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HENKIN-RAMIREZ FORMULAS WITH WEIGHT FACTORS

by B. BERNDTSSON and M. ANDERSSON

Introduction.

The method of explicit formulas for solving the $\bar{\partial}$-equation has been much in use in later years, starting with the work of Henkin [5] and Ramirez [8]. These formulas are based on the construction of integral kernels, the so called Henkin-Ramirez or Cauchy-Leray kernels, which can be constructed in any strictly pseudoconvex domain, although more elementarily so for the case of a convex domain.

Notwithstanding the great success of this method, the resulting kernels are not always well suited in applications. This is perhaps most clearly seen in one complex variable where the Henkin-Ramirez kernel always becomes $K(\zeta, z) = c\,\frac{d(\zeta - z)}{\zeta - z}$, i.e. the classical Cauchy kernel. It is of course only very rarely that this kernel gives good estimates. Therefore it could be of interest to find modifications of the kernels which contain "weight factors". One type of such kernels has been given by Dautov and Henkin [4]. They use weights which behave like a power of the distance to the boundary (see also [2] and [1] for the case of the ball).

The aim of this note is to show a rather general (and simple) method for constructing formulas with weights. The method is based on a representation of the kernels as "Laplace transforms" of "oscillatory integrals". This was inspired by a similar representation of the Bergman kernel as a Fourier integral operator by Boutet de Monvel and Sjöstrand. Our construction is however much more elementary, and we don't know the theory of Fourier integral operators well enough to know whether there is more than a superficial analogy.

As special cases we obtain weights of the Dautov-Henkin type, weights with polynomial decrease in $\mathbf{C}^n$ and weights which behave roughly like $\exp\text{-}\varphi$ where φ is a convex function. The plan of the paper is this: In section 1 we give the basic construction. In section 2 we write out the formulas more explicitly and modify the construction to find more general weights. Finally in section 3 we study some examples in $\mathbf{C}^n$ and show that we get minimal solutions in certain L^2-spaces, and also indicate some (known) estimates in other norms. We have no essentially new estimates, but believe that the construction in itself may have some interest and hope to come back to the question of estimates later.

As general background references we quote [11] and [7]. The latter reference also contains a good bibliography.

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Section 1.

First we briefly recall the classical construction. In the space $\mathbf{C}^n \times \mathbf{C}^n = \{(\xi, \eta)\,;\ \ \xi \in \mathbf{C}^n,\ \eta \in \mathbf{C}^n\}$ we consider the differential form $\mu = \langle \xi, \eta \rangle^{-n} \omega'(\xi) \wedge \omega(\eta)$. Here

$$\omega'(\xi) = \sum_{j=1}^{n} (-1)^{j-1} \xi_j \bigwedge_{i \neq j} d\xi_i ,$$

$$\omega(\eta) = d\eta_1 \wedge \ldots d\eta_n \quad \text{and} \quad \langle \xi, \eta \rangle = \sum_1^n \xi_j \eta_j .$$

The form μ is well defined on $E = \{(\xi, \eta)\,;\ \langle \xi, \eta \rangle \neq 0\}$, and one can readily verify that $d\mu = 0$. Next, let D be domain in $\mathbf{C}^n$, and consider a map $s = (s_1, \ldots, s_n) : \overline{D} \times \overline{D} \longrightarrow \mathbf{C}^n$ satisfying the condition $\langle s, \zeta - z \rangle \neq 0$ for $\zeta \neq z$ (we use (ζ, z) as coordinates on $\overline{D} \times \overline{D}$).

To be more precise, we assume s to be of class C^1 and satisfy

$$|s(\zeta, z)| \leqslant C|\zeta - z| \quad \text{and} \quad |\langle s, \zeta - z \rangle| \geqslant c\,|\zeta - z|^2 \qquad (1)$$

uniformly for $\zeta \in \overline{D}$ and z in any compact subset of D. Then we define the map $\psi : \overline{D} \times \overline{D} - \Delta \longrightarrow E$, $\psi(\zeta, z) = (s(\zeta, z),\ \zeta - z)$ (where Δ is the diagonal in $\overline{D} \times \overline{D}$), and set $K = \psi^* \mu$, the pullback of μ to $\overline{D} \times \overline{D} \backslash \Delta$. Let $K_{p,q}$ be the component of K of bidegree (p, q) in z and $(n-p, n-q-1)$ in ζ. If we suppose that s also satisfies the condition

$$s(\zeta, \cdot) \text{ is for } \zeta \in \partial D \text{ fixed holomorphic in } D, \tag{2}$$

then the following theorem holds:

THEOREM (see e.g. [11]). – *Let f be a (p, q)-form in $C^1(\overline{D})$ such that $\overline{\partial} f = 0$, $q \geqslant 1$. Then $u = C_{p,q,n} \int_{\zeta \in D} \wedge K_{p,q-1}$ satisfies the equation $\overline{\partial} u = f$ (provided s satisfies (1) and (2)).*

One sees that for $n = 1$ $K_{0,0} = \dfrac{d\zeta}{\zeta - z}$ regardless of the choice of s. Thus we get the classical Cauchy formula

$$f = \overline{\partial} \frac{1}{2\pi i} \int_D \frac{f \wedge d\zeta}{\zeta - z} \tag{3}$$

if f is a $(0, 1)$ form. One obtains other solution formulas simply by multiplying with a function $F(\zeta, z)$, if F is holomorphic in z, $F = 1$ for $\zeta = z$ and of, say, class C^1. Thus

$$f = \overline{\partial} \frac{1}{2\pi i} \int_D F(\zeta, z) \frac{f \wedge d\zeta}{\zeta - z}.$$

This is seen writing $F(\zeta, z) = 1 + (\zeta - z)\, g(\zeta, z)$ and comparing with (3). In several variables such a simple procedure is clearly not possible, since the singularities of the kernel $K_{p,q}$, are more complicated. It turns out that one has to add lower order terms, although we shall do this in an implicit way.

We start by considering the form on $\mathbf{C}^n \times \mathbf{C}^n$

$$A = \exp \langle \xi, \eta \rangle\, \omega(\xi) \wedge \omega(\eta).$$

Let $s = (s_1, \ldots, s_n) : \overline{D} \times \overline{D} \longrightarrow \mathbf{C}^n$ satisfy condition (1) as before. We introduce another map $Q : (Q_1, \ldots, Q_n) : \overline{D} \times \overline{D} \longrightarrow \mathbf{C}^n$, which is to satisfy the sole condition of being holomorphic in $z \in D$ for $\zeta \in D$ fixed (and be of, say, class C^1). Then define

$$\psi : (\overline{D} \times \overline{D} \backslash \Delta) \times (0, \infty) \longrightarrow \mathbf{C}^n \times \mathbf{C}^n$$

by $$\psi(\zeta, z, t) = (Q + ts, \zeta - z)$$

and put $N = \psi^* A$. We can write $N = N_t + N'$ where N_t is that component of N which contains dt. As A is a holomorphic form of maximal degree $dA = 0$. It follows that $dN = 0$ (for $\zeta \neq z$), and sorting out the terms that contain dt we get

$$d_{\zeta,z}N_t = -d_t N'. \tag{4}$$

Next we define K as

$$K = \int_0^\infty N_t . \tag{5}$$

In order for this definition to make sense we make the temporary assumption that $\operatorname{Re}\langle s, \zeta - z\rangle < 0$ (for $\zeta \neq z$), but we will see later that this is not necessary. Then we can differentiate under the sign of integration and get

$$\begin{aligned} d_{\zeta,z}K = \int_0^\infty d_{\zeta,z} N_t = -\int_0^\infty d_t N' = N'|_{t=0} \\ = \exp\langle Q, \zeta - z\rangle\ \omega(Q) \wedge \omega(\zeta - z) = P \end{aligned} \tag{6}$$

outside the diagonal in $D \times D$. (We use the convention of putting dt to the far right side when integrating). The point is that P does not contain s, so it is by the assumption on Q holomorphic in z, and the components of positive degree in $d\bar{z}$ are zero. Note that for $Q \equiv 0$ we get

$$N_t = -\exp t\langle s, \zeta - z\rangle (t^{n-1}\,\omega'(s) \wedge \omega(\zeta - z) \wedge dt)$$

so that K is $-(n-1)!$ times the usual Cauchy-Leray form. In general

$$N_t = -\exp\langle Q, \zeta - z\rangle\ \exp t\langle s, \zeta - z\rangle\ (t^{n-1}\omega'(s) \wedge \omega(\zeta - z) \wedge dt + \sum_{k=0}^{n-2} t^k a_k \wedge dt) \tag{7}$$

where a_k are forms that do not contain t. Hence K is essentially $\exp\langle Q, \zeta - z\rangle$ times the Cauchy-Leray form plus "lower order terms" (i.e. terms whose singularity $\langle s, \zeta - z\rangle^{-(k+1)}$ is of lower order). We can now prove

THEOREM 1 (Koppelman's formula). – *Let f be a (p, q)-form in $C^1(\bar{D})$. Then*

$$f = C_{p,q,n} \left\{ \int_{\partial D} f \wedge K_{p,q} + (-1)^{p+q+1} \left(\int_D \bar{\partial} f \wedge K_{p,q} - \bar{\partial}_z \int_D f \wedge K_{p,q-1} \right) \right\} \tag{8}$$

for $q > 0$ *and*

$$f = C_{p,n} \left\{ \int_{\partial D} f \wedge K_{p,0} + (-1)^{p+1} \int_D \bar{\partial} f \wedge K_{p,0} - \int_D f \wedge P_{p,0} \right\} \qquad (9)$$

for $q = 0$. *Here* $K_{p,q}$ *is the component of* K *which is of bidegree* (p, q) *in* z *and* $(n-p, n-q-1)$ *in* ζ, *and similarily for* P.

Proof. – The proof is of course completely parallell to the proof of the classical Koppelman's formula. However we think it is simpler to repeat the whole proof, rather than just indicate the necessary modifications.

Let ϕ be a smooth (p, q) form with compact support in D. We have to show that the integral $\int_D \phi \wedge f$ equals the integral of the right hand side of (8) (or (9)) against ϕ. We may then replace $K_{p,q}$ (resp $P_{p,q}$) by K (resp P), since no other component gives a contribution for bidegree reasons. Put

$$D_\epsilon = D \times D - \{(\zeta, z) \in D \times D\,;\ |\zeta - z| < \epsilon\},$$

and consider

$$\int_{\partial(D \times D)} \phi(z) \wedge f(\zeta) \wedge K(\zeta, z) = I\,.$$

If ϵ is small enough compared to the distance from $\operatorname{supp}\phi$ to ∂D, we can apply Stokes theorem to D_ϵ and get

$$I = \int_{D_\epsilon} d\phi \wedge f \wedge K + (-1)^{p+q} \int_{D_\epsilon} \phi \wedge df \wedge K + \int_{D_\epsilon} \phi \wedge f \wedge P \qquad (10)$$
$$+ \int_{|\zeta - z| = \epsilon} \phi \wedge f \wedge K\,.$$

It is easy to see that in (7) all the forms a_k, and $\omega'(s)$, have coefficients that are $0(|s|) = 0(|\zeta - z|)$. Hence

$$K = 0(|s| / |\langle s\,; \zeta - z\rangle|^n) = 0(|\zeta - z|^{1-2n})$$

according to (1), uniformly for all z in the support of ϕ. Hence the integrals in the first three terms in (10) are absolutely integrable when $\epsilon \longrightarrow 0$. To see how the fourth term behaves we first need to study K more carefully.

From (7) follows that

$$K = -(n-1)!\,(\exp\langle Q, \zeta - z\rangle)\,\langle s, z - \zeta\rangle^{-n}\,\omega'(s) \wedge \omega(\zeta - z) + T_1 \qquad (11)$$

where the coefficients in T_1 are $0(|\zeta - z|^{2-2n})$. The first term in (11) can be written

$$-(n-1)!\,\langle s,\zeta-z\rangle^{-n}\,\omega'(s)\wedge\omega(\zeta-z)+T_2 \qquad (12)$$

where $T_2 = 0(|\zeta - z|^{2-2n})$. Hence, to compute

$$\lim_{\epsilon\to 0}\int_{|\zeta-z|=\epsilon}\phi\wedge f\wedge K,$$

we can replace K by the first term in (12), i.e. the classical Cauchy-Leray form times a constant. But then it is well known that the limit equals $C_{p,q,n}\int_D \phi\wedge f$.

This fact, under the sole assumption (1), is implicit in [11], and also in [9] where however only the boundary values are considered. Since there appears to be no proof published we give one at the end of this section.

Finally, an application of Stokes theorem with respect to the variable z yields

$$\int_{D\times D} d\phi\wedge f\wedge K = (-1)^{p+q+1}\int_D \phi\wedge d_z\int_D f\wedge K.$$

We also notice that we can replace d by $\bar\partial$ everywhere since K is of degree at least n in $d\zeta$ and dz together, and so is $\phi\wedge f$. Collecting, we have then

$$\int_D \phi\wedge\int_{\partial D} f\wedge K = \int_D \phi\wedge\{(-1)^{p+q}\int_D \bar\partial f\wedge K - \bar\partial_z\int_D f\wedge K\}$$
$$+\int_D \phi\wedge\int_D f\wedge P + C_{p,q,n}\int_D \phi\wedge f.$$

This completes the proof of Theorem 1 if we remember that $P_{p,q} = 0$ for $q > 0$.

From Koppelman's formula we get in the usual way:

THEOREM 2. – *Suppose s satisfies (1) and (2), and that f is a (p,q)-form with coefficients in $C^1(\bar D)$, with $q > 1$, such that $\bar\partial f = 0$. Then $u = (-1)^{p+q}C_{p,q,n}\int_D f\wedge K_{p,q-1}$ solves the equation $\bar\partial u = f$.*

Proof. – Because of condition (2) the pullback of $K_{p,q}$ to $\zeta\in\partial D$ is zero for $q \geq 1$. Hence the theorem follows from Koppelman's formula.

As seen in the proof our kernel K is

$$K = -(n-1)!\,e^{\langle Q,\zeta-z\rangle}K' + T$$

where K$'$ is the classical Henkin-Ramirez kernel and T is "lower order terms". This type of weights is however too special for many purposes. We shall see that, essentially, the exponential function can be replaced by any holomorphic function. This we will do in section 2 after having written K more explicitly.

Proof that

$$\lim_{\epsilon\to 0}\int_{|\zeta-z|=\epsilon}\phi\wedge f\wedge k = \mathrm{C}_{p,q,n}\int\phi\wedge f.$$

As mentioned before it suffices to consider the case $\mathrm{Q}\equiv 0$. Suppose s satisfies (1). We may also assume $\langle s,\zeta-z\rangle>0$ for $\zeta\neq z$ since otherwise we replace s by $s\,\dfrac{\overline{\langle s,\zeta-z\rangle}}{|\langle s,\zeta-z\rangle|}$ which does not change the kernel (see [11] or Lemma 4, section 2). We still have $ds=0(1)$.

Now, let $b=\overline{\zeta}-\overline{z}$ be the "Bochner-Martinelli section", and put $s_\lambda=\lambda s+(1-\lambda)b$, $0<\lambda<1$. Consider the map

$$h(\zeta,z,\lambda):\ \overline{\mathrm{D}}\times\overline{\mathrm{D}}\times[0,1]\longrightarrow\mathbf{C}^n$$

defined by $h(\zeta,z,\lambda)=s_\lambda(\zeta,z)$. Put $\mathrm{H}=h^*\mu$, the pullback of μ to $\overline{\mathrm{D}}\times\overline{\mathrm{D}}\times[0,1]$. Then $d\mathrm{H}=0$ for $\zeta\neq z$.

Applying Stokes theorem to the integral

$$\mathrm{I}_\epsilon=\int_{\partial(\{|\zeta-z|=\epsilon\}\times[0,1])}\phi\wedge f\wedge\mathrm{H}$$

we get

$$\mathrm{I}_\epsilon=\int_{\{|\zeta-z|=\epsilon\}\times[0,1])}d(\phi\wedge f)\wedge\mathrm{H}. \tag{*}$$

On the other hand, since the boundary of $\{|\zeta-z|=\epsilon\}$ is zero (remember ϕ has compact support) we have

$$\mathrm{I}_\epsilon=\int_{|\zeta-z|=\epsilon}\phi\wedge f\wedge\mathrm{K}(s)-\int_{|\zeta-z|=\epsilon}\phi\wedge f\wedge\mathrm{K}(b) \tag{**}$$

where $\mathrm{K}(s)=k$ and $\mathrm{K}(b)$ is the kernel defined by the choice $s=b$.

Observe that in (*) only occur terms of H which contain $d\lambda$. This shows that

$$|\mathrm{H}|\leqslant\left(\frac{|s_\lambda|(|s|+|b|)}{(\lambda\langle s,\zeta-z\rangle+(1-\lambda)|\zeta-z|^2)^n}=0(|\zeta-z|^{2-2n})\right).$$

Since the surface measure of $\{|\zeta-z|=\epsilon\}$ is $0(\epsilon^{2n-1})$ we get

$\lim_{\epsilon \to 0} I_\epsilon = 0$. In view of (**) this shows that it suffices to prove our claim for the case $s = b$. Then it is proved easily by making the substitution $\zeta - z = x$ or consulting the litterature [11].

It is clear from the proof that (1) actually can be relaxed considerably.

Section 2.

Remember that $K = \int_0^\infty N_t$ where

$$N = \exp(\langle Q, \zeta - z\rangle + t\langle s, \zeta - z\rangle)\, \omega(Q + ts) \wedge \omega(\zeta - z)$$

and N_t is the component of N which contains dt. With Q and s we associate the $(1,0)$-forms

$$\sum_1^n s_j d(\zeta_j - z_j) \quad \text{and} \quad \sum_1^n Q_j d(\zeta_j - z_j)$$

which we also denote by s and Q respectively.

LEMMA 3. – *Let* $(a_1, \ldots, a_n)$ *be complex numbers and set* $\omega'(a, \xi) = \Sigma(-1)^{j-1} a_j \bigwedge_{i \neq j} d\xi_i$. *Then*

$$\omega'(a, \xi) \wedge \omega(\eta) = C_n\, \Sigma a_j d\eta_j \wedge (\Sigma d\xi_j \wedge d\eta_j)^{n-1} \tag{13}$$

where $C_n = (-1)^{n(n-1)/2}/(n-1)!$.

Proof. – Define a vector $a = \Sigma a_j \dfrac{\partial}{\partial \xi_j}$. Then $\omega'(a, \xi) = a \lrcorner\, \omega(\xi)$ where $\lrcorner$ denotes interior multiplication of a form with a vector. Now

$$n!\, \omega(\xi) \wedge \omega(\eta) = (-1)^{n(n-1)/2} (\Sigma d\xi_j \wedge d\eta_j)^n .$$

Taking the interior product of both sides with a and using the fact that interior multiplication is an antiderivation we get

$$n!\, \omega'(a, \xi) \wedge \omega(\eta) = (-1)^{n(n-1)/2} n\, \Sigma a_j d\eta_j \wedge (\Sigma d\xi_j \wedge d\eta_j)^{n-1} .$$

This is the assertion.

Now observe that

$$N_t = \exp(\langle Q, \zeta - z\rangle + t\langle s, \zeta - z\rangle)\, dt \wedge \omega'(s, Q + ts) \wedge \omega(\zeta - z)$$

so Lemma 3 gives

$$\begin{aligned} \mathrm{N}_t &= \mathrm{C}_n \exp(\langle \mathrm{Q}, \zeta - z\rangle + t\langle s, \zeta - z\rangle)\, dt \wedge s \wedge (d\mathrm{Q} + t\,ds)^{n-1} \\ &= \mathrm{C}_n \exp(\langle \mathrm{Q}, \zeta - z\rangle + t\langle s, \zeta - z\rangle)\, dt \wedge s \wedge \sum_{k=0}^{n-1} \binom{n-1}{k} (d\mathrm{Q})^k \\ &\qquad \wedge (ds)^{n-1-k}\, t^{n-k-1} \end{aligned}$$

The definition of K now shows that

$$\mathrm{K} = -\,\mathrm{C}_n \exp\langle \mathrm{Q}, \zeta - z\rangle \sum_{k=0}^{n-1} \frac{(n-1)!}{k!}\, \frac{s \wedge (d\mathrm{Q})^k \wedge (ds)^{n-1-k}}{\langle s, \zeta - z\rangle^{n-k}} \cdot \qquad (14)$$

For the associated "projection kernel", P, we get

$$\begin{aligned} \mathrm{P} &= \exp\langle \mathrm{Q}, \zeta - z\rangle\, \omega(\mathrm{Q}) \wedge \omega(\zeta - z) \\ &= (-1)^{n(n-1)/2}/n!\, \exp\langle \mathrm{Q}, \zeta - z\rangle\, (d\mathrm{Q})^n . \end{aligned} \qquad (15)$$

Before continuing let us note that since we are only interested in components of bidegree $= n$ in $d\zeta$ and dz together we can replace d by $\bar\partial$ everywhere in (14) and (15).

LEMMA 4. – *Let* $\varphi : \bar{\mathrm{D}} \times \bar{\mathrm{D}} \longrightarrow \mathrm{C}\setminus\{0\}$ *be any* C^1*-function. Then if we replace* s *by* φs *in (14) we obtain the same kernel.*

Proof. – $\varphi s \wedge (d\varphi s)^j = \varphi s \wedge (d\varphi \wedge s + \varphi ds)^j = \varphi^{j+1} s \wedge (ds)^j$ since $s \wedge s = 0$.

This shows that our kernels have the same homogeneity property as the usual Henkin-Ramirez kernels, and thus we can remove the previous assumption $\operatorname{Re}\langle s, \zeta - z\rangle < 0$.

One simple choice of Q is as follows. Let φ be a convex function in D and put $\mathrm{Q}_j = -\,2\partial\varphi/\partial\zeta_j$. The inequality

$$\varphi(z) - \varphi(\zeta) > 2 \operatorname{Re} \Sigma\, \frac{\partial\varphi}{\partial\zeta_j}(z_j - \zeta_j) \qquad (16)$$

shows that in this case the weight $\exp\langle \mathrm{Q}, \zeta - z\rangle$ satisfies

$$|\exp\langle \mathrm{Q}, \zeta - z\rangle| \leqslant \exp\varphi(z)\, \exp - \varphi(\zeta) .$$

Hence our kernel gives a solution u to $\bar\partial u = f$ for which we can estimate $|u| \exp - \varphi$ with $|f| \exp - \varphi$. Of course, the precise form of the estimates will depend on the choice of s and also involve the Hessian $\partial\bar\partial\varphi$ and we shall not pursue these questions here.

Instead we shall look for more general weights. Replace Q in (14) by λQ where λ is a positive parameter and denote the result $K^{(\lambda)}$. Let g be a function, or even a distribution, on $[0,\infty)$ and set $\widetilde{K}=\int_0^\infty K^{(\lambda)}e^{-\lambda}g(\lambda)\,d\lambda/a$ where $a=\int_0^\infty e^{-\lambda}g(\lambda)\,d\lambda$. Here of course we have to assume that the integrals converge and that $a\neq 0$. Let $G(\alpha)=\int_0^\infty e^{-\alpha\lambda}g(\lambda)\,d\lambda$ be the Laplace transform of g, and normalize so that $a=G(1)=1$. Then (14) gives

$$\widetilde{K}=-C_n\sum_{k=0}^{n-1}\frac{(n-1)!}{k!}G^{(k)}(\langle Q,z-\zeta\rangle+1)\frac{s\wedge(dQ)^k\wedge(ds)^{n-1-k}}{\langle s,\zeta-z\rangle^{n-k}} \tag{17}$$

and if we define $\widetilde{P}$ in a corresponding way, (15) gives

$$\widetilde{P}=(-1)^{n(n-1)/2}/n!\;G^{(n)}(\langle Q,z-\zeta\rangle+1)(dQ)^n. \tag{18}$$

Now, conversely, suppose G is a holomorphic function of one variable in a simply connected domain that contains the image of $\overline{D}\times\overline{D}$ under the map $(\zeta,z)\longrightarrow\langle Q,z-\zeta\rangle+1$, and that $G(1)=1$. Then we can define $\widetilde{K}$ and $\widetilde{P}$ using (17) and (18). We then get the principal result of this paper.

THEOREM 5. – *With* G *as above Koppelman's formula (Theorem 1) holds with* K *and* P *replaced by* $\widetilde{K}$ *repectively* $\widetilde{P}$.

Proof. – In case G is a nice entire function, e.g. a polynomial, this is clear from the above. Namely if we take g to be a combination of derivatives of the Dirac measure at the origin, and use Koppelman's formula for each $K^{(\lambda)}$. The general case follows since G can be approximated by polynomials uniformly on the image of

$$(\zeta,z)\longrightarrow\langle Q,z-\zeta\rangle+1.$$

Of course, one could also verify by direct computation that $\widetilde{K}$ and $\widetilde{P}$ satisfy the required identities and then repeat the proof of Koppelman's formula.

Since the kernel K is the special case of the above construction with $G(\alpha)=\exp(1-\alpha)$, we will drop the tildes in the sequel and write simply K and P for the kernels in (17) and (18).

For each choice of Q and G Theorem 5 gives a solution operator for the $\bar\partial$-equation and representation formulas for holomorphic functions (provided s satisfies (1) and (2)). We shall now consider some concrete examples.

Example 1. – Let φ be a negative convex function in $\overline{D}$ and set $Q_j = \dfrac{1}{\varphi}\dfrac{\partial\varphi}{\partial\zeta_j}$.

$$\langle Q, z-\zeta\rangle + 1 = \frac{\langle\partial\varphi, z-\zeta\rangle + \varphi}{\varphi}$$

so by inequality (16)

$$\operatorname{Re}(\langle Q, z-\zeta\rangle + 1) \geqslant \frac{\varphi(z) + \varphi(\zeta)}{2\varphi(\zeta)} > 0 .$$

Hence $G(\alpha) = \alpha^{-N}$; $N > 0$, will do in Theorem 4, and we obtain the kernel

$$K = -C_n \sum_{k=0}^{n-1} \gamma_k \left(\frac{\varphi}{\langle\partial\varphi, z-\zeta\rangle + \varphi}\right)^{N+k} \frac{s \wedge (dQ)^k \wedge (\bar\partial s)^{n-1-k}}{\langle s, z-\zeta\rangle^{n-k}} \cdot \quad (19)$$

Note that

$$(dQ)^k = \left[\frac{1}{\varphi}\, d\Sigma\, \frac{\partial\varphi}{\partial\zeta_j}\, d(\zeta_j - z_j) - \frac{1}{\varphi^2}\, d\varphi \wedge \Sigma\, \frac{\partial\varphi}{\partial\zeta_j}\, d(\zeta_j - z_j)\right]$$

so that all coefficients in this form are $0(|\varphi|^{-k-1})$. Hence we can relax the assumption that φ be strictly negative on $\overline{D}$, and only assume $\varphi \leqslant 0$, (replace φ by $\varphi - \epsilon$ and let $\epsilon \downarrow 0$). In particular, if D is a convex domain with C^2-boundary we can choose $\varphi = \rho - \epsilon$ where $D = \{\zeta \in C^n ; \rho(\zeta) < 0\}$ where ρ is convex defining function for D. Then $dQ = \dfrac{1}{\varphi}\, d\Sigma\, \dfrac{\partial\varphi}{\partial\zeta_j}\, d(\zeta_j - z_j)$ when restricted to $z \in \partial D$, since $d\rho = 0$ there.

Hence the coefficients of $(dQ)^k$ are then $0(|\varphi|^{-k})$, which implies that after letting $\epsilon \longrightarrow 0$ we get a kernel, K, which restricted for $\zeta \in \partial D$ vanishes, even if s does not satisfy condition 2. Moreover in this case the representation formula for holomorphic functions which we get from Koppelman's formula will contain only the kernel P and no integral over the boundary.

One way to make suitable choices of s is as follows. Let $A = (A_{jk})$ be a C^1-function from $\overline{D} \times \overline{D}$ with values in the space of positively semidefinite hermitian matrices. Suppose A is (uniformly) positively definite on compacts in $D \times D$ and moreover that for $\zeta \in \partial D$

$$\Sigma\, A_{jk}(\zeta_j - z_j)\,(\overline{\zeta}_k - \overline{z}_k) > 0 \quad \text{for} \quad z \in D\,.$$

Then $s_j = \Sigma\, A_{jk}(\overline{\zeta}_k - \overline{z}_k)$ will satisfy (1). As A we can take:

i) $A = I =$ identity, $s = \overline{\zeta} - \overline{z}$.

ii) in $D = \{\zeta\,;\ \rho(\zeta) < 0\}$ $\ A = A(\zeta) = -\rho(\zeta)\,I + \left(\frac{\partial\rho}{\partial\zeta_j}\,\frac{\partial\rho}{\partial\overline{\zeta}_k}\right)$.

Then for $\zeta \in \partial D$ $s = \langle \overline{\partial}\rho\,;\ \overline{\zeta} - \overline{z}\rangle\, \partial\rho$ which by Lemma 4 is equivalent to $\partial\rho$, and so is holomorphic in $z \in D$.

iii) $A = A(z) = -\rho(z)\,I + \left(\frac{\partial\rho}{\partial z_j}\,\frac{\partial\rho}{\partial\overline{z}_k}\right)$.

The matrices in ii) and iii) have for ζ resp z near ∂D roughly the same behaviour as the coefficient matrix of the Bergman metric. ii) always gives a solution operator, whereas iii) will do in case the weight is zero on ∂D. To conclude this example let us show that iii) gives a particularily simple formula in case of (0, 1)-forms. Thus we shall consider $K_{0,0}$, the component of K of bidegree (0,0) in z. Then

$$(\overline{\partial}s)_{0,0} = -\,\Sigma\, A_{jk}\, d\zeta_j \wedge d\overline{\zeta}_k = -\,\mathcal{A}$$

$$(\overline{\partial}Q)_{0,0} = -\,\partial\overline{\partial}\,\log -\frac{1}{\rho}\,,$$

$$\langle s\,,\zeta - z\rangle = \Sigma\, A_{jk}(\zeta_j - z_j)\,(\overline{\zeta}_k - \overline{z}_k) = \|\zeta - z\|_A^2$$

and

$$s_{0,0} = \partial_\zeta \|\zeta - z\|_A^2\,.$$

Hence (19) becomes

$$K = \sum_{k=0}^{n-1} \gamma'_k \left(\frac{\rho}{\langle\partial\rho\,, z - \zeta\rangle + \rho}\right)^{N+k} \frac{\partial_\zeta \|\zeta - z\|_A^2 \wedge \left(\partial\overline{\partial}\log -\frac{1}{\rho}\right)^k \wedge \mathcal{A}^{n-1-k}}{\|\zeta - z\|_A^{2(n-k)}} \tag{20}$$

For $z \in \partial D$ $\ \mathcal{A} = \Sigma\, \frac{\partial\rho}{\partial z_j}\,\frac{\partial\rho}{\partial\overline{z}_k}\, d\zeta_j \wedge d\overline{\zeta}_k$

and

$$s = \partial_\zeta \|\zeta - z\|_A^2 = \Sigma\, \frac{\partial\rho}{\partial\overline{z}_k}\,(\overline{\zeta}_k - \overline{z}_k)\ \Sigma\, \frac{\partial\rho}{\partial z_j}\, d\zeta_j\,.$$

This implies $s \wedge \mathcal{A} = 0$ so

$$K_{00} = C\left(\frac{\rho}{\langle\partial\rho\,, z - \zeta\rangle + \rho}\right)^{N+n-1} \frac{\partial_\zeta \|\zeta - z\|_A^2 \wedge \left(\partial\overline{\partial}\log -\frac{1}{\rho}\right)^{n-1}}{\|\zeta - z\|_A^2} \tag{21}$$

The corresponding projection operator is

$$\mathrm{P}_{00} = \mathrm{C}\left(\frac{\rho}{\langle \partial\rho , z - \zeta \rangle + \rho}\right)^{\mathrm{N}+n} \left(\partial\bar{\partial} \log - \frac{1}{\rho}\right)^{n} . \tag{22}$$

It is not difficult to see that, if D is strictly convex, we can take limits when $z \longrightarrow \partial\mathrm{D}$ in Koppelman's formula and so obtain

PROPOSITION 6. – *Let* D *be a convex* C^2*-domain with defining function* ρ. *Let* f *be a* $\bar{\partial}$*-closed* $(0,1)$*-form in* $\mathrm{C}^2(\bar{\mathrm{D}})$, *and* u *a holomorphic function in* $\mathrm{C}^1(\bar{\mathrm{D}})$. *Then*

$$v(z) = \mathrm{C}_n \int_{\mathrm{D}} f \wedge \mathrm{K}_{00} \tag{*}$$

is a solution to the equation $\bar{\partial} v = f$ *and* $u(z) = \mathrm{C}_n \int_{\mathrm{D}} u\, \mathrm{P}_{00}$ for $z \in \mathrm{D}$. K_{00} *and* P_{00} *are given in (20) and (22). If* D *is strictly convex,* v *has boundary values given by (*) with* K_{00} *as in (21).*

This kernel easily gives the L^1-estimates on the boundary, that first were obtained by Skoda [9] and Henkin [6]. Notice that the difference in dependence on the "tangential" and "normal" parts of f is exhibited by the form $\partial\bar{\partial} \log - \frac{1}{\rho} = \frac{-1}{\rho} \partial\bar{\partial}\rho + \frac{1}{\rho^2} \partial\rho \wedge \bar{\partial}\rho$. When D is the unit ball and $\rho = |\zeta|^2 - 1$

$$\frac{\rho}{\langle \partial\rho , z - \zeta \rangle + \rho} = \frac{1 - |\zeta|^2}{1 - \bar{\zeta} \cdot z}$$

and $\left(\partial\bar{\partial} \log - \frac{1}{\rho}\right)^n = \text{const. } (1 - |\zeta|^2)^{-(n+1)}$.

Hence, for $\mathrm{N} = 1$, $\mathrm{P}_{0,0}$ is just the Bergman kernel, and the solution to the $\bar{\partial}$-equation given by (20) in the interior and (21) on the boundary is the minimal solution in L^2. This solution has been found earlier by Charpentier [2], in the interior and on the boundary already in [9]. Intuitively (22) $(\mathrm{N} = 1)$ could be viewed as an approximate Bergman kernel, in the same way as the Cauchy-Leray kernel is an approximate Szegö kernel.

As mentioned in the introduction, kernels with similar behaviour as (20) have been found earlier in [4]. See also [3] for a different method.

A similar proposition also holds for strictly pseudoconvex domains, just put $Q_j = \frac{h_j}{\rho}$, where $\{h_j\}$ is the section of Henkin and Ramirez (see [11]), and proceed in the same way,

Example 2. – Let $f_1, \ldots, f_p$ be holomorphic in D and of class $C^1(\bar{D})$. Then one can write

$$f_k(z) - f_k(\zeta) = \int_0^1 \frac{d}{dt} f_k(\zeta + t(z - \zeta))\, dt = \Sigma g_j^k (z_j - \zeta_j) \qquad (23)$$

(if D is convex), where the $g_j^k : s$ are holomorphic in ζ and z. Then set

$$Q_j = \frac{\sum_k \overline{f_k(\zeta)}\, g_j^k}{\Sigma |f_k(\zeta)|^2 + \epsilon} .$$

We get

$$\langle Q, z - s \rangle + 1 = \frac{\Sigma \overline{f_k(\zeta)}\, f_k(z) + \epsilon}{\Sigma |f_k(\zeta)|^2 + \epsilon} .$$

If we then take $G(\alpha) = \alpha^N$, we get weights which may be of use in two connections. Firstly, if w is a $\bar{\partial}$-closed form which vanishes to a high enough degree on the set of common zeros of $f_1, \ldots, f_p$, we can let $\epsilon \longrightarrow 0$ and get a solution to $\bar{\partial} u = w$ which also vanishes there (we even get $u = \Sigma f_k u_k$ for some forms u_k).

Secondly, we can use the representation formula for holomorphic functions to solve a division problem. Namely Koppelman's formula gives in this case

$$f = C_n \left\{ \int_{\partial D} f K_{0,0} - \int_D f P_{0,0} \right\} \qquad (24)$$

provided f vanishes sufficiently rapidly on the common zeros of $f_1, \ldots, f_p$. This gives an explicit representation of f as $f = f_k\, g_k$ with g_k holomorphic.

Similar formulas exist for strictly pseudoconvex domains, although (23) then must be replaced by a less elementary analog. Finally, the boundary integral in (24) can be suppressed if we combine this method with the one in Example 1, but we don't go into details since we only aim at indicating possible applications.

Added in proof. – This choice of Q also turns out to give kernels supported on the set of common zeros of $f_1, \ldots, f_p$. A more detailed analysis will be the object of a forthcoming paper "A formula for interpolation and division in $\mathbf{C}^n$", by one of the authors.

Section 3.

In this section we use Theorem 5 to derive some kernels, by means of which we obtain explicit solution formulas for the equation $\bar{\partial} u = f$, where f is a $\bar{\partial}$-closed $(p, q+1)$-form in $\mathbf{C}^n$. Such formulas have previously been obtained by Skoda in [10]. Our kernels give roughly the same estimates as in [10], which are essentially the best possible. Moreover in the case when f is a (0,1)-form, the solutions will be minimal in certain L^2-spaces. At the end of this section we briefly discuss the case when f has growth of infinite order.

Now, consider formula (17). We choose $Q_j = \dfrac{\zeta_j}{1 + |\zeta|^2}$ and $s_j = \bar{\zeta}_j - \bar{z}_j$. Furthermore, for each non-negative integer m, we may take $G(\alpha) = \alpha^m$.

We define the kernels K^m as those which are obtained from formula (17), with the special choices of Q, s and G stated above.

Since $G^k(\alpha)$ equals $\dfrac{m!\, \alpha^{m-k}}{(m-k)!}$ when $j \leqslant m$ and zero otherwise, and $\langle Q, z - \zeta \rangle = \dfrac{\bar{\zeta} \cdot z + 1}{|\zeta|^2 + 1}$, the explicit expression for K^m is

$$K^m = c \sum_{k=0}^{\min(m, n-1)} \binom{m}{k} \left(\frac{1 + \bar{\zeta} \cdot z}{1 + |\zeta|^2} \right)^{m-k} \frac{\partial |\zeta - z|^2 \wedge (\bar{\partial} \partial |\zeta - z|^2)^{n-1-k} \wedge (\bar{\partial} Q)^k}{|\zeta - z|^{2(n-k)}}$$

Keeping in mind formula (18), the associated projection forms P^m can be written

$$P^m = c \cdot \left(\frac{1 + \bar{\zeta} \cdot z}{1 + |\zeta|^2} \right)^{m-n} (\bar{\partial} Q)^n$$

when $m \geqslant n$. When $m < n$ the projection forms are identically zero. A simple computation shows that the component $P^m_{0,0}$ of bidegree $(0,0)$ in z has the simple appearence

$$P^m_{\sigma, 0} = c \cdot \frac{(1 + \bar{\zeta} \cdot z)^{m-n}}{(1 + |\zeta|^2)^{m+1}} (\bar{\partial} \partial |\zeta|^2)^n$$

when $m \geqslant n$.

Now we shall see that the Koppelman's formula is still valid in this case although D, being $\mathbf{C}^n$, is not bounded. Since

$$|\partial|\zeta - z|^2| \leqslant |\zeta - z|, \quad |\bar{\partial} Q| \leqslant (1 + |\zeta|^2)^{-1}$$

and $|\bar{\partial}\partial|\zeta - z|^2| \leqslant 1$ K^m can be estimated by

$$R_m(\zeta, z) = C \sum_{k=0}^{\min(m,n-1)} \frac{|1 + \bar{\zeta} z|^{m-k}}{(1 + |\zeta|^2)^m} \, \frac{1}{|\zeta - z|^{2n-2k-1}} \, .$$

For a smooth (p, q)-form f the following "Koppelman's formula" holds.

PROPOSITION 7. – *Suppose*

$$\int |\bar{\partial} f(\zeta)| \, R^m(\zeta, z) < \infty \tag{25}$$

for every fixed z.

a) *If* $q > 0$ *and* $\int |f(\zeta)| \, R^m(\zeta, z) < \infty$ *for every fixed* z, *then*

$$f(z) = \int \bar{\partial} f(\zeta) \wedge K^m_{p,q}(\zeta, z) - \bar{\partial}_z \int f(\zeta) \wedge K^m_{p,q-1}(\zeta, z)$$

b) *If* $q = 0$ *and* $\int |f(\zeta)| \, |P^m(\zeta, z)| < \infty$ *for every fixed* z, *then*

$$f(z) = \int \bar{\partial} f(\zeta) \wedge K^m_{p,0} \, (\zeta, z) + \int f(\zeta) \wedge P^m_{p,0}(\zeta, z) \, .$$

In particular (25) is satisfied when f *is* $\bar{\partial}$-*closed, and hence* $\int f \wedge K^m_{(p,q-1)}$ *is a solution to* $\bar{\partial} u = f$ *if the hypothesis of a) is satisfied.*

Proof. – Choose a $\varphi \in C_0^\infty(\mathbf{C}^n)$ such that $\varphi \equiv 1$ for $|\zeta| < 1$ and $\varphi \equiv 0$ when $|\zeta| > 2$. Put $\varphi_R(\zeta) = \varphi\left(\frac{\zeta}{R}\right)$. The Koppelman's formula implies, if $q > 0$

$$\varphi_R f = \int \varphi_R \bar{\partial} f \wedge K^m_{p,q} + \int \bar{\partial}\varphi_R \wedge f \wedge K^m_{p,q} - \bar{\partial}_z \int \varphi_R f \wedge K^m_{p,q-1} \, .$$

Since $|\bar{\partial}\varphi_R| \leqslant \text{const} \, \frac{1}{R}$ the assumptions about f and $\bar{\partial} f$ imply that $\varphi_R f$ and the first two integrals converge uniformly to f, $\int \bar{\partial} f \wedge K^m_{p,q}$ and 0, respectively when z belongs to a compact set. Hence a) follows. In a similar way b) is proved.

By L^2_{m+1} we denote the space $L^2\left(\frac{d\lambda(z)}{(1+|z|^2)^{m+1}}\right)$ and by A_{m+1}, the intersection of L^2_{m+1} and the space of entire functions.

THEOREM 8. –

a) $\Pi_m u(z) = \int u(\zeta)\, P^m_{0,0}(\zeta, z)$ *is the orthogonal projection of* L^2_{m+1} *onto* A_{m+1}.

b) *Assume* f *is a smooth* $(0,1)$*-form satisfying*

$$\int |f(\zeta)|\, R^m(\zeta, z) < \infty$$

for every fixed z. *If* $\bar\partial u = f$ *has a solution in* L^2_{m+1}, *the minimal one* v *is given by* $v(z) = \int f(\zeta) \wedge K^m_{0,0}(\zeta, z)$.

Proof. – a) In fact A_{m+1} is nothing but the space of polynomials of degree at most $m-n$. Since $\zeta \longrightarrow (1 + \overline{\zeta}, z)^{m-n}$ belongs to L^2_{m+1}, $\Pi_m u$ is defined and is in A_{m+1} when u is in L^2_{m+1}. By the appearent anti-symmetry property of $P^m_{0,0}$ $\frac{d\lambda(z)}{(1+|z|^2)^{m+1}}$ an application of the Fubini Theorem shows that Π_m is self-adjoint. Hence $u - \Pi_m u$ is orthogonal to A_{m+1}.

b) Suppose u is a solution in L^2_{m+1}. By Prop. b)

$$u = \int \bar\partial u \wedge K^m_{0,0} + \int u P_{0,0} = \int f \wedge K^m_{0,0} + \int u P_{0,0},$$

and hence from a) we get that $v = \int f \wedge K^m_{0,0}$ is the minimal solution.

As was mentioned above, when $f \in C^1_{(p,q)}$, or even $f \in L^1_{\mathrm{loc}}$, one gets essentially the same estimates as in [10], with appropriate choices of m. We state two theorems to this effect, the first of which is formulated in terms of the growth-function

$$E_f(t) = t^{-2n} \int_{|\zeta|<t} |f(\zeta)|\, d\lambda(\zeta),$$

which is useful in many applications.

THEOREM 9. – *Suppose* $f \in L^1_{\mathrm{loc}}(p, q+1)$, $\bar\partial f = 0$, *and for some* $\alpha \geqslant -2n$ $E_f(t) \leqslant (1+t)^\alpha$. *Then there exists a solution* u, $u \in L^1_{\mathrm{loc}(p,q)}$, *to* $\bar\partial u = f$ *satisfying*

$$E_u(t) \leqslant A_{n,\alpha}(1+t)^{\alpha+1}(1 + \log(1+t)).$$

THEOREM 9′. – *Suppose* f *is in* $C^1_{(p,q+1)}$, $\bar\partial f = 0$, *and for some* $\alpha \geqslant -2n$ $|f(z)| \leqslant (1+|z|)^\alpha$. *Then there exists a solution* $u \in C^1_{(p,q)}$ *to* $\bar\partial u = f$ *satisfying*

$$|u(z)| \leqslant A_{\alpha,n}(1+|z|)^{\alpha+1}(1+\log(1+|z|).$$

The proofs are nothing but straight-forward estimates, and we omit them. Given $\alpha \geqslant -2n$, one only has to choose m such that $m > \alpha + n \geqslant m + \min(n-1, m) - n$.

The logarithm actually occurs only when α is an integer, and the forms $\bar\partial z_1^i \log(1+|z|^2)$ show that the logarithm cannot be dispensed with. When $n > 1$ and f is a $(0,1)$-form a small additional argument shows that Theorem 9′ holds for all $\alpha \in R$.

As was pointed out in section 2, if $\varphi(\zeta)$ is convex in C^n, then $2\mathrm{Re}\,\langle \partial\varphi, z-\zeta\rangle \leqslant \varphi(z) - \varphi(\zeta)$. With the same proof as in Proposition 6 we obtain

PROPOSITION 10. – *If* $f \in C^1_{(0,q+1)}$, *is* $\bar\partial$*-closed and*

$$\int \sum_{p<n} |f(\zeta)| \frac{e^{-\varphi(\zeta)}\,|\bar\partial\partial\varphi|^p}{p!\,|\zeta - z|^{2n-2p-1}} < \infty$$

for every fixed z, *then a solution* u *to* $\bar\partial u = f$, *satisfying*

$$|u(z)| \leqslant e^{\varphi(z)} \int \sum_{p<n} \frac{|f(\zeta)|\,e^{-\varphi(\zeta)}\,|2\bar\partial\partial\varphi|^p}{p!\,|\zeta - z|^{2n-2p-1}}$$

is given by

$$u(z) = \int e^{\langle 2\partial\varphi, z-\zeta\rangle} \sum_{j<n} \wedge \frac{\partial|\zeta - z|^2 \wedge (2\bar\partial\partial\varphi)^p \wedge (\bar\partial\partial|\zeta - z|^2)^{n-1-p}}{p!\,|\zeta - z|^{2n-2p}}.$$

In the special case when $\varphi = |\zeta|^2/2$ the solution is minimal in $L^2(e^{-|z|^2}d\lambda)$, as one can see by inspection of the corresponding projection operator. However, in general these estimates are quite awkward, and the problem to obtain good estimates in the case of growth of infinite order remains.

BIBLIOGRAPHIE

[1] B. BERNDTSSON, Integral formulas for the $\partial\bar{\partial}$-equation and zeros of bounded holomorphic functions in the unit ball, *Math. Ann.,* 249 (1980), 163-176.

[2] P. CHARPENTIER, Solutions minimales de l'équation $\bar{\partial}u = f$ dans la boule et dans le polydisque, *Ann. Inst. Fourier,* 30, 4 (1980), 121-153.

[3] A. CUMENGE, Extension holomorphe dans des classes de Hardy, Thèse, Université Paul Sabatier de Toulouse, 1980.

[4] S.V. DAUTOV, G.M. HENKIN, Zeros of holomorphic functions of finite order and weighted estimates for solutions of the $\bar{\partial}$-equation (russian), *Math. Sb.,* 107 (1979), 163-174.

[5] G.M. HENKIN, Integral representation of a function in a strictly pseudo-convex domain and applications to the $\bar{\partial}$-problem (russian), *Mat. Sb.,* 82 (1970), 300-308.

[6] G.M. HENKIN, Solutions with bounds of the equations of H. Lewy and Poincaré-Lelong. Construction of functions of the Nevanlinna class with given zeros in a strictly pseudoconvex domain, *Soviet Math. Dokl.,* 16 (1976), 3-13.

[7] G.M. HENKIN, The Lewy equation and analysis on pseudoconvex manifolds, *Russian Math. Surveys,* 32 (1977), 59-130.

[8] E. RAMIREZ DE ARELLANO, Ein Divisionsproblem und Randintegraldarstellungen in der komplexen Analysis, *Math. Ann.,* 184 (1970), 172-187.

[9] H. SKODA, Valeurs au bord pour les solutions de l'opérateur d'' et caractérisation des zéros des fonctions de la classe de Nevanlinna, *Bull. Soc. Math. France,* 104 (1976), 225-299.

[10] H. SKODA, d''-cohomologie à croissance lente dans $\mathbf{C}^n$, *Ann. Scient. Ec. Norm. Sup.*, 4 (1971), 97-120.

[11] N. ØVRELID, Integral representation formulas and L^p-estimates for the $\bar{\partial}$-equation, *Math. Scand.*, 29 (1971), 137-160.

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