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THE DEFICIENCY OF ENTIRE FUNCTIONS WITH FEJÉR GAPS

par Takafumi MURAI

1. Introduction.

We say that a sequence $n_1 < n_2 < \ldots$ of positive integers is a Fejér gap series if $\sum_{k=1}^{\infty} 1/n_k < \infty$. We say that an entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ has Fejér gaps if $S(f) = \{n \geqslant 1 \,; c_n \neq 0\}$ is a Fejér gap series. There is a classical result by Fejér [4] and Biernacki [1]: An entire function with Fejér gaps takes any complex value infinitely often. It is one of the themes of gap series to extend or improve this result. We are concerned with extending this result from the point of view of the Nevanlinna theory. The purpose of this paper is to show:

THEOREM. – *An entire function with Fejér gaps has no finite deficient value.*

Our theorem improves the above result and Kövari's result [9]: An entire function $f(z)$ has no finite Borel exceptional value if $S(f) = (n_k)_{k=1}^{\infty}$ satisfies $\varliminf_{k\to\infty} n_k\,\eta(k)/\log\log k = \infty$ for some positive increasing function $\eta(r)$ in an interval $(0,\infty)$ with $\int_0^{\infty} \eta(r)\,dr < \infty$.

We say that an entire function $f(z)$ has Fabry gaps if $S(f) = (n_k)_{k=1}^{\infty}$ satisfies $\lim_{k\to\infty} k/n_k = 0$. The Fabry gap condition is weaker than the Fejér gap condition. Hence it is natural to ask whether the assertion of our theorem is valid when the Fejér gap condition is replaced by the Fabry gap condition. We shall answer this question in the negative. Clunie [3] constructed a sequence $S = (n_k)_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty} 1/n_k = \infty$ such that any entire function $f(z)$ with $S(f) \subset S$ has no finite Borel exceptional value. Hence it seems difficult to investigate the value-distribution of entire functions with $\lim_{k \to \infty} k/n_k = 0$ and $\sum_{k=1}^{\infty} 1/n_k = \infty$.

For the deficiency of entire functions of finite lower order with gaps, various results are known. It is interesting to compare our theorem with the following result [6,11]: An entire function $f(z)$ satisfies $\Delta(f) \leqslant C\rho(f)\,D(f)$, where $\Delta(f)$ denotes the sum of deficiencies for complex values, C an absolute constant, $\rho(f)$ the lower order and $D(f)$ the infimum over all $D > 0$ such that $(e^{int})_{n \in S(f)}$ is incomplete in the space of square integrable functions in an interval $(-D, D)$. Let us note that an entire function $f(z)$ with Fejér gaps satisfies $D(f) = 0$. Hence this inequality gives the assertion of our theorem under the additional condition $\rho(\cdot) < \infty$. This remark was first given by Fuchs [6]. Our theorem gives a new information about $\Delta(\cdot)$ in the case $\rho(\cdot) = \infty$ and $D(\cdot) = 0$.

2. Notation and lemmas.

2.1. Throughout the paper, we use C, C', C'', C_0 for absolute constants. The value of C, C' or C'' differs in general from one occasion to another. We denote by D_r, S_r $(r > 0)$ the open disk with center 0 and radius r, and its boundary, respectively. Let $\mathbf{C}$ denote the complex plane.

For an entire function $g(z) = \sum_{n=0}^{\infty} c_n z^n$, the maximum modulus and the maximum term associated with $r > 0$ are defined by

$$M(r, g) = \max\{|g(z)|\,;\, z \in S_r\},$$

$\mu(r, g) = \max\{|c_n r^n|\,;\, n \geqslant 1\}$, respectively. The characteristic function is defined by $m(r, g) = (1/2\pi)\int_0^{2\pi} \log^+ |g(re^{it})|\,dt$ $(r > 0)$, where $\log^+ x = \max\{\log x, 0\}$ $(x > 0)$. The counting function at $a \in \mathbf{C}$ is defined by $N(r, a, g) = \int_0^r n(x, a, g)/x\,dx$

$(r > 0)$, where $n(x, a, g)$ denotes the number of roots (counted according to the multiplicity) of $g(z) = a$ in $D_x - \{0\}$. The deficiency at $a \in \mathbf{C}$ is defined by $\delta(a, g) = 1 - \limsup_{r \to \infty} N(r, a, g)/m(r, g)$.

We say that a set E in $(0, \infty)$ is of finite logarithmic measure if $\int_E 1/(1+r)\, dr < \infty$. For two real-valued functions $A(r)$, $B(r)$ in $(0, \infty)$, we say that $A(r) \leqslant B(r)$ holds log-finely (l.f.) if this inequality holds outside a set of finite logarithmic measure.

For a function $P(t)$ in an interval $[0, 2\pi)$, we put $m(P) = (1/2\pi) \int_0^{2\pi} \log^+ |P(t)|\, dt$. The conjugate function of $P(t)$ is defined by

$$\widetilde{P}(t) = \lim_{\epsilon \to 0} (-1/2\pi) \int_\epsilon^\pi \{P(t+s) - P(t-s)\} \cot(s/2)\, ds$$

([16] p. 131).

For a sequence $S = (n_k)_{k=1}^\infty$ of positive integers, we denote by $\omega(r, S)$ $(r > 0)$ the number of all integers k with $n_k < r$ and put $\Omega(r, S) = \int_0^r \omega(x, S)/x\, dx$ $(r > 0)$. We easily see that S is a Fejér gap series if and only if $\int_0^\infty \omega(x, S)/x^2\, dx < \infty$.

2.2. Here are some lemmas necessary for the proof of our theorem.

LEMMA 1 ([7] p. 1). – *Let* $g(z)$ *be an entire function with* $g(0) \neq 0$. *Then*

$$\begin{aligned} N(r, 0, g) &= (1/2\pi) \int_0^{2\pi} \log |g(re^{it})|\, dt - \log |g(0)| \\ &= m(r, g) - m(r, 1/g) - \log |g(0)| \quad (r > 0). \end{aligned} \tag{1}$$

LEMMA 2 ([7] p. 22). – *Let* $g(z)$ *be the same as above and* $b_1, \ldots, b_m$ *all its zeros (counted according to the multiplicity) in* D_R. *Then, for any* $0 < r < R$,

$$\frac{\partial}{\partial t} \log |g(z)| = \operatorname{Re} \Phi(z; R, g) + \sum_{k=1}^m \{\operatorname{Re} \phi(z; b_k) + \operatorname{Re} \psi(z; R, b_k)\} \quad (z = re^{it}), \tag{2}$$

where

$$\begin{cases} \Phi(z\,;\mathrm{R}\,,g) = (iz/2\pi)\displaystyle\int_0^{2\pi} \log|g(\mathrm{R}e^{is})|\,\{2\mathrm{R}e^{is}/(\mathrm{R}e^{is}-z)^2\}\,ds \\ \phi(z\,;a) = iz/(z-a)\,, \quad \psi(z\,;\mathrm{R}\,,a) = iz\overline{a}/(\mathrm{R}^2-\overline{a}z)\,. \end{cases} \tag{3}$$

LEMMA 3 ([15]). – *For an entire function* $g(z)$ *and* $\epsilon > 0$,

$$\log \mu(r\,,g) \geqslant (1-\epsilon)\log \mathrm{M}(r\,,g) \quad \text{(l.f.)}\,. \tag{4}$$

LEMMA 4. – *Let* $\mathrm{A}(r)$ *be a non-decreasing function in* $(0\,,\infty)$ *with* $\mathrm{A}(r) \geqslant 1$ $(r>0)$ *and let* $\rho(r)$ *be a positive function in* $(0\,,\infty)$ *such that* $\int_0^\infty \rho(r)\,dr < \infty$ *and* $\rho(r+h) \leqslant \mathrm{C}\rho(r)$ $(r>0\,,\,0<h\leqslant 1)$ *for some* $\mathrm{C}>0$. *Then, for any* $q>1$,

$$\mathrm{A}\{r + r\rho(\log \mathrm{A}(r))\} \leqslant q\mathrm{A}(r) \quad \text{(l.f.)}\,. \tag{5}$$

This lemma is analogous to Lemma 3 in [10] and hence we omit the proof.

LEMMA 5. – *Let* $\mathrm{A}(r)$ *be the same as above and let* $\gamma(r)$ $(\not\equiv 0)$ *be a non-negative non-decreasing function in* $(0\,,\infty)$ *such that* $\int_0^\infty \gamma(r)/r^2\,dr < \infty$. *We put* $\Gamma(r) = \int_0^r \gamma(x)/x\,dx$ $(r>0)$ *and, for every* $r>0$, *define* v_r *by* $\Gamma(v_r) = \mathrm{A}(r)$. *Then, for any* $q>1$,

$$\mathrm{A}\{r + r\Gamma(v_r)/v_r\} \leqslant q\,\mathrm{A}(r) \quad \text{(l.f.)}\,. \tag{6}$$

Since $\Gamma(r)$ is continuous and increasing in $(r_0\,,\infty)$ $(\Gamma(r_0)=1)$ and $\lim_{r\to\infty}\Gamma(r)=\infty$, v_r is well-defined. We define $\alpha(r)$ by $\Gamma(\alpha(r)) = r$ and put $\rho(r) = e^r/\alpha(e^r)$. We have

$$\int_0^\infty \rho(r)\,dr = \int_1^\infty 1/\alpha(r)\,dr = \int_{\alpha(1)}^\infty \gamma(x)/x^2\,dx < \infty\,.$$

Since $\alpha(r)$ is increasing, we have $\rho(r+h)\leqslant e\rho(r)$ $(r>0, 0<h\leqslant 1)$. Hence $\rho(r)$ satisfies the conditions in Lemma 4. Since

$$\rho(\log \mathrm{A}(r)) = \mathrm{A}(r)/\alpha(\mathrm{A}(r)) = \Gamma(v_r)/v_r\,,$$

Lemma 4 gives (6).

LEMMA 6 ([16] p. 134). – *For an integrable function* $\mathrm{P}(t)$ *in* $[0, 2\pi)$ *and* $x>0$,

$$\operatorname{meas}(t\in[0,2\pi)\,;\,|\widetilde{\mathrm{P}}(t)|\geqslant x) \leqslant (\mathrm{C}/x)\int_0^{2\pi} |\mathrm{P}(s)|\,ds\,, \tag{7}$$

where «meas» *denotes the* 1*-dimensional Lebesgue measure and* C *a constant.*

LEMMA 7. — *For any non-constant polynomial* $g(z)$, $m(1, g'/(gd_g)) \leqslant C$, *where* d_g *denotes the degree of* $g(z)$ *and* C *a constant.*

Given a non-constant polynomial $g(z)$, we write

$$g(z) = \alpha \prod_{k=1}^{d} (z - y_k e^{-i\theta_k}) \quad (d = d_g, y_k \geqslant 0, \theta_k \in [0, 2\pi)).$$

For a while, we assume that $y_k \neq 1$ $(1 \leqslant k \leqslant d)$. Put

$$P(t) = \sum_{k=1}^{d} 1/(d\{1 - y_k e^{i(t-\theta_k)}\}).$$

Then $m(1, g'/(gd)) = m(P)$. Let us express $P(t)$ as follows:

$$P(t) = \sum_{k=1}^{d} = \Sigma_1 + \Sigma_2 + \Sigma_3 = P_1(t) + P_2(t) + P_3(t),$$

where Σ_1 denotes the summation over all k with $y_k < 1/2$ or $y_k > 2$, Σ_2 the summation over all k with $1/2 \leqslant y_k < 1$ and $\Sigma_3 = \sum_{k=1}^{d} - \Sigma_1 - \Sigma_2$. Then

$$m(P) \leqslant m(P_1) + m(P_2) + m(P_3) + \log 3$$

([7] p. 5). Since $|P_1(t)| \leqslant 2$ $(t \in [0, 2\pi))$, we have $m(P_1) \leqslant \log 2$. To estimate $m(P_2)$, let us write:

$$\begin{aligned} P_2(t) &= \Sigma_2 (1 - y_k)/(dP_k(t)) + \Sigma_2\, y_k \{1 - \cos(t - \theta_k)\}/(dP_k(t)) \\ &\qquad + \Sigma_2\, i y_k \sin(t - \theta_k)/(dP_k(t)) \\ &= P_{21}(t) + P_{22}(t) + iP_{23}(t), \end{aligned}$$

where

$$P_k(t) = Q(y_k e^{i(t-\theta_k)}) = |1 - y_k e^{i(t-\theta_k)}|^2.$$

Then $m(P_2) \leqslant m(P_{21}) + m(P_{22}) + m(P_{23}) + \log 3$. Note that

$$\int_0^{\infty} \eta/(\eta^2 + r^2)\, dr = \pi/2 \quad (\eta > 0).$$

Since

$$\begin{aligned} (1 - y_k)/P_k(t) &\leqslant (1 - y_k)/\{(1 - y_k)^2 + y_k^2 \sin^2(t - \theta_k)\} \\ &\leqslant \pi\eta_k/\{\eta_k^2 + (t - \theta_k)^2\} \\ &\qquad (\eta_k = \pi(1 - y_k)/(2y_k),\ |t - \theta_k| \leqslant \pi/2) \end{aligned}$$

for all k occuring in Σ_2, we have

$$m(\mathrm{P}_{21}) \leqslant \log\{(1/2\pi)\int_0^{2\pi} |\mathrm{P}_{21}(t)|\,dt + 1\} \leqslant \log(\pi+1).$$

Since

$$y_k\{1-\cos(t-\theta_k)\}/\mathrm{P}_k(t)$$
$$\leqslant y_k\{1-\cos(t-\theta_k)\}/\{2y_k(1-\cos(t-\theta_k))\} = 1/2$$

for all k occuring in Σ_2, we have $m(\mathrm{P}_{22}) = 0$. Note that, for any $\theta \in [0,2\pi)$, $\mathrm{V}(z) = \mathrm{V}(re^{it}) = 2r\sin(t-\theta)/\mathrm{Q}(ze^{-i\theta})$ is a conjugate harmonic function in D_1 of

$$\mathrm{U}(z) = \mathrm{U}(re^{it}) = (1-r^2)/\mathrm{Q}(ze^{-i\theta}).$$

We have

$$\mathrm{V}(re^{it}) = (-1/2\pi)\int_0^{\pi} \{\mathrm{U}(re^{i(t+s)}) - \mathrm{U}(re^{i(t-s)})\}\cot(s/2)\,ds$$
$$(0<r<1) \qquad ([16]\text{ p. }103).$$

Hence, putting $\mathrm{R}(t) = (1/2d)\,\Sigma_2(1-y_k^2)/\mathrm{P}_k(t)$, we have $\mathrm{P}_{23}(t) = \widetilde{\mathrm{R}}(t)$. Note that $\int_0^{2\pi} |\mathrm{R}(t)|\,dt \leqslant \pi$. Lemma 6 shows that, with a constant C', $\operatorname{meas}(\mathrm{E}_j) \leqslant \mathrm{C}'2^{-j}$ $(j \geqslant 1)$, where $\mathrm{E}_j = \{t\in[0,2\pi)\,;\,2^{j-1} \leqslant |\widetilde{\mathrm{R}}(t)| < 2^j\}$, and hence

$$m(\mathrm{P}_{23}) = m(\widetilde{\mathrm{R}}) = (1/2\pi)\sum_{j=1}^{\infty}\int_{\mathrm{E}_j} \log|\widetilde{\mathrm{R}}(t)|\,dt \leqslant \mathrm{C}'\sum_{j=1}^{\infty} j2^{-j}.$$

Consequently, we obtain $m(\mathrm{P}_2) \leqslant \mathrm{C}''$. We have similarly $m(\mathrm{P}_3) \leqslant \mathrm{C}''$; in estimating $\mathrm{P}_{33}(t)$ analogously to $\mathrm{P}_{23}(t)$, we use

$$y\sin(t-\theta)/\mathrm{Q}(ye^{i(t-\theta)}) = -(1/y)\sin(\theta-t)/\mathrm{Q}((1/y)\,e^{i(\theta-t)}),$$

so that the estimate for $\mathrm{P}_{33}(t)$ follows from that for $\mathrm{P}_{23}(t)$. Thus $m(1, g'/(gd)) \leqslant \mathrm{C}$ for some constant $\mathrm{C} > 0$.

To remove the assumption that $y_k \neq 1$ $(1 \leqslant k \leqslant d)$, we choose a sequence $(\delta_j)_{j=1}^{\infty}$ of positive numbers tending to 1 so that $\delta_j y_k \neq 1$ $(1 \leqslant k \leqslant d,\, j \geqslant 1)$ and put

$$g_j(z) = \alpha \prod_{k=1}^{d} (z - \delta_j y_k e^{-i\theta_k}).$$

Then $$m(1, g'/(gd)) = \lim_{j\to\infty} m(1, g_j'/(g_j d)) \leqslant \mathrm{C}.$$

LEMMA 8. – *For any trigonometric polynomial* $\mathrm{P}(t) = \Sigma\,\hat{\mathrm{P}}(k)\,e^{ikt}$ *with* n *non-zero coefficients,*

$$m(\mathrm{P}) \geqslant \log^+ \max |\hat{\mathrm{P}}(k)| - \mathrm{C}n\,, \tag{8}$$

where C *is a constant.*

Let C' be the constant in Lemma 7. We put $\mathrm{C} = \mathrm{C}' + \log 2$ and inductively prove (8). In the case $n = 1$, (8) evidently holds. Suppose that (8) holds in the case $n - 1$. Let $\mathrm{P}(t)$ be a trigonometric polynomial with n non-zero coefficients. Considering $e^{imt}\,\mathrm{P}(t)$ with a suitable $m \geqslant 0$ if necessary, we may assume that $\hat{\mathrm{P}}(k) = 0 \quad (k < 0)$ and $\hat{\mathrm{P}}(0) \neq 0$. Choose j so that $|\hat{\mathrm{P}}(j)| = \max\limits_k |\hat{\mathrm{P}}(k)|$. Considering $e^{idt}\,\mathrm{P}(-t) \quad (d = d_{\mathrm{P}})$ if necessary, we may assume that $2j \geqslant d$. Then we have

$$m(\mathrm{P}'/j) = m\,\{\mathrm{P}(\mathrm{P}'/(\mathrm{P}d))\,(d/j)\} \leqslant m(\mathrm{P}) + m(\mathrm{P}'/(\mathrm{P}d)) + \log 2\,.$$

Since $m(\mathrm{P}'/(\mathrm{P}d)) = m(1\,, g'/(gd)) \left(g(z) = \sum\limits_{k=0}^{d} \hat{\mathrm{P}}(k)\,z^k\right)$, Lemma 7 gives $m(\mathrm{P}) \geqslant m(\mathrm{P}'/j) - \mathrm{C}$. Since $\mathrm{P}'(t)/j$ is a trigonometric polynomial with $n - 1$ non-zero coefficients such that

$$\max_k |\hat{\mathrm{P}}'(k)/j| \geqslant |\hat{\mathrm{P}}(j)| = \max_k |\hat{\mathrm{P}}(k)|\,,$$

the assumption of our induction gives

$$m(\mathrm{P}) \geqslant \{\log^+ \max_k |\hat{\mathrm{P}}'(k)/j| - \mathrm{C}(n-1)\} - \mathrm{C} \geqslant \log^+ \max_k |\hat{\mathrm{P}}(k)| - \mathrm{C}n\,.$$

LEMMA 9. — *For any Fejér gap series* T, *there exists a Fejér gap series* S *which contains* T *as a subsequence such that, with a constant* $\mathrm{C} > 0$,

$$\sqrt{r} \leqslant \omega(r\,,\mathrm{S})\,,\quad \omega(r\,,\mathrm{S}) \leqslant \mathrm{C}\Omega(r\,,\mathrm{S}) \quad (r \geqslant 2)\,. \tag{9}$$

We easily see that $\mathrm{T} \cup \bigcup\limits_{j=0}^{\infty} [2^{j-1}\,, 2^{j-1} + [2^{j/2}])$ is also a Fejér gap series, where $[x]$ denotes the integral part of x. From the beginning, we may assume that $\omega(r\,,\mathrm{T}) \geqslant \sqrt{r} \quad (r \geqslant 2)$. Hence it is sufficient to construct a Fejér gap series $\mathrm{S}(\supset \mathrm{T})$ satisfying the second inequality in (9). Let $\sigma_j \quad (j \geqslant 10)$ denote the number of integers in $\mathrm{T} \cap [2^{j-1}\,, 2^j)$. We may also assume that $\sigma_j \geqslant 100$ $(j \geqslant 10)$. Put $\tau_j = \min\left\{\sigma_j + \left[\sum\limits_{\ell=1}^{\infty} \beta^\ell \sigma_{j+\ell}\right], 2^{j-1}\right\} \quad (\beta = 2/5)$. Then we have

$$\sum_{j=10}^{\infty} \tau_j 2^{-j} \leqslant \sum_{j=10}^{\infty} 2^{-j} \sum_{\ell=0}^{\infty} \beta^\ell \sigma_{j+\ell} = \sum_{j=10}^{\infty} \sigma_j 2^{-j} \sum_{\ell=0}^{j-10} (2\beta)^\ell$$

$$\leqslant 5 \sum_{j=10}^{\infty} \sigma_j 2^{-j} \leqslant 5 \int_0^{\infty} (1/r)\, d\omega(r\,,\mathrm{T})$$

$$= 5 \int_0^{\infty} \omega(r\,,\mathrm{T})/r^2\, dr < \infty\,. \tag{10}$$

Let us prove that $\tau_{j+1} \leqslant 3\tau_j$ $(j \geqslant 10)$. If $\tau_j = 2^{j-1}$, then $\tau_{j+1} \leqslant 2^j \leqslant 3\tau_j$. If $\tau_j = \sigma_j + \left[\sum_{\ell=1}^{\infty} \beta^{\ell} \sigma_{j+\ell} \right]$, then

$$3\tau_j \geqslant 3 \left[\sum_{\ell=1}^{\infty} \beta^{\ell} \sigma_{j+\ell} \right] \geqslant (5/2) \sum_{\ell=1}^{\infty} \beta^{\ell} \sigma_{j+\ell} \geqslant \tau_{j+1} \, .$$

Thus $\tau_{j+1} \leqslant 3\tau_j$ $(j \geqslant 10)$. Now, to T, we add $\mathrm{T}^c \cap [1,2^9)$ and add arbitrarily $\tau_j - \sigma_j$ integers in $\mathrm{T}^c \cap [2^{j-1}, 2^j)$ for all $j \geqslant 10$, and call the resulting sequence S. Then (10) shows that S is a Fejér gap series. Let us write simply $\omega(r) = \omega(r, \mathrm{S})$, $\Omega(r) = \Omega(r, \mathrm{S})$ $(r > 0)$. If $2^{j-1} \leqslant r < 2^j$ $(j \geqslant 10)$, then

$$\begin{aligned} \omega(r) &\leqslant \omega(2^j) \leqslant \omega(2^{j-3}) + \tau_{j-2} + \tau_{j-1} + \tau_j \leqslant \omega(2^{j-3}) + 39\tau_{j-3} \\ &\leqslant 40\omega(2^{j-3}) \leqslant 40\omega(r/e) \leqslant 40 \int_{r/e}^{r} \omega(x)/x \, dx \leqslant 40\Omega(r) \, . \end{aligned}$$

If $2 \leqslant r < 2^9$, then $\omega(r) \leqslant 2^9 \leqslant 2^{10}\,\Omega(2) \leqslant 2^{10}\,\Omega(r)$. Hence S satisfies the second inequality in (9) with $\mathrm{C} = 2^{10}$.

3. Proposition.

3.1. For the proof of our theorem, we show the following proposition, which is interesting in itself.

PROPOSITION. – *For an entire function $f(z)$ with Fejér gaps and $\epsilon > 0$,*

$$m(r, f) \geqslant (1 - \epsilon) \log \mathrm{M}(r, f) \quad (\text{l.f.}) \, . \tag{11}$$

In this section, we shall prove our proposition. Given an entire function $f(z)$ with Fejér gaps, we express $f(z)$, with the aid of Lemma 9, as follows:

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \quad (0 = n_0 < n_1 < n_2 < \cdots) \, , \tag{12}$$

where $\mathrm{S} = (n_k)_{k=1}^{\infty}$ is a Fejér gap series satisfying (9). Without loss of generality, we may assume that $a_0 = 1$. We write simply

$$\left. \begin{aligned} & m(r) = m(r, f) \, , \ \mathrm{M}(r) = \mathrm{M}(r, f) \, , \ \mu(r) = \mu(r, f) \\ & \omega(r) = \omega(r, \mathrm{S}) \, , \ \Omega(r) = \Omega(r, \mathrm{S}) \quad (r > 0) \, . \end{aligned} \right\} \tag{13}$$

We need

LEMMA 10. – *Let* $\omega^*(r)$ *be a non-negative non-decreasing function in* $(0,\infty)$ *such that* $4\omega(r) \leqslant \omega^*(r)$ $(r>0)$,

$$\lim_{r\to\infty} \omega^*(r)/\omega(r) = \infty \quad \textit{and} \quad \int_0^\infty \omega^*(r)/r^2\,dr < \infty .$$

We put $\Omega^*(r) = \int_0^r \omega^*(x)/x\,dx$ $(r>0)$ *and, for every* $r>0$, *define* u_r *by* $\Omega^*(u_r) = 5\log\mu(r)$. *We put* $\Lambda(r) = \Sigma^* |a_k| r^{n_k}$ $(r>0)$, *where* Σ^* *denotes the summation over all* k *with* $n_k > u_r$. *Then* $\Lambda(r) \leqslant 1$ l.f..

Applying Lemma 5 to $A(r) = 5\log\mu(r)$, $\gamma(r) = \omega^*(r)$, $q=2$, we have $\mu(r + r\sigma_r^*) \leqslant \mu(r)^2$ l.f., where $\sigma_r^* = \Omega^*(u_r)/u_r$. Hence

$$\begin{aligned}\Lambda(r) &\leqslant \Sigma^* |a_k| (r + r\sigma_r^*)^{n_k} (1+\sigma_r^*)^{-n_k} \leqslant \mu(r)^2\, \Sigma^* (1+\sigma_r^*)^{-n_k} \\ &\leqslant \{\mu(r)^2/\sigma_r^*\} \exp\{(-1+o(1))\,\Omega^*(u_r)\} \leqslant \mu(r)^{-3+o(1)}\, u_r \\ &\leqslant M(r)^{-2+o(1)}\, u_r \quad (r \longrightarrow \infty) \quad (\text{l.f.}) . \qquad (14)\end{aligned}$$

Since $u_r \leqslant \omega(u_r)^2 \leqslant C\Omega(u_r)^2 = o\{\Omega^*(u_r)^2\} = o\{M(r)\}$, (14) gives the required inequality.

3.2. Since $\Lambda(r) \leqslant 1$ l.f., we have $\max\{|a_k| r^{n_k}; n_k \leqslant u_r\} = \mu(r)$ l.f.. Put $f_*(z) = \Sigma_* a_k z^{n_k}$, $P_r(t) = f_*(re^{it})$ $(r>0)$, where Σ_* denotes the summation over all k with $n_k \leqslant u_r$. Then Lemmas 3 and 8 show that

$$\begin{aligned} m(P_r) &\geqslant \log^+ \max\{|a_k| r^{n_k}; n_k \leqslant u_r\} - C\omega(u_r) \\ &= \log\mu(r) - o\{\Omega^*(u_r)\} = (1-o(1))\log M(r) \quad (r \longrightarrow \infty) \quad (\text{l.f.}) . \end{aligned} \qquad (15)$$

Let $\epsilon > 0$. Then Lemma 10 and (15) show that

$$m(P_r) \geqslant (1-\epsilon^2)\log(M(r)+1) ,$$

$\Lambda(r) \leqslant 1$ and $(M(r)+1)^{1-\epsilon} \geqslant M(r)^{1-2\epsilon} + 1$ hold outside a set F of finite logarithmic measure. Put

$$H_r = \{t \in [0,2\pi) ; \log^+ |P_r(t)| \geqslant (1-\epsilon)\log(M(r)+1)\} \quad (r>0) .$$

Then, for any $r \notin F$,

$$\begin{aligned} 2\pi(1-\epsilon^2)\log(M(r)+1) &\leqslant 2\pi m(P_r) = \left\{ \int_{H_r} + \int_{H_r^c} \right\} \log^+ |P_r(t)|\,dt \\ &\leqslant \text{meas}(H_r)\log(M(r)+1) + \text{meas}(H_r^c)(1-\epsilon)\log(M(r)+1) , \end{aligned}$$

and hence $\operatorname{meas}(H_r) \geqslant 2\pi(1-\epsilon)$. This shows that, for any $t \in H_r$ $(r \notin F)$,

$$\log^+ |f(re^{it})| \geqslant \log^+ \{|P_r(t)| - \Lambda(r)\} \geqslant \log\{(M(r)+1)^{1-\epsilon} - 1\}$$
$$\geqslant (1-2\epsilon)\log M(r).$$

Consequently,

$$m(r) \geqslant (1/2\pi)\int_{H_r} \log^+ |f(re^{it})|\,dt \geqslant (1-2\epsilon)(1-\epsilon)\log M(r) \quad (r \notin F).$$

This completes the proof of our proposition.

4. Proof of Theorem.

4.1. In this section, we shall give the proof of our theorem. Given an entire function $f(z)$ with Fejér gaps, we express $f(z)$ in the form (12) with a Fejér gap series $S = (n_k)_{k=1}^{\infty}$ satisfying (9). Without loss of generality, it is sufficient to prove $\delta(0,f) = 0$. We may assume that $a_0 = 1$. We write simply $n(r) = n(r,0,f)$, $N(r) = N(r,0,f)$ $(r > 0)$. We use the notation in (13) and the functions $\omega^*(r)$, $\Omega^*(r)$, u_r, $\Lambda(r)$ in Lemma 10. Let $\widetilde{\omega}(r)$ be a non-negative non-decreasing function in $(0,\infty)$ such that $4\omega(r) \leqslant 2\widetilde{\omega}(r) \leqslant \omega^*(r)$ $(r > 0)$ and

$$\lim_{r\to\infty} \widetilde{\omega}(r)/\omega(r) = \lim_{r\to\infty} \omega^*(r)/\widetilde{\omega}(r) = \infty.$$

We put $\sigma_r = \Omega(u_r)/u_r$, $\widetilde{\sigma}_r = \widetilde{\Omega}(u_r)/u_r$, $\sigma_r^* = \Omega^*(u_r)/u_r$ $(r > 0)$, where $\widetilde{\Omega}(r) = \int_0^r \widetilde{\omega}(x)/x\,dx$. With every $r > 0$, we associate $\widetilde{r} = r + r\widetilde{\sigma}_r$, $\hat{r} = r + 2r\widetilde{\sigma}_r$, $r^* = r + r\sigma_r^*$. Then $r \leqslant \widetilde{r} \leqslant \hat{r} \leqslant r^*$ $(r > 0)$. Our method requires us to study the lower bound of $f(z)$. To do this, we show

LEMMA 11. — *We have, with a constant* $C_0 > 0$,

$$\xi_{\theta r} = \max\{|f(\widetilde{r}e^{i(t+\theta)})|\,;\,|t| \leqslant \sigma_r\} \geqslant \exp\{-C_0\Omega(u_r)\} \quad \textit{for all}$$
$$\theta \in [0,2\pi) \ (\text{l.f.}). \qquad (16)$$

Let $\chi(t)$ be the even function in $(-\infty,\infty)$ defined by $\chi(t) = 1 - t$ $(0 \leqslant t < 1)$ and $\chi(t) = 0$ $(t \geqslant 1)$. For $U > 0$ and a positive integer n, we put $\chi_{Un}(t) = \chi_U(t) + (1/in)\chi'_U(t)$, where $\chi_U(t) = U\chi(Ut)$. Then $\int_{-\infty}^{\infty} \chi_{Un}(t)\,dt = 1$, $\int_{-\infty}^{\infty} e^{int}\chi_{Un}(t)\,dt = 0$,

$\int_{-\infty}^{\infty} |\chi_{Un}(t)|\,dt \leqslant 1 + 2U/n$ and the support of $\chi_{Un}(t)$ is contained in $[-1/U, 1/U]$. Given $r > 0$, we put

$$X_U(t) = \chi_{Un_1} * \chi_{Un_2} * \cdots * \chi_{Un'}(t) \quad (U > 0),$$

where n' denotes the largest integer in S with $n' \leqslant u_r$. Then $\int_{-\infty}^{\infty} X_U(t)\,dt = 1$, $\int_{-\infty}^{\infty} e^{int} X_U(t)\,dt = 0 \quad (n \in S,\ n \leqslant n')$,

$$\begin{aligned}\int_{-\infty}^{\infty} |X_U(t)|\,dt &\leqslant \prod_{n_k \leqslant n'} (1 + 2U/n_k) = \exp\{\int_0^{u_r} \log(1 + 2U/x)\,d\omega(x)\} \\ &= \exp\{\omega(u_r)\log(1 + 2U/u_r) \\ &\qquad + 2U\int_0^{u_r} \omega(x)/(x(x + 2U))\,dx\} \\ &\leqslant \exp\{\omega(u_r)\log(1 + 2U/u_r) + \Omega(u_r)\} \\ &\leqslant \exp\{C\Omega(u_r)(1 + U/u_r)\}\end{aligned}$$

and the support of $X_U(t)$ is contained in $[-2\omega(u_r)/U, 2\omega(u_r)/U]$.

We recall the polynomial $f_*(z) = \Sigma_* a_k z^{n_k}$ in the section 3. For any $\theta \in [0, 2\pi)$, we have, with a constant $C' > 0$,

$$\begin{aligned}1 &= |\int_{-\infty}^{\infty} f_*(\tilde{r}e^{i(t+\theta)})\,X_U(t)\,dt| \\ &\leqslant \max\{|f_*(\tilde{r}e^{it})|\,;\ |t - \theta| \leqslant 2\omega(u_r)/U\} \int_{-\infty}^{\infty} |X_U(t)|\,dt \\ &\leqslant \max\{|f_*(\tilde{r}e^{it})|\,;\ |t - \theta| \leqslant C'\Omega(u_r)/U\}\exp\{C\Omega(u_r)(1 + U/u_r)\}.\end{aligned}$$

Putting $U = C'u_r$, we have, with a constant $C'' > 0$,

$$\max\{|f_*(\tilde{r}e^{it})|\,;\ |t - \theta| \leqslant \sigma_r\} \geqslant \exp\{-C''\Omega(u_r)\}. \tag{17}$$

Since

$$\begin{aligned}\Lambda(\tilde{r}) &= \Sigma^* |a_k| \{r^*(1 + \tilde{\sigma}_r)/(1 + \sigma_r^*)\}^{n_k} \\ &\leqslant \mu(r)^2\ \Sigma^* \{1 - (1 - o(1))\,\sigma_r^*\}^{n_k} \\ &\leqslant \{\mu(r)^2/\sigma_r^*\}\exp\{-(1 - o(1))\,\Omega^*(u_r)\} \leqslant \mu(r)^{-3+o(1)} \\ &\qquad\qquad \leqslant M(r)^{-2+o(1)} \quad \text{l.f.},\end{aligned}$$

we have $\Lambda(\tilde{r}) = o\{1/M(r)\} = o(\exp\{-C''\Omega(u_r)\})$ l.f.. Hence (17) gives (16).

Proposition, Lemmas 5 and 11 show that

$$m(r^*) \leqslant 2m(r).\ \log M(r) \leqslant 2m(r),\ \xi_{\theta r} \geqslant \exp\{-C_0\Omega(u_r)\} \quad (\theta \in [0, 2\pi)) \tag{18}$$

hold outside a set G of finite logarithmic measure. Let $\mathcal{R}_r$, Γ_r $(r > 0)$ denote all zeros (counted according to the multiplicity) of $f(z)$ in D_r and in $D_{\hat{r}} - D_r$, respectively. Put

$$\begin{cases} f_r(z) = f(z) \prod_{a \in \Gamma_r} \{(\hat{r}^2 - \bar{a}z)/(\hat{r}(z-a))\} \\ g_r(z) = f_r(z) \exp\{C_0 \Omega(u_r)\} \quad (r > 0)\,. \end{cases} \tag{19}$$

Since $|f_r(z)| \geqslant |f(z)|$ $(z \in S_{\widetilde{r}})$, we have

$$\max\{|g_r(\widetilde{r}e^{it})|\,;\,|t-\theta| \leqslant \sigma_r\} \geqslant \xi_{\theta r} \exp\{C_0\Omega(u_r)\} \geqslant 1 \qquad (\theta \in [0,2\pi)\,,\ r \notin G)\,. \tag{20}$$

LEMMA 12. — *If*

$$\lim_{r\to\infty,\, r\notin G} m(\widetilde{r}, 1/g_r)/m(r) = 0\,, \tag{21}$$

then $\delta(0,f) = 0$.

Note that $\Omega(u_r) = o\{m(r)\}$ $(r \longrightarrow \infty, r \notin G)$. Since $m(\widetilde{r}, 1/f_r) \leqslant m(\widetilde{r}, 1/g_r) + C_0\Omega(u_r)$ and $m(\widetilde{r}, f_r) \geqslant m(\widetilde{r}) \geqslant m(r)$, (21) gives

$$\limsup_{r\to\infty,\, r\notin G} m(\widetilde{r}, 1/f_r)/m(\widetilde{r}, f_r) \leqslant \lim_{r\to\infty,\, r\notin G} m(\widetilde{r}, 1/g_r)/m(r) = 0\,. \tag{22}$$

We have

$$\begin{aligned} \log|f_r(0)| &= \sum_{a\in\Gamma_r} \log(\hat{r}/|a|) \leqslant n(\hat{r})\log(\hat{r}/r) \leqslant Cn(\hat{r})\,\widetilde{\sigma}_r \\ &\leqslant C\,\{\int_{\hat{r}}^{r^*} n(x)/x\,dx\}\,\widetilde{\sigma}_r/\log(r^*/\hat{r}) \leqslant C'N(r^*)\,(\widetilde{\sigma}_r/\sigma_r^*) \\ &= o\{m(r^*)\} = o\{m(\widetilde{r}, f_r)\} \quad (r \longrightarrow \infty, r \notin G)\,. \end{aligned} \tag{23}$$

By Lemma 1, (22) and (23), we have

$$\begin{aligned} &\lim_{r\to\infty,\, r\notin G} N(\widetilde{r}, 0, f_r)/m(\widetilde{r}, f_r) \\ &= 1 - \lim_{r\to\infty,\, r\notin G} \{m(\widetilde{r}, 1/f_r) + \log|f_r(0)|\}/m(\widetilde{r}, f_r) = 1\,. \end{aligned}$$

Since $N(\widetilde{r}, 0, f_r) \leqslant N(\widetilde{r})$ and $m(\widetilde{r}, f_r) \geqslant m(\widetilde{r})$, this gives $\delta(0,f) = 0$.

4.2. By Lemma 12, it is sufficient to prove (21). Given $r \notin G$, we put $W = \{t \in [0,2\pi)\,;\,|g_r(\widetilde{r}e^{it})| < 1\}$. We may assume that W is not empty. Then W is a finite union of open intervals each of which has length at most $2\sigma_r$ by (20). We write $W = \bigcup_{\mu=1}^{\nu} I_\mu$, where I_μ's are mutually disjoint open intervals. Put $I_\mu = (\alpha_\mu, \beta_\mu)$, $\gamma_\mu = (\alpha_\mu + \beta_\mu)/2$ $(1 \leqslant \mu \leqslant \nu)$. Then

$$|g_r(\tilde{r}e^{i\alpha_\mu})| = |g_r(\tilde{r}e^{i\beta_\mu})| = 1 \quad (1 \leqslant \mu \leqslant \nu).$$

By Lemma 2, we have

$$\begin{aligned}
2\pi m(\tilde{r}, 1/g_r) &= -\int_W \log|g_r(\tilde{r}e^{it})|\,dt = -\sum_{\mu=1}^{\nu} \int_{I_\mu} \log|g_r(\tilde{r}e^{it})|\,dt \\
&= -\sum_{\mu=1}^{\nu} \int_{I_\mu} (t-\gamma_\mu)\frac{\partial}{\partial t}\log|f_r(\tilde{r}e^{it})|\,dt \\
&= -\sum_{\mu=1}^{\nu} \int_{I_\mu} (t-\gamma_\mu)\,\Phi(t)\,dt \\
&\quad -\sum_{\mu=1}^{\nu}\sum_{a\in\mathfrak{A}_r} \int_{I_\mu} (t-\gamma_\mu)\,\phi_a(t)\,dt - \sum_{\mu=1}^{\nu}\sum_{a\in\mathfrak{A}_r} \int_{I_\mu} (t-\gamma_\mu)\,\psi_a(t)\,dt \\
&\quad (= L_{r1} + L_{r2} + L_{r3},\ \text{say}),
\end{aligned} \tag{24}$$

where $\Phi(t) = \operatorname{Re}\Phi(\tilde{r}e^{it}; \hat{r}, f_r)$, $\phi_a(t) = \operatorname{Re}\phi(\tilde{r}e^{it}; a)$ and

$$\psi_a(t) = \operatorname{Re}\psi(\tilde{r}e^{it}; \hat{r}, a).$$

First we estimate $|L_{r1}|$. By Lemma 1 and $|f_r(z)| = |f(z)|$ $(z \in S_{\hat{r}})$, we have

$$\begin{aligned}
|L_{r1}| &\leqslant \sigma_r \int_0^{2\pi} |\Phi(t)|\,dt \\
&\leqslant \sigma_r \int_0^{2\pi} \{(\tilde{r}\hat{r}/\pi)\int_0^{2\pi} |\log|f(\hat{r}e^{is})||/|\hat{r}e^{is} - \tilde{r}e^{it}|^2\,ds\}\,dt \\
&\leqslant (\sigma_r\tilde{r}\hat{r}/\pi)\int_0^{2\pi} |\log|f(\hat{r}e^{is})||\,\{\int_0^{2\pi} 1/|\hat{r}e^{is} - \tilde{r}e^{it}|^2\,dt\}\,ds \\
&\leqslant (C\sigma_r/\tilde{\sigma}_r)\int_0^{2\pi} |\log|f(\hat{r}e^{is})||\,ds \\
&\leqslant 4\pi C m(\hat{r})\,(\sigma_r/\tilde{\sigma}_r) \leqslant C' m(r)\,(\sigma_r/\tilde{\sigma}_r).
\end{aligned} \tag{25}$$

Next we estimate $|L_{r2}|$. We have

$$|L_{r2}| = |\sum_{a\in\mathfrak{A}_r}\sum_{\mu=1}^{\nu} \int_{I_\mu} (t-\gamma_\mu)\,\phi_a(t)\,dt|$$

$$= \left| \sum_{a \in \mathfrak{A}_r} \sum_{\mu=1}^{\nu} \{(1/2) \int_{\mathrm{I}_\mu} (t-\gamma_\mu)^2 \phi_a'(t)\,dt - (|\mathrm{I}_\mu|^2/8)(\phi_a(\beta_\mu) - \phi_a(\alpha_\mu))\} \right|$$

$$\leqslant \sum_{a \in \mathfrak{A}_r} \sum_{\mu=1}^{\nu} (|\mathrm{I}_\mu|/2)^2 \int_{\mathrm{I}_\mu} |\phi_a'(t)|\,dt$$

$$\leqslant \sigma_r^2 \sum_{a \in \mathfrak{A}_r} \int_0^{2\pi} |\phi_a'(t)|\,dt$$

$$\leqslant \sigma_r^2 \sum_{a \in \mathfrak{A}_r} \int_0^{2\pi} \widetilde{r}^2/|\widetilde{r}e^{it} - a|^2\,dt$$

$$\leqslant \mathrm{C}n(r)\,\sigma_r^2/\widetilde{\sigma}_r \leqslant \mathrm{CN}(\widetilde{r})\,\sigma_r^2/\{\widetilde{\sigma}_r \log(\widetilde{r}/r)\} \leqslant \mathrm{C}'m(r)\,(\sigma_r/\widetilde{\sigma}_r)^2\,. \tag{26}$$

We have analogously as in (26)

$$|\mathrm{L}_{r3}| \leqslant \mathrm{C}'m(r)\,(\sigma_r/\widetilde{\sigma}_r)^2\,. \tag{27}$$

By (24), (25), (26) and (27), we have, with a constant $\mathrm{C}'' > 0$, $m(\widetilde{r}, 1/g_r)/m(r) \leqslant \mathrm{C}''\sigma_r/\widetilde{\sigma}_r$. Letting $r \longrightarrow \infty$ $(r \notin \mathrm{G})$, we have (21). This completes the proof of our theorem.

5. An entire function with Fabry gaps such that $\delta(0,\cdot) = 1$.

5.1. In this section, we shall show that the assertion of our theorem is not valid when the Fejér gap condition is replaced by the Fabry gap condition. We construct an entire function $g_\infty(z)$ with Fabry gaps such that $\delta(0, g_\infty) = 1$.

For an entire function $g(z) = \sum_{n=0}^{\infty} c_n z^n$ and a non-negative integer d, we put $pr(g,d)(z) = \sum_{n=0}^{d} c_n z^n$. The outline of our construction is as follows. An entire function e^{z^p} $(p \geqslant 1)$ does not take zero and $\mathrm{S}(e^{z^p}) = \{pn\,;\, n \geqslant 1\}$. When $d > 0$ is sufficiently large, $g(z) = pr(e^{z^p}, d)(z)$ behaves like e^{z^p} in a given disk D_{r_0}. Let $g^*(z) = g(z)\exp(z/r_0)^q$ $(q > d)$. Then $pr(g^*, d)(z) = g(z)$. We choose q sufficiently large. Then $g^*(z)$ behaves like e^{z^p} in D_{r_0} since $\lim_{q\to\infty} \{\exp(r/r_0)^q - 1\} = 0$ $(r < r_0)$, and $g^*(z)$ behaves like $\exp(z/r_0)^q$ in $(\bar{\mathrm{D}}_{r_0})^c$ in the sense of the deficiency since $\lim_{q\to\infty} \exp(r/r_0)^q = \infty$ $(r > r_0)$. Repeating this discussion, we construct an entire function which locally behaves like $\exp(\eta z)^p$ for some $\eta > 0$ and some integer $p \geqslant 1$ in the sense of the deficiency.

To express this argument concretely, we inductively define two sequences $r_0 < r'_0 < r_1 < r'_1 \cdots$, $q_0 < q_1 < \cdots$ and two sequences $(g_k(z))_{k=0}^{\infty}$, $(h_k(z))_{k=0}^{\infty}$ of entire functions such that, for every $k \geqslant 0$, $h_k(z) \neq 0$ $(z \in \mathrm{S}_{r_0} \cup \ldots \cup \mathrm{S}_{r_k} \cup \mathrm{D}^c_{r'_k})$. Let $r_0 = 1$, $r'_0 = 2$, $q_0 = 1$, $g_0(z) \equiv 0$, $h_0(z) = e^z$.

Suppose that $r_0 < r'_0 < \cdots < r_{m-1} < r'_{m-1}$, $q_0 < \cdots < q_{m-1}$, $(g_k(z))_{k=0}^{m-1}$, $(h_k(z))_{k=0}^{m-1}$ are defined so that, for any $0 \leqslant k \leqslant m-1$, $h_k(z)$ satisfies the above condition. Then r_m, r'_m, $g_m(z)$, $h_m(z)$ are defined in the following manner. Let $r_m = r'_{m-1} + 1$. We choose a positive integer d_m so that, with $g_m(z) = pr(h_{m-1}, d_m)(z)$:

$$d_m > d_{m-1} + q_{m-1} \tag{28}$$

$$|g_m(z) - h_{m-1}(z)| \leqslant 2^{-m} \quad (z \in \mathrm{D}_{r_m}) \tag{29}$$

$$n(r_\ell, 0, h_{m-1}) = n(r_\ell, 0, g_m) = \lim_{r \downarrow r_\ell} n(r, 0, g_m) \quad (0 \leqslant \ell \leqslant m). \tag{30}$$

Such a choice is possible, since $pr(h_{m-1}, d)(z)$ converges uniformly to $h_{m-1}(z)$ in $\overline{\mathrm{D}}_{r_m}$ when d tends to infinity. Let r'_m $(r'_m > r_m)$ be a number such that $g_m(z) \neq 0$ in $\mathrm{D}^c_{r'_m}$. We choose a positive integer q_m so that, with $h_m(z) = g_m(z) \exp(z/r'_{m-1})^{q_m}$:

$$d_m/q_m \leqslant (1/10)\, 2^{-m} \tag{31}$$

$$\mathrm{M}(r_{m-1}, g_m) \{\exp(r_{m-1}/r'_{m-1})^{q_m} - 1\} \leqslant 2^{-m} \tag{32}$$

$$\{t \in [0, 2\pi) ; |h_m(re^{it})| \geqslant \exp(2d_m^2 r)\} \text{ contains } [q_m/(10 d_m)] \text{ intervals of length } \pi/(2q_m) \ (r \geqslant r_m). \tag{33}$$

To see that such a choice is possible, we show that (33) holds as long as q_m is sufficiently large. Put $h^*_q(z) = |g_m(z) \exp(z/r'_{m-1})^q|$ $(q \geqslant 1)$. Given $r \geqslant r_m$, we choose $t_r \in [0, 2\pi)$ so that

$$\mathrm{M}(r, g_m) = |g_m(re^{it_r})| \ (\geqslant 1).$$

Since $g_m(z)$ is a polynomial of degree d_m,

$$|g_m(re^{it})| \geqslant 1/2 \quad (t \in [t_r - 1/(2d_m), t_r + 1/(2d_m)]).$$

Let $q'_m = 100d_m$. Then

$$\{t \in [t_r - 1/(2d_m), t_r + 1/(2d_m)]; \cos qt \geqslant 1/\sqrt{2}\}$$

contains $[q/(2\pi d_m)] - 1$ ($\geqslant [q/(10d_m)]$) intervals of length $\pi/(2q)$ $(q \geqslant q'_m)$. Hence $\{t \in [0, 2\pi); h_q(re^{it}) \geqslant (1/2)\exp\{(r/r'_{m-1})^q/\sqrt{2}\}\}$ contains $[q/(10d_m)]$ intervals of length $\pi/(2q)$ $(q \geqslant q'_m)$. Let q''_m ($\geqslant q'_m$) be an integer such that

$$(1/2)\exp\{(x/r'_{m-1})^q/\sqrt{2}\} \geqslant \exp(2d_m^2 x) \quad (x \geqslant r_m, q \geqslant q''_m).$$

Then $h_q^*(z)$ satisfies the property in (33) for all $q \geqslant q''_m$. Hence the above choice of q_m is possible. Thus we define the above four sequences.

5.2. Now we show that $g_\infty(z) = \lim\limits_{m\to\infty} g_m(z)$ exists and satisfies the required conditions. First we prove that $g_\infty(z)$ is an entire function. Given $k \geqslant 1$, we have, for any $z \in \mathrm{D}_{r_k}$, $m \geqslant k+1$,

$$\begin{aligned}|g_{m+1}(z) - g_m(z)| &\leqslant |g_{m+1}(z) - h_m(z)| + |h_m(z) - g_m(z)| \\ &\leqslant 2^{-m-1} + \mathrm{M}(r_{m-1}, g_m)\{\exp(r_{m-1}/r'_{m-1})^{q_m} - 1\} \leqslant 2^{-m+1},\end{aligned}$$

according to (29) and (32). Hence $g_\infty(z) = \sum\limits_{m=0}^{\infty}\{g_{m+1}(z) - g_m(z)\}$ converges uniformly in D_{r_k}. Since $k \geqslant 1$ is arbitrary, $g_\infty(z)$ is an entire function. Next we prove that $g_\infty(z)$ has Fabry gaps, that is, $\lim\limits_{r\to\infty} \omega(r)/r = 0$, where $\omega(r) = \omega(r, \mathrm{S}(g_\infty))$ $(r > 0)$. By (28) and (31), we have

$$\begin{cases} pr(g_\infty, d_{m+1})(z) = g_{m+1}(z) = pr(h_m, d_{m+1})(z) \\ \mathrm{S}(h_m) = \{\ell q_m + n; \ell \geqslant 0, n \in \mathrm{S}(g_m)\} \cup \{\ell q_m; \ell \geqslant 1\} \quad (m \geqslant 1). \end{cases} \tag{34}$$

Let r be a number such that $d_m < r \leqslant d_{m+1}$ for some $m \geqslant 2$ and let ℓ be a positive integer such that $d_m + (\ell - 1)q_m < r \leqslant d_m + \ell q_m$. If $\ell > 1$, then

$$\omega(r)/r = \omega(r, \mathrm{S}(h_m))/r \leqslant \ell(d_m + 1)/\{(\ell - 1)q_m\} \leqslant 2^{-m-1},$$

according to (31). Since $d_{m+1} > d_m + q_m$, we have, in particular, $\omega(d_{m+1})/d_{m+1} \leqslant 2^{-m-1}$. This inequality is valid with m replaced by any positive integer. If $\ell = 1$, then

$$\omega(r)/r = \omega(d_m)/r \leqslant \omega(d_m)/d_m \leqslant 2^{-m}.$$

Thus $\omega(r)/r \leqslant 2^{-m}$ $(d_m < r \leqslant d_{m+1})$. This shows that

$$\lim_{r\to\infty} \omega(r)/r = 0 .$$

Finally, we prove $\delta(0, g_\infty) = 1$. Let r be a number such that $r_m < r \leqslant r_{m+1}$ for some $m \geqslant 1$. Rouché's theorem shows $n(r, 0, g_\infty) \leqslant \limsup_{k\to\infty} n(r, 0, g_k)$. Since zeros of $h_k(z)$ equal those of $g_k(z)$ for all $k \geqslant 1$, (30) gives

$$n(r, 0, g_k) \leqslant n(r_{m+1}, 0, g_k) = n(r_{m+1}, 0, g_m) = d_m \quad (k \geqslant m).$$

Hence

$$\mathrm{N}(r, 0, g_\infty) \leqslant n(r, 0, g_\infty) \log r \leqslant \lim_{k\to\infty} n(r, 0, g_k) \log r \leqslant d_m \log r . \tag{35}$$

Since $|g_\infty(z) - g_{m+2}(z)| \leqslant \sum_{k=m+2}^{\infty} |g_{k+1}(z) - g_k(z)| \leqslant 1$ and

$$|g_{m+2}(z) - h_{m+1}(z)| \leqslant 1 \quad (z \in \mathrm{S}_r),$$

we have

$$m(r, g_\infty) \geqslant m(r, g_{m+2}) - \log 2 \geqslant m(r, h_{m+1}) - \log 4 . \tag{36}$$

Now we study the lower bound of $m(r, h_{m+1})$. Since

$$|g_{m+1}(z) - h_m(z)| \leqslant 1 \quad (z \in \mathrm{S}_r),$$

(33) holds with $h_m(z)$, $\exp(2d_m^2 r)$ replaced by $g_{m+1}(z)$, $\exp(d_m^2 r)$, respectively. Note that, for any interval Y^* in $[0, 2\pi)$ of length $\pi/(2q_m)$, $\{t \in \mathrm{Y}^* ; \cos q_{m+1} t \geqslant 0\}$ contains $[q_{m+1}/(4q_m)] - 1$ intervals of length π/q_{m+1}. Since

$$\begin{aligned} \mathrm{Y} &= \{t \in [0, 2\pi) ; |h_{m+1}(re^{it})| \geqslant \exp(d_m^2 r)\} \\ &\supset \{t \in [0, 2\pi) ; |g_{m+1}(re^{it})| \geqslant \exp(d_m^2 r),\ \cos q_{m+1} t \geqslant 0\} , \end{aligned}$$

Y contains $[q_m/(10d_m)]\,\{[q_{m+1}/(4q_m)] - 1\}$ intervals of length π/q_{m+1}. This shows that $\mathrm{meas}(\mathrm{Y}) \geqslant \mathrm{C}/d_m$ for some constant $\mathrm{C} > 0$. Thus

$$m(r, h_{m+1}) \geqslant (1/2\pi) \int_{\mathrm{Y}} \log^+ |h_{m+1}(re^{it})|\, dt \geqslant \mathrm{C} d_m r/(2\pi) . \tag{37}$$

By (35), (36) and (37), we have

$$\mathrm{N}(r, 0, g_\infty)/m(r, g_\infty) \leqslant d_m \log r/\{(\mathrm{C} d_m r/2\pi) - \log 4\} \qquad (r_m < r \leqslant r_{m+1}).$$

This gives $\lim_{r\to\infty} \mathrm{N}(r, 0, g_\infty)/m(r, g_\infty) = 0$ and hence $\delta(0, g_\infty) = 1$. Thus we know that the assertion of our theorem does not hold with Fejér gaps replaced by Fabry gaps.

6. Application of our method.

Our method also yields that:

(*) An entire function with Fejér gaps takes any complex value infinitely often in a given sector.

This assertion improves Hayman's result [8]: An entire function $f(z)$ takes any complex value infinitely often in a given sector if $S(f) = (n_k)_{k=1}^{\infty}$ satisfies $k(\log k)(\log\log k)^{\alpha}/n_k = O(1)$ for some $\alpha > 2$. Our method is applicable to the equidistribution theory in sectors.

In this section we prove (*). Let $f(z)$ be an entire function with Fejér gaps. In the case of $\rho(f) < \infty$, the required assertion is already known [8]. Hence we may assume that $\rho(f) = \infty$. Without loss of generality, it is sufficient to show that $f(z)$ takes 0 infinitely often in $\Gamma_\alpha = \{z\,;\,|\arg z| < \alpha\}$ $(0 < \alpha \leqslant 1)$. For the sake of simplicity, we work only with $f(0) = 1$.

Now we assume that $f(z)$ takes 0 a finite number of times in Γ_α and show a contradiction. Let $G_{\beta r}(z)$ $(0 < \beta < \pi, r > 0)$ be Green's function of $\Gamma_\beta(r) = \Gamma_\beta \cap D_r$ with pole at $r/2$. Then our assumption gives

$$N(\alpha, r) = (1/2\pi)\int_{\partial\Gamma_\alpha(r)} \frac{\partial}{\partial n} G_{\alpha r}(z)\log|f(z)|\,|dz| - \log|f(r/2)| = O(1) \quad (r \longrightarrow \infty), \tag{38}$$

where $\partial/\partial n$ denotes the inner normal derivative and $\partial\Gamma_\alpha(r)$ the boundary of $\Gamma_\alpha(r)$. Note that $N(\beta, r) \leqslant N(\alpha, r)$ $(0 < \beta \leqslant \alpha)$. We have, with two positive constants η^*, H^* depending on α,

$$\begin{cases} 0 \leqslant \dfrac{\partial}{\partial n} G_{\beta r}(z) \leqslant H^*/r & (z \in \partial\Gamma_\beta(r)) \\ \dfrac{\partial}{\partial n} G_{\beta r}(z) \geqslant \eta^*/r & (z \in \partial\Gamma_\beta(r),\ |\arg z| \leqslant \beta/2) \end{cases} \tag{39}$$

for all β with $\alpha/4 \leqslant \beta \leqslant \alpha/2$ [12]. Hence we have

$$\begin{aligned} N(\beta, r) \geqslant \eta \int_{-\beta/2}^{\beta/2} \log^+ |f(re^{it})|\,dt - H\int_{-\beta}^{\beta} \log^+ 1/|f(re^{it})|\,dt \\ - (H/r)\int_0^r \{\log^+ 1/|f(xe^{i\beta})| + \log^+ 1/|f(xe^{-i\beta})|\}\,dx - \log M(r/2) \\ (= \Xi_+(\beta, r) - \Xi_-(\beta, r) - \xi(\beta, r) - \log M(r/2), \text{ say}) \end{aligned} \tag{40}$$

for all β with $\alpha/4 \leqslant \beta \leqslant \alpha/2$, where $\eta = \eta^*/(2\pi)$ and $H = H^*/(2\pi)$. Our proposition shows that $\Xi_+(\alpha/4, r) \geqslant (\alpha\eta/8) \log M(r)$ l.f.. The argument given in the section 4 yields that $\Xi_-(\alpha/2, r) = o\{\log M(r)\}$l.f.. Since

$$\int_0^\pi \xi(\beta, r)\, d\beta = (2\pi H/r) \int_0^r m(x, 1/f)\, dx \leqslant (2\pi H/r) \int_0^r \log M(x)\, dx\,,$$

we can choose $\beta_r (\alpha/4 \leqslant \beta_r \leqslant \alpha/2)$ so that

$$\xi(\beta_r, r) \leqslant (8\pi H/\alpha r) \int_0^r \log M(x)\, dx\,.$$

Thus we have

$$\begin{aligned} N(\alpha, r) \geqslant N(\beta_r, r) &\geqslant \Xi_+(\alpha/4, r) - \Xi_-(\alpha/2, r) - \xi(\beta_r, r) - \log M(r/2) \\ &\geqslant (\eta\alpha/8) \log M(r) - o\{\log M(r)\} \\ &\quad - (8\pi H/\alpha r) \int_0^r \log M(x)\, dx - \log M(r/2) \quad \text{(l.f.)}\,. \end{aligned} \tag{41}$$

Let $\epsilon = \eta\alpha^2/(65\pi H)$. Since $\rho(f) = \infty$, there exists a set U of infinite logarithmic measure in $(0, \infty)$ such that

$$\lim_{r\to\infty,\, r\in U} \log M((1-\epsilon) r)/\log M(r) = 0\,.$$

Then

$$\limsup_{r\to\infty,\, r\in U} (1/r) \int_0^r \log M(x)\, dx/\log M(r)$$

$$\leqslant \limsup_{r\to\infty,\, r\in U} \{(1-\epsilon) \log M((1-\epsilon) r) + \epsilon \log M(r)\}/\log M(r) \leqslant \epsilon\,.$$

Hence (41) gives $\limsup_{r\to\infty} N(\alpha, r) = \infty$, which contradicts (38). This completes the proof of (*).

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