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VECTOR BUNDLES ON MANIFOLDS WITHOUT DIVISORS AND A THEOREM ON DEFORMATIONS

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Introduction.

The motivation for this paper was to gather some information on holomorphic vector bundles on some non-algebraic compact complex manifolds, especially manifolds without divisors. As a first step, we treat the case of 2-bundles. Examples of such 2-bundles are given by extensions

$$0 \to L \to E \to M \otimes \mathcal{I}_Z \to 0 \qquad (*)$$

where L and M are line bundles and $\mathcal{I}_Z$ is the ideal sheaf of a 2-codimensional locally complete intersection. On a projective algebraic manifold every 2-bundle is of this form, however L, M and Z are not uniquely determined by E. In sharp contrast to this, on a manifold without divisors, the « devissage » $(*)$ is uniquely determined for an indecomposable bundle E (cf. Theorem 2.2). On the other hand, on such highly non-algebraic manifolds there might exist 2-bundles without any such devissage; we call them non-filtrable. More precisely, E admits a devissage if and only if there exists a line bundle L such that $E \otimes L^*$ has non-trivial sections.

In order to prove the existence of non-filtrable bundles on 2-tori with Picard number zero, we prove (in § 3) some general theorems on the deformation of vector bundles and projective bundles which might be of independant interest. Roughly speaking, any deformation of a vector bundle on a compact complex space is composed of a deformation of det (E) and a deformation of the associated projective bundle **P**(E); for a precise formulation see Theorem 3.4. As a corollary we get: If

$\dim H^2(X, \text{End } E) = \dim H^2(X,\mathcal{O}_X)$, then the basis of the versal deformation of E is smooth. We use this last fact to deform a certain filtrable bundle on a 2-torus into a non-filtrable one (Proposition 4.9).

In an Appendix we gather some facts on algebraic dimension and Picard number of 2-tori.

Notations. – By a vector bundle on a complex space X we always mean a holomorphic vector bundle which we consider as a locally free $\mathcal{O}_X$-module of constant (finite) rank. The dimensions of cohomology groups are denoted by $h^q(X,\mathcal{F}) := \dim H^q(X,\mathcal{F})$.

1. Filtration of bundles.

In this section we collect some more or less well known facts about vector bundles which are extensions of the form

$$0 \to L \to E \to M \otimes \mathcal{J}_Z \to 0,$$

where L and M are line bundles and Z is a 2-codimensional locally complete intersection.

1.1. If E is a vector bundle of rank r on a smooth curve, then there exists a (not uniquely determined) filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = E,$$

where E_k is a subbundle of rank k (cf. Atiyah [1]). On a complex manifold of dimension > 1 this is no longer true. Instead of subbundles one has to consider coherent subsheaves $\mathcal{F} \subset E$. Such subsheaves are always torsion-free. The following facts are well known:

a) Let $\mathcal{F}$ be a coherent sheaf on a complex manifold X. Then the set

$$\text{Sing}(\mathcal{F}) = \{x \in X : \mathcal{F}_x \text{ is not a free } \mathcal{O}_{X,x}\text{-module}\}$$

is analytic of codimension $\geqslant 1$.

If $\mathcal{F}$ is torsion-free, $\text{Sing}(\mathcal{F})$ is of codim $\geqslant 2$. If $\mathcal{F}$ is reflexive, i.e. $\mathcal{F} = \mathcal{F}^{**}$, then codim $\text{Sing}(\mathcal{F}) \geqslant 3$. If $\mathcal{F}$ is reflexive and has rank 1, it is locally free, i.e. a line bundle.

b) Let E be a vector bundle on a complex manifold X and $\mathscr{F} \subset$ E a coherent subsheaf. Then the set Sing (E/$\mathscr{F}$) is equal to the set

$$S = \{x \in X : \mathscr{F}_x \text{ is not a direct summand of } E_x\}$$

and $\mathscr{F} | X \backslash S$ is a subbundle of $E | X \backslash S$.

c) For every $\mathscr{F} \subset E$ we denote by $\hat{\mathscr{F}}$ the following coherent subsheaf of E : Let $p : E \to E/\mathscr{F}$ be the canonical projection and Tors (E/$\mathscr{F}$) the torsion submodule of E/$\mathscr{F}$. Define

$$\hat{\mathscr{F}} : = p^{-1} (\text{Tors} (E/\mathscr{F})).$$

Then $\mathscr{F} \subset \hat{\mathscr{F}} \subset E$ and $\hat{\mathscr{F}}$ coincides with $\mathscr{F}$ outside an analytic set of codimension $\geqslant 1$. The quotient $E/\hat{\mathscr{F}}$ is torsion-free, hence $\hat{\mathscr{F}}$ is a subbundle of E outside an analytic set of codimension $\geqslant 2$.

1.2. Definition. — *A vector bundle* E *of rank* r *on a complex manifold* X *is called filtrable if there exists a filtration*

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}_r = E$$

where $\mathscr{F}_k$ *is a coherent subsheaf of rank* k.

Of course every vector bundle on a compact algebraic manifold is filtrable, but we will prove the existence of bundles on certain non-algebraic manifolds which are not filtrable.

Remark. — According to 1.1.*c*) we may assume all quotients $E/\mathscr{F}_k$ to be torsion-free. In that case the $\mathscr{F}_k/\mathscr{F}_{k-1}$ are torsion-free of rank 1 and $L_k : = (\mathscr{F}_k/\mathscr{F}_{k-1})^{**}$ are line bundles. Moreover,

$$\det E \cong L_1 \otimes \ldots \otimes L_r.$$

This last formula comes from the fact that $0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}_r = E$ is a filtration of subbundles outside a set of codimension $\geqslant 2$.

1.3. Lemma. — *Let* $\mathscr{F} \subset E$ *be a coherent subsheaf of rank* 1 *of a vector bundle on a complex manifold. If* $E/\mathscr{F}$ *is torsion-free, then* $\mathscr{F}$ *is locally free.*

Proof. — It suffices to show that $\mathscr{F}$ is reflexive. Let $\mathscr{F}^{**} \to E$ denote the bidual of the inclusion morphism $\mathscr{F} \to E$ and consider the sheaf $\hat{\mathscr{F}} : = \text{Im} (\mathscr{F}^{**} \to E)$. Then $\hat{\mathscr{F}}/\mathscr{F} \subset E/\mathscr{F}$ is a torsion sheaf, hence $\hat{\mathscr{F}}/\mathscr{F} = 0$. Since $\mathscr{F}^{**} \to E$ is a monomorphism, $\mathscr{F}^{**} \cong \hat{\mathscr{F}} \cong \mathscr{F}$, q.e.d.

1.4. COROLLARY. — *A vector bundle of rank* 2 *on a connected complex manifold* X *is filtrable if and only if there exists a line bundle* L *on* X *such that* $\Gamma(X,L^* \otimes E) \neq 0$.

1.5. COROLLARY. — *On a complex manifold* X *let* E *be a vector bundle,* L *a line bundle and* $\alpha : L \to E$ *a sheaf monomorphism. Then*

$$\mathrm{Supp}\,(\mathrm{Tors}\,(E/\mathrm{Im}\,(L \xrightarrow{\alpha} E)))$$

is (*empty or*) *of pure codimension* 1.

Proof. — Set $\mathscr{F} := \mathrm{Im}\,(L \to E)$ and define $\hat{\mathscr{F}} \subset E$ as in 1.1.*c*). $\mathscr{F}$ is isomorphic to L and $\hat{\mathscr{F}}$ is locally free by Lemma 1.3. The inclusion map $\mathscr{F} \hookrightarrow \hat{\mathscr{F}}$ may be considered as a section of the line bundle $\mathscr{F}^* \otimes \hat{\mathscr{F}}$, hence $\mathrm{Supp}\,(\hat{\mathscr{F}}/\mathscr{F})$ has pure codimension 1. But

$$\mathrm{Tors}\,(E/\mathrm{Im}\,(L \to E)) \cong \hat{\mathscr{F}}/\mathscr{F}\,.$$

1.6. PROPOSITION. — *For every filtrable* 2*-bundle* E *on a complex manifold* X *there exist line bundles* L, M *on* X *and a* 2*-codimensional* (*possibly empty*) *analytic subspace* $Z \subset X$ *such that* E *fits into an exact sequence*

$$0 \to L \xrightarrow{\alpha} E \xrightarrow{\beta} M \otimes \mathscr{I}_Z \to 0\,.$$

Proof. — Let $0 \subset \mathscr{F} \subset E$ be a filtration such that $E/\mathscr{F}$ is torsion free. By Lemma 1.3 the sheaf $L := \mathscr{F}$ is locally free of rank 1. Let $\alpha : L \to E$ be the inclusion map. Set $M := (E/\mathscr{F})^{**}$. The image of the natural inclusion map

$$E/\mathscr{F} \to (E/\mathscr{F})^{**} = M$$

is of the form $M \otimes \mathscr{I}_Z$, where $\mathscr{I}_Z$ is the ideal sheaf of a subspace $Z \subset X$ of codimension $\geqslant 2$. But Z may also be defined by the vanishing of $\alpha \in \Gamma(X,L^* \otimes E)$, hence is a locally complete intersection of codimension $= 2$ (or empty). The morphism β is the quotient map $E \to E/\mathscr{F}$ composed with the isomorphism $E/\mathscr{F} \xrightarrow{\sim} M \otimes \mathscr{I}_Z$.

1.7. *Notation.* — We call an exact sequence

$$(*) \qquad 0 \to L \to E \to M \otimes \mathscr{I}_Z \to 0$$

as in Proposition 1.6 a *devissage* of E. We have

$$\det\,(E) \cong L \otimes M\,,$$

in particular $c_1(\mathrm{E}) = c_1(\mathrm{L}) + c_1(\mathrm{M})$. The bundle $\mathrm{L}^* \otimes \mathrm{E}$ has a section vanishing on the subspace Z. Hence

$$c_2(\mathrm{L}^* \otimes \mathrm{E}) = \text{dual class of } [\mathrm{Z}].$$

Since $\mathrm{E}^* \cong \mathrm{E} \otimes \det \mathrm{E}^*$, we can tensor $(*)$ by $\mathrm{L}^* \otimes \mathrm{M}^*$ to get the dual devissage

$$0 \to \mathrm{M}^* \to \mathrm{E}^* \to \mathrm{L}^* \otimes \mathscr{I}_\mathrm{Z} \to 0.$$

1.8. Recall that a vector bundle E on a compact complex connected manifold is *simple* if $\mathrm{End}\,(\mathrm{E}) = \mathbf{C}$. This is equivalent to the fact that every non-zero endomorphism is invertible. If rank $\mathrm{E} = 2$ and E is not simple, then E is filtrable. In fact, if $\sigma : \mathrm{E} \to \mathrm{E}$ is a non-zero, non-invertible endomorphism, then $\mathrm{Ker}\,\sigma \subset \mathrm{E}$ is a subsheaf of rank 1.

1.9. LEMMA. — *Let* E *be an indecomposable* 2*-bundle on a compact connected complex manifold* X *and* $\sigma \in \mathrm{End}\,\mathrm{E}$ *a non-invertible endomorphism. Then* $\sigma^2 = 0$.

Proof. — Consider the eigenvalues λ_1, λ_2 of σ. (Since X is compact connected, the eigenvalues of σ in all fibres of E are the same.) Necessarily $\lambda_1 = \lambda_2$, otherwise the eigenspaces would define a decomposition of E. Since $\det(\sigma) = 0$, we have $\lambda_1 = \lambda_2 = 0$, which implies $\sigma^2 = 0$.

1.10. In general, the devissage of a 2-bundle is not uniquely determined. However we shall discuss conditions which guarantee uniqueness.

Let X be a compact connected complex manifold and L, L' line bundles on X. Following Atiyah [1] we shall write $\mathrm{L}' \leqslant \mathrm{L}$ if there exists a non-zero morphism $\mathrm{L}' \to \mathrm{L}$.

We call a devissage $\mathrm{L} \rightarrowtail \mathrm{E} \twoheadrightarrow \mathrm{M} \otimes \mathscr{I}_\mathrm{Z}$ of a 2-bundle E *maximal*, if for every other devissage $\mathrm{L}' \rightarrowtail \mathrm{E} \twoheadrightarrow \mathrm{M}' \otimes \mathscr{I}_\mathrm{Z}$, we have $\mathrm{L}' \leqslant \mathrm{L}$.

1.11. PROPOSITION. — *Let* E *be a non-simple, indecomposable* 2*-bundle on a compact connected complex manifold* X. *Then* E *admits a uniquely determined maximal devissage*

$$0 \to \mathrm{L} \to \mathrm{E} \to \mathrm{M} \otimes \mathscr{I}_\mathrm{Z} \to 0.$$

This maximal devissage is characterized by the fact that $\mathrm{M} \leqslant \mathrm{L}$.

Proof. – Let $\sigma : E \to E$ be a non-zero, non-invertible endomorphism. Let $L := \operatorname{Ker} \sigma$. Since $E/\operatorname{Ker} \sigma \cong \operatorname{Im} \sigma$ is torsion-free, L is a line bundle by Lemma 1.3. We may write $\operatorname{Im} \sigma \cong M \otimes \mathscr{I}_Z$, where M is a line bundle and $Z \subset X$ a subspace of codimension 2. By Lemma 1.9 we have $\operatorname{Im} \sigma \subset \operatorname{Ker} \sigma$, hence there exists a monomorphism $M \otimes \mathscr{I}_Z \to L$, which extends to a monomorphism $M \to L$. So we get a devissage

$$0 \to L \xrightarrow{\alpha} E \xrightarrow{\beta} M \otimes \mathscr{I}_Z \to 0$$

with $M \leqslant L$.

We will now show that a devissage with $M \leqslant L$ is the uniquely determined maximal devissage.

i) Maximality. Let $f : L' \to E$ be any non-zero morphism. If $\beta \circ f : L' \to M \otimes \mathscr{I}_Z$ is non-zero, then $L' \leqslant M \leqslant L$. If however $\beta \circ f = 0$, we have $L' \cong \operatorname{Im} f \subset \operatorname{Im} \alpha \cong L$, i.e. $L' \leqslant L$ in every case.

ii) Uniqueness. Let

$$0 \to L' \xrightarrow{f} E \to M' \otimes \mathscr{I}_{Z'} \to 0$$

be a second maximal devissage. Then $L' \leqslant L \leqslant L'$, hence $L' \cong L$. If $\beta \circ f : L' \to M \otimes \mathscr{I}_Z$ is non-zero, the composite map

$$L' \xrightarrow{\beta \circ f} M \otimes \mathscr{I}_Z \to M \to L$$

is non-zero, hence an isomorphism. This implies in particular $Z = \varnothing$ and $\beta \circ f : L' \to M$ is an isomorphism. But then $E = L \oplus L'$, which was excluded. So necessarily $\beta \circ f = 0$ and we get a factorization

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & E \\ & & & {\scriptstyle g}\nwarrow & \uparrow{\scriptstyle f} \\ & & & & L' \end{array}$$

Since $L' \cong L$, g is an isomorphism. This implies that the two devissages are isomorphic, q.e.d.

2. Vector bundles of rank 2 on manifolds without divisors.

2.1. Let L and L′ be two line bundles on a complex connected manifold and f : L′ → L a non-zero morphism. Let D be the zero divisor of f. Then $L \cong L' \otimes [D]$, where [D] denotes the line bundle associated to D. Therefore, if X is a complex connected manifold without divisors, the relation $L' \leqslant L$ implies $L' \cong L$.

Recall that a compact connected complex manifold without divisors has algebraic dimension zero, i.e. the only meromorphic functions are constant. The converse is not true (think of blow-ups !), however a torus has algebraic dimension zero if and only if it admits no divisors.

We will now give a rough classification of 2-bundles on manifolds without divisors.

2.2. THEOREM. – *Let* X *be a compact connected complex manifold without divisors. Then we have the following classification of vector bundles of rank* 2 *on* X :

I. *Filtrable bundles.*

1) *Indecomposable bundles.*

A filtrable 2*-bundle is indecomposable if and only if its devissage is uniquely determined.*

i) *Simple bundles. They have a devissage*

$$0 \to L \to E \to M \otimes \mathscr{I}_Z \to 0$$

with $L \ncong M$ *and endomorphism ring* $\operatorname{End} E \cong \mathbf{C}$.

ii) *Non-simple bundles. Their devissage is*

$$0 \to L \to E \to L \otimes \mathscr{I}_Z \to 0$$

and $\operatorname{End} E \cong \mathbf{C}[\varepsilon]$, $\varepsilon^2 = 0$, *is the ring of dual numbers.*

2) *Decomposable bundles*

i) *Bundles of the form* $E \cong L \oplus M$ *with* $L \ncong M$.

In this case End E $\cong$ **C** $\oplus$ **C** *with componentwise multiplication.*

ii) *Bundles of the form* $E \cong L \oplus L$.

In this case End $E \cong M_2(\mathbf{C})$ *is the full matrix ring.*

II. *Non-filtrable bundles.*

These bundles are all simple, i.e. End E = **C**.

Proof. – *a*) Let E be an indecomposable 2-bundle on X with a devissage

$$0 \to L \xrightarrow{\alpha} E \xrightarrow{\beta} M \otimes \mathcal{I}_Z \to 0.$$

We will show that the devissage is uniquely determined and that the assertions in 1i), ii) hold.

i) Suppose $M \ncong L$. Let $f : L' \to E$ be any monomorphism of a line bundle L' in E. We claim that

$$\beta \circ f : L' \to M \otimes \mathcal{I}_Z$$

is zero. Otherwise Z would be empty (since X has no divisors) and $\beta \circ f : L' \to M$ an isomorphism. But this would imply $E \cong L \oplus M$, contradicting the indecomposability of E. Therefore f factorizes as follows

$$\begin{array}{ccc} L & \xrightarrow{\alpha} & E \\ & \nwarrow^{g} & \uparrow{\scriptstyle f} \\ & & L' \end{array}$$

and g is necessarily an isomorphism. This implies the uniqueness of the devissage.

Tensoring the dual devissage $M^* \rightarrowtail E^* \twoheadrightarrow L^* \otimes \mathcal{I}_Z$ by E, we get an exact sequence

$$0 \to M^* \otimes E \to E^* \otimes E \to L^* \otimes E \otimes \mathcal{I}_Z \to 0,$$

which implies

$$\dim \operatorname{End} E \leqslant \dim \Gamma(X, M^* \otimes E) + \dim \Gamma(X, L^* \otimes E \otimes \mathcal{I}_Z).$$

The uniqueness of the devissage of E implies

$$\Gamma(X,M^*\otimes E) = 0 \quad \text{and} \quad \Gamma(X,L^*\otimes E\otimes \mathcal{I}_Z) \subset \Gamma(X,L^*\otimes E) \cong \mathbf{C},$$

hence dim End E $= 1$, i.e. E is simple.

ii) If $M \cong L$, denote by ε the composed morphism

$$E \to M \otimes \mathcal{I}_Z \hookrightarrow M \xrightarrow{\sim} L \to E.$$

Obviously $\varepsilon \neq 0$ and $\varepsilon^2 = 0$. In particular E is non-simple and the uniqueness of the devissage follows from Proposition 1.10. Since $\mathbf{C}[\varepsilon] \subset$ End E, it remains to be shown that dim End E $\leqslant 2$. To see this, we use the same inequality as above

$$\dim \operatorname{End} E \leqslant \dim \Gamma(X,M^*\otimes E) + \dim \Gamma(X,L^*\otimes E\otimes \mathcal{I}_Z).$$

Since $M \cong L$, the uniqueness of the devissage implies $\dim \Gamma(X,M^*\otimes E) = 1$ and $\dim \Gamma(X,L^*\otimes E\otimes \mathcal{I}_Z) \leqslant 1$, hence dim End E $\leqslant 2$. Therefore End E $\cong \mathbf{C}[\varepsilon]$.

b) It is clear that the devissage of a decomposable bundle $E \cong L \oplus M$ is not uniquely determined. Furthermore

$$\operatorname{End}(E) \cong \operatorname{End}(L) \oplus \operatorname{End}(M) \oplus \operatorname{Hom}(L,M) \oplus \operatorname{Hom}(M,L),$$

which gives the endomorphism rings as asserted in 2i), ii).

c) That non-filtrable 2-bundles are simple follows from 1.7. This completes the proof of Theorem 2.2. We now look at a relative situation.

2.3. THEOREM. – *Let* X *be a compact complex manifold without divisors,* S *a Stein manifold with* $H^2(S,\mathbf{Z}) = 0$ *and* E *a vector bundle of rank* 2 *on* $X \times S$. *For* $s \in S$ *denote by* i_s *the inclusion map*

$$i_s : X \xrightarrow{\sim} X \times \{s\} \hookrightarrow X \times S$$

and $E_s := i_s^* E$. *Suppose that* E_s *is filtrable and indecomposable for all* $s \in S$ *(i.e. belongs to class* I.1 *in the classification of Theorem* 2.2). *Then there exist line bundles* $L \to X \times S$, $M \to X \times S$ *and a subspace* $Z \subset X \times S$ *of codimension* 2 *which is flat over* S, *such that* E *fits into* *an exact sequence*

$$0 \to \mathbf{L} \to \mathbf{E} \to \mathbf{M} \otimes \mathscr{I}_Z \to 0$$

whose restriction to every fibre $X \times \{s\}$ *is the uniquely determined devissage of* E_s.

Proof. – Let $\mathbf{L} \to X \times \mathrm{Pic}(X)$ be the universal line bundle. Consider the bundle

$$\mathbf{L}^* \boxtimes \mathbf{E} \to X \times (\mathrm{Pic}(X) \times S).$$

Let $p: X \times (\mathrm{Pic}(X) \times S) \to \mathrm{Pic}(X) \times S$ be the projection. By the semicontinuity theorem the set

$$S' := \{(\xi,s) \in \mathrm{Pic}(X) \times S : H^0(p^{-1}(\xi,s), L_\xi^* \otimes E_s) \neq 0\}$$

is analytic. Since the devissage of every bundle E_s is uniquely determined, the projection $q: S' \to S$ is bijective, hence biholomorphic if we provide S' with the structure of a reduced subspace of $\mathrm{Pic}(X) \times S$. Let $\varphi: S \to S' \subset \mathrm{Pic}(X) \times S$ be the inverse map of q and define the line bundle $\mathrm{L} \to X \times S$ by

$$\mathrm{L} := (\mathrm{id}_X \times \varphi)^* \mathbf{L}.$$

For every $s \in S$, the vector space $\mathrm{Hom}(L_s, E_s)$ is one-dimensional, hence the direct image sheaf

$$\pi_* \mathcal{H}om(\mathrm{L},\mathrm{E}),$$

where $\pi: X \times S \to S$ is the projection, is locally free of rank 1 on S. The hypothesis $H^2(S,\mathbb{Z}) = 0$ implies $\pi_* \mathcal{H}om(\mathrm{L},\mathrm{E}) \cong \mathcal{O}_S$. Let $\alpha: \mathrm{L} \to \mathrm{E}$ be the morphism corresponding to a global non-vanishing section of $\pi_* \mathcal{H}om(\mathrm{L},\mathrm{E})$. The restriction $\alpha_s: L_s \to E_s$ of α to any fibre $\pi^{-1}(s)$ is up to a constant factor the unique monomorphism of a line bundle into E_s. The image $\alpha(\mathrm{L})$ is a direct summand of E outside a set of codimension 2. Corollary 1.5 implies that $\mathrm{E}/\alpha(\mathrm{L})$ is torsion free. Define the line bundle $\mathrm{M} \to X \times S$ by

$$\mathrm{M} := (\mathrm{E}/\alpha(\mathrm{L}))^{**}.$$

Then $\mathrm{E}/\alpha(\mathrm{L}) \cong \mathrm{M} \otimes \mathscr{I}_Z$ for a certain 2-codimensional subspace $Z \subset X \times S$. Since Z is locally a complete intersection whose intersection with every fibre $\pi^{-1}(s)$ is 2-codimensional, Z is flat over S. The morphism $\alpha : L \to E$ together with the quotient map $E \to E/\alpha(L) \cong M \otimes \mathscr{I}_Z$ gives the desired exact sequence

$$0 \to L \to E \to M \otimes \mathscr{I}_Z \to 0.$$

2.4. Theorem 2.3 implies the following: Let $E \to X \times S$ be a vector bundle as in Theorem 2.3 and

$$0 \to L_s \to E_s \to M_s \otimes \mathscr{I}_{Z_s} \to 0$$

the unique devissage of E_s. Then

$$s \mapsto [L_s] \quad \text{and} \quad s \mapsto [M_s]$$

define holomorphic maps $S \to \mathrm{Pic}(X)$. Moreover there is a holomorphic map

$$S \to D(X), \quad s \mapsto Z_s,$$

where $D(X)$ denotes the Douady space of all compact analytic subspaces of X, cf. [4].

3. Deformations of vector bundles and projective bundles.

3.1. Holomorphic fibre bundles with fibre $\mathbf{P}_{r-1}$ and structure group $PGL(r,\mathbf{C})$ on a complex space X (we will call them briefly projective $(r-1)$-bundles or $\mathbf{P}_{r-1}$-bundles) are classified by $H^1(X,PGL(r,\mathcal{O}))$. Every vector bundle E of rank r on X gives rise to a projective $(r-1)$-bundle $\mathbf{P}(E)$. The relevant exact sequence is

$$0 \to \mathcal{O}^* \to GL(r,\mathcal{O}) \to PGL(r,\mathcal{O}) \to 0,$$

to which is associated the exact cohomology sequence

$$H^1(X,GL(r,\mathcal{O})) \to H^1(X,PGL(r,\mathcal{O})) \to H^2(X,\mathcal{O}^*).$$

Thus if $H^2(X,\mathcal{O}^*) = 0$ (in particular if X is a curve or $\mathbf{P}_n$) every projective bundle is of the form $\mathbf{P}(E)$ where E is a vector bundle (cf. Atiyah [1]).

For general X this is no longer true. However we will show that if P_0 is a projective bundle associated to a vector bundle, then any small deformation of P_0 also comes from a vector bundle.

3.2. THEOREM. – *Let* E_0 *be a vector bundle of rank* r *on a compact complex space* X. *Let* $P \to X \times S$ *be a deformation of* $\mathbf{P}(E_0)$ *over the germ* (S,0). *Then there exists a deformation* $E \to X \times S$ *of the vector bundle* E_0 *such that* $P \cong \mathbf{P}(E)$. *Moreover one can choose* E *such that* det E *is a trivial deformation of* det E_0. *With this supplementary condition* E *is uniquely determined.*

Proof. – The deformation P is given by a cocycle

$$\xi \in H^1(X \times S, PGL(r, \mathcal{O}))$$

which can be represented by a cochain

$$(g_{ij}) \in C^1(\mathcal{U} \times S, GL(r, \mathcal{O})),$$

where $\mathcal{U} = (U_i)_{i \in I}$ is a suitable open covering of X. We may assume all intersections $U_i \cap U_j$ to be simply connected. We may further assume that

$$(g_{ij}(0)) \in C^1(\mathcal{U}, GL(r, \mathcal{O}))$$

is a cocycle defining the vector bundle E_0. Therefore there exists a cochain

$$(c_{ijk}) \in C^2(\mathcal{U} \times S, \mathcal{O}^*)$$

with

$$c_{ijk}(0) = 1$$

and

$$g_{ij}g_{jk} = c_{ijk}g_{ik} \quad \text{on} \quad (U_i \cap U_j \cap U_k) \times S.$$

Since the $U_i \cap U_j$ are simply connected, there exist functions

$$\gamma_{ij} \in \mathcal{O}^*((U_i \cap U_j) \times S)$$

with

$$\det g_{ij} = \gamma_{ij}^r.$$

We define

$$\tilde{g}_{ij} := g_{ij}\frac{\gamma_{ij}(0)}{\gamma_{ij}}, \in \mathrm{GL}(r,\mathcal{O}((\mathrm{U}_i \cap \mathrm{U}_j)\times \mathrm{S})).$$

We have then

$$\tilde{g}_{ij}(0) = g_{ij}(0)$$

and

$$\det \tilde{g}_{ij}(s) = \gamma_{ij}(0)^r \quad \text{for all} \quad s \in \mathrm{S}.$$

We will show that $(\tilde{g}_{ij})$ is a cocycle, i.e.

$$(*) \qquad \tilde{g}_{ij}\tilde{g}_{jk} = \tilde{g}_{ik}.$$

Indeed, we have $\tilde{g}_{ij}\tilde{g}_{jk} = \tilde{c}_{ijk}\tilde{g}_{ik}$ with a cochain

$$\tilde{c}_{ijk} \in \mathrm{C}^2(\mathscr{U}\times \mathrm{S},\mathcal{O}^*), \qquad \tilde{c}_{ijk}(0) = 1.$$

Then taking determinants we get

$$\gamma_{ij}(0)^r\gamma_{jk}(0)^r = (\tilde{c}_{ijk})^r\gamma_{ik}(0)^r.$$

On the other hand $\gamma_{ij}(0)^r\gamma_{jk}(0)^r = \gamma_{ik}(0)^r$, hence

$$(\tilde{c}_{ijk})^r = 1.$$

Since $\tilde{c}_{ijk}(0) = 1$, this implies $\tilde{c}_{ijk} = 1$ as an element of $\mathcal{O}^*((\mathrm{U}_i \cap \mathrm{U}_j \cap \mathrm{U}_k)\times \mathrm{S})$. Thus we have proved the cocycle relation $(*)$. The cocycle

$$(\tilde{g}_{ij}) \in \mathrm{Z}^1(\mathscr{U}\times \mathrm{S},\mathrm{GL}(r,\mathcal{O}))$$

defines the desired deformation E of E_0 for which $\mathbf{P}(\mathrm{E}) \cong \mathrm{P}$ and $\det \mathrm{E}$ is the trivial deformation of $\det \mathrm{E}_0$.

Uniqueness. Let $\mathrm{E}' \to \mathrm{X}\times \mathrm{S}$ be another deformation of E_0 with $\mathbf{P}(\mathrm{E}') \cong \mathrm{P}$. Then $\mathrm{E}' \cong \mathrm{E}\otimes \mathrm{L}$, where $\mathrm{L} \to \mathrm{X}\times \mathrm{S}$ is a deformation of the trivial line bundle. If both $\det \mathrm{E}$ and $\det \mathrm{E}' = (\det \mathrm{E})\otimes \mathrm{L}^r$ are trivial deformations of $\det \mathrm{E}_0$, it follows that L^r is trivial. Since L_0 is trivial, L must be trivial itself.

3.3. Given a vector bundle E of rank r on a complex space X, we have a canonical injection

$$\mathcal{O}_X \to \mathit{End}\, E, \; f \mapsto f.\mathrm{id}_E .$$

This injections splits by the map

$$\varphi \mapsto \frac{1}{r}\,\mathrm{trace}\,(\varphi)$$

and we get a direct sum decomposition

$$\mathit{End}\, E \cong \mathcal{O}_X \oplus \mathit{End}_0\, E,$$

where $\mathit{End}_0\, E$ is the sheaf of endomorphisms of trace zero. In particular, we have for any $q \in \mathbf{N}$

$$H^q(X, \mathrm{End}\, E) \cong H^q(X, \mathcal{O}_X) \otimes H^q(X, \mathrm{End}_0\, E).$$

Consider the projective bundle $\mathbf{P}(E)$ associated to E. If X is compact, the versal deformation of $\mathbf{P}(E)$ exists and the tangent space of the basis of the versal deformation is $H^1(X, \mathrm{End}_0\, E)$.

3.4. THEOREM. — *Let* E_0 *be a vector bundle on the compact complex space* X. *Let* $E' \to X \times \Sigma$ *be a deformation of* E_0 *such that* $\mathbf{P}(E') \to X \times \Sigma$ *is the versal deformation of* $\mathbf{P}(E_0)$. *Let* $L \to X \times \Pi$ *be the versal deformation of the trivial line bundle on* X. *Then the exterior tensor product*

$$L \boxtimes E' \to X \times (\Pi \times \Sigma)$$

is the versal deformation of E_0.

Remarks. — *a*) The deformation $E' \to X \times \Sigma$ exists by Theorem 3.2.

b) The versal deformation $L \to X \times \Pi$ of the trivial line bundle can be obtained as follows: Choose cocycles $(h_{ij}^{\mu}) \in Z^1(\mathcal{U}, \mathcal{O}_X)$, $\mu = 1, \ldots, m$ whose cohomology classes form a basis of $H^1(X, \mathcal{O}_X)$. Then $\Pi = (\mathbf{C}^m, 0)$ and

$$g_{ij} := \exp\left(\sum_{\mu=1}^{m} t_\mu h_{ij}^{\mu}\right),$$

where $t_1, \ldots, t_m$ are the coordinates in $\mathbf{C}^m$, is the cocycle defining L.

Proof of Theorem 3.4. – Let

$$E \to X \times S$$

be the versal deformation of E_0. Then $\mathbf{P}(E) \to X \times S$ is a deformation of $\mathbf{P}(E_0)$, hence there exists a map $\alpha : S \to \Sigma$ such that

$$\mathbf{P}(E) \cong \alpha^*\mathbf{P}(E') = \mathbf{P}(\alpha^*E').$$

Then there exists a deformation $M \to X \times S$ of the trivial line bundle such that

$$E \cong M \otimes \alpha^*E'.$$

By the versal property of $L \to X \times \Pi$, there exists a map $\beta : S \to \Pi$ such that $M \cong \beta^*L$. Thus, letting

$$f := (\beta,\alpha) : S \to \Pi \times \Sigma,$$

we have

$$E \cong f^*(L \otimes E').$$

On the other hand, by the versal property of $E \to X \times S$, there exists a map $g : \Pi \times \Sigma \to S$ such that

$$L \boxtimes E' \cong g^*E.$$

Therefore $E \cong (g \circ f)^*E$, which implies

$$(dg)_0 \circ (df)_0 = d(g \circ f)_0 = id_{T_0S}.$$

Consider the diagram

$$T_0S \xrightarrow{(df)_0} T_{(0,0)}(\Pi \times \Sigma) \xrightarrow{(dg)_0} T_0S.$$

Since $T_0\Pi = H^1(X,\mathcal{O}_X)$, $T_0\Sigma = H^1(X, \mathrm{End}_0\, E)$ and $T_0S = H^1(X, \mathrm{End}\, E)$, we have

$$\dim T_0S = \dim T_{(0,0)}(\Pi \times \Sigma),$$

hence $(df)_0$ and $(dg)_0$ are isomorphisms. This implies that $f : S \to \Pi \times \Sigma$ is an isomorphism of germs and

$L \boxtimes E' \to X \times (\Pi \times \Sigma)$ isomorphic to the versal deformation $E \to X \times S$, q.e.d.

3.5. Corollary. — *Let* E *be a vector bundle on a compact complex space* X *such that*

$$\dim H^2(X, \mathrm{End}\, E) = \dim H^2(X, \mathcal{O}_X).$$

Then the basis S *of the versal deformation of* E *is smooth.*

Proof. — The hypothesis implies $H^2(X, \mathrm{End}_0\, E) = 0$. Therefore the basis Σ of the versal deformation of $\mathbf{P}(E)$ is smooth, so $S = \Pi \times \Sigma$ is also smooth.

3.6. Corollary. — *Let* X *be a smooth compact complex surface with trivial canonical bundle (for example a torus or a* K3*-surface) and* E *be a simple vector bundle on* X. *Then the basis of the versal deformation of* E *is smooth.*

Proof. — By Serre duality

$$H^2(X, \mathrm{End}\, E) \cong H^0(X, \mathrm{End}\, E)^* \cong \mathbf{C}$$

and

$$H^2(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X)^* \cong \mathbf{C}.$$

Therefore we can apply Corollary 3.5.

4. Vector bundles on tori with trivial Néron-Severi group.

4.1. Recall the theorem of Riemann-Roch for a (smooth, compact complex) surface X. If E is a vector bundle of rank r on X, we have

$$\chi(X,E) = r\chi(X,\mathcal{O}_X) + \frac{1}{2}(c_1(X)c_1(E) + c_1(E)^2) - c_2(E),$$

$$\chi(X,\mathcal{O}_X) = \frac{1}{12}(c_1(X)^2 + c_2(X)).$$

In particular we can apply Riemann-Roch to the endomorphism bundle *End* E. Since

$$c_1(\text{End E}) = 0,$$
$$c_2(\text{End E}) = r^2 c_2(\text{E}) - (r-1)c_1(\text{E})^2,$$

we get

$$\chi(\text{X}, \text{End E}) = r^2\chi(\text{X},\mathcal{O}_\text{X}) + (r-1)c_1(\text{E})^2 - r^2c_2(\text{E}).$$

4.2. The Néron-Severi group of a surface X is defined by

$$\text{NS(X)} := \text{Im}\,(\text{H}^1(\text{X},\mathcal{O}^*_\text{X}) \xrightarrow{c_1} \text{H}^2(\text{X},\text{Z})).$$

In the following we shall deal with surfaces X (especially tori) having $\text{NS(X)} = 0$. If in addition X is Kähler, then X has no divisors, in particular its algebraic dimension is zero. (Hopf surfaces always have $\text{NS(X)} = 0$, whereas their algebraic dimension may be zero or one.)

4.3. PROPOSITION. – *Let* X *be a Kähler surface with* $\text{NS(X)} = 0$. *Then for any vector bundle* E *of rank* 2 *on* X *we have* $c_2(\text{E}) \geqslant 0$.

Proof. – Since $\text{NS(X)} = 0$, we have $c_1(\text{X}) = c_1(\text{E}) = 0$, hence

$$\chi(\text{X,E}) = \frac{1}{6}c_2(\text{X}) - c_2(\text{E}),$$
$$\chi(\text{X},\mathcal{O}_\text{X}) = \frac{1}{12}c_2(\text{X}).$$

Since for a surface with algebraic dimension zero we have $\chi(\text{X},\mathcal{O}_\text{X}) \geqslant 0$ (cf. [3], part 6, Prop. 1.5), it follows $c_2(\text{X}) \geqslant 0$. We consider first the case that E is not filtrable. Then

$$\text{H}^0(\text{X,E}) = 0 \text{ and } \text{H}^2(\text{X,E})^* \cong \text{H}^0(\text{X,E}^*\otimes\text{K}_\text{X}) = 0,$$

hence

$$0 \leqslant h^1(\text{X,E}) = -\chi(\text{X,E}) = c_2(\text{E}) - \frac{1}{6}c_2(\text{X}) \leqslant c_2(\text{E}).$$

If E is filtrable we have a devissage

$$0 \to \text{L} \to \text{E} \to \text{M} \otimes \mathcal{J}_\text{Z} \to 0.$$

Since $c_1(L) = c_1(M) = 0$, $c_2(E)$ is the dual class of Z, hence non-negative.

4.4. We consider now bundles on a two-dimensional torus X. Since the tangent bundle of X is trivial, we have

$$\chi(X,\mathcal{O}_X) = 0.$$

Serre duality gives

$$h^2(X,E) = h^0(X,E^*)$$

for every vector bundle E on X.

4.5. PROPOSITION. — *Let* E *be a simple vector bundle of rank* r *on a two-dimensional torus* X *with*

$$c_1(E) = c_2(E) = 0.$$

Then E *is homogeneous, i.e. invariant under translations.*

Proof. — Since $h^0(X, \mathrm{End}\, E) = h^2(X, \mathrm{End}\, E) = 1$, we have by Riemann-Roch $h^1(X, \mathrm{End}\, E) = 2$, hence

$$h^1(X, \mathrm{End}_0\, E) = h^1(X, \mathrm{End}\, E) - h^1(X,\mathcal{O}_X) = 0.$$

By Theorem 3.4 the versal deformation of E is given by

$$E \boxtimes L \to X \times \Pi,$$

where $L \to X \times \Pi$ is the versal deformation of the trivial line bundle.

We now construct a family $F \to X \times X$ in the following way: Let

$$a : X \times X \to X$$

be the addition map $a(x,y) := x + y$ and define

$$F := a^*E.$$

By versality, we get a map

$$\varphi : (X,0) \to \Pi$$

of space germs such that

$$F|X \times (X,0) \cong \varphi^*(E \boxtimes L).$$

Let $\tau_x : X \to X$ be the translation $y \mapsto x + y$. Then for x in a sufficiently small neighborhood of $0 \in X$ we have

$$\tau_x^* E \cong E \otimes L_{\varphi(x)}.$$

Taking determinants, we get

$$\tau_x^* (\det E) \cong (\det E) \otimes L^2_{\varphi(x)}.$$

Since $\det E$ is a topologically trivial line bundle, it is homogeneous, which shows that $L^2_{\varphi(x)}$ is the trivial line bundle. Since $L_{\varphi(0)}$ is trivial, $L_{\varphi(x)}$ itself is trivial. Hence $\tau_x^* E \cong E$ for all sufficiently small x. Since every neighborhood of zero generates X, the bundle E is homogeneous.

4.6. COROLLARY — *Every* 2*-bundle* E *on a* 2*-dimensional torus with* $c_1(E) = c_2(E) = 0$ *is filtrable.*

Proof. — By (1.8) we may assume that E is simple. Then E is homogeneous by Proposition 4.5. By a theorem of Matsushima ([5], Prop. 3.2) E is filtrable.

4.7. PROPOSITION. — *Let* X *be a two-dimensional torus with* $NS(X) = 0$. *A two-bundle* E *on* X *is induced by a representation*

$$\sigma : \pi_1(X) \to GL(2,C)$$

if and only if $c_2(E) = 0$.

Remark. — If one drops the hypothesis $NS(X) = 0$, the result does not necessarily hold. Oda [7] has constructed a 2-bundle E on an algebraic 2-dimensional torus with $c_1(E) = c_2(E) = 0$ which does not admit a connection, hence is not induced by a representation.

Proof of Proposition 4.7. — A bundle induced by a representation of $\pi_1(X)$ possesses an integrable connection, hence all its Chern classes are zero (cf. Atiyah [2]).

Conversely suppose $c_2(E) = 0$. Then by Corollary 4.6, E is filtrable (since automatically $c_1(E)=0$). We now distinguish two cases.

i) If E is decomposable, it is a sum of two topologically trivial line bundles, hence induced by a representation (Appell-Humbert).

ii) If E is indecomposable, we have a devissage

$$0 \to L \to E \to M \otimes \mathcal{J}_Z \to 0$$

with $L, M \in \mathrm{Pic}_0(X)$. Since $c_2(E) = 0$, Z must be empty. We have (by Riemann-Roch)

$$\dim H^1(X, M^* \otimes L) = \begin{cases} 2 & \text{if } L \cong M, \\ 0 & \text{if } L \cong M. \end{cases}$$

Since E is indecomposable, the second possibility is excluded and we have an exact sequence

$$0 \to L \to E \to L \to 0.$$

The extensions of L by L are classified by

$$H^1(X, \mathrm{Hom}(L,L)) = H^1(X, \mathcal{O}).$$

Now the translations operate trivially on $H^1(X,\mathcal{O})$, which shows that E is homogeneous, hence induced by a representation [5].

4.8. *Example of a non-filtrable bundle.* — Let X be a two dimensional torus with $NS(X) = 0$. Let $L, M \in \mathrm{Pic}_0(X) = \mathrm{Pic}(X)$ be two line bundles on X with $L \ncong M$ and $Z \subset X$ a subspace consisting of two simple points. Consider a 2-bundle E_0 on X which is an extension

$$0 \to L \to E_0 \to M \otimes \mathcal{J}_Z \to 0.$$

We will show that in the versal deformation of E_0 there occur non-filtrable bundles.

Let us first convince ourselves that there is such a bundle E_0. The extensions of $M \otimes \mathcal{J}_Z$ by L are classified by the group $\mathrm{Ext}^1(M \otimes \mathcal{J}_Z, L)$. There is an exact sequence

$$\begin{aligned} 0 &\to H^1(X, \mathit{Hom}(M \otimes \mathcal{J}_Z, L)) \to \mathrm{Ext}^1(M \otimes \mathcal{J}_Z, L) \to \\ &\to \Gamma(X, \mathit{Ext}^1(M \otimes \mathcal{J}_Z, L)) \to H^2(X, \mathit{Hom}(M \otimes \mathcal{J}_Z, L)). \end{aligned}$$

Since Z has codimension 2, we have

$$\mathit{Hom}(M \otimes \mathcal{J}_Z, L) \cong M^* \otimes L.$$

By Serre duality $H^2(X,M^*\otimes L) \cong H^0(X,M\otimes L^*)^* = 0$, hence by Riemann-Roch $H^1(X,M^*\otimes L) = 0$. On the other hand, since Z is a locally complete intersection consisting of discrete points.

$$\mathcal{E}xt^1(M\otimes\mathcal{I}_Z,L) \cong \mathcal{O}_Z,$$

which proves

$$\mathrm{Ext}^1(M\otimes\mathcal{I}_Z,L) \cong \Gamma(X,\mathcal{O}_Z) \cong \mathbf{C}\oplus\mathbf{C}.$$

By Serre [8], the sheaf corresponding to an extension $\xi\in \mathrm{Ext}^1(M\otimes\mathcal{I}_Z,L)$ is locally free if and only if its image in $\mathbf{C}\oplus\mathbf{C}$ under the above isomorphism has both coordinates different from zero. Extensions ξ_1, ξ_2 which differ only by a constant factor $\lambda\in\mathbf{C}^*$ give rise to isomorphic sheaves.

4.9. Proposition. — *On a two-torus* X *with* $NS(X) = 0$ *there exist non-filtrable vector bundles* E *of rank* 2 *with* $c_2(E) = 2$.

Proof. — Let E_0 be a 2-bundle with devissage

$$0 \to L \to E_0 \to M\otimes\mathcal{I}_Z \to 0$$

as in 4.8. By Theorem 2.2 this bundle is simple, hence the basis (V,0) of its versal deformation $E \to X\times V$ is smooth (Corollary 3.6). The dimension of V equals $h^1(X, \mathrm{End}\, E_0)$ and can be calculated by Riemann-Roch : We have $\chi(X,\mathcal{O}_X) = 0$ and $c_1(E_0) = 0$, hence

$$h^1(X, \mathrm{End}\, E_0) = h^0(X, \mathrm{End}\, E_0) + h^2(X, \mathrm{End}\, E_0) + 4c_2(E) = 2 + 8 = 10.$$

Since small deformations of simple bundles are simple and have the same Chern classes, this dimension is invariant under small deformations. This implies that the versal deformation of E_0 is also versal in neighboring points.

Suppose now that all bundles E_s, $s\in V$, are filtrable. Then they belong all (for s sufficiently close to 0) to class I.1.i) of the classification of Theorem 2.2. By Theorem 2.3 there exist deformations $\mathcal{L} \to X\times V$ and $\mathcal{M} \to X\times V$ of L resp. M and a two-codimensional subspace $\mathcal{Z}\subset X\times V$, flat over V, such that E fits into an exact sequence

$$0 \to \mathcal{L} \to E \to M\otimes\mathcal{I}_{\mathcal{Z}} \to 0.$$

Since $Z = \mathscr{X}_0$ consists of two simple points, also $\mathscr{X}_s$ consists of two simple points for s sufficiently near 0. We can define a holomorphic map

$$\varphi : V \to \mathrm{Pic}_0(X) \times \mathrm{Pic}_0(X) \times S^2X$$

by

$$s \mapsto (\mathscr{L}_s, \mathscr{M}_s, \mathscr{X}_s).$$

Since $\dim(\mathrm{Pic}_0(X) \times \mathrm{Pic}_0(X) \times S^2X) = 8$,

$$S := \varphi^{-1}(L,M,Z)$$

is a subgerm of V of dimension $\geqslant 2$ and we get a family

$$0 \to q^*L \to E|X \times S \to q^*M \otimes \mathscr{I}_{Z \times S} \to 0,$$

where $q : X \times S \to X$ is the projection. This family of extensions defines a holomorphic map

$$\psi : S \to \mathrm{Ext}^1(M \otimes \mathscr{I}_Z, L) \cong \mathbf{C}^2.$$

Since $0 \notin \psi(S)$, we have an associated map

$$\bar{\psi} : S \to \mathbf{P}(\mathrm{Ext}^1(M \otimes \mathscr{I}_Z, L)) \cong \mathbf{P}_1.$$

If $\bar{\psi}(s) = \bar{\psi}(s')$, then $E_s \cong E_{s'}$. Since $\dim S \geqslant 2$, the fibres of $\bar{\psi}$ have dimension $\geqslant 1$. Thus there exists a 1-dimensional subgerm $C \subset S$, such that $E|X \times C$ is a trivial deformation of E_0. But this is a contradiction to the versality of the deformation $E \to X \times V$. Hence there must exist non-filtrable bundles E_s in this deformation, q.e.d.

Appendix

Picard number and algebraic dimension of tori.

1. *Generalities.* Let X be a compact complex connected manifold of dimension n. Its algebraic dimension $a(X)$ is defined as the transcendence degree of its field of meromorphic functions. As is well known, $a(X) \leqslant n$. We denote by $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*)$ the group of isomorphism classes of holomorphic line bundles on X, and by

$$\mathrm{Pic}_0(X) = \mathrm{Ker}\,(H^1(X,\mathcal{O}^*) \xrightarrow{c_1} H^2(X,\mathbf{Z}))$$

the subgroup of line bundles with vanishing first Chern class. The Néron-Severi group NS(X) is defined by the exact sequence

$$0 \to \mathrm{Pic}_0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0.$$

Hence we can write

$$\mathrm{NS}(X) = \mathrm{Im}\,(H^1(X,\mathcal{O}^*) \xrightarrow{c_1} H^2(X,\mathbf{Z})).$$

The rank of NS(X) is called the Picard number of X and is denoted by $\rho(X)$:

$$\rho(X) = \mathrm{rank}_{\mathbf{Z}}\, \mathrm{NS}(X).$$

Assume now that X is a Kähler manifold and consider the Hodge decomposition

$$H^2(X,\mathbf{C}) = H^{20}(X,\mathbf{C}) \oplus H^{11}(X,\mathbf{C}) \oplus H^{02}(X,\mathbf{C}).$$

Denote by $j\colon H^2(X,\mathbf{Z}) \to H^2(X,\mathbf{C})$ the map induced by the inclusion $\mathbf{Z} \hookrightarrow \mathbf{C}$. Then the famous Lefschetz Theorem on (1,1)-classes reads

$$\mathrm{NS}(X) = j^{-1}(H^{11}(X,\mathbf{C})).$$

So, denoting as usual $\dim_{\mathbf{C}} H^{11}(X,\mathbf{C})$ by $h^{11}(X)$, we have

(i) $$\rho(X) \leqslant h^{11}(X).$$

Equality does not necessarily hold, however we have

(ii) $\rho(X) = h^{11}(X) \Rightarrow X$ projective algebraic

(iii) $\rho(X) = 0 \Rightarrow a(X) = 0$.

2. *The case of tori.* Suppose now X is a torus,

$$X = V/\Gamma,$$

where V is a vector space of dimension n over $\mathbf{C}$ and $\Gamma \subset V$ a lattice of rank $2n$. One has a natural isomorphism

$$H^2(X,\mathbf{Z}) \cong \mathrm{Alt}^2_{\mathbf{Z}}(\Gamma,\mathbf{Z})$$

of $H^2(X,\mathbf{Z})$ with the space of alternating integer-valued 2-forms on Γ. Let

$$H(V,\Gamma) = \{H : H \text{ hermitian form on } V \text{ with } \operatorname{Im} H(\Gamma \times \Gamma) \subset \mathbf{Z}\}.$$

Since the imaginary part $\operatorname{Im} H$ of a hermitian form H is an alternating 2-form which determines completely H, we may consider $H(V,\Gamma)$ as a subgroup of $\operatorname{Alt}^2_{\mathbf{Z}}(\Gamma,\mathbf{Z}) \cong H^2(X,\mathbf{Z})$. With this identification one has by the theorem of Appell-Humbert (cf. Mumford [6])

$$NS(X) = H(V,\Gamma).$$

Following Weil [9], let us call Riemann form of X any hermitian form $H \in H(V,\Gamma)$ which is positive semi-definite. Then the algebraic dimension of X is given by

$$a(X) = \max \{\operatorname{rank} H : H \text{ Riemann form of } X\}.$$

In order to be able to make explicit calculations, we introduce coordinates. Let $V = \mathbf{C}^n$ and let Γ be the lattice generated by the vectors $\gamma_1, \ldots, \gamma_{2n} \in \mathbf{C}^n$, which we consider as column vectors. Define the $n \times 2n$ period matrix

$$\Pi := (\gamma_1, \ldots, \gamma_{2n}).$$

Then $H(V,\Gamma)$ is identified with the space of all hermitian $n \times n$ matrices A for which

$$(*) \qquad \operatorname{Im}({}^t\Pi A \bar{\Pi}) \in \mathbf{Z}^{2n \times 2n}.$$

3. *Examples.* In this section we consider two-dimensional tori. We want to give examples for all possible pairs $(a(X),\rho(X))$. For these examples we consider tori determined by period matrices of the form

$$\Pi = \begin{pmatrix} 1 & 0 & ip & ir \\ 0 & 1 & iq & is \end{pmatrix} = (I,iP); \quad P = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathbf{R}^{2\times 2}.$$

An hermitian 2×2 matrix can be written as

$$A = \begin{pmatrix} x & u + iv \\ u - iv & y \end{pmatrix}, \qquad x, y, u, v \in \mathbf{R}.$$

The condition ($*$) above becomes

(i) $v \in \mathbf{Z}$, $(ps - qr)v \in \mathbf{Z}$.

(ii) $px + qu \in \mathbf{Z}$, $pu + qy \in \mathbf{Z}$,
$rx + su \in \mathbf{Z}$, $ru + sy \in \mathbf{Z}$.

Obviously the conditions (i) are independent of (ii) and yield a contribution of 1 or 0 to the Picard number of X, according as $ps - qr$ is rational or not. Since $ps - qr \neq 0$, the system

(iii) $px + qu = n_1$, $pu + gy = n_3$,
$rx + su = n_2$, $ru + sy = n_4$,

has at most one solution for fixed $(n_1,n_2,n_3,n_4) \in \mathbf{Z}^4$. Hence the group of triples (x,y,u) satisfying (ii) is isomorphic to the group of those $(n_1,n_2,n_3,n_4) \in \mathbf{Z}^4$ for which (iii) has a solution. But this system has a solution if and only if the value of u deduced from the first pair of equations is the same as that deduced from the second pair, that is if and only if

(iv) $n_1 r - n_2 p + n_3 s - n_4 q = 0$.

The subgroup of $\mathbf{Z}^4$ defined by this equation has rank equal to

$$4 - \operatorname{rank}_{\mathbf{Q}}(p,q,r,s).$$

Summing up, we have proved

Proposition. — *Let* Γ *be the lattice in* $\mathbf{C}^2$ *spanned by the columns of the matrix*

$$\begin{pmatrix} 1 & 0 & ip & ir \\ 0 & 1 & iq & is \end{pmatrix}, \qquad p, q, r, s \in \mathbf{R}.$$

Then the Picard number of the torus $X = \mathbf{C}^2/\Gamma$ *is given by the formula*

$$\rho(X) = 4 - \operatorname{rank}_{\mathbf{Q}}(p,q,r,s) + \begin{cases} 1 & \text{if } ps - qr \in \mathbf{Q}, \\ 0 & \text{if } ps - qr \notin \mathbf{Q}. \end{cases}$$

Since for a two-torus X we have $h^{11}(X) = 4$, from (App. 1), (i) - (iii) follow the following restrictions for the Picard number:

$$\begin{aligned} 0 \leqslant \rho(X) \leqslant 3, &\quad \text{if} \quad a(X) = 0, \\ 1 \leqslant \rho(X) \leqslant 3, &\quad \text{if} \quad a(X) = 1, \\ 1 \leqslant \rho(X) \leqslant 4, &\quad \text{if} \quad a(X) = 2. \end{aligned}$$

Besides these there are no other restrictions as is shown by the following examples. In the table we give the matrix P determining the period matrix $\Pi = (I, iP)$ of the required torus.

	$a = 0$	$a = 1$	$a = 2$
$\rho = 0$	$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{3} & \sqrt{5} \end{pmatrix}$	impossible	impossible
$\rho = 1$	$\dfrac{1}{\sqrt{6}-\sqrt{5}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{3} & \sqrt{5} \end{pmatrix}$	$\begin{pmatrix} 1 & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}$	$\begin{pmatrix} -\sqrt{2} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$
$\rho = 2$	$\begin{pmatrix} 1 & -\sqrt[3]{2} \\ \sqrt[3]{2} & 1 \end{pmatrix}$	$\begin{pmatrix} \sqrt[3]{2} & 1 \\ 0 & \sqrt[3]{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$
$\rho = 3$	$\begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \sqrt[3]{2} & 0 \\ 0 & \sqrt[3]{2} \end{pmatrix}$
$\rho = 4$	impossible	impossible	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The values of $\rho(X)$ follow from the proposition. We leave it as an exercise to the reader to verify the values of $a(X)$ by determining the maximal rank of a Riemann form.

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