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STOCHASTIC HARMONIC MORPHISMS: FUNCTIONS MAPPING THE PATHS OF ONE DIFFUSION INTO THE PATHS OF ANOTHER

by L. CSINK and B. ØKSENDAL

1. Introduction.

Let D be a domain of the complex plane **C** and let $g: \mathrm{D} \to \mathbf{C}$ be (non-constant) analytic. If B_t^x denotes the Brownian motion in **C** starting at $x \in \mathrm{D}$, then a famous theorem of P. Lévy states that – up to the exit time of D – $g(\mathrm{B}_t^x)$ is after a change of time scale Brownian motion starting at $g(x)$. A proof of the Lévy theorem based on stochastic integrals can be found in Mc Kean [14]. Bernard, Campbell and Davie [1] extended this result to $\mathbf{R}^n$, giving a characterization of the functions which, in the sense above, preserve the paths of Brownian motion.

In this article we investigate the following more general situation: Let $(\mathrm{X}_t,\Omega,\mathrm{P}^x)$, $(\mathrm{Y}_t,\hat{\Omega},\hat{\mathrm{P}}^y)$ be diffusions on sets $\mathscr{V} \subset \mathbf{R}^d$, $\mathscr{W} \subset \mathbf{R}^p$ respectively.

Let $\mathrm{U} \subset \mathscr{V}$ be open and $\varphi: \mathrm{U} \to \mathscr{W}$ continuous, non-constant. When will φ map the paths of X_t into the paths of Y_t? In Section 2 we give a precise formulation of this problem. Intuitively we consider the processes $\varphi(\mathrm{X}_t)$ up to the exit time for X_t from U combined with Y_t from then on, and ask whether this process, after a change of time scale, can be identified with the Y_t-process.

In Section 3 we state and prove the main result of this paper (Theorem 1). This result gives several characterizations of such functions φ. One of these characterizations is the following:

$$\text{(ii)} \qquad \mathscr{A}[f \circ \varphi](x) = \lambda(x)\hat{\mathscr{A}}[f](\varphi(x)); \qquad x \in \mathrm{U}$$

for all smooth functions f, where $\mathcal{A}$ and $\hat{\mathcal{A}}$ denote the characteristic operators of X_t and Y_t, respectively, and $\lambda(x) \geqslant 0$ is continuous, positive except on a set with empty X-fine interior.

In Section 4 we give some examples and applications of Theorem 1 : *a*) First we illustrate how the Lévy theorem (and the Bernard, Campbell, Davie-extension) follows from this result (Corollary 1). *b*) Second, if we apply the result to the special case when $\mathcal{V} = \mathcal{W}$ and $\varphi(x) = x$, we obtain that if two diffusions have the same hitting distributions, then one of them can be obtained from the other by a change of time scale (Corollary 2). This was proved for more general Markov processes by Blumenthal, Getoor and Mc Kean [3], [4]. *c*) Another characterization obtained in Theorem 1 is that

$$\text{(iv)} \quad \hat{\mathcal{A}}[f] \equiv 0 \quad \text{in} \quad W \Rightarrow \mathcal{A}[f \circ \varphi] \equiv 0 \quad \text{in} \quad \varphi^{-1}(W)$$

for all open sets $W \subset \mathcal{W}$ and all smooth functions f. In other words, if f is harmonic in W with respect to the process Y_t then $f \circ \varphi$ should be harmonic in $\varphi^{-1}(W)$ with respect to X_t. In the context of harmonic spaces such functions are called *harmonic morphisms*. They have been studied by Constantinescu and Cornea [5], Fuglede [11], [12], Sibony [17] and others. So the functions φ above represent stochastic versions of the harmonic morphisms, and we find it natural to call them *stochastic harmonic morphisms*. In Corollary 3 we prove that such functions are finely continuous and finely open. The last property has been established by Constantinescu and Cornea [5] in the non-probabilistic setting of Brelot harmonic spaces. *d*) Theorem 1 can also be used to answer converted types of problems, such as : Given a class of functions φ, find all diffusions X_t, Y_t (if any) such that the functions φ map the paths of X_t into the paths of Y_t. If such diffusions can be found, they might be useful in the investigation of the properties of the functions φ. For example, on the basis of the many interesting applications of Brownian motion to complex analysis due to the Lévy theorem, (see for example B. Davis [8]) it is natural to ask :

Are there any other diffusions X_t, Y_t in $\mathbf{C}$ than Brownian motion such that all analytic functions φ map the paths of X_t into the paths of Y_t? We give a negative answer to this question (Corollary 4).

In the case when $X_t = Y_t$ this problem was studied (and answered in the negative) for more general processes (continuous strong Markov processes) by Øksendal and Stroock [16].

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2. Definitions and precise formulation of the problem.

Let (A_t,Ω',R^x) and (B_t,Ω'',S^x) be stochastic processes on some topological space E (the state space).

Let $\tau: \Omega' \to [0,\infty]$ be a random time. Then we define a stochastic process $C_t = C_t(.) : \Omega' \times \Omega'' \to E$ called the *τ-welding of* A_t *and* B_t, as follows

$$(2.1)\quad C_t(\omega',\omega'')=\begin{cases} A_t(\omega'); & t < \tau(\omega') \\ B_{t-\tau(\omega')}(\omega''); & t \geqslant \tau(\omega'), \quad (\omega',\omega'') \in \Omega' \times \Omega'' \end{cases}$$

with probability law Q^x defined by (with $0 \leqslant t_1 < t_2 < \cdots < t_n$)

$$(2.2)\quad Q^x[C_{t_1} \in E_1, \ldots, C_{t_n} \in E_n, t_k \leqslant \tau < t_{k+1}]$$
$$= \int_{\Omega'} \chi_{E_1}(C_{t_1}) \ldots \chi_{E_k}(C_{t_k})\chi_{[t_k,t_{k+1})}(\tau) . S^{A_\tau}[B_{t_{k+1}-\tau} \in E_{k+1}, \ldots, B_{t_n-\tau} \in E_n]\, dR^x,$$

where χ_K denotes the characteristic function (indicator function) of the set K and E_i denote Borel sets in E.

For a more general construction of this kind, see Stroock and Varadhan [18], Theorem 6.1.2.

We will apply this to the following situation:

Let (X_t,Ω,P^x) and $(Y_t,\hat{\Omega}.,\hat{P}^y)$ be diffusions on Borel sets $\mathscr{V} \subset \mathbf{R}^d$ and $\mathscr{W} \subset \mathbf{R}^p$, respectively, in the sense of Dynkin [9], [10]. Let U be an open, connected subset of $\mathscr{V}$ with closure $\bar{U} \subset \mathscr{V}$ and let $\varphi : \bar{U} \to \mathscr{V}$ be a continuous function.

Let $\tau = \tau_U = \inf\{t>0; X_t \notin U\}$ be the (first) exit time of U for X_t. Let $\psi: \varphi(\bar{U}) \to \bar{U}$ be a right inverse of φ, i.e. a measurable function such that $\varphi(\psi(y)) = y$ for all $y \in \varphi(\bar{U})$. Then we define the stochastic process $A_t(.) : \Omega \to \varphi(\bar{U})$ for $t \leqslant \tau$ as follows:

$$A_t(\omega) = \varphi(X_t(\omega)); \qquad \omega \in \Omega, \qquad 0 \leqslant t \leqslant \tau$$

with probability law (for $y \in \varphi(\bar{U})$)

$$\text{(2.3)} \quad P'^y[A_{t_1} \in E_1, \ldots, A_{t_n} \in E_n] = P^{\psi(y)}[X_{t_1} \in \varphi^{-1}(E_1), \ldots, X_{t_n} \in \varphi^{-1}(E_n), t_n \leqslant \tau],$$

where $0 \leqslant t_1 < \ldots < t_n$ and E_i are Borel sets.

Now let Z_t be the τ_U-welding of A_t and Y_t:

$$\text{(2.4)} \quad Z_t(\omega,\hat{\omega}) = \begin{cases} \varphi(X_t(\omega)); & t < \tau(\omega); \quad (\tau = \tau_U) \\ Y_{t-\tau(\omega)}(\hat{\omega}); & t \geqslant \tau(\omega); \quad (\omega,\hat{\omega}) \in \Omega \times \hat{\Omega} \end{cases}$$

with probability law $\tilde{P}^y$ according to (2.2):

$$\text{(2.5)} \quad \tilde{P}^y[Z_{t_1} \in E_1, \ldots, Z_{t_n} \in E_n, t_k \leqslant \tau < t_{k+1}]$$
$$= \int_\Omega \chi_{\varphi^{-1}(E_1)}(X_{t_1}) \ldots \chi_{\varphi^{-1}(E_k)}(X_{t_k}) \chi_{[t_k,t_{k+1})}(\tau) \cdot \hat{P}^{\varphi(X_\tau)}[Y_{t_{k+1}-\tau} \in E_{k+1}, \ldots, Y_{t_n-\tau} \in E_n]\, dP^x.$$

Intuitively, the process Z_t is obtained by « gluing » together $\varphi(X_t)$ up to the exit time τ of U with Y_t for $t \geqslant \tau$. We are now ready to state a precise formulation of our problem:

Characterize the functions φ *such that* Z_t *– possibly after a change of time scale – coincides with (i.e. has the same finite-dimensional distribution as)* Y_t, *for any choice of right inverse* ψ *of* φ.

If φ has this property, we will say that φ maps the paths of X_t into the paths of Y_t.

In the following E^x, $\tilde{E}^y$ and $\hat{E}y$ will denote the expectation operator with respect to the measures P^x, $\tilde{P}^y$ and $\hat{P}^y$, respectively, and τ_F, $\tilde{\tau}_G$ and $\hat{\tau}_H$ will be the (first) exit times from the sets F, G and H for the processes X_t, Z_t and Y_t, respectively.

The following connection between $\tilde{E}^y$ and $E^{\psi(y)}$ will be crucial:

LEMMA 1. – Let $G \subset \varphi(\bar{U})$ *be open,* $g : \bar{G} \to \mathbf{R}$ *continuous. Then*

$$\text{(2.6)} \qquad \tilde{E}^y[g(Z_{\tilde{\tau}_G})] = E^{\psi(y)}[\hat{g} \circ \varphi(X_{\tau_H})],$$

where $H = \varphi^{-1}(G)$ *and*

$$\hat{g}(y) = \hat{E}^y[g(Y_{\hat{\tau}_G})]$$

is the Y_t*-harmonic extension of* $g \mid \partial G$ *to* G ($g \mid \partial G$ *is the restriction of* g *to the boundary* ∂G *of* G).

Proof. – Since $\tilde{\tau}_G \geqslant \tau_H$ we have

$$\begin{aligned}\tilde{E}^y[g(Z_{\tilde{\tau}_G})] &= \tilde{E}^y[g(Z_{\tilde{\tau}_G}) . \chi_{\{\tilde{\tau}_G = \tau_H\}}] + \tilde{E}^y[g(Z_{\tilde{\tau}_G}) . \chi_{\{\tilde{\tau}_G > \tau_H\}}] \\ &= \tilde{E}^y[g(Z_{\tilde{\tau}_G}) . \chi_{\partial H \setminus L}(X_{\tau_H})] + \tilde{E}^y[g(Z_{\tilde{\tau}_G}) . \chi_L(X_{\tau_H})],\end{aligned}$$

where $L = \{x \in \partial H; \varphi(x) \in G\} = \{x \in \partial H \cap \partial U; \varphi(x) \in G\}$. This gives, using (2.5) and putting $x = \psi(y)$:

$$\begin{aligned}&\tilde{E}^y[g(Z_{\tilde{\tau}_G})] \\ &= \int_{\partial H \setminus L} g(\varphi(v)) . P^x[X_{\tau_H} \in dv] + \int_L \hat{E}^{\varphi(v)}[g(Y_{\hat{\tau}_G})] . P^x[X_{\tau_H} \in dv] \\ &= \int_{\partial H \setminus L} g(\varphi(v)) . P^x[X_{\tau_H} \in dv] + \int_L \hat{g}(\varphi(v)) . P^x[X_{\tau_H} \in dv] \\ &= \int_{\partial H} \hat{g}(\varphi(v)) . P^x[X_{\tau_H} \in dv] = E^x[\hat{g}(\varphi(X_{\tau_H}))],\end{aligned}$$

since $\hat{g} = g$ on $\partial H \setminus L$.

3. The main result.

If (A_t, Ω', P) is a stochastic process in $\mathcal{U} \subset \mathbf{R}^k$ and $E \subset \mathcal{U}$ is a Borel set then the *hitting distribution* of A_t on E is the measure $d\mu(y) = P[A_T \in dy]$, where $T = \inf\{t > 0; A_t \in E\}$ is the first hitting time of E. In other words,

$$\int f(y)\, d\mu(y) = E[f(A_T)]; \qquad f \text{ bounded, continuous.}$$

A Borel set $V \subset \mathcal{V}$ is called X-*finely open* if the exit time τ_V from V is positive a.s., for every starting point $x \in V$. A Borel set $E \subset \mathcal{V}$ is called *polar* (for X) if

$$P^x[\exists t > 0; X_t \in E] = 0 \qquad \text{for all } x,$$

i.e. X_t does not hit E, a.s. The Y-finely open and Y-polar sets in $\mathscr{W}$ are defined similarly.

Let $\mathscr{A}$, $\hat{\mathscr{A}}$ and A, $\hat{A}$ denote the characteristic operators and the infinitesimal generators of X_t, Y_t, respectively. We will assume throughout that X_t and Y_t are diffusions in the sense of Dynkin [9], [10], although some of the implications proved below can be obtained under weaker hypotheses.

We will need that $\mathscr{A}[f \circ \varphi] \in C(\bar{U})$ (the real continuous functions on $\bar{U}$) for all $f \in C^2(\mathscr{W})$ (the twice continuously differentiable functions on $\mathscr{W}$), or at least for all f in a class of functions which is pointwise boundedly dense in $C(\mathscr{W})$. This will give that $A[f \circ \varphi] = \mathscr{A}[f \circ \varphi] \in C(\bar{U})$ for all $f \in C^2(\mathscr{W})$, by Theorem 5.5, p. 143 in Dynkin [9]. For example, it will suffice to assume that $\varphi \in C^2(\mathscr{V})$.

We will also assume one of the following two conditions: Either:

(A) φ is not X-finely locally constant, i.e. $\varphi^{-1}(y)$ does not contain non-empty X-finely open sets, for $y \in \mathscr{W}$.

Or

(B) The points in $\varphi(U)$ are polar for Y.

The assumption (A) or (B) is only needed in the implication (i) $\Rightarrow$ (ii).

We refer the reader to Blumenthal and Getoor [2] for information about potential theory associated with Markov processes.

We are now ready to state and prove the main result of this paper:

THEOREM 1. — *The following are equivalent:*

(i) Z_t *and* Y_t *have the same hitting distributions, for any choice of right inverse* ψ *of* φ.

(ii) *For all* $f \in C^2(\mathscr{W})$, $x \in U$ *we have*

$$\mathscr{A}[f \circ \varphi](x) = \lambda(x) . \hat{\mathscr{A}}[f](\varphi(x)),$$

where $\lambda(x) \geqslant 0$ *is continuous,* $\lambda(x) > 0$ *except possibly on an* X*-finely nowhere dense set.*

(iii) Z_t *coincides with* Y_t *after a change of time scale. More precisely, there exists a continuous function* $\lambda(x) \geqslant 0$ *on* $\bar{U}$ *with* $\lambda(x) > 0$ *except* *possibly on a set with empty* X*-fine interior such that if we define* (*with* $\tau = \tau_U$)

$$\sigma_t(\omega) = \begin{cases} \int_0^t \lambda(X_u)\,du; & t \leqslant \tau \\ \int_0^\tau \lambda(X_u)\,du + t - \tau; & t > \tau \end{cases}$$

and let β_t *be the inverse of* σ_t, *then* Z_{β_t} *is a Markov process equivalent to* Y_t (*i.e.* Z_{β_t} *has the same finite-dimensional distributions as* Y_t).

(iv) *For all open sets* $W \subset \mathscr{W}$ *and* $f \in C^2(\mathscr{W})$ *we have*

$$\hat{\mathscr{A}}[f] \equiv 0 \text{ in } W \Rightarrow \mathscr{A}[f \circ \varphi] \equiv 0 \text{ in } \varphi^{-1}(W).$$

Proof. – (i) ⇒ (ii): Suppose Z_t and Y_t have the same hitting distributions.

First we observe that in this situation assumption (B) actually implies assumption (A): Choose $y \in \varphi(U)$. If $\varphi^{-1}(y)$ contains an X-finely open set G then

$$P^x[\exists t>0; X_t \in G] = 1 \qquad \text{for all} \qquad x \in G.$$

Hence $\tilde{P}^y[\exists t>0; Z_t = y] = 1$, so $\{y\}$ is not polar for Y, using (i).

Therefore in the proof of (i) ⇒ (ii) it will be enough to assume that (A) holds.

Let W be a neighbourhood of $y \in \varphi(U)$. Let $f \in C^2(\mathscr{W})$. Then letting $D = \varphi^{-1}(W)$, we get from Lemma 1

$$(3.1) \quad \frac{\hat{E}^y[f(Y_{\hat{\tau}_W})] - f(y)}{\hat{E}^y[\hat{\tau}_W]} = \frac{\tilde{E}^y[f(Z_{\tilde{\tau}_W})] - f(y)}{\hat{E}^y(\hat{\tau}_W]}$$
$$= \frac{E^x[\hat{f} \circ \varphi(X_{\tau_D})] - f(\varphi(x))}{E^x[\tau_D]} \cdot \frac{E^x[\tau_D]}{\hat{E}^y[\hat{\tau}_W]},$$

where $\hat{f}$ denotes the Y-harmonic extension of $f \mid \partial W$ to W and $x = \psi(y)$.

By our assumption (A) on φ the set $F = \varphi^{-1}(y)$ does not contain a non-empty X-finely open set.

Therefore the point x is a fine boundary point of F.

Then $\tau_D \downarrow y$ as $W \downarrow y$. From Corollary p. 133 in Dynkin I [9] we have

$$E^x[f \circ \varphi(X_{\tau_0})] - f \circ \varphi(x) = E^x\left[\int_0^{\tau_0} \mathcal{A}[f \circ \varphi](X_t)\, dt\right].$$

So, by continuity of $\mathcal{A}[f \circ \varphi]$ we obtain

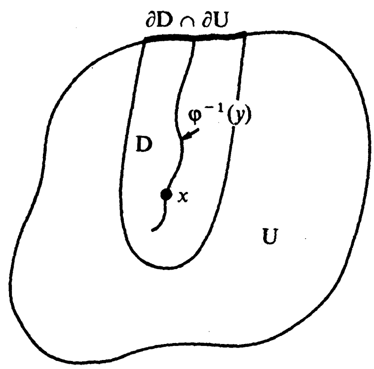
$$\lim_{W \downarrow y} \frac{E^x[f \circ \varphi(X_{\tau_D})] - f \circ \varphi(x)}{E^x[\tau_D]} = \mathcal{A}[f \circ \varphi](x).$$

From this we get

$$\text{(3.2)} \quad \lim_{W \downarrow y} \frac{E^x[\hat{f} \circ \varphi(X_{\tau_D})] - f \circ \varphi(x)}{E^x[\tau_D]}$$
$$= \mathcal{A}[f \circ \varphi](x) + \lim_{W \downarrow y} \frac{1}{E^x[\tau_D]} \cdot \int_{\partial U} (\hat{f} \circ \varphi - f \circ \varphi)(u)\, d\mu_D^x(u),$$

where μ_D^x is the hitting distribution of X_t^x on ∂D, using that

$$u \in \partial D \backslash \partial U \Rightarrow \varphi(u) \in \partial W \Rightarrow \hat{f} \circ \varphi(u) - f \circ \varphi(u) = 0.$$



Let g be any positive, bounded smooth (i.e. C^2) function on $\mathcal{V}$ such that $g \equiv 0$ in a neighbourhood of x. Then, again from Corollary p. 133 in Dynkin [9]:

$$E^x[\tau_D]^{-1} \cdot \int_{\partial U} g(u)\, d\mu_D^x(u) \leqslant E^x[\tau_D]^{-1} . (E^x[g(X_\tau)] - g(x))$$
$$= E^x[\tau_D]^{-1} . E^x\left[\int_0^{\tau_D} \mathcal{A}[g](X_t)\, dt\right] \to \mathcal{A}[g](x) = 0$$

as $D \downarrow F$ i.e. $W \downarrow y$.

In particular, this holds if g is a positive constant, hence for any constant and then also for any bounded, smooth function on ∂U. This proves that

$$\text{(3.3)} \qquad \lim_{W\downarrow y} \frac{1}{E^x[\tau_D]} \cdot \int_{\partial U} (\hat{f} \circ \varphi - f \circ \varphi)(u)\, d\mu_D^x(u) = 0.$$

Combining (3.1)-(3.3) we get that

$$\text{(3.4)} \qquad \mathcal{A}[f \circ \varphi](x) = \lambda(x) \hat{\mathcal{A}}[f](\varphi(x)),$$

where
$$\lambda(x) = \lim_{W\downarrow y} \frac{\hat{E}^y[\hat{\tau}_W]}{E^x[\tau_D]}; \qquad 0 \leqslant \lambda(x) < \infty.$$

(If $\lambda(x) = \infty$ then $\hat{\mathcal{A}}[f](\varphi(x)) = 0$ for all f, so $y = \varphi(x)$ is a trap for Y_t, hence for Z_t. Then $\varphi^{-1}(y)$ contains a non-empty X-finely open set. Consequently, assuming (A) we obtain $\lambda(x) < \infty$).

We want to prove that $\lambda(x) > 0$ except possibly on a set with empty X-fine interior. Suppose that $B \subset U$ is X-finely open such that $\lambda(x) \equiv 0$ in B.

Then $\qquad \mathcal{A}[f \circ \varphi](x) \equiv 0$ in B, $\qquad$ for all f.

Therefore $f \circ \varphi(x) = \displaystyle\int_{\partial B} (f \circ \varphi)\, d\mu_B^x$, for all f.

Choose a bounded sequence $\{f_n\}$ of C^2 functions such that

$$f_n(y) \to 1 \quad (\text{where } y = \varphi(x)) \qquad \text{and} \qquad f_n \to 0 \quad \text{on} \quad \varphi(\partial B)\backslash\{y\}.$$

Then $1 = \lim_{n\to\infty} \displaystyle\int_{\partial B} (f_n \circ \varphi)\, d\mu_B^x(F)$, where $F = \varphi^{-1}(y)$. So $\varphi \equiv y$ on ∂B.

Since the same must hold for any finely open subset of B, we conclude that $\varphi \equiv y$ in B. This contradicts our assumption (A) on φ. Thus we have proved that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Assume that (ii) holds.

Define

$$\sigma_t(\omega) = \begin{cases} \displaystyle\int_0^t \lambda(X_u)\, du; & t \leqslant \tau \\ \displaystyle\int_0^\tau \lambda(X_u)\, du + t - \tau; & t > \tau \end{cases}$$

where $\tau = \tau_U$ is the first exit time of U for X_t, as before. Note that $t \to \sigma_t$ is strictly increasing for a.a. ω, since $\lambda(x) > 0$ except possibly on a set F with empty X-fine interior (X_t exits from F immediately, a.s.). Let β_t be the inverse of σ_t. Then if we put

$$\bar{X}_t = X_{\beta_t},$$

and let $\bar{\mathcal{A}}$ denote the characteristic operator of $\bar{X}_t$, we have $\mathcal{D}_{\bar{\mathcal{A}}}(x) = \mathcal{D}_{\mathcal{A}}(x)$ for all x and, if $\lambda(x) > 0$,

$$\mathcal{A}g(x) = \lambda(x).\bar{\mathcal{A}}g(x); \qquad g \in \mathcal{D}_{\mathcal{A}},$$

where $\mathcal{D}_{\bar{\mathcal{A}}}$ and $\mathcal{D}_{\mathcal{A}}$ denote the domain of definition of $\bar{\mathcal{A}}$ and $\mathcal{A}$, respectively. (See Dynkin I [9], p. 324.)

So from (ii) we obtain that

$$\hat{\mathcal{A}}[f](\varphi(x)) = \bar{\mathcal{A}}[f \circ \varphi](x)$$

for all x such that $\lambda(x) > 0$.

By continuity this identity holds for all $x \in U$. In particular,

$$\hat{A}[f](\varphi(x)) = \bar{A}[f \circ \varphi](x), \qquad x \in U, \tag{3.5}$$

where $\hat{A}$ and $\bar{A}$ denote the infinitesimal generators of Y_t and $\bar{X}_t$, respectively.

Let $T = \bar{\tau}_U$ be the first exit time of U for $\bar{X}_t$. Define M_t as the T-welding of $\varphi(\bar{X}_t)$ and Y_t:

$$M_t = \begin{cases} \varphi(\bar{X}_t), & t \leqslant T \\ Y_{t-T}^{\varphi(\bar{X}_T)}, & t > T \end{cases}$$

Let $\check{P}^y$ denote the probability law of M_t, $\check{E}^y$ the expectation. Since $T = \beta^{-1}(\tau)$ we see that $M_t = Z_{\beta_t}$. So we have to prove that M_t and Y_t have the same finite-dimensional distributions.

Let g be a smooth function on $\mathcal{W}$. Then

$$\frac{d}{dt}[\hat{E}^y(g(Y_t))] = \hat{A}[\hat{E}^y(g(Y_t))] = \hat{E}^y[\hat{A}g(Y_t)]$$

and

$$\hat{E}^y[g(Y_0)] = g(y). \tag{3.6}$$

On the other hand, if $y = \varphi(x)$ then

$$(3.7)\quad \check{E}^y[g(M_t)] = E^x[g(\varphi(\bar{X}_t)\cdot\chi_{[t,\infty)}(T)] + \int \hat{E}^{\varphi(\bar{X}_T)}[g(Y_{t-T})]\,dP^x,$$

and therefore

$$\begin{aligned}(3.8)\quad & \frac{d}{dt}\{\check{E}^y[g(M_t)]\} \\ &= \frac{d}{dt}\{E^x[g(\varphi(\bar{X}_t))\cdot\chi_{[t,\infty)}(T)]\} + \int \frac{d}{dt}\{\hat{E}^{\varphi(\bar{X}_T)}[g(Y_{t-T})]\}\,dP^x \\ &= E^x[\bar{A}[g\circ\varphi](\bar{X}_t)\cdot\chi_{[t,\infty)}(T)] + \int \hat{E}^{\varphi(\bar{X}_T)}[\hat{A}g(Y_{t-T})]\,dP^x \\ &= E^x[\hat{A}g(\varphi(\bar{X}_t))\cdot\chi_{[t,\infty)}(T)] + \int \hat{E}^{\varphi(\bar{X}_T)}[\hat{A}g(Y_{t-T})]\,dP^x \\ &= \check{E}^y[\hat{A}g(M_t)].\end{aligned}$$

Moreover, $\check{E}^y[g(M_0)] = g(y)$.

So the two functions $V_t:\ C^2(\mathscr{W}) \to \mathbf{R}$ and $W_t:\ C^2(\mathscr{W}) \to \mathbf{R}:\ t > 0$ defined by

$$V_t g = \hat{E}y[g(Y_t)] \qquad \text{and} \qquad W_t g = \check{E}^y[g(M_t)]; \qquad g \in C^2(\mathscr{W})$$

both satisfy the equation in u_t

$$\frac{d}{dt}u_t(g) = u_t(\hat{A}(g)), \qquad u_0 g = g(y), \qquad g \in C^2(\mathscr{W}).$$

By uniqueness (see for example Dynkin I [9], p. 28, where the equation $\frac{d}{dt}u_t = \hat{A}u_t$ is considered, the same proof applies to get the above case), we must have $V_t = W_t$, i.e.

$$(3.9)\qquad \hat{E}^y[g(Y_t)] = \check{E}^y[g(M_t)]; \qquad y \in \mathscr{W},$$

for all smooth, and hence all bounded measurable g on $\mathscr{W}$.

Similarly we get that for $t_1, t \geqslant 0$, g_1, g smooth

$$\begin{aligned}(3.10)\quad & \frac{d}{dt}\{\hat{E}^y[g_1(Y_{t_1})\cdot g(Y_{t_1+t})]\} \\ &= \int g_1(v)\frac{d}{dt}\{\hat{E}^v[g(Y_t)]\}\hat{P}^y(Y_{t_1}\in dv) \\ &= \int g_1(v)\hat{E}^v[\hat{A}g(Y_t)]\hat{P}^y(Y_{t_1}\in dv) = \hat{E}^y[g_1(Y_{t_1})\cdot\hat{A}g(Y_{t_1+t})].\end{aligned}$$

So the function $a_t : C^2(\mathcal{W}) \to \mathbf{R}$ defined by

$$a_t(g) = \hat{E}^y[g_1(Y_{t_1})g(Y_{t_1+t})]; \qquad t \geqslant 0, \qquad g \in C^2(\mathcal{W})$$

is the unique solution of the equation

$$\frac{d}{dt}u_t(g) = u_t(\hat{A}(g)), \qquad u_0 g = \hat{E}^y[g_1(Y_{t_1})g(Y_{t_1})]; \qquad g \in C^2(\mathcal{W}).$$

But we claim that the same equation is satisfied by

$$b_t(g) = \check{E}^y[g_1(M_{t_1})g(M_{t_1+t})].$$

To see this, we first consider

$$(3.11) \quad \frac{d}{dt}\{\check{E}^y[g_1(M_{t_1})g(M_{t_1+t})\cdot\chi_{[0,t_1)}(T)]\}$$
$$= \int \chi_{[0,t_1)}(s)\cdot\frac{d}{dt}\{\hat{E}^{\varphi(v)}[g_1(Y_{t_1-s})g(Y_{t_1+t-s})]\}P^x(\bar{X}_T \in dv, T \in ds)$$
$$= \int \chi_{[0,t_1)}(s)\{\hat{E}^{\varphi(v)}[g_1(Y_{t_1-s})\hat{A}g(Y_{t_1+t-s})]P^x(\bar{X}_T \in dv, T \in ds)$$
$$= \check{E}^y[g_1(M_{t_1})\hat{A}g(M_{t_1+t})\chi_{[0,t_1)}(T)].$$

Similarly,

$$(3.12) \quad \frac{d}{dt}\{\check{E}^y[g_1(M_{t_1})g(M_{t_1+t})\cdot\chi_{[t_1,t_1+t)}(T)]\}$$
$$= \check{E}^y[g_1(M_{t_1})\hat{A}g(M_{t_1+t})\chi_{[t_1,t_1+t)}(T)].$$

Finally, when $y = \varphi(x)$ we get using (2.5)

$$(3.13.) \quad \frac{d}{dt}\{\check{E}^y[g_1(M_{t_1})g(M_{t_1+t})\cdot\chi_{[t_1+t,\infty)}(T)]\}$$
$$= \frac{d}{dt}\{E^x[g_1(\varphi(\bar{X}_{t_1})).g(\varphi(\bar{X}_{t_1+t}))\chi_{[t_1+t,\infty)}(T)]\}$$
$$= E^x[g_1(\varphi(\bar{X}_{t_1})).\bar{A}[g\circ\varphi](\bar{X}_{t_1+t})\chi_{[t_1+t,\infty)}(T)]$$
$$= E^x[g_1(\varphi(\bar{X}_{t_1})).\hat{A}g(\varphi(\bar{X}_{t_1+t}))\cdot\chi_{[t_1+t,\infty)}(T)].$$

So combining (3.11), (3.12) and (3.13) we obtain

$$\frac{d}{dt}b_t(g) = \frac{d}{dt}\{\check{E}^y[g_1(M_{t_1})g(M_{t_1+t})]\} = b_t\hat{A}g.$$

And from (3.9) we have

$$b_0(g) = \check{E}^y[g_1(M_{t_1})g(M_{t_1})] = \hat{E}^y[g_1(Y_{t_1})g(Y_{t_1})].$$

So by uniqueness we must have $a_t(g) = b_t(g)$, i.e.

$$\hat{E}^y[g_1(Y_{t_1})g(Y_{t_1+t})] = \check{E}^y[g_1(M_{t_1})g(M_{t_1+t})]; \qquad g \in C^2(\mathscr{W}).$$

Using induction on this argument we obtain

$$\hat{E}^y[g_1(Y_{t_1}) \dots g_n(Y_{t_n})] = \check{E}^y[g_1(M_{t_1}) \dots g_n(M_{t_n})]. \tag{3.14}$$

So $\{Y_t\}$ and $\{M_t\}$ have the same finite-dimensional distributions.

Since $\{Y_t\}$ is a Markov process w.r.t. the σ-algebras $\hat{\mathscr{F}}_t$ generated by $\{Y_s; s \leqslant t\}$, it follows from (3.14) that $\{M_t\}$ is a Markov process w.r.t. the σ-algebras $\mathscr{F}_t$ generated by $\{M_s; s \leqslant t\}$, by the following well-known argument:

If $0 \leqslant t_1 < \cdots < t_k \leqslant t \leqslant t + s$ and g, $h_j (1 \leqslant j \leqslant k)$ are bounded Borel measurable functions from $\mathscr{W}$ to $\mathbf{R}$, then, if

$$h = h_1(M_{t_1}) \dots h_k(M_{t_k})$$

we have by (3.14) and the Markov property of Y_t:

$$\begin{aligned} \check{E}^y[h.g(M_{t+s})] &= \hat{E}^y[h_1(Y_{t_1}) \dots h_k(Y_{t_k})g(Y_{t+s})] \\ &= \hat{E}^y[\hat{E}(h_1(Y_{t_1}) \dots h_k(Y_{t_k})g(Y_{t+s}) \mid \hat{\mathscr{F}}_t)] \\ &= \hat{E}^y[h_1(Y_{t_1}) \dots h_k(Y_{t_k})\hat{E}^{Y_t}[g(Y_s)]] = \check{E}^y[h.\check{E}^{M_t}[g(M_s)]]. \end{aligned}$$

This implies that

$$\check{E}^y[g(M_{t+s}) \mid \mathscr{F}_t] = \check{E}^{M_t}[g(M_s)],$$

so M_t is a Markov process. This proves (iii).

(iii) ⇒ (iv): Assume (iii). Then if $f \in C^2(\mathscr{W})$ and $W \subset \mathscr{W}$ is open, we have

$$\tilde{E}^y[f(Z_{\tilde{\tau}_W})] = \hat{E}^y[f(Y_{\hat{\tau}_W})].$$

From Lemma 1 we have, letting $V = \varphi^{-1}(W)$,

$$E^x[\hat{f} \circ \varphi(X_{\tau_V})] = \tilde{E}^y[f(Z_{\tilde{\tau}_W})], \tag{3.15}$$

where $\hat{f}$ is the Y-harmonic extension of $f \mid \partial W$ to W.

If $\hat{\mathcal{A}}[f] \equiv 0$ in W, then $\hat{f} = f$ in W (see Corollary, p. 133 in Dynkin [9]).

So if $y = \varphi(x)$ we have

$$\begin{aligned} E^x[f \circ \varphi(X_{\tau_V})] = E^x[\hat{f} \circ \varphi(X_{\tau_V})] &= \tilde{E}^y[f(Z_{\tilde{\tau}_W})] \\ &= \hat{E}^y[f(Y_{\hat{\tau}_W})] = \hat{f}(y) = f(y) = f \circ \varphi(x). \end{aligned}$$

This implies that $\mathcal{A}[f \circ \varphi](x) = 0$, and (iv) is proved.

(iv) ⇒ (i): Assume (iv) holds. Then if W is open in $\mathcal{W}$ and $\hat{f}$ denotes the Y-harmonic extension of $f \mid \partial W$ to W, we have that $\hat{f} \circ \varphi$ is X-harmonic in $V = \varphi^{-1}(W)$. Therefore

$$\hat{f} \circ \varphi(x) = E^x[\hat{f} \circ \varphi(X_{\tau_V})].$$

Using Lemma 1 we obtain, with $y = \varphi(x)$,

$$\hat{E}^y[f(Y_{\hat{\tau}_W})] = \hat{f} \circ \varphi(x) = E^x[\hat{f} \circ \varphi(X_{\tau_V})] = \tilde{E}^y[f(Z_{\tilde{\tau}_W})],$$

so Y_t and Z_t have the same hitting distributions.

This completes the proof of the theorem.

For the statements (ii) and (iv) in Theorem 1 the requirement that φ be continuously extendable to ∂U seems unnatural. And it turns out that if we only assume $\varphi \in C^2(U)$ then (ii) actually implies some kind of « stochastic boundary continuity » of φ, in the following sense:

THEOREM 2. — *Let* $V \subset \mathcal{V}$ *be open,* $\varphi \in C^2(V)$. *Assume that*

$$\mathcal{A}[f \circ \varphi](x) = \lambda(x).\hat{\mathcal{A}}[f](\varphi(x))$$

for all $f \in C^2(\mathcal{W})$ *and all* $x \in V$, *where* $\lambda(x) \geqslant 0$ *is continuous on* V, $\lambda(x) > 0$ *except possibly on an* X*-finely nowhere dense set. Then for all* $x \in V$

(3.16) $\lim_{t \uparrow \tau} \varphi(X_t)$ *exists a.s.* P^x *on* $\{\sigma_\tau < \infty\}$,

where $\tau = \tau_V$ *and* $\sigma_t = \int_0^t \lambda(X_u)\, du;\ t \leqslant \tau$.

Proof. — Fix $x \in V$. We apply Theorem 1 to an increasing sequence of open sets U_n, $\bar{U}_n \subset V$ and $\bigcup_{n=1}^{\infty} U_n = V$.

Then if, as before, $\beta_t = \sigma_t^{-1}$ and $M_t^{(n)} = Z_{\beta_t}^{(n)}$ with probability law $\tilde{P}_n = \tilde{P}_n^x$ is the σ_{τ_n}-welding of $\varphi(X_{\beta_t})$ and Y_t (with $\tau_n = \tau_{U_n}$) we have that $M_t^{(n)}$ for each n has the same finite-dimensional distributions w.r.t. $\tilde{P}_n$ as Y_t w.r.t. $\hat{P} = \hat{P}^y$, $y = \varphi(x)$. Choose $\varepsilon > 0$. We can regard $\hat{\Omega}$ as the space of continuous $\mathbf{R}^p$-valued functions on $[0,\infty)$.

If we equip $\hat{\Omega}$ with the topology of uniform convergence on bounded intervals, then by Prohorov's theorem (see for example Stroock and Varadhan [18], Theorem 1.1.3) there exists a compact $\hat{K} \subset \hat{\Omega}$ such that

$$\hat{P}(\hat{K}) \geqslant 1 - \varepsilon.$$

Let $0 < h$, $T < \infty$ and put

$$N_h = \sup \{ |Y_s(\hat{\omega}) - Y_t(\hat{\omega})| ;\ |s-t| \leqslant h, 0 \leqslant s,t \leqslant T, \hat{\omega} \in \hat{K} \}.$$

Then by compactness of $\hat{K}$,

$$\lim_{h \downarrow 0} N_h = 0.$$

Now let

$$W_n = \{(\omega,\hat{\omega}); |M_s^{(n)} - M_t^{(n)}| \leqslant N_h \text{ for all } 0 \leqslant s,\ t \leqslant T,\ |s-t| \leqslant h,\ h > 0\}.$$

Then

$$\tilde{P}_n(W_n) \geqslant \hat{P}(K) \geqslant 1 - \varepsilon \qquad \text{for all } n.$$

In particular,

$$1 - \varepsilon \leqslant \tilde{P}_n(|M_s^{(n)} - M_t^{(n)}| \leqslant N_n \text{ for all } 0 \leqslant s,\ t \leqslant T \wedge \sigma_{\tau_n},\ |s-t| \leqslant h,\ h > 0) = P^x(S_n),$$

where

$$S_n = \{\omega; |\varphi(X_{\beta(s)}) - \varphi(X_{\beta(t)})| \leqslant N_h \quad \text{for all} \quad 0 \leqslant s, \quad t \leqslant T \wedge \sigma_{\tau_n},\ |s-t| \leqslant h, h > 0\}.$$

So if

$S = \bigcap_{n=1}^{\infty} S_n$, we have

$$P^x(S) = \lim_{n \to \infty} P^x(S_n) \geqslant 1 - \varepsilon.$$

Since ε was arbitrary, this implies that

$$\lim \varphi(X_{\beta_t}) \text{ exists a.s. when } t \uparrow T \wedge \sigma_\tau .$$

Since T was arbitrary, we conclude that

$$\lim_{t \uparrow \tau} \varphi(X_t) \quad \text{exists a.s. on} \quad \{\sigma_\tau < \infty\},$$

as asserted.

We now observe that if $\varphi \in C^2(V)$, $\tau = \tau_V$ and

$$\varphi(X_\tau) = \lim_{t \uparrow \tau} \varphi(X_t) \quad \text{exists a.s. on} \quad \{\sigma_\tau < \infty\},$$

then we can define the σ_τ-welding of $\varphi(X_{\beta_t})$ and Y_t in the same way as before (section 2).

Thus we obtain a more general version of Theorem 1, Theorem 1′, where we drop the assumption that φ can be extended continuously to ∂U and replace (i) by

(i′) For any open set $V \subset U$, $\bar{V} \subset U$, the σ_{τ_V}-welding Z_t^V *of* $\varphi(X_t)$ and Y_t has the same hitting distributions as Y_t, for any choice of right inverse ψ of φ.

4. Applications.

In this section we give some examples and applications of Theorem 1.

a) The Lévy theorem : Apply Theorem 1 to the case when X_t, Y_t are Brownian motion processes on $\mathbf{R}^d$ and $\mathbf{R}^p$, respectively, where d, $p \geqslant 1$. Since the characteristic operator of the Brownian motion is $\frac{1}{2}\Delta$, where Δ is the Laplacian, condition (ii) of Theorem 1 becomes

$$\Delta[f \circ \varphi](x) = \lambda(x).\Delta[f](\varphi(x)); \qquad x \in U \tag{4.1}$$

which is equivalent to

$$\begin{cases} \lambda(x) = |\nabla\varphi_i(x)|^2; \; 1 \leqslant i \leqslant p, \text{ where } \varphi = (\varphi_1,\ldots,\varphi_p); \\ \hfill x \in U \\ \nabla\varphi_i.\nabla\varphi_j = 0 \text{ when } i \neq j; \\ \qquad 1 \leqslant i,\ j \leqslant p \ (\textit{here.denotes the scalar product}) \\ \Delta\varphi_j = 0 \text{ for } 1 \leqslant j \leqslant p. \end{cases} \tag{4.2}$$

If $d = p = 2$ then (4.2) is equivalent to say that φ is analytic (or conjugate analytic), as assumed in the original Lévy theorem, For general d, p condition (4.2) was obtained by Bernard, Campbell and Davie [1], using stochastic integrals, as necessary and sufficient for a continuous function φ to be « Brownian path preserving » (BPP).

So in the Brownian motion case the equivalence of (ii) and (iii) in Theorem 1 can be formulated as follows :

COROLLARY 1 (The Bernard-Campbell-Davie extension of the Lévy theorem). – *Let* $U \subset \mathbf{R}^d$ *be open and* $\varphi : U \to \mathbf{R}^p$, $\varphi \in C^2(U)$. *Let* (B_t, Ω, P^x), $(\hat{B}_t, \hat{\Omega}, \hat{P}^y)$ *be Brownian motion process in* $\mathbf{R}^d$ *and* $\mathbf{R}^p$, *respectively.*

Then the following are equivalent :

(I) $\qquad \varphi = (\varphi_1, \ldots, \varphi_p)$ *satisfies* (4.2).

(II) *If we define*

$$\sigma_t = \sigma_t(\omega) = \int_0^t |\nabla \varphi_1(B_s)|^2 \, ds ,$$

then σ_t *is strictly increasing, for a.a.* ω, *and*

$$\varphi(B_\tau) = \lim_{t \uparrow \tau} \varphi(B_t) \qquad \textit{exists a.e. on} \qquad \{\omega ; \sigma(t) < \infty\}$$

where τ *is the exit time of* U *for* B_t. *And the process* $M_t(\omega,\hat{\omega})$; $t \geqslant 0$, $(\omega,\hat{\omega}) \in \Omega \times \hat{\Omega}$ *defined by*

$$M_t(\omega,\hat{\omega}) = \begin{cases} \varphi(B_{\sigma_t^{-1}}) & t < \sigma(\tau) \\ \varphi(B_\tau) + \hat{B}_{t-\sigma(\tau)}; & t \geqslant \sigma(\tau) \end{cases}$$

with probability measure $P^x \times \hat{P}^0$ *coincides with Brownian motion in* $\mathbf{R}^p$.

Proof. – (II) $\Rightarrow$ (I) follows directly from (iii) $\Rightarrow$ (ii) in Theorem 1′, since the assumption in (II) that σ_t is strictly increasing replaces the assumption in (iii) that $\lambda(x) > 0$ except possibly on an X-finely nowhere dense set.

(I) $\Rightarrow$ (II) : Note that if (I) holds then the critical points of φ constitute a set with empty fine interior, in fact a polar set (see Fuglede [11], p. 116). So (II) follows from Theorem 1′.

b) Diffusions with the same hitting distributions.

Put $\mathscr{V} = \mathscr{W}$ and define $\varphi(x) = x$ for $x \in \mathscr{V}$. Then the equivalence of (i) and (iii) in Theorem 1 gives the following:

COROLLARY 2. — *Two diffusions* X_t, Y_t *on* $\mathscr{V} \subset \mathbf{R}^d$ *have the same hitting distributions if and only if one can be transformed into the other by a change of time scale, or more precisely: There exists a continuous function* $\lambda(x) \geqslant 0$ *on* $\mathscr{V}$, $\lambda(x) > 0$ *except possibly on a set with empty* X*-fine interior, such that if we define*

$$\sigma_t = \int_0^t \lambda(X_u)\, du; \qquad t \geqslant 0$$

then $X_{\sigma_t^{-1}}$ *and* Y_t *have the same finite-dimensional distributions.*

This is a diffusion version of the more general result (valid for Hunt processes) due to Blumenthal, Getoor and McKean [3], [4].

c) Harmonic morphisms.

If X_t is a diffusion on an open set $\mathscr{V} \subset \mathbf{R}^d$ with characteristic operator $\mathscr{A}$, then the set of functions

$$\mathscr{H}_{\mathscr{V}} = \{f \in C^2(\mathscr{V});\ \mathscr{A} f = 0 \text{ in } \mathscr{V}\}$$

constitutes a $\mathfrak{P}$-harmonic space ([6]). So the functions $\varphi: U \to \mathscr{W}$ which map the paths of X_t into the paths of a diffusion Y_t on $\mathscr{W} \subset \mathbf{R}^p$ are by the equivalence of (iii) and (iv) in Theorem 1 exactly the *harmonic morphisms* from the harmonic space associated with X to the harmonic space associated with Y. This notion was introduced by Constantinescu and Cornea [5] in the general setting of harmonic spaces, and it has also been studied by Fuglede [11], [12], Ishihara [13] and Sibony [17] (for a stochastic interpretation of harmonic *maps* between Riemannian manifolds, see Darling [7] and Meyer [15]).

In view of the general correspondence between harmonic spaces and Markov processes (see [6]) it seems natural to conjecture that such a stochastic interpretation of harmonic morphisms can be extended to more general Markov processes.

As an application we note the following immediate consequence of Theorem 1:

COROLLARY 3. — *Let* $\varphi \in C^2(U)$ *be a stochastic harmonic morphism* (*i.e.* φ *satisfies* (iv) *of Theorem* 1).

(I) *Then* φ *is* $X - Y$ *finely continuous.*

(II) *Assume, in addition, that either*

(A) φ *is not* X-*finely locally constant or*

(B) *the points of* $\varphi(U)$ *are polar for* Y.

Then φ *is* $X - Y$ *finely open.*

Remark. — The conclusion in (II), under the assumption (B), was proved by Constantinescu and Cornea [5] (Theorem 3.5), in the (non-probabilistic) setting of $\mathfrak{P}$-harmonic spaces.

Proof of Corollary 3.

(I) Let $W \subset \mathscr{W}$ be a Borel set, let $x \in U \cap \varphi^{-1}(W)$ and $y = \varphi(x)$. Then if x is not in the X-fine interior of $\varphi^{-1}(W)$, X_t leaves $\varphi^{-1}(W)$ immediately, a.s.

Therefore $\varphi(X_t)$ leaves W immediately, a.s.

But then the hitting distribution on $\mathscr{W}\backslash W$ for Z_t is the unit point mass at y, δ_y. Since (iv) $\Rightarrow$ (i) in Theorem 1 without the assumptions (A) or (B), the hitting distribution for Y_t on $\mathscr{W}\backslash W$ is δ_y as well. So if we let

$$T = \inf\{t>0; Y_t \notin W\},$$

then $T < \infty$ and $Y_T = y$ a.s. $\hat{P}^y$.

So y is *regular* for $\mathscr{W}\backslash W$ w.r.t. Y_t by Theorem 11.4 in Blumenthal and Getoor [2], i.e. $\hat{P}^y[T=0]=1$.

Hence W is not Y-finely open.

(II) Choose V finely open in U. Then for all $x \in V$, X_t stays in V for a positive period of time a.s. P^x. So Z_t stays in $\varphi(V)$ for a positive period of time a.s. $\tilde{P}^y$, when $y = \varphi(x)$. By (iii) of Theorem 1 the same must hold for Y_t w.r.t. $\hat{P}^y$, so $\varphi(V)$ is Y-finely open.

d) A converse of the Lévy theorem.

Finally we give an example to illustrate how Theorem 1 can be used in the investigation of problems where the function (or class Φ of functions) φ is given and one asks for all diffusions X_t, Y_t such that φ maps the paths of X_t into the paths of Y_t. We think that this can be a fruitful point of view in the investigation of properties of this class of functions.

In our example we choose as our function class Φ the family of all analytic functions φ on a fixed open set $U \subset \mathbf{C}$, the complex plane. From the Lévy theorem we know that if $X_t = Y_t = B_t$, the Brownian motion, then every $\varphi \in \Phi$ maps the paths of X_t into those of Y_t. The next result says that this is essentially the only pair of diffusions X_t, Y_t with this property:

COROLLARY 4 (Converse of the Lévy theorem). – *Let X_t, Y_t be diffusion processes on U and $\mathbf{C}$, respectively, where $U \subset \mathbf{C}$ is open. Suppose that for all non-constant analytic $\varphi : U \to \mathbf{C}$ the τ-welding of $\varphi(X_t)$ and Y_t has the same hitting distributions as Y_t, where $\tau = \tau_U$ is the first exit time of U for X_t. Then X_t and Y_t is the Brownian motion on U and $\mathbf{C}$ respectively, modulo a change of time scale.*

Remark. – In the case when we assume $X_t = Y_t$, this result is a consequence of a result obtained in [16], valid for all path-continuous Markov processes X_t.

Proof of Corollary 4. – Let

$$\mathcal{A} = a_{11}\frac{\partial^2}{\partial x^2} + a_{12}\frac{\partial^2}{\partial x\,\partial y} + a_{22}\frac{\partial^2}{\partial y^2} + b_1\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y}$$

and

$$\hat{\mathcal{A}} = A_{11}\frac{\partial^2}{\partial x^2} + A_{12}\frac{\partial^2}{\partial x\,\partial y} + A_{22}\frac{\partial^2}{\partial y^2} + B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y}$$

be the characteristic operators of X_t, Y_t respectively. Then if $\varphi(x,y) = u(x,y) + iv(x,y) : U \to \mathbf{C}$ is analytic we obtain from equation (ii) in Theorem 1 and the Cauchy-Riemann equations that

1) $a_{11}\cdot u_x^2 + a_{12}u_xu_y + a_{22}\cdot u_y^2 = \lambda_\varphi(x,y)A_{11}(u,v)$

2) $-\,2a_{11}u_x\cdot u_y + a_{12}[u_x^2 - u_y^2] + 2a_{22}\cdot u_xu_y = \lambda_\varphi(x,y)A_{12}(u,v)$

3) $a_{11}\cdot u_y^2 - a_{12}u_xu_y + a_{22}\cdot u_x^2 = \lambda_\varphi(x,y)A_{22}(u,v)$

4) $(a_{11}-a_{22})u_{xx} + a_{12}u_{xy} + b_1u_x + b_2u_y = \lambda_\varphi(x,y)B_1(u,v)$

5) $(a_{22}-a_{11})u_{xy} - a_{12}u_{yy} - b_1u_y + b_2u_x = \lambda_\varphi(x,y)B_2(u,v)$.

Applying this to $u(x,y) = c + x$, $v(x,y) = d + y$, we obtain

1)' $$a_{11}(x,y) = \lambda_{c,d}(x,y)A_{11}(c+x,d+y)$$

2)' $$a_{12}(x,y) = \lambda_{c,d}(x,y)A_{12}(c+x,d+y)$$

3)' $$a_{22}(x,y) = \lambda_{c,d}(x,y)A_{22}(c+x,d+y)$$

4)' $$b_1(x,y) = \lambda_{c,d}(x,y)B_1(c+x,d+y)$$

5)' $$b_2(x,y) = \lambda_{c,d}(x,y)B_2(c+x,d+y).$$

So $A_{11}(c,d)$, $A_{12}(c,d)$, $A_{22}(c,d)$, $B_1(c,d)$ and $B_2(c,d)$ are all proportional. Therefore, by performing a time change on Y_t, we may assume they are constants. Performing a time change on X_t, we obtain that a_{ij}, b_i are constants also, $1 \leqslant i, j \leqslant 2$. From 1) we obtain that $\lambda_{C\varphi} = C^2\lambda_\varphi$ when C is constant, but if this is applied to 4) and 5) with $C = -1$, we obtain $B_1 = B_2 = 0$. So by 4)' and 5)' we also have $b_1 = b_2 = 0$. Therefore 4) and 5) are reduced to

4)'' $$(a_{11} - a_{22})u_{xx} + a_{12}u_{xy} = 0$$

5)'' $$(a_{22}-a_{11})u_{xy} - a_{12}u_{yy} = 0.$$

With $u(x,y) = xy$ 4)'' gives $a_{12} = 0$ and 5)'' gives $a_{11} = a_{22}$. So $A_{12} = 0$ also and $A_{11} = A_{22}$. That completes the proof of Corollary 4.

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