta < \rho_\beta$ the coefficient of (11) is non-zero; so in this domain we must have $\omega\left(\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_{m-2}}, \frac{\partial}{\partial\phi_\beta}\right) = 0$ which remains valid on $r_\beta = \rho_\beta$ by continuity. Hence $\omega$ induces, on the boundary of each $T^{m-2} \times D_\beta$, the zero form. By Stoke's theorem this contradicts (10). Thus we have proved that

$$\operatorname{Im}\{H^m(A_m(\mathcal{F})) \longrightarrow H^m(A_1(\mathcal{F}))\}$$

has infinite dimension.

## 3. Remarks.

The proof of (C) given above is inspired by the work of Schwarz [4]. We note that it is possible to use analogous ideas to construct some more smooth (non-analytic) codimension $c$ foliations on $T^m$ with $E_\mu^{\mu,o}(\mathcal{F})$ infinite dimensional. One can also ensure that the singular set of such foliations is nowhere dense. (The *singular set* is made up of those points where the infinitesimal transformations of the foliation fail to span the tangent space. In [3] it is proved that for a compact foliated manifold with empty singular set, $E_2(\mathcal{F})$ is finite dimensional.) In another direction we can prove that if a smooth manifold $M^m$ admits one smooth codimension $c$ foliation, then it admits another with $E_\mu^{\mu,o}(\mathcal{F})$ infinite dimensional.

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