ANNALES DE L'INSTITUT FOURIER

JOHN N. MATHER

Foliations of surfaces I: an ideal boundary

Annales de l'institut Fourier, tome 32, nº 1 (1982), p. 235-261
<http://www.numdam.org/item?id=AIF_1982__32_1_235_0>

Ann. Inst. Fourier
32, 1 (1982), 235-261

FOLIATIONS OF SURFACES : I, AN IDEAL BOUNDARY

by John N. MATHER(*)

Consider a foliation F defined in U-p, where p is a point in a surface S, and U is a neighborhood of p in S. The point p is often called an "isolated singularity" of F. We will define an "ideal boundary" $\beta = \beta(F_p)$ associated in a topologically invariant way to the germ of F at p, in this paper.

In subsequent work, we will use β to study the topology of F in a neighborhood of p.

For simplicity, we will suppose in most of this paper that F is a foliation of the punctured plane, i.e., the plane with the origin deleted. We will study the ideal boundary associated to the "singularity" of F at the origin. Obviously, this involves no loss of generality.

Here is an outline of the definition of $\beta = \beta(F_0)$ when F is a foliation of the punctured plane. A leaf of F which is not compact is homeomorphic to the real line, with respect to the foliation topology. Hence, it has two ends (which we call leaf-ends). We let $\mathcal{E} = \mathcal{E}(F_0)$ denote the set of all leaf-ends which converge to the origin. The set $\mathcal{E}$ has a natural cyclic order (§ 2).

We will say that distinct elements e, e' in a cyclicly ordered set Σ are neighbors if the orientation of $ee''e'$ with respect to the cyclic order on Σ is independent of the choice of $e'' \in \Sigma - \{e, e'\}$.

Example. – Consider the circle with the standard cyclic order. Remove a small open interval. The endpoints of the interval are neighbors in the resulting set.

(*) Partially supported by an NSF grant.

We will construct β by identifying neighbors in $\mathcal{E}$ and filling in the holes by a process analogous to Dedekind's method of constructing the set of real numbers from the set of rational numbers. The cyclic order on $\mathcal{E}$ induces a cylic order on β (§ 3).

Obviously, when $\mathcal{E}$ is empty, so is β. When $\mathcal{E}$ is finite, but not empty, β is one point. Part of the main theorem of this paper (§ 12) states that when $\mathcal{E}$ is infinite, there is a cyclic order preserving bijection of β onto the circle.

We let P denote the punctured plane, and let

$$\widetilde{P} = \widetilde{P}(F_0) = P \amalg \beta(F_0)$$

(disjoint union). We will define a topology on $\widetilde{P}$, as follows. In the case $\mathcal{E}$ is empty, we provide $\widetilde{P}$ with the same topology as P. In the case $\mathcal{E}$ is finite, there is a bijection of $\widetilde{P}$ onto the plane, which is the identity on P and sends β to 0, and we topologize $\widetilde{P}$ so that this bijection is a homeomorphism.

In the case $\mathcal{E}$ is infinite, we consider the set $\mathcal{J}$ of all Jordan curves Γ (i.e., homeomorphic images of the circle in the plane) such that:

a) $0 \in \Gamma$, and

b) there is an open arc $V \subset \Gamma$ with $0 \in V$ such that each of the two components of $V - 0$ is contained in a leaf of F. (The leaves for the two components will usually be different.)

For $\Gamma \in \mathcal{J}$, we let U_Γ be the open subset of the plane bounded by Γ. We define a subset $\widetilde{U}_\Gamma$ of $\widetilde{P}$, as follows. We let $\widetilde{U}_\Gamma \cap P = U_\Gamma$. Denote the ends of the components of $V - 0$ which converge to 0 by e' and e''. By condition b), $e', e'' \in \mathcal{E}$. To define $\widetilde{U}_\Gamma \cap \beta$, we consider two cases, depending on whether $\pi(e') = \pi(e'')$ or not, where $\pi : \mathcal{E} \longrightarrow \beta$ denotes the natural mapping. In either case, we will need to consider an auxiliary element $e \in \mathcal{E}$ satisfying:

c) all points sufficiently near e on the leaf of which e is an end are in U_Γ, and

d) $\pi(e) \neq \pi(e')$ and $\pi(e) \neq \pi(e'')$.

Case 1. – $\pi(e') = \pi(e'')$. If $e \in \mathcal{E}$ satisfying c) and d) exists, we let $\widetilde{U}_\Gamma \cap \beta = \beta - \pi(e')$. If no such e exists, we let $\widetilde{U}_\Gamma \cap \beta = \phi$.

Case 2. – $\pi(e') \neq \pi(e'')$. If $e \in \mathcal{E}$ satisfying c) and d) exists, we let $\widetilde{U}_\Gamma \cap \beta$ be the set of $x \in \beta$ such that $\pi(e')\, x\, \pi(e'')$ has the same orientation as $\pi(e')\,\pi(e)\,\pi(e'')$, with respect to the cyclic order on β. This orientation is independent of the choice of e (§ 5). If no such e exists, we let $\widetilde{U}_\Gamma \cap \beta = \phi$.

(In fact, when $\pi(e') \neq \pi(e'')$, there always exists $e \in \mathcal{E}$ satisfying c) and d), by the main theorem of this paper. However, this fact will not be proved until we prove the main theorem of this paper, and in the meantime we need a definition of $\widetilde{U}_\Gamma$.)

We provide $\widetilde{P}$ with the topology generated by all open sets in P and all sets $\widetilde{U}_\Gamma$, $\Gamma \in \mathcal{J}$. Our main theorem (§ 12) states that when $\mathcal{E}$ is infinite, there is a homeomorphism of topological pairs: $(\widetilde{P}, \beta) = (S^1 \times [0, \infty), S^1 \times 0)$, where S^1 denotes the circle and $[0, \infty)$ the set of non-negative numbers.

In Chapter I of this paper, we will give the definition of β and $\widetilde{P}$ in more detail.

The main tool we use in the proof of the main theorem is the Poincaré-Bendixson theory. Chapter II contains an outline of the results we need from this theory. All the ideas of Chapter II come from Bendixson's paper.

Chapter III contains the statement and proof of the main theorem.

Our original motivation was to study foliations of the plane in a neighborhood of infinity. Of course, our results apply to this case. Haefliger and Reeb have made a beautiful study of planar foliations in [2]. Even though we do not use any of their results directly, their ideas inspired this paper.

Since our results are purely topological, they apply in many cases. For example, if S is a surface of finite connectivity, and e is an end of S, then there is a deleted neighborhood of e which is homeomorphic to a punctured plane [3], [5]. If F is a foliation of S, then our theorem applies to the germ of F in a neighborhood of e.

I would like to thank the referee for numerous improvements in the exposition.

CHAPTER I

DEFINITION OF THE IDEAL BOUNDARY

1. Definition of Foliations on Surfaces.

By a *surface,* we will mean a Hausdorff, second countable topological space, each point of which has a neighborhood homeomorphic to the plane.

Following Haefliger, we will define a foliation on a surface S in the following way. A *chart* will mean a pair (U, ϕ), where U is an open set in S, and ϕ is a homeomorphism of U onto an open set in the plane. We identify the plane with $\mathbf{R}^2$. Two charts (U, ϕ) and (V, ψ) will be said to be *compatible* if the *transition mapping* $\xi = \phi\psi^{-1} : \psi(V \cap U) \longrightarrow \phi(V \cap U)$ satisfies the condition that the germ of ξ at each point has the form $\xi(x, y) = (\xi_1(x, y), \xi_2(y))$. In other words, the second component of $\xi(x, y)$ depends only on y locally in a neighborhood of each point in the domain of ξ.

An *atlas* is a family $\{(U_i, \xi_i)\}_i$ of mutually compatible charts such that $\bigcup_i U_i = S$. Two atlases are said to be *equivalent* if their union is an atlas; this is an equivalence relation. A *foliation* is an equivalence class of atlases.

A foliation is said to be *defined* by an atlas if it is the equivalence class of the atlas. In this case, any chart in the atlas is said to be a *chart for the foliation.*

Let F be the foliation defined by an atlas $\{(U_i, \phi_i)\}_i$. By the *foliation topology,* or F-*topology,* we mean the topology on S in which a subset V is open if and only if $\phi_i(V \cap U_i)$ intersects each horizontal line $\{y = \text{constant}\}$ in an open set. This topology is independent of the atlas chosen to define the foliation.

A connected component of S with respect to the F-topology is called a *leaf* of the foliation.

2. Ends of leaves.

We will use the following terminology. An open arc, i.e., a topological space homeomorphic to an open interval, has two "ends". A closed arc, i.e., a topological space homeomorphic to a closed interval, has two "endpoints". A half open arc, i.e., a topological space homeomorphic to a half-open interval, has one "end" and one "endpoint". The use of the word "end" here agrees with the usage in the topological "theory of ends".

Let F be a foliation of a surface S. Each non-compact leaf L of F is homeomorphic to $\mathbf{R}$ with respect to the F-topology. Hence, it has two ends. If $u : \mathbf{R} \longrightarrow \mathrm{L}$ is a homeomorphism, and $\mathrm{T} \in \mathbf{R}$, then $\{u(t) : t \geqslant \mathrm{T}\}$ and $\{u(t) : t \leqslant \mathrm{T}\}$ will be called *half-leaves.* Each such half-leaf has just one end, which is one of the ends of L. Thus, the end of $\{u(t) : t \geqslant \mathrm{T}\}$ (resp. $\{u(t) : t \leqslant \mathrm{T}\}$) is the end of L which corresponds to $+\infty$ (resp. $-\infty$).

By the *Riemann sphere,* we will mean the plane with ∞ adjoined, topologized in the usual way.

Suppose F is a foliation of the punctured plane P. Suppose L is a non-compact leaf of F, and e is an end of L. We define the *limit set* $\lim e$ of e to be the set of x *in the Riemann sphere* such that any neighborhood of x meets any half-leaf in L whose end is e.

Obviously, $\lim e$ is a compact, connected, non-empty subset of the Riemann sphere.

DEFINITION of $\mathcal{E}$. – *We will say that* e *converges to* 0 *(resp.* ∞*) if* $\lim e = 0$ *(resp.* ∞*). We let* $\mathcal{E} = \mathcal{E}(\mathrm{F}_0)$ *denote the set of all ends of leaves of* F *which converge to the origin.*

In the rest of this section, we will show how to define a cyclic order on $\mathcal{E}$.

DEFINITION. – *Let* Σ *be a set. By a cyclic order on* Σ *we will mean a subset* P *of the set of ordered triples of distinct elements of* Σ, *which satisfies the following conditions:*

a) *Let* x_1, x_2, x_3 *be three distinct elements of* Σ *and let* π *be a permutation of* $\{1, 2, 3\}$. *In the case* π *is an even permutation,* $(x_1, x_2, x_3) \in \mathrm{P}$ *if and only if* $(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) \in \mathrm{P}$.

In the case π *is an odd permutation* $(x_1, x_2, x_3) \in \mathrm{P}$ *if and only if* $(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) \notin \mathrm{P}$.

b) *Let* x_1, x_2, x_3, x_4 *be any four distinct elements of* Σ. *If* $(x_1, x_2, x_3) \in \mathrm{P}$ *and* $(x_1, x_3, x_4) \in \mathrm{P}$, *then* $(x_1, x_2, x_4) \in \mathrm{P}$.

Relation with total orders. — If Σ is a set and $x \in \Sigma$, then we may define a one-one correspondence between cyclic orders on Σ and total orders on $\Sigma - x$, as follows. If P is a cyclic order on Σ, define a total order on $\Sigma - x$ by setting $y < z$ whenever $(x, y, z) \in \mathrm{P}$.

DEFINITION. — *If* x, y, z *are distinct elements of* Σ, *we will say that* xyz *has positive orientation (with respect to* P*) if* $(x, y, z) \in \mathrm{P}$; *otherwise, we will say* xyz *has negative orientation.*

Example. — If x, y, z are three distinct points on the circle, S^1, we will say that xyz has positive orientation if a point which traverses the circle in the counter-clockwise direction meets x, y, and z in that order.

We will need the following classical result from plane topology.

LEMMA 2. — *If* $\mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n$ *are closed arcs in the plane which have a common end-point but no other intersections, the plane can be mapped homeomorphically onto itself, so that the arcs are mapped onto segments radiating from a point. Moreover, in any two such homeomorphisms, the cyclic order of the image segments is the same or reversed.*

This is exercise 3 on p. 170 of [4]. It is an easy consequence of the preceding material in [4]. This material includes the Schoenflies theorem [4, Chapt. VII, Theorem 4.1], but the Schoenflies theorem by itself does not seem to be enough to give this result.

Definition of the cyclic order on $\mathcal{E}$. — Consider three distinct elements $e_1, e_2, e_3 \in \mathcal{E}$. Each e_i is the end of some half-leaf Λ_i and $\overline{\Lambda}_i = 0 \cup \Lambda_i$. Each $\overline{\Lambda}_i$ is a closed arc in the plane, having 0 as one endpoint. Two different $\overline{\Lambda}_i$ do not intersect, unless they lie on the same leaf. However, in the latter case, we can arrange that they do not intersect by shrinking them suitably. Thus, in any case, we get three closed arcs $\overline{\Lambda}_1, \overline{\Lambda}_2, \overline{\Lambda}_3$, which intersect only at the origin.

By Lemma 2, there is an orientation preserving homeomorphism ϕ of the plane onto itself, such that $\phi(0) = 0$, and $\phi(\overline{\Lambda}_1)$, $\phi(\overline{\Lambda}_2)$, $\phi(\overline{\Lambda}_3)$ are segments of straight lines in the plane. One endpoint of each such straight line segment is the origin, and we may choose ϕ so that the other endpoint x_i lies on the unit circle. We decree that $e_1 e_2 e_3$ has positive orientation if $x_1 x_2 x_3$ does. A well-known argument shows that the orientation of $x_1 x_2 x_3$ is independent of ϕ, as long as ϕ is chosen to be orientation preserving.

3. Dedekind completion.

It is obvious that the Dedekind process of filling in the holes can be applied to cyclicly ordered sets as well as totally ordered sets. In this section, we define this process in detail.

Given a total order on a set Σ, we may define a cyclic order on it by decreeing that xyz has positive orientation if $x < y < z$ or $z < x < y$ or $y < z < x$. We will say that the cyclic order so defined is *induced* from the total order on Σ.

Now suppose Σ is a cyclicly ordered set. Any total order on Σ which induces the given cyclic order will be said to be *admissible.* A total order on Σ for which there is neither a greatest nor a least element will be said to be *unbounded.* We define the *Dedekind completion* $\hat{\Sigma}$ of Σ to be the disjoint union of Σ and all unbounded, admissible total orders on Σ.

Relation with Dedekind cuts. – Consider distinct admissible total orders $<$ and $<'$ on Σ. Let

$$\mathrm{P} = \{x \in \Sigma : \ \exists y \in \Sigma,\ x < y,\ y <' x\}$$

$$\mathrm{P}' = \{x \in \Sigma : \ \exists y \in \Sigma,\ y < x,\ x <' y\}.$$

LEMMA 3. – a) *$x < y$ and $y <' x$ if and only if $x \in \mathrm{P}$ and $y \in \mathrm{P}'$.*

b) $\Sigma = \mathrm{P} \amalg \mathrm{P}'$ *(disjoint union).*

Proof. – a) "Only if" is obvious. To prove "if", we consider $x \in \mathrm{P}$, $y \in \mathrm{P}'$. By hypothesis, there exists $w, z \in \Sigma$ such that $x < w$, $w <' x$, $z < y$, and $y <' z$. We wish to prove $x < y$. Suppose the contrary, i.e., $y \leqslant x$. Then $z < y \leqslant x < w$, so zyw and zxw have positive orientation. Since zyw has positive orientation and $y <' z$, we obtain $y <' w <' z$. Since zxw has positive orientation and $w <' x$, we obtain $w <' z <' x$. Hence, $y <' w <' z <' x$. Hence, yzx has positive orientation, which contradicts $z < y \leqslant x$. This contradiction shows $x < y$. The same argument (with the order reversed) shows $y <' x$.

b) The fact that $\mathrm{P} \cap \mathrm{P}' = \phi$ follows immediately from the fact that if $x \in \mathrm{P}$ and $y \in \mathrm{P}'$, then $x < y$.

To prove $\Sigma = \mathrm{P} \cup \mathrm{P}'$, suppose there exists $x \in \Sigma - (\mathrm{P} \cup \mathrm{P}')$. We will show that $y < z \Longrightarrow y <' z$, for any $y, z \in \Sigma$.

When either y or z is x, this follows immediately from the definitions of P and P'. Otherwise, one of the following cases holds: $x < y < z$, $y < x < z$, or $y < z < x$. In the second case, we obtain $y <' x <' z$. In the first case, $x <' y$, $x <' z$, and xyz has positive orientation, so $x <' y <' z$. The third case is treated like the first. In any case we obtain $y <' z$.

Thus, $y < z \Longrightarrow y <' z$, so $<$ and $<'$ are the same, contrary to assumption. □

Since Σ is the disjoint union of P and P' and $x \in \mathrm{P}$, $y \in \mathrm{P}'$ imply $x < y$, we obtain that $(\mathrm{P}, \mathrm{P}')$ is a Dedekind cut for the order $<$. Likewise, $(\mathrm{P}', \mathrm{P})$ is a Dedekind cut for the order $<'$. Conversely, if $(\mathrm{P}, \mathrm{P}')$ is any Dedekind cut for the admissible order $<$, we can define a second admissible order $<'$ on Σ, by setting $x <' y$ if $x \in \mathrm{P}'$, $y \in \mathrm{P}$, or $x, y \in \mathrm{P}$ and $x < y$, or $x, y \in \mathrm{P}'$ and $x < y$. Thus, for any admissible total order $<$ on Σ, the other admissible total orders are in one-one correspondence with the Dedekind cuts for $<$.

The cyclic order on $\hat{\Sigma}$. – In § 2, we explained how to define a total order on $\Sigma - x$ for any $x \in \Sigma$. We can extend this to a total order on Σ in two ways: either make x the largest element or the smallest element. The two total orders associated to x in this way are admissible.

Let x_1, x_2, x_3 be three distinct elements of $\hat{\Sigma}$ and let $<_1, <_2, <_3$ be three admissible total orders associated to x_1, x_2, x_3 in the following way. Each x_i is either an unbounded, admissible total order on Σ or an element of Σ. In the first case, we let $<_i = x_i$. In the second case, we let $<_i$ be one of the two total orders we just associated to x_i. We will assume that these orders are chosen so that they are distinct. This is always possible, but a wrong choice may lead two of these orders to be the same in the case two of the x_i's are neighbors in Σ.

Let (P, P') (resp. (Q, Q')) be the Dedekind cut for the order $<_1$ defined by the order $<_2$ (resp. $<_3$). In other words,

$$P = \{x \in \Sigma : \exists y \in \Sigma\,,\ x <_1 y\,,\ y <_2 x\}\,,$$

etc. We will say $x_1 x_2 x_3$ has positive orientation if $P \subset Q$. Otherwise, $Q \subset P$, since (P, P') and (Q, Q') are both Dedekind cuts for the order $<_1$, and we will say $x_1 x_2 x_3$ has negative orientation. It is easy to check that the conditions for a cyclic order are satisfied.

Properties of the Dedekind completion. – It is easily checked that the inclusion $\Sigma \subset \hat{\Sigma}$ preserves cyclic order, and $\hat{\Sigma} = \hat{\hat{\Sigma}}$.

4. Definition of β.

Let Σ be a set provided with a cyclic order. If $x, y \in \Sigma$, we will say that y is the successor of x and x is the predecessor of y if xyz has positive orientation for any $z \in \Sigma - \{x, y\}$. In this case, we will say x and y are *neighbors.*

We let $\overline{\Sigma}$ be the set obtained from Σ by identifying any two neighbors in Σ. Two elements $x, y \in \Sigma$ project to the same element of $\overline{\Sigma}$ if and only if there are at most finitely many $z \in \Sigma$ such that xyz is positively oriented or there are at most finitely many $z \in \Sigma$ such that xyz is negatively oriented.

We define a cyclic order on $\overline{\Sigma}$ in the obvious way. Let $\overline{x}, \overline{y}, \overline{z}$ be three distinct elements of $\overline{\Sigma}$ and let x, y, z be elements of Σ which project onto them. We will say $\overline{x}\,\overline{y}\,\overline{z}$ is positively oriented if xyz is.

In general, $\overline{\Sigma}$ can have neighbors, so we can have $\overline{\overline{\Sigma}} \neq \overline{\Sigma}$. Moreover, it may happen that $\hat{\overline{\Sigma}} \neq \overline{\hat{\Sigma}}$.

DEFINITION. – *If* F *is a foliation of the punctured plane, we set* $\beta = \beta(\mathrm{F}_0) = \hat{\overline{\mathcal{E}}}(\mathrm{F}_0)$.

From our main theorem, it follows that there are no neighbors in $\overline{\mathcal{E}}$, so $\overline{\mathcal{E}} = \overline{\overline{\mathcal{E}}}$.

We let $\pi : \mathcal{E} \longrightarrow \beta$ be the composition of the projection $\mathcal{E} \longrightarrow \overline{\mathcal{E}}$ with the inclusion $\overline{\mathcal{E}} \longrightarrow \hat{\overline{\mathcal{E}}}$.

5. Topology on $\widetilde{\mathrm{P}}$.

Let F be a foliation of P. When we defined the topology on $\widetilde{\mathrm{P}} = \widetilde{\mathrm{P}}(\mathrm{F}_0)$ in the introduction, we remarked that the orientation of $\pi(e')\,\pi(e)\,\pi(e'')$ was independent of the choice of e as long as conditions c) and d) of the introduction were satisfied. In this section, we note a stronger result.

Let e', e'' be distinct members of $\mathcal{E}$. Let Λ', Λ'' be half-leaves whose ends are e' and e''. We will think of e' and e'' as the germs of $\overline{\Lambda}'$ and $\overline{\Lambda}''$ at the origin. Then $e' \cup e''$ separates the germ at the origin of the plane into two germs of open sets, U and V.

THEOREM 5. – *Let* $e \in \mathcal{E}$ *be distinct from* e' *and* e''. *The orientation of* $e'ee''$ *depends only on whether* $e \subset \mathrm{U} \cup 0$ *or* $e \subset \mathrm{V} \cup 0$, *and it is opposite in one case from what it is in the other. Here, we think of* e *as the germ of* $\overline{\Lambda}$ *at the origin, where* Λ *is half-leaf whose end is* e.

Proof. – Immediate from Lemma 2. □

If x, y are elements of a cyclicly ordered set Σ, we define the open interval (xy) to be the set of $z \in \mathrm{Z}$ such that xzy is positively oriented. The family of open intervals in Σ is the basis of a topology, called the *topology associated to the cyclic order on* Σ.

From the definition of the topology $\widetilde{\mathrm{P}}$ and Theorem 5, it follows easily that the topology on β induced from the topology on $\widetilde{\mathrm{P}}$ is the same as the topology associated to the cyclic order on β.

CHAPTER II

POINCARE-BENDIXSON THEORY

6. Poincaré index.

Let U be an open subset of the plane and let F be a foliation of U. The well-known Poincaré index $i(\mathrm{F}, \gamma)$ is defined for every closed curve γ in U (i.e., continuous mapping γ of the circle S^1 into U). We will sketch the definition in this section.

For the definition, we need the following result.

LEMMA 6. – *If* F *is an orientable foliation of a surface* S, *then there is a flow* α *on* S *whose trajectories are the leaves of* F.

Recall that a flow α on S is a continuous action $\alpha : \mathbf{R} \times \mathrm{S} \longrightarrow \mathrm{S}$ of $\mathbf{R}$ on S. Its trajectories are the sets $\alpha(\mathbf{R} \times x)$, for $x \in \mathrm{S}$.

Proof. – We will consider measures μ defined on the σ-algebra $\mathcal{B}$ of Borel sets associated to the foliation topology. In other words, $\mathcal{B}$ is the smallest σ-algebra which contains the F-open sets. We will say such a measure is F-*locally regular* if it has no atoms, each F-open set has positive measure, each F-compact set has finite measure, and whenever (U, ϕ) is a chart for the foliations and $[a, b] \times [c, d] \subset \phi(\mathrm{U})$, we have that $\mu(\phi^{-1}([a, b] \times t))$ is a continuous function of $t \in [c, d]$.

We can construct an F-locally regular measure μ on $\mathcal{B}$ by using charts for the foliation to construct it locally, and using a partition of unity to fit the locally constructed measures together. We will say μ is F-*regular,* if in addition to being F-locally regular, every half-leaf for F has infinite measure. It is easy to see that if μ is F-locally regular, then there is a continuous function g on S such that $g\mu$ is F-regular.

Thus, there is an F-regular measure μ on S. To such a measure and an orientation of F, we may associate a flow α, as follows. Consider $u \in \mathrm{U}$ and let L be the leaf which contains u. Then α is uniquely defined among possible actions by requiring that $\alpha(u, t)$ move in the positive direction on L as t increases, and for small $\epsilon > 0$, $\mu\{\alpha(u, t) : 0 \leqslant t \leqslant \epsilon\} = \epsilon$. In this way, we may construct a continuous action α of $\mathbf{R}$ on S whose trajectories are the leaves of F. □

The definition of the Poincaré index may be given as follows. Suppose there is a homotopy $\Gamma : S^1 \times I \longrightarrow U$ such that $\Gamma(\theta, 0) = \gamma(\theta)$ and for each fixed θ, the mapping $t \longrightarrow \Gamma(\theta, t)$ is an embedding of I into a single leaf of F. In this case, we define $i(F, \gamma)$ to be the degree of the mapping

$$\theta \longrightarrow \frac{\gamma(\theta) - \Gamma(\theta, 1)}{\|\gamma(\theta) - \Gamma(\theta, 1)\|}$$

of S^1 into S^1. It may be seen that this number is independent of the choice of Γ, when Γ exists.

When F is orientable, Γ exists. It may be constructed by using an action α of $\mathbf{R}$ on U whose trajectories are the leaves of U.

In general, let γ^2 be the closed curve obtained by tracing over γ twice. Then $i(F, \gamma^2)$ may be defined in the above way, i.e. Γ exists for γ^2 in place of γ. In the case F is not orientable, this may be seen by letting $\widetilde{U}$ be the two fold covering space of U such that the pull-back $\widetilde{F}$ of F to $\widetilde{U}$ is orientable. Then γ^2 is the image of a closed curve in $\widetilde{U}$ and the leaves of $\widetilde{F}$ are the trajectories of an action α of $\mathbf{R}$ on $\widetilde{U}$. Using these data, one constructs Γ easily.

When $i(F, \gamma)$ may be defined as above, it is easy to see that $i(F, \gamma) = i(F, \gamma^2)/2$. In general, we define $i(F, \gamma)$ be this formula.

The Poincaré index $i(F, \gamma)$ depends only on F and the homology class of γ; it defines a homomorphism $i_F : H_1(U, \mathbf{Z}) \longrightarrow \mathbf{Q}$, whose values are integers, or half-integers.

7. The case when there is a leaf connecting 0 and ∞.

Let F be a foliation of P. When $\mathcal{E}(F_0) = \emptyset$, any neighborhood of the origin contains a compact leaf, which separates the origin from infinity. This follows from the proof of Théorème VI in [1]. See also p. 256 of [3]. On p. 257 of [3], Kerékjártó gives a complete topological classification of germs F_0 of foliations for which $\mathcal{E}(F_0) = \emptyset$. We will not discuss this case in this paper.

Suppose $\mathcal{E} = \mathcal{E}(F_0) \neq \emptyset$. Let $e \in \mathcal{E}$ and let L be a leaf whose end is e. Let Γ be a Jordan curve in the plane which encloses the origin, and intersects L. Let U be the open subset of the plane bounded by Γ. Let h be a homeomorphism of U onto the plane which is the identity in a neighborhood of the origin. Let L_0 be the component of $U \cap L$ which has e as an end. Then $h(L_0)$ is a leaf of $h_*(F|U)$, one of whose ends converges to the origin, and the other of whose ends coverges to ∞. We will say that such a leaf *connects* 0 *and* ∞.

We have equality of germs: $(h_*F)_0 = F_0$. Thus, we have shown that if $\mathcal{E}(F_0) \neq \emptyset$, then F_0 is the germ of a foliation having a leaf connecting 0 and ∞.

The following result is a consequence of the well known Poincaré-Bendixson theorem, suitably formulated. However, since it seems to be slightly different from anything in the literature, it seems advisable to sketch a proof.

THEOREM 7. – *If* F *has a leaf connecting* 0 *and* ∞ *then every end of a leaf of* F *converges to* 0 *or to* ∞.

Reduction of Theorem 7 to the orientable case. – Let F′ be the foliation of P obtained by pulling back F *via* the mapping $z \longrightarrow z^2$ of the Riemann sphere onto itself, where z denotes the complex coordinate $z = x + iy$. The foliation F′ is always orientable, and it is easily seen that Theorem 7 for F′ implies Theorem 7 for F.

Proof in the orientable case.

DEFINITION. – *Let* F *be a foliation of a surface* S. *By an open (resp. half-open, resp. closed) transversal to* F, *we mean an open (resp. half-open, resp. closed) arc* η *in* S *such that for each* $u \in \eta$, *there is a chart* (U, ϕ) *for* F *such that* $u \in U$ *and*

$$\phi(U \cap \eta) \subset x_0 \times \mathbf{R} \subset \mathbf{R}^2,$$

for some $x_0 \in \mathbf{R}$.

Thus, ϕ takes F to the horizontal foliation and η to an interval in a vertical line.

We will need the following basic result due to Bendixson.

LEMMA 7. – *Suppose* F *is an orientable foliation of a connected open subset* U *of the plane. Let* L^* *be any leaf of* F *and* η *any transversal to* F. *Suppose* L^* *meets* η *in more than one point. Let* A *and* B *be two successive (on* L^**) points of* $L^* \cap \eta$. *Let* L_0^* *and* η_0 *be the closed arcs of* L^* *and* η *with endpoints* A *and* B. *Then* $i(F, L_0^* \cup \eta_0) = \pm 1$.

Proof. – By Lemma 6, there is a flow α on U whose trajectories are the leaves of F. Let $\gamma : S^1 \longrightarrow L_0^* \cup \eta_0$ be a homeomorphism. Using α, we may construct a homotopy $\Gamma : S^1 \times I \longrightarrow U$ having the properties required for the definit i on of $i(F, \gamma)$ and such that the image of Γ lies in $\overline{R}$, where R is the open subset of the plane bounded by $L_0^* \cup \eta_0$ (see Fig. 7). By the

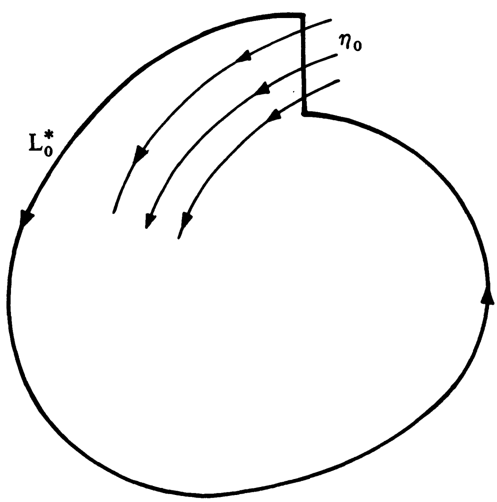

Fig. 7

Schoenflies theorem, there is an isotopy of $\overline{R}$ onto the unit disk in the plane. Using this and the definition of $i(F, \gamma)$ in terms of Γ, we see that $i(F, \gamma) = +1$ or -1 according to whether γ traverses $L_0^* \cup \eta_0$ in the positive or negative direction. □

Now we return to the proof of Theorem 7 and suppose F is an orientable foliation of P with a leaf L connecting 0 and ∞. Let e be an end of a leaf L^* of F. Since $\lim e$ is connected and non-void, it is enough to prove $\lim e \cap P = \phi$.

Suppose the contrary. Let $x \in \lim e \cap P$. Let η be a transversal to F containing x. If $x \notin L$, we choose η so $\eta \cap L = \phi$ and x is in the interior of η. If $x \in L$, we choose η so x is an endpoint of η, $\eta \cap L = x$, and L^* meets η infinitely often. Let A and B be two successive points on L^* of $L^* \cap \eta$. Let L_0^* and η_0 be the closed arcs of L^* and η with endpoints A and B. Then $(L_0^* \cup \eta_0) \cap L = \phi$, so $L_0^* \cup \eta_0$ is null-homotopic in P, and $i(F, L_0^* \cup \eta_0) = 0$. But, Lemma 7 implies $i(F, L_0^* \cup \eta_0) = \pm 1$. This contradiction shows that $\lim e \cap P = \phi$. □

8. Neighbors.

Let F be a foliation of P. Using the method of Bendixson [1, Théorème V], we may prove:

THEOREM 8.1. – *Let $e, e' \in \mathcal{E} = \mathcal{E}(F_0)$ be neighbors. There exists a compact set D in the plane and a homeomorphism ϕ of D onto the rectangle $\{-1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$, such that $\phi(0) = 0$, $\phi(F|D - 0)$ is the foliation by horizontal lines, and e, e' are the ends of the half-leaves, Λ, Λ', where*

$\Lambda = \phi^{-1}\{-1 \leqslant x < 0, y = 0\}$ *and* $\Lambda' = \phi^{-1}\{0 < x \leqslant 1, y = 0\}$.

(See Fig. 8.1.)

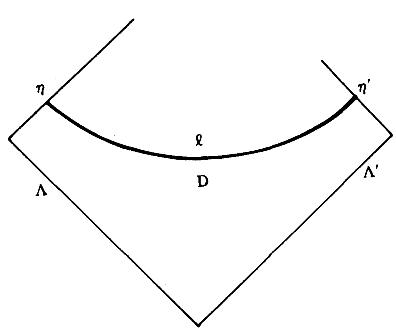

Fig. 8.1

The statement that $\phi(F|D-0)$ is the foliation by horizontal lines means that for each point $x \in D - 0$, there is a chart (U, ψ) for F such that $\phi | U \cap (D-0) = \psi | U \cap (D-0)$.

Proof. – Let Λ and Λ' be half-leaves whose ends are e and e'. Then $\Lambda' \cup \Lambda \cup 0$ separates any sufficiently small neighborhood for the origin into two open sets U and V, such that the origin is adherent to both of them. It follows from Theorem 5 and the hypothesis that e and e' are neighbors, that one of these open sets, say U, has the property that there is no half-leaf in U whose end converges to the origin.

Let η and η' be half-open transversals to F such that the endpoint of η (resp. η') is the endpoint of Λ (resp. Λ'). We suppose in addition that η (resp. η') is on the same side of Λ (resp. Λ') as U (Fig. 8.1).

Let $u \in \eta$ and let λ be the half-leaf on the same side of η and Λ, having u as endpoint. Bendixson shows [1, Théorème V] that for u sufficiently near to the endpoint of Λ, λ intersects η' in a point v. Moreover, he shows that as u tends to the endpoint of Λ, v converges to the endpoint of Λ'.

Here, we sketch Bendixson's proof. We choose a closed arc γ joining a point of η to a point of η' and not otherwise meeting $\eta \cup \Lambda \cup 0 \cup \Lambda' \cup \eta'$. Let η_0 (resp. η'_0) be the arc of η (resp. η') between the endpoint of Λ (resp. Λ') and the point $\eta \cap \gamma$ (resp. $\eta' \cap \gamma$). Let R be the open subset of the plane bounded by the Jordan curve $\Lambda \cup 0 \cup \Lambda' \cup \eta'_0 \cup \gamma \cup \eta_0$. We will suppose there is a neighborhood of the origin in the plane which has the same intersection with R as with U. It is always possible to choose γ so this holds.

We will suppose F has a leaf joining 0 and ∞. There is no loss of generality in supposing this, by the discussion at the beginning of § 7, since $\mathcal{E}(F_0) \neq \phi$, and Theorem 8.1 is an assertion about the germ F_0.

Choose $u \in \eta_0$. The first step in Bendixson's argument is to show λ intersects $\gamma \cup \eta'_0$. For, if not, it follows from Theorem 7 that the end of λ converges to the origin. Since $\lambda \subset \overline{R}$ in the case $\lambda \cap (\gamma \cup \eta'_0) = \phi$, it follows that points on λ near the end of λ lie in U. But this contradicts the defining property of U.

Let v be the point where λ intersects $\gamma \cup \eta_0'$. As u approaches the endpoint of Λ on η_0, the point v moves monotonically on $\gamma \cup \eta_0'$ towards the endpoint of Λ'. Thus, v approaches some limit w in $\gamma \cup \eta_0'$. The last step in Bendixson's argument is to show that w is the endpoint of Λ'.

Suppose otherwise. Let ℓ be the arc on λ joining u and v. Let ζ be the half-leaf emanating from w in the same direction as ℓ from v. (This is meaningful, since v approaches arbitrarily close to w.) We have $\zeta \subset \overline{R}$, since $\ell \subset \overline{R}$, and any neighborhood of any point in ζ meets ℓ, if u is sufficiently close to the endpoint of Λ. Hence, Theorem 7 implies that the end of ζ converges to the origin. Since $\zeta \subset \overline{R}$, it follows that points on ζ near the end of ζ are in U. This contradicts the defining property of U.

This contradiction shows that v converges to the endpoint of Λ'. This completes Bendixson's argument.

Choose u so that $v \in \eta_0'$. Let η_1 (resp. η_1') be the arc of η (resp. η') whose endpoints are the endpoint of Λ and u (resp. the endpoint of Λ' and v). Let D_0 be the open subset of the plane bounded by the Jordan curve $\ell \cup \eta_1' \cup \Lambda' \cup 0 \cup \Lambda \cup \eta_1$. Let $D = \overline{D}_0$.

We may choose a simply connected open neighborhood U of $D - 0$ in the punctured plane. Then $F|U$ is orientable, so by Lemma 6, there is a flow of α in U whose trajectories are the leaves of $F|U$.

To construct ϕ, we choose a homeomorphism ϕ of η_1 onto $\{x = -1,\ 0 \leqslant y \leqslant 1\}$ which takes the endpoint of Λ to $(-1, 0)$. We construct a flow β on $\{-2 < x < 2,\ 0 \leqslant y \leqslant 1\} - (0,0)$, whose trajectories are the horizontal lines $y = \text{const.} \neq 0$, and the two sets $\{-2 < x < 0,\ y = 0\}$ and $\{0 < x < 2,\ y = 0\}$. We construct β so the x coordinate is increasing for the flow and a point in η_1 (other than the endpoint of Λ) takes the same time to reach η_1' under α as the corresponding point in $\{x = -1,\ 0 < y \leqslant 1\}$ takes to reach $\{x = 1,\ 0 < y \leqslant 1\}$ under β. We then extend ϕ to a homeomorphism of D onto the rectangle $\{-1 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$, which takes the flow α into the flow β. It is easily seen that ϕ has the required properties. □

THEOREM 8.2. – *If $\mathcal{E}$ is finite, but non-empty, then the number $|\mathcal{E}|$ of elements in $\mathcal{E}$ uniquely determines the topological type of the germ of* F *at the origin.*

Proof. – If $|\mathcal{E}| \geqslant 3$, this is a corollary of Theorem 8.1, and the other cases may be deduced by taking a finite covering of the punctured plane. □

The topological type of F for $|\mathcal{E}| = 1, 2, 3, 4$ is indicated in Fig. 8.2.

Theorems 8.1 and 8.2 are stated in [3], pp. 257-258.

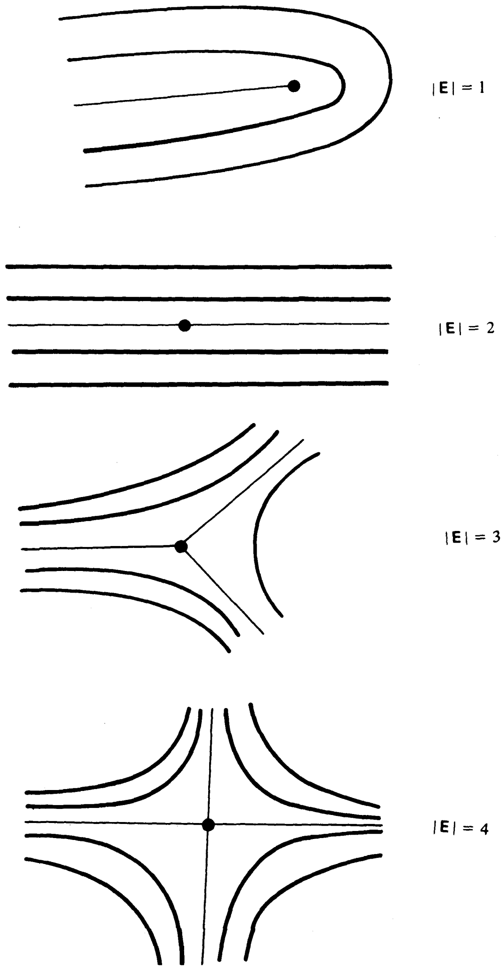

Fig. 8.2

9. Nodal regions.

Let F be a foliation of P. Let L be a leaf both of whose ends converge to the origin. Then $L \cup 0$ is a Jordan curve. Let R be the open subset of the plane which it bounds. Bendixson [1] calls R a *closed nodal region.*

THEOREM 9.1 (Bendixson [1], p. 19). – *Every end* e' *of a leaf* L' *in a closed nodal region converges to the origin.*

Proof. – Let L be the leaf which bounds the closed nodal region R. Let Γ be a simple closed curve in the plane enclosing the origin, and touching L in exactly one point. Let U be the open subset of the plane bounded by Γ. By Theorem 7, any end of any leaf of $F|U$ converges to the origin or to Γ. For a leaf in R, the second case is obviously excluded. □

Let η be a transversal to F. Let A and B be two points on η, and let Λ_A and Λ_B be two half-leaves whose endpoints are A and B and which lie on the same side of η. Suppose that Λ_A and Λ_B do not meet η except in their endpoints and that their ends converge to 0. Let η_0 be the closed arc in η bounded by A and B. Bendixson [1] called the region R in the plane bounded by $\Lambda_A \cup 0 \cup \Lambda_B \cup \eta_0$ an *open nodal region.*

THEOREM 9.2. – *Let* C *be a point on* η_0 *and let* Λ_C *be the half-leaf whose endpoint is* C *and which lies on the same side of* η_0 *as* Λ_A *and* Λ_B. *Then* Λ_C *does not meet* η_0 *except in* C, *and the end of* Λ_C *converges to the origin. Both ends of a leaf which lies entirely in* R *converge to the origin.*

In Fig. 9, we show an example where a leaf lies entirely in R.

Proof. – Suppose to the contrary that Λ_C does meet η_0 a second time. Let D be the first such point of intersection. Let ℓ be the closed arc in Λ_C joining C and D; let η_1 be the closed arc in η_0 joining C and D. Then $i(F, \eta_1 \cup \ell) = \pm \frac{1}{2}$, but $\eta_1 \cup \ell$ is null-homotopic in P, a contradiction.

The remaining statements follow from Theorem 7. □

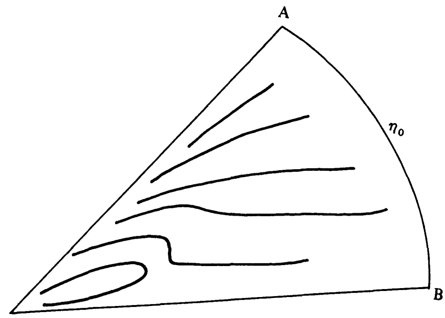

Fig. 9

10. On the structure of $\mathcal{E}$.

Let F be a foliation of P, and let $\mathcal{E} = \mathcal{E}(F_0)$.

Let Λ and Λ' be half-leaves whose ends e and e' converge to the origin. Let ζ be an arc in the plane, joining the endpoints of Λ and Λ' and not otherwise meeting $\Lambda \cup 0 \cup \Lambda'$. Let U be the open subset of the plane which the Jordan curve $\Lambda \cup 0 \cup \Lambda' \cup \zeta$ bounds. We will suppose that if Λ'' is a half-leaf not meeting $\Lambda \cup 0 \cup \Lambda' \cup \zeta$, with end $e'' \in \mathcal{E}$, then $ee''e'$ has positive orientation if and only if $\Lambda'' \subset U$.

THEOREM 10. – *One of the following alternatives holds:*

a) *There are (at most) finitely many* $e'' \in \mathcal{E}$ *such that* $ee''e'$ *has positive orientation, or*

b) U *contains an (open or closed) nodal region.*

In case b), U may contain both open and closed nodal regions.

Proof. – Let Σ be the set of points σ on ζ such that σ is the endpoint of a half-leaf lying in $\Lambda \cup U \cup \Lambda' \cup \zeta$ and meeting ζ only in σ, whose end converges to the origin.

Suppose Σ is infinite. Let τ be a point of accumulation of Σ and let η be a transversal through τ. Then η crosses infinitely many leaves which pass through Σ near their point of intersection with η. Two such leaves and a suitable arc of η determine an open nodal region. By shrinking it suitably, we may arrange that it is in U. Hence, b) holds when Σ is infinite.

Suppose Σ is finite. Let σ'' and σ''' be two successive points of Σ on ζ, counting from the endpoint of Λ to the endpoint of Λ'. Let Λ'' and Λ''' be the half-leaves in $\Lambda \cup \Lambda' \cup U \cup \zeta$ of which σ'' and σ''' are the endpoints. Let e'' and e''' be the ends of Λ'' and Λ'''.

Let ζ_0 be the closed arc in ζ whose endpoints are σ'' and σ'''. Let R be the open subset of the plane bounded by the Jordan curve $\Lambda'' \cup 0 \cup \Lambda''' \cup \zeta_0$. Obviously, $R \subset U$.

Suppose there exists $e^{(iv)} \in \mathcal{E}$ such that $e''e^{(iv)}e'''$ has positive orientation. Let $L^{(iv)}$ be the leaf of which $e^{(iv)}$ is the end. Points on $L^{(iv)}$ near $e^{(iv)}$ lie in R, since $e''e^{(iv)}e'''$ has positive orientation. Furthermore, $L^{(iv)}$ does not meet ζ_0, for otherwise Σ would have a point between σ'' and σ''' on ζ. Thus $L^{(iv)}$ lies in R. By Theorem 7, applied to $F|U$ for a suitable open disk U, the limit set of both ends of $L^{(iv)}$ is the origin. Hence, $L^{(iv)}$ determines a closed nodal region contained in R. Thus, b) holds in this case.

Suppose, on the other hand, that for each pair σ'', σ''' of successive points on Σ, there is no $e^{(iv)}$ as above. Then each $e'' \in \mathcal{E}$ such that $ee''e'$ has positive orientation corresponds to a point of Σ. It follows that the set of such e'' is finite, i.e. a) holds. □

11. Extensions of leaf-ends.

Let F be a foliation of P and let $e \in \mathcal{E} = \mathcal{E}(F_0)$. We will define the extension $\mathrm{Ext}(e)$ of e, as follows.

Let L be the leaf whose end is e. Consider an open transversal η meeting L in a single point x. Let Λ be the half-leaf in L whose endpoint is x and whose end is e. Let U_η be the union of all half-leaves which have an endpoint on η and which emanate from the same side of η as Λ. Let $\mathrm{Ext}(e) = \bigcap_\eta \overline{U}_\eta$, where the intersection is taken over all open transversals meeting L in exactly one point, and the closure is taken in the Riemann sphere.

Obviously, $\mathrm{Ext}(e)$ is a closed, connected set in the Riemann sphere, and $0 \in \mathrm{Ext}(e)$.

For any $e \in \mathcal{E}$, let L_e denote the leaf whose end is e.

THEOREM 11 [1, p. 25]. – *Let* K *be a compact subset of* P. *There are at most finitely many* $e \in \mathcal{E}$ *such that* L_e *meets* K *and* Ext(e) *also meets* K.

Proof. – Let Γ be a Jordan curve which separates the origin from K. For each $e \in \mathcal{E}$ such that L_e intersects Γ, let x_e be the first such point of intersection on L_e, counting from e. Let Σ be the set of such x_e for which L_e and Ext(e) meet K.

Were the conclusion of the theorem false, then Σ would be infinite. Suppose this is the case and let x be an accumulation point of Σ. Let η be an open transversal containing x. Let $x_{e(1)}, x_{e(2)}, x_{e(3)}$ be three points of Σ near x, and let Λ_i (for $i = 1, 2, 3$) be the half-leaf whose end is e_i, whose endpoint y_i is on η, and which does not otherwise intersect η. Suppose y_2 lies between y_1 and y_3 on η. Let η_0 be the subarc of η whose endpoints are y_1 and y_3. Then $\Lambda_1 \cup 0 \cup \Lambda_3 \cup \eta_0$ bounds an open nodal region R. We may suppose $\overline{R} \cap K = \phi$, by taking Γ small enough.

By Theorem 9.2, $U_{\eta(0)} \subset R$. Since $\mathrm{Ext}(e_2) \subset \overline{U}_{\eta(0)}$ and $\overline{R} \cap K = \phi$, it follows that $\mathrm{Ext}(e_2) \cap K = \phi$, which contradicts the assumption that $x_{e(2)} \in \Sigma$. This contradiction proves the theorem. □

COROLLARY 11.1. – *There are (at most) countably many* $e \in \mathcal{E}$ *such that* $\mathrm{Ext}(e) \neq 0$. □

COROLLARY 11.2. – *There are at most countably many* $e \in \mathcal{E}$ *which have neighbors.*

Proof. – Immediate from Theorem 8.1 and Corollary 11.1. □

CHAPTER III

PROOF OF THE MAIN THEOREM

12. Main theorem.

If F *is a foliation of the punctured plane, and* $\mathcal{E}(F_0)$ *is infinite, then there is a homeomorphism of topological pairs*

$$(\widetilde{P}(F_0), \beta(F_0)) \approx (S^1 \times [0, \infty), S^1 \times 0),$$

whose restriction to $\beta(F_0)$ *is cyclic order preserving.*

In this section, we will construct a homeomorphism

$$h_0 : P \xrightarrow{\approx} S^1 \times (0, \infty).$$

In § 13, we will show that h_0 extends to a homeomorphism

$$h : \widetilde{P} \xrightarrow{\approx} S^1 \times [0, \infty).$$

Construction of h_0. – Let $\mathcal{L}$ be the set of leaves of F having ends which are members of $\mathcal{E}$ and have neighbors in $\mathcal{E}$. By Corollary 11.2, $\mathcal{L}$ is (at most) countable. Hence there exists a countable set $\mathcal{L}'$ of leaves of F, disjoint from $\mathcal{E}$, such that $\cup \mathcal{L}'$ is a dense subset of P. Let $\mathcal{E}'$ be the set of ends of members of $\mathcal{L}'$ which converge to the origin. No member of $\mathcal{E}'$ has a neighbor in $\mathcal{E}$.

From Theorem 10 and the hypothesis that $\mathcal{E}$ is infinite, it follows that there exists at least one (open or closed) nodal region. Every leaf meeting a nodal region has at least one end which converges to the origin. Since $\cup \mathcal{L}'$ is dense in P, we obtain that $\mathcal{E}'$ is infinite. Obviously, $\mathcal{E}'$ is countable. Let $\mathcal{E}' = \{e_1, e_2, e_3, \ldots\}$ be an enumeration of the members of $\mathcal{E}'$.

We construct a sequence $\Gamma_1, \Gamma_2, \ldots$ of Jordan curves enclosing the origin and a sequence $\Lambda_1, \Lambda_2, \ldots$ of half-leaves, as follows. We let Γ_1 be a Jordan curve enclosing the origin, and Λ_1 a half-leaf whose end is e_1, whose endpoint lies on Γ_1, and which does not otherwise intersect Γ_1. Suppose, for the inductive step, that

$\Gamma_1, \ldots, \Gamma_{i-1}, \Lambda_1, \ldots, \Lambda_{i-1}$ have been chosen, the various Λ_j, $1 \leqslant j \leqslant i-1$, are mutually disjoint, and that e_j is the end of Λ_j. We then choose a Jordan curve Γ_i enclosing the origin, contained in the intersection of the disk of radius i^{-1} centered at the origin with the open set bounded by Γ_{i-1}, such that Γ_i meets each Λ_j, $1 \leqslant j \leqslant i-1$ in exactly one point and meets the leaf whose end is e_i. Such a Jordan curve exists by the Lemma 2 applied to $\bigcup_{j=1}^{i-1} \overline{\Lambda}_j$. We define Λ_i to be the half-leaf whose end is e_i and which meets Γ_i in just one point, its endpoint.

We will construct a sequence $\theta_1, \theta_2, \ldots$ of distinct points in S^1 such that the following properties hold.

a) The one-one correspondence $\theta_j \longleftrightarrow \Lambda_j \cap \Gamma_i$ between $\{\theta_1, \ldots, \theta_i\}$ and $\{\Lambda_1 \cap \Gamma_i, \ldots, \Lambda_i \cap \Gamma_i\}$ preserves cyclic order, where the first set is provided with the cyclic order induced from that on the circle, and the second set is provided with the cyclic order induced from that on Γ_i.

b) θ_i bisects the component of $S^1 - \{\theta_1, \ldots, \theta_{i-1}\}$ which contains it.

The construction is by induction. Let θ_1 be any point of S^1. Let θ_2 be the diametrically opposed point. Assuming $\theta_1, \ldots, \theta_{i-1}$ have been constructed, θ_i is uniquely determined by conditions a) and b). The existence of θ_i satisfying a) and b) is a consequence of the following remark: The one-one correspondence

$$\Lambda_j \cap \Gamma_{i-1} \longleftrightarrow \Lambda_j \cap \Gamma_i$$

between $\{\Lambda_1 \cap \Gamma_{i-1}, \ldots, \Lambda_{i-1} \cap \Gamma_{i-1}\}$ and $\{\Lambda_1 \cap \Gamma_i, \ldots, \Lambda_{i-1} \cap \Gamma_i\}$ preserves cyclic order. This follows from plane topology and the following facts: Γ_i is a Jordan curve in the open set bounded by the Jordan curve Γ_{i-1}, and each Λ_j, $1 \leqslant j \leqslant i-1$, intersects both Γ_{i-1} and Γ_i in exactly one point.

Using the Schoenflies theorem, one sees easily that there is a homeomorphism h_0 of P onto $S^1 \times (0, \infty)$ which sends Γ_i onto $S^1 \times i^{-1}$ and Λ_i onto $\theta_i \times (0, i^{-1}]$, for each $i = 1, 2, \ldots$.

In § 13, we will show that any such homeomorphism h_0 extends to a homeomorphism h of $\widetilde{\mathrm{P}}$ onto $S^1 \times [0, \infty)$.

13. Extension of h_0.

LEMMA 13.1. – $\{\theta_1, \theta_2, \ldots\}$ *is dense in* S^1.

Proof. – In view of the condition b) in the definition of the sequence $\theta_1, \theta_2, \ldots$, it is enough to prove that each component of $S^1 - \{\theta_1, \ldots, \theta_i\}$ contains some θ_ℓ, for each $i = 1, 2, \ldots$. In view of condition a) in this definition, we may restate the existence of such θ_ℓ in terms of the Γ_i and Λ_i, as follows.

Consider a component Γ_i^0 of $\Gamma_i - \{\Lambda_1 \cap \Gamma_i, \ldots, \Lambda_i \cap \Gamma_i\}$. Let $x = \Lambda_j \cap \Gamma_i$ and $y = \Lambda_k \cap \Gamma_i$ be the endpoints of $\overline{\Gamma}_i^0$. We will suppose x and y are chosen so that xzy has positive orientation, with respect to the cyclic order on Γ_i, for any $z \in \Gamma_i^0$. It is enough to prove that there exists $e_\ell \in \mathcal{E}'$ with $e_j e_\ell e_k$ positively oriented. For, then $(\Lambda_j \cap \Gamma_\ell, \Lambda_\ell \cap \Gamma_\ell, \Lambda_k \cap \Gamma_\ell)$ has positive orientation, with respect to the cyclic order on Γ_ℓ, and so θ_ℓ lies in the component of $S^1 - \{\theta_1, \ldots, \theta_i\}$ bounded by θ_j and θ_k.

There are an infinite number of $e \in \mathcal{E}$ such that $e_j e e_k$ has positive orientation. For, if there were only a finite number, then e_j would have a successor and e_k a predecessor, contrary to the fact that no member of $\mathcal{E}'$ has a neighbor in $\mathcal{E}$. Thus, alternative b) in Theorem 10 (for e_j and e_k in place of e and e') holds. Consider the nodal region R whose existence is guaranteed by alternative b) in Theorem 10.

Since $\cup \mathcal{L}'$ is dense in P, it intersects the open set R. Let $z \in (\cup \mathcal{L}') \cap R$. Let L be the leaf which contains z. By Theorems 9.1 and 9.2 there is a half-leaf $\Lambda \subset L$, with endpoint z, such that $\Lambda \subset R$ and its end e converges to the origin. Since $L \in \mathcal{L}'$ and e converges to the origin, $e \in \mathcal{E}'$. Let $e = e_\ell$. In view of the property of R stated in Theorem 10, $e_j e_\ell e_k$ is positively oriented. □

Since no member of $\mathcal{E}'$ has a neighbor in $\mathcal{E}$, $\pi : \mathcal{E}' \longrightarrow \beta$ is injective.

LEMMA 13.2. – $\pi(\mathcal{E}')$ *is dense in* β.

Proof. – Since β is the Dedekind completion of $\overline{\mathcal{E}} = \pi(\mathcal{E})$, this amounts to the assertion that if $e, e' \in \pi(\mathcal{E})$ and $\pi(e) \neq \pi(e')$, then there exists $e_i \in \mathcal{E}'$ such that $\pi(e)\,\pi(e_i)\,\pi(e')$ is positively oriented.

Since $\pi(e) \neq \pi(e')$, there are an infinite number of $e'' \in \mathcal{E}$ such that $ee''e'$ has positive orientation. Exactly the same argument as in the proof of Lemma 13.1 then shows that there exists $e_i \in \mathcal{E}'$ such that ee_ie' has positive orientation. Since no element of $\mathcal{E}'$ has a neighbor in $\mathcal{E}$, it follows that $\pi(e)\,\pi(e_i)\,\pi(e')$ has positive orientation. □

For $e_i \in \mathcal{E}'$, we define $h(\pi(e_i)) = \theta_i$. This defines a cyclic-order preserving bijection between $\pi(\mathcal{E}')$ and $\{\theta_1, \theta_2, \ldots\}$. Since $\pi(\mathcal{E}')$ is dense in the Dedekind complete set β, and $\{\theta_1, \theta_2, \ldots\}$ is dense in S^1, it follows that h extends uniquely to a cyclic order preserving bijection $h : \beta \longrightarrow S^1$.

We let $h : \widetilde{P} \longrightarrow S^1 \times [0, \infty)$ be h_0 on P and as just defined on β.

Proof that h *is a homeomorphism.* – Given three positive integers i, j, k we define V_{ijk} to be the set of $(\theta, t) \in S^1 \times [0, \infty)$ such that $\theta_j \theta_\ell \theta_k$ is positively oriented and $0 \leqslant t < i^{-1}$. We let $U_{ijk} = h_0^{-1}(V_{ijk})$. Then $U_{ijk} = U_\Gamma$ for a suitable $\Gamma \in \mathcal{J}$ (cf. the definition of the topology on $\widetilde{P}$ in the introduction). We let $\widetilde{U}_{ijk} = \widetilde{U}_\Gamma$.

Choose $\ell > i$ such that $\theta_j \theta_\ell \theta_k$ is positively oriented. The half-leaf $\Lambda_\ell = h_0^{-1}(\theta_\ell \times (0, \ell^{-1}])$ lies in U_{ijk}, and $e_j e_\ell e_k$ is positively oriented. It follows from the definition of $\widetilde{U}_{ijk}$ that $\widetilde{U}_{ijk} \cap \beta$ consists of all $x \in \beta$ such that $\pi(e_j)\, x\, \pi(e_k)$ is positively oriented. On the other hand, $h^{-1}(V_{ijk}) \cap \beta$ is the same set, since $h \mid \beta$ is cyclic order preserving. Hence, $h(\widetilde{U}_{ijk}) = V_{ijk}$.

From the fact that the θ_i are dense in S^1, it follows that the collection of sets V_{ijk}, together with the open subsets of $S^1 \times (0, \infty)$, form a basis of the topology of $S^1 \times [0, \infty)$. From the fact that $\pi(\mathcal{E}')$ is dense in β, it follows that the collection of sets $\widetilde{U}_{ijk}$, together with the open subsets of P, form a basis of the topology of $\widetilde{P}$. Hence h is a homeomorphism. □

BIBLIOGRAPHIE

[1] I. Bendixson, Sur les courbes définies par des équations différentielles, *Acta Math.*, 24 (1901), 1-88.

[2] A. Haefliger and G. Reeb, Variétés (non séparées) à une dimension et structures feuilletées du plan, *Enseignement Math.*, 3 (1957), 107-125.

[3] B. von Kerékjártó, *Vorlesungen über Topologie*, Berlin Springer-Verlag (1923).

[4] M.H.A. Newman, *Elements of the Topology of Plane Sets of Points*, Cambridge Univ. Press (1939).

[5] I. Richards, On the classification of noncompact surfaces, *Trans. Amer. Math. Soc.*, 106 (1963), 259-269.

Manuscrit reçu le 15 juillet 1981.

John N. Mather,
Princeton University
Department of Mathematics
Fine Hall
Box 37
Princeton N.J. 08544 (USA).