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DENSITY QUESTIONS IN THE CLASSICAL THEORY OF MOMENTS

by Ch. BERG and J.P.R. CHRISTENSEN

0. Introduction.

Let μ be a positive Borel measure on the real line having moments of all orders. The set $\mathcal{P}$ of polynomials in one variable is a subset of $\mathcal{L}^p(\mathbf{R}, \mu)$ for any number $p \in [1, \infty[$. We are interested in the following question:

For which p *and* μ *is* $\mathcal{P}$ *dense in* $\mathcal{L}^p(\mathbf{R}, \mu)$?

This question was answered completely long ago for $p = 1$ and $p = 2$, cf. Akhiezer [1] pp. 42-49:

Let V_μ denote the set of positive measures on the line having the same sequence of moments as μ. Then $\mathcal{P}$ is dense in $\mathcal{L}^1(\mathbf{R}, \mu)$ if and only if μ is an extreme point of V_μ (Naimark [5]), and $\mathcal{P}$ is dense in $\mathcal{L}^2(\mathbf{R}, \mu)$ if and only if μ is N-*extremal* (M. Riesz [7]). In particular $\mathcal{P}$ is dense in $\mathcal{L}^2(\mathbf{R}, \mu)$ and a fortiori in $\mathcal{L}^p(\mathbf{R}, \mu)$ for $1 \leqslant p \leqslant 2$ if μ is *determinate.*

We prove below that if μ is *indeterminate* then $\mathcal{P}$ is not dense in $\mathcal{L}^p(\mathbf{R}, \mu)$ for any $p > 2$. Furthermore, there exist indeterminate measures μ for which $\mathcal{P}$ is dense in $\mathcal{L}^p(\mathbf{R}, \mu)$ for $1 \leqslant p < 2$ but not for $p = 2$. Finally there exist determinate measures μ for which $\mathcal{P}$ is not dense in $\mathcal{L}^p(\mathbf{R}, \mu)$ for any $p > 2$, and there exist determinate measures for which $\mathcal{P}$ is dense in $\mathcal{L}^p(\mathbf{R}, \mu)$ for every $p \geqslant 1$.

In the indeterminate case V_μ is a compact convex metrisable set in the vague topology and we exhibit two dense subsets: (a) The measures having positive C^∞-density with respect to Lebesgue measure. (b) The set of extreme points of V_μ which are continuous and singular with respect to Lebesgue measure.

It is well-known that an indeterminate and N-extremal measure is discrete with mass in countably many points, which are the zeroes of a certain entire function. We prove that if the mass in a finite number n of these points is removed, and if arbitrary positive masses are added at n points different from the remaining points, the resulting measure is again indeterminate and N-extremal.

1. Preliminaries.

In the following we give a summary of some well-known facts from the theory of moments, cf. [1], [8]. Let $\mathscr{M}^*$ denote the set of positive Radon measures on $\mathbf{R}$ having moments of all orders and let $\mathscr{M}_1^*$ be the subset of probability measures. The set $\mathscr{M}^*$ inherits the vague topology from the set of positive Radon measures on $\mathbf{R}$ and is metrisable, cf. [2]. Two measures $\mu, \nu \in \mathscr{M}^*$ are called *equivalent* if

$$s_n = \int x^n \, d\mu(x) = \int x^n \, d\nu(x) \quad \text{for} \quad n = 0, 1, \ldots .$$

With each equivalence class of measures is thus associated a sequence $s = (s_n)_{n \geq 0}$ of moments. The classical theorem of Hamburger states that there is a one-to-one correspondence between the set of equivalence classes of measures and the set of positive semidefinite sequences, i.e. real sequences $(s_n)_{n \geq 0}$ for which $(s_{i+j})_{0 \leq i, j \leq n}$ is a positive semidefinite matrix for all $n \geq 0$. For $\mu \in \mathscr{M}^*$ the equivalence class in $\mathscr{M}^*$ containing μ is denoted V_μ. The set V_μ is clearly convex, and it is compact in the vague topology. For the last statement cf. [1] p. 31-32.

A measure $\mu \in \mathscr{M}^*$ (or the corresponding moment sequence) is called *determinate* if V_μ is a singleton, and *indeterminate* if V_μ consists of more than one measure.

Assume that $\mu \in \mathscr{M}_1^*$ has infinite support and let s be the corresponding moment sequence. Let $\mathscr{P}$ denote the vector space of polynomials with complex coefficients and let $L : \mathscr{P} \longrightarrow \mathbf{C}$ denote the linear form uniquely determined by

$$L(x^n) = s_n \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

Putting

$$(p, q) = L(p\bar{q}) \quad \text{for} \quad p, q \in \mathscr{P},$$

$(\cdot,\cdot)$ is an inner product on $\mathscr{P}$, and $(\mathscr{P},(\cdot,\cdot))$ is isometrically imbedded in the Hilbert space $L^2(\mathbf{R},\sigma)$ for all measures $\sigma \in V_\mu$. The associated orthonormal polynomials $(P_n)_{n\geqslant 0}$ (cf. [1] p. 3) depend only on s or V_μ. The function

$$h_n(x,y) = \sum_{k=0}^{n} P_k(x)\,P_k(y)\,,\quad x,y \in \mathbf{C} \tag{1}$$

is the reproducing kernel for the space of polynomials of degree $\leqslant n$, i.e.

$$p(x) = \int p(y)\,h_n(x,y)\,d\mu(y),\ x \in \mathbf{C} \tag{2}$$

for all polynomials p of degree $\leqslant n$.

If μ is *indeterminate* the series

$$h(x,y) = \sum_{k=0}^{\infty} P_k(x)\,P_k(y)\,,\quad x,y \in \mathbf{C}$$

converges uniformly on compact subsets of $\mathbf{C}\times\mathbf{C}$ to an entire holomorphic function. For $x \in \mathbf{C}$ and $\sigma \in V_\mu$ the function $h(x,\cdot)$ is in $\mathscr{L}^2(\mathbf{R},\sigma)$ and

$$\lim_{n\to\infty} \sum_{k=0}^{n} P_k(x)\,P_k = h(x,\cdot)\ \text{ in } \mathscr{L}^2(\mathbf{R},\sigma).$$

This implies that for all $p \in \mathscr{P}$

$$p(x) = \int p(y)\,h(x,y)\,d\sigma(y) \quad \text{for} \quad x \in \mathbf{C},\ \sigma \in V_\mu. \tag{3}$$

The function h is called the *reproducing kernel* associated with μ (or V_μ).

For an indeterminate $\mu \in \mathscr{M}_1^*$ we recall Nevanlinna's parametrization of V_μ, cf. [1] p. 98:

There is a one-to-one correspondence $\varphi \longleftrightarrow \sigma_\varphi$ between functions $\varphi \in \mathscr{N} \cup \{\infty\}$ and measures $\sigma_\varphi \in V_\mu$ established by

$$\int \frac{d\sigma_\varphi(x)}{x-z} = -\,\frac{A(z)\,\varphi(z) - C(z)}{B(z)\,\varphi(z) - D(z)}\,,\quad z \in \mathbf{C}\backslash\mathbf{R}\,, \tag{4}$$

where A, B, C, D are certain entire functions, and $\mathscr{N}$ is the class of Nevanlinna-Pick functions. A function φ belongs to $\mathscr{N}$ if it is holomorphic in $\mathbf{C}\backslash\mathbf{R}$ and satisfies $\varphi(\overline{z}) = \overline{\varphi(z)}$ and $\operatorname{Im}\varphi(z) \geqslant 0$

for $\operatorname{Im} z > 0$. On $\mathscr{N}$ we consider the topology of uniform convergence on compact subsets of $\mathbf{C} \setminus \mathbf{R}$. This topology is extended to $\mathscr{N} \cup \{\infty\}$ by defining a subset $G \subseteq \mathscr{N} \cup \{\infty\}$ such that $\infty \in G$ to be open, if $G \cap \mathscr{N}$ is open in $\mathscr{N}$, and if there exist a compact subset $K \subseteq \mathbf{C} \setminus \mathbf{R}$ and a number $L > 0$ such that

$$\{\varphi \in \mathscr{N} \mid \forall z \in K : |\varphi(z)| > L\} \subseteq G .$$

Then it is easy to see that $\mathscr{N} \cup \{\infty\}$ is a Hausdorff space and that the correspondence $\varphi \longleftrightarrow \sigma_\varphi$ is a *homeomorphism* between $\mathscr{N} \cup \{\infty\}$ and V_μ. It follows that $\mathscr{N}$ is locally compact and that $\mathscr{N} \cup \{\infty\}$ is the one-point compactification of $\mathscr{N}$.

The N-extremal measures in V_μ are precisely the measures σ_t for $t \in \mathbf{R} \cup \{\infty\}$, when t is identified with the constant t function in $\mathscr{N}$. A measure $\sigma_\varphi \in V_\mu$ is called *canonical* of *order* $m \geqslant 0$ if $\varphi \in \mathscr{N}$ is a real rational function of degree m. The N-extremal measures in V_μ are the canonical measures of order zero and σ_∞.

Concerning the density of $\mathscr{P}$ in $\mathscr{L}^p(\mathbf{R}, \mu)$, where $\mu \in \mathscr{M}^*$, we first remark that if $\mathscr{P}$ is dense in $\mathscr{L}^p(\mathbf{R}, \mu)$, then $\mathscr{P}$ is also dense in $\mathscr{L}^r(\mathbf{R}, \mu)$ for any $r \in [1, p]$. This is an immediate consequence of the fact that $\left(\int |f(x)|^r \, d\mu(x)\right)^{1/r}$ is an increasing function of $r \in [1, \infty[$, when μ is a probability measure. We next remark that if $\mathscr{P}$ is dense in $\mathscr{L}^p(\mathbf{R}, \mu)$ then also in $\mathscr{L}^p(\mathbf{R}, \nu)$ for $\nu = f \, d\mu$ if $f \in \mathscr{L}_+^\infty(\mu)$.

The following fundamental result is due to M. Riesz [7]: The set $\mathscr{P}$ is dense in $\mathscr{L}^2(\mathbf{R}, \mu)$ if and only if the measure $(1 + x^2)^{-1} \, d\mu(x)$ is determinate. Cf. also [3] p. 84.

In the following we need certain dense subsets of V_μ in the indeterminate case.

THEOREM 1. – *Let* $\mu \in \mathscr{M}_1^*$ *be indeterminate.*

a) *The set of measures* $\sigma \in V_\mu$ *which are canonical of some order is dense in* V_μ.

b) *The set of measures* $\sigma \in V_\mu$ *of the form* $\sigma = f(x) \, dx$, *where* f *is a positive* C^∞*-function, is dense in* V_μ.

Proof. – By the homeomorphism between V_μ and $\mathscr{N} \cup \{\infty\}$ the results follow if we prove that the corresponding sets of functions $\varphi \in \mathcal{N} \cup \{\infty\}$ are dense in $\mathcal{N} \cup \{\infty\}$. The functions $\varphi \in \mathcal{N}$ are given by the integral representation

$$\varphi(z) = \alpha z + \beta + \int_{-\infty}^{\infty} \frac{tz+1}{t-z}\, d\tau(t),$$

where $\alpha \geqslant 0$, $\beta \in \mathbf{R}$ and τ is a positive finite measure.

The set of functions $\varphi \in \mathcal{N}$ corresponding to (α, β, τ), where τ is a finite measure concentrated in finitely many points, is easily seen to be dense in $\mathcal{N} \cup \{\infty\}$, and this gives (a).

We next consider the set of functions $\varphi \in \mathcal{N}$ corresponding to (α, β, τ), where $\tau = g(t)\,dt$ and $g \in \mathcal{S}$ such that $g(t) > 0$ for all $t \in \mathbf{R}$, $\mathcal{S}$ being the Schwartz space of rapidly decreasing C^∞-functions.

This set is dense in $\mathcal{N} \cup \{\infty\}$, and we prove that $\sigma_\varphi \in V_\mu$ corresponding to such $\varphi \in \mathcal{N}$ is of the form

$$\sigma_\varphi = f(x)\,dx\,,$$

where f is a positive C^∞-function, and (b) follows.

Suppose therefore that

$$\varphi(z) = \alpha z + \beta + \int \frac{tz+1}{t-z}\, g(t)\,dt\,,$$

where $g \in \mathcal{S}$ and $g(t) > 0$ for all $t \in \mathbf{R}$, and let $\sigma_\varphi \in V_\mu$ be the corresponding measure such that (4) holds.

The Poisson kernel and the conjugate Poisson kernel for the half-plane $\operatorname{Im} z > 0$ are denoted P and Q respectively, so we have (cf. [9])

$$P_y(t) = \frac{1}{\pi}\,\frac{y}{t^2+y^2}\,, \quad Q_y(t) = \frac{1}{\pi}\,\frac{t}{t^2+y^2} \quad \text{for } t \in \mathbf{R},\ y > 0.$$

We can write φ in the form

$$\varphi(z) = \alpha z + \widetilde{\beta} + \int \frac{\widetilde{g}(t)}{t-z}\, dt\,,$$

with $\widetilde{\beta} = \beta - \int t g(t)\,dt$ and $\widetilde{g}(t) = (1+t^2)\,g(t) \in \mathcal{S}$, and hence for $z = x + iy$, $y > 0$

$$\begin{aligned}\varphi(z) &= \alpha z + \widetilde{\beta} - \pi Q_y * \widetilde{g}(x) + i\pi P_y * \widetilde{g}(x)\\ &= \alpha z + \widetilde{\beta} - \pi P_y * (\mathfrak{H}\widetilde{g} - i\widetilde{g})\,(x)\,,\end{aligned}$$

where

$$\mathfrak{H}g(x) = \lim_{\delta \to 0^+} \frac{1}{\pi} \int_{|u|>\delta} \frac{\widetilde{g}(x-u)}{u}\, du$$

denotes the Hilbert transform of $\widetilde{g}$. Since $\widetilde{g} \in \mathscr{S}$, $\mathfrak{H}\widetilde{g}$ is a C^∞-function tending to zero at infinity. Therefore $P_y * (\mathfrak{H}\widetilde{g} - i\widetilde{g})$ tends uniformly to $\mathfrak{H}\widetilde{g} - i\widetilde{g}$ as $y \longrightarrow 0$, and we find

$$\varphi^+(x) := \lim_{y \to 0^+} \varphi(x+iy) = \alpha x + \widetilde{\beta} - \pi \mathfrak{H}\widetilde{g}(x) + i\pi\widetilde{g}(x). \tag{5}$$

The functions A, B, C and D are entire functions, real on the real axis, and they satisfy

$$A(z)\,D(z) - B(z)\,C(z) = 1 \quad \text{for all } z \in \mathbf{C}. \tag{6}$$

Since

$$\operatorname{Im} \varphi^+(x) = \pi\widetilde{g}(x) > 0 \quad \text{for all} \quad x \in \mathbf{R},$$

we conclude that

$$B(x)\,\varphi^+(x) - D(x) \neq 0 \quad \text{for all} \quad x \in \mathbf{R},$$

and therefore

$$-\frac{A(z)\,\varphi(z) - C(z)}{B(z)\,\varphi(z) - D(z)} \longrightarrow -\frac{A(x)\,\varphi^+(x) - C(x)}{B(x)\,\varphi^+(x) - D(x)}$$

as $y \longrightarrow 0$, uniformly for x in compact subsets of $\mathbf{R}$.

It follows from the Stieltjes-Perron inversion formula that

$$f(x) = -\frac{1}{\pi} \operatorname{Im} \frac{A(x)\,\varphi^+(x) - C(x)}{B(x)\,\varphi^+(x) - D(x)}, \quad x \in \mathbf{R}$$

is density with respect to Lebesgue measure of the measure σ_φ, i.e.

$$\sigma_\varphi = f(x)\,dx.$$

Using (5) and (6) we find

$$f(x) = \frac{\widetilde{g}(x)}{[B(x)\,(\alpha x + \widetilde{\beta} - \pi\mathfrak{H}\widetilde{g}(x)) - D(x)]^2 + \pi^2 B(x)^2\,\widetilde{g}(x)^2},$$

which is a C^∞-function > 0 for all $x \in \mathbf{R}$. □

Remark. – A similar but simpler calculation leads to the following expression for the measure $\sigma_{\beta+i\gamma}$ corresponding to the constant function $\varphi(z) = \beta + i\gamma \in \mathscr{N}$, where $\beta \in \mathbf{R}$, $\gamma > 0$:

$$\sigma_{\beta+i\gamma} = \frac{\gamma}{\pi} \{(\beta B(x) - D(x))^2 + \gamma^2 B(x)^2\}^{-1} dx,$$

which has a positive analytic density with respect to Lebesgue measure.

The set $\mathscr{P}$ is not dense in $\mathscr{L}^1(\mathbf{R}, \sigma_{\beta+i\gamma})$ for any $\beta \in \mathbf{R}$, $\gamma > 0$. To see this we notice that

$$\sigma_{\beta_1+i\gamma_1} = \frac{\gamma_1}{\gamma_2} \frac{(\beta_2 B(x) - D(x))^2 + \gamma_2^2 B(x)^2}{(\beta_1 B(x) - D(x))^2 + \gamma_1^2 B(x)^2} \sigma_{\beta_2+i\gamma_2},$$

and the density function is bounded on $\mathbf{R}$ as is easily seen. The next lemma then shows the assertion.

LEMMA 1. – *Let* $\mu \in \mathscr{M}_1^*$ *be indeterminate and suppose that* $\sigma_1, \sigma_2 \in V_\mu$ *are different measures such that* $\sigma_2 = \varphi d\sigma_1$ *with* $\varphi \in \mathscr{L}^\infty(\mathbf{R}, \sigma_1)$. *Then* $\mathscr{P}$ *is not dense in* $\mathscr{L}^1(\mathbf{R}, \sigma_1)$.

Proof. – If we assume that $\mathscr{P}$ is dense in $\mathscr{L}^1(\mathbf{R}, \sigma_1)$ we get immediately that $\mathscr{P}$ is also dense in $\mathscr{L}^1(\mathbf{R}, \sigma_2)$ and in

$$\mathscr{L}^1(\mathbf{R}, 1/2(\sigma_1 + \sigma_2)).$$

By Naimark's theorem (cf. [1] p. 47)(*) this implies that σ_1, σ_2 and $1/2(\sigma_1 + \sigma_2)$ are all extreme points of V_μ. This is however in contradiction with $\sigma_1 \neq \sigma_2$. □

2. Density results in the indeterminate case.

Let $\mu \in \mathscr{M}_1^*$ be indeterminate and let $\sigma \in V_\mu$. By the theorem of M. Riesz [7], cf. [1] p. 43, it is known that $\mathscr{P}$ is dense in $\mathscr{L}^2(\mathbf{R}, \sigma)$ if and only if σ is N-extremal, and by a theorem of Naimark [5], cf. [1] p. 47, $\mathscr{P}$ is dense in $\mathscr{L}^1(\mathbf{R}, \sigma)$ if and only if σ is an extreme point of V_μ. The set of N-extremal measures $\sigma \in V_\mu$ is a compact subset of the set of extreme points of V_μ, which by Theorem 1 (a) is dense in V_μ, since canonical measures are extreme points of V_μ, cf. Corollary 2 below.

Our first main result is that there are no measures $\sigma \in V_\mu$ such that $\mathscr{P}$ is dense in $\mathscr{L}^p(\mathbf{R}, \sigma)$ when $p > 2$.

(*) The formulation in [1] contains a misprint: L_σ^2 shall be replaced by L_σ^1.

THEOREM 2. – *Let* $\mu \in \mathcal{M}_1^*$ *be indeterminate and let* $p > 2$. *Then* $\mathcal{P}$ *is not dense in* $\mathcal{L}^p(\mathbf{R}, \mu)$.

Proof. – Let q be the dual exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then $1 < q < 2$. For arbitrary $x \in \mathbf{C}$ fixed, $y \mapsto (x - y)\, h(x, y)$ is a function in $\mathcal{L}^q(\mathbf{R}, \mu)$. In fact, by Hölder's inequality we have

$$\int |(x - y)\, h(x, y)|^q \, d\mu(y) \leqslant \left(\int |h(x, y)|^2 \, d\mu(y)\right)^{q/2} \left(\int |x - y|^{\frac{2q}{2-q}} \, d\mu(y)\right)^{\frac{2-q}{2}} < \infty .$$

The reproducing property (3) of h gives

$$\int (x - y)\, p(y)\, h(x, y)\, d\mu(y) = 0 \quad \text{for all} \quad p \in \mathcal{P},$$

so assuming $\mathcal{P}$ dense in $\mathcal{L}^p(\mathbf{R}, \mu)$, we necessarily have

$$(x - y)\, h(x, y) = 0 \quad \mu\text{-a.e.}.$$

Since the function is continuous we get

$$(x - y)\, h(x, y) = 0 \quad \text{for} \quad x \in \mathbf{C},\ y \in \operatorname{supp}(\mu),$$

in particular

$$h(x, y) = 0 \quad \text{for} \quad x \in \mathbf{C} \backslash \mathbf{R},\ y \in \operatorname{supp}(\mu),$$

and by continuity

$$h(y, y) = \sum_{k=0}^{\infty} \mathrm{P}_k(y)^2 = 0 \quad \text{for} \quad y \in \operatorname{supp}(\mu),$$

which is a contradiction. □

THEOREM 3. – *Let* $\mu \in \mathcal{M}_1^*$ *be indeterminate and let* h *be the associated reproducing kernel. For all* N-*extremal measures* $\sigma \in \mathrm{V}_\mu$ *and all* $x \in \operatorname{supp}(\sigma)$ *the set of zeroes of the entire function* $y \mapsto h(x, y)$ *is precisely the set* $\operatorname{supp}(\sigma) \backslash \{x\}$. *In particular* $h(x, y) \neq 0$ *for* $x \in \mathbf{C} \backslash \mathbf{R}$, $y \in \mathbf{R}$.

Proof. – Let σ be a N-extremal measure. It is known that σ has the form $\sigma = \sum_{n=1}^{\infty} a_n \epsilon_{x_n}$ and $a_n h(x_n, x_n) = 1$ for all $n \in \mathbf{N}$, cf. [1] p. 114. For $p \in \mathcal{P}$ we have

$$p(x_n) = \int h(x_n, y)\, p(y)\, d\sigma(y) = h(x_n, x_n)\, p(x_n)\, a_n + \int_{\mathbf{R}\setminus\{x_n\}} h(x_n, y)\, p(y)\, d\sigma(y),$$

so that

$$\int_{\mathbf{R}\setminus\{x_n\}} h(x_n, y)\, p(y)\, d\sigma(y) = 0 .$$

This shows that the function

$$g(y) = \begin{cases} 0 & \text{for } y = x_n, \\ h(x_n, y) & \text{for } y \neq x_n, \end{cases}$$

is orthogonal to $\mathscr{P}$, which is dense in $\mathscr{L}^2(\mathbf{R}, \sigma)$, and hence $h(x_n, x_m) = 0$ for all $m \neq n$.

For $x, z \in \mathbf{C}\setminus\operatorname{supp}(\sigma)$ the function

$$g_{x,z}(y) = \frac{(x - y)\, h(x, y)}{z - y}$$

belongs to $\mathscr{L}^2(\mathbf{R}, \sigma)$. Writing $p \in \mathscr{P}$ as

$$p(y) = p(z) + (y - z)\, r(y)$$

with $r \in \mathscr{P}$, we find using (3)

$$\int g_{x,z}(y)\, p(y)\, d\sigma(y) = p(z)\, k_{x,z},$$

where

$$k_{x,z} = \int g_{x,z}(y)\, d\sigma(y) .$$

The constant $k_{x,z}$ is non-zero, because otherwise $g_{x,z}$ would be orthogonal to $\mathscr{P}$ and hence zero σ-a.e., so that

$$h(x, x_n) = 0 \quad \text{for} \quad n = 1, 2, \ldots .$$

By the reproducing property this implies that $p(x) = 0$ for all $p \in \mathscr{P}$ which is absurd.

Therefore the function

$$y \longmapsto \frac{(x - y)\, h(x, y)}{z - y} - k_{x,z}\, h(z, y),$$

which is in $\mathscr{L}^2(\mathbf{R}, \sigma)$, is orthogonal on $\mathscr{P}$ and thus zero for $y = x_1, x_2, \ldots$. This shows that

$$(x - x_n)\, h(x, x_n) = k_{x,z}(z - x_n)\, h(z, x_n) \quad \text{for} \quad n = 1, 2, \ldots .$$

If $h(x, x_n) = 0$ for some $x \in \mathbf{C} \setminus \mathrm{supp}(\sigma)$ and some n, we therefore get $h(z, x_n) = 0$ for all $z \in \mathbf{C} \setminus \mathrm{supp}(\sigma)$, and letting z tend to x_n we get $h(x_n, x_n) = 0$, which is a contradiction.

The last assertion follows from the fact that for every $y \in \mathbf{R}$ there exists a N-extremal measure $\sigma \in \mathrm{V}_\mu$ such that $y \in \mathrm{supp}(\sigma)$.

□

COROLLARY 1. – *Let* $\sigma \in \mathscr{M}_1^*$ *be* N-*extremal and let* h *be the associated reproducing kernel. For* $x, z \in \mathbf{C} \setminus \mathrm{supp}(\sigma)$ *the function*

$$y \longmapsto \frac{(x-y)\, h(x, y)}{(z-y)\, h(z, y)}$$

is constant for $y \in \mathrm{supp}(\sigma)$ *with value*

$$k_{x,z} = \int \frac{(x-t)\, h(x, t)}{z-t}\, d\sigma(t).$$

THEOREM 4. – *Let* $\mu \in \mathscr{M}_1^*$ *be indeterminate and canonical of order* $m \geqslant 1$. *Then* $\mathscr{P}$ *is dense in* $\mathscr{L}^p(\mathbf{R}, \mu)$ *for* $1 \leqslant p < 2$ *but not for* $p = 2$.

Proof. – Since a canonical measure of order $m \geqslant 1$ is not N-extremal, $\mathscr{P}$ is not dense in $\mathscr{L}^2(\mathbf{R}, \mu)$.

Let $z_1, \ldots, z_m$ be m different numbers in $\mathbf{C} \setminus \mathbf{R}$ and define a new measure σ by

$$\sigma = \left(\prod_{k=1}^{m} |x - z_k|^{-2}\right) d\mu(x).$$

It follows by [1] p. 121 that $\mathscr{P}$ is dense in $\mathscr{L}^2(\mathbf{R}, \sigma)$. To see that $\mathscr{P}$ is dense in $\mathscr{L}^p(\mathbf{R}, \mu)$ for $p \in [1, 2[$, if suffices to show that given a continuous function $f: \mathbf{R} \longrightarrow \mathbf{C}$ with compact support, there exists a polynomial R such that

$$\int |f(x) - \mathrm{R}(x)|^p\, d\mu(x)$$

is arbitrarily small. However, by Hölder's inequality

$$\int |f(x) - \mathrm{R}(x)|^p\, d\mu(x) = \int |f(x) - \mathrm{R}(x)|^p \prod_{k=1}^{m} |x - z_k|^2\, d\sigma(x)$$

$$\leqslant \left(\int |f(x) - \mathrm{R}(x)|^2\, d\sigma(x)\right)^{\frac{p}{2}} \left(\int \prod_{k=1}^{m} |x - z_k|^{\frac{4}{2-p}}\, d\sigma(x)\right)^{\frac{2-p}{2}},$$

which can be made arbitrarily small since $\mathscr{P}$ is dense in $\mathscr{L}^2(\mathbf{R}, \sigma)$. □

COROLLARY 2. – *Let* $\mu \in \mathscr{M}_1^*$ *be indeterminate. A measure* $\sigma \in V_\mu$, *which is canonical of some order* $\geqslant 0$, *is an extreme point of* V_μ.

Remark. – A canonical measure μ of order $m \geqslant 0$ is discrete and its support is the zero-set of an entire holomorphic function, hence a countable discrete set. This follows from Nevanlinna's parametrization. However, for the case $m = 0$ the discreteness property is a consequence of $\mathscr{P}$ being dense in $\mathscr{L}^2(\mathbf{R}, \mu)$.

In fact, suppose μ is indeterminate and such that $\mathscr{P}$ is dense in $\mathscr{L}^2(\mathbf{R}, \mu)$. For a continuous function f with compact support the function

$$F(x) = \int f(y)\, h(x, y)\, d\mu(y), \quad x \in \mathbf{C}$$

is entire, and f-F is orthogonal to $\mathscr{P}$. Therefore $f(x) = F(x)$ for all $x \in \operatorname{supp}(\mu)$. Since any continuous function with compact support equals an entire function on $\operatorname{supp}(\mu)$ we conclude that $\operatorname{supp}(\mu)$ is a discrete subset of $\mathbf{R}$.

It would be interesting to know if a similar result holds for $1 < p < 2$, i.e. if $\mathscr{P}$ being dense in $\mathscr{L}^p(\mathbf{R}, \mu)$ for an indeterminate μ implies that $\operatorname{supp}(\mu)$ is discrete. Such a result can definitely not be true for $p = 1$, as the following shows. We recall that a measure μ is *continuous* if $\mu(\{x\}) = 0$ for all $x \in \mathbf{R}$.

THEOREM 5. – *Let* $\mu \in \mathscr{M}_1^*$ *be indeterminate. The set of measures* $\sigma \in V_\mu$, *which are continuous and for which* $\mathscr{P}$ *is dense in* $\mathscr{L}^1(\mathbf{R}, \sigma)$, *is a dense* G_δ*-subset of* V_μ.

Proof. – The set of measures $\sigma \in V_\mu$ for which $\mathscr{P}$ is dense in $\mathscr{L}^1(\mathbf{R}, \sigma)$ is equal to the set $ex(V_\mu)$ of extreme points of V_μ, hence a dense G_δ-set, cf. [6].

We claim that the set $c(V_\mu)$ of continuous measures in V_μ is a dense G_δ-set. By Baire's theorem then follows that $ex(V_\mu) \cap c(V_\mu)$ is a dense G_δ-set.

To see that $c(V_\mu)$ is a G_δ-set we choose a decreasing sequence of continuous functions $\varphi_n : \mathbf{R} \longrightarrow [0, 1]$ such that $\varphi_n(0) = 1$,

$\operatorname{supp}(\varphi_n) \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right]$ and define

$$\Phi_n(\sigma) = \max_{x\in \mathbf{R}} \sigma * \varphi_n(x), \quad \Phi(\sigma) = \inf_{n\in \mathbf{N}} \Phi_n(\sigma) \quad \text{for} \quad \sigma \in V_\mu .$$

It is easy to see that $\Phi_n : V_\mu \longrightarrow [0, \infty[$ is continuous and that Φ is upper semicontinuous. It follows that

$$c(V_\mu) = \{\sigma \in V_\mu \mid \Phi(\sigma) = 0\}$$

is a G_δ-set.

That finally $c(V_\mu)$ is dense follows from Theorem 1 (b). □

Remark. – The set $s(V_\mu)$ of measures in V_μ which are singular with respect to Lebesgue measure is a dense G_δ-set. By Baire's theorem follows that

$$ex(V_\mu) \cap c(V_\mu) \cap s(V_\mu)$$

is a dense G_δ-set in V_μ, i.e. *the set of continuous singular extreme point of* V_μ *is a dense* G_δ-set.

The density of $s(V_\mu)$ follows from Theorem 1 (a). To see that $s(V_\mu)$ is a G_δ-set we choose a probability measure τ on $\mathbf{R}$ with a positive density, e.g. $\tau = \frac{1}{\pi}(1 + x^2)^{-1} dx$, and define

$$\Psi : V_\mu \longrightarrow \mathbf{R} \quad \text{by} \quad \Psi(\sigma) = \|\sigma - \tau\|,$$

the norm being the total variation. Then Ψ is lower semicontinuous and $s(V_\mu) = \{\sigma \in V_\mu \mid \Psi(\sigma) = 2\}$ is a G_δ-set.

3. Density results in the determinate case.

Let $\mu \in \mathscr{M}^*$ be determinate. By the theorem of M. Riesz [7] $\mathscr{P}$ is dense in $\mathscr{L}^p(\mathbf{R}, \mu)$ for $1 \leqslant p \leqslant 2$. There are of course many examples of measures μ for which $\mathscr{P}$ is dense in $\mathscr{L}^p(\mathbf{R}, \mu)$ for all $p \in [1, \infty[$. The following simple general result in this direction is stated and proved for the sake of completeness.

THEOREM 6. – *Let* $\mu \in \mathscr{M}^*$ *and suppose that there exists a number* $\alpha > 0$ *such that*

$$\int e^{\alpha|x|}\, d\mu(x) < \infty .$$

Then $\mathscr{P}$ is dense in $\mathscr{L}^p(\mathbf{R}, \mu)$ for all $p \in [1, \infty[$.

Remark. – By Theorem 2 we see that μ is determinate. This is also a simple consequence of Carleman's criterion, cf. e.g. [3] p. 80.

The following proof of Theorem 6 is inspired by [4]. Let $p \in [1, \infty[$, define $q \in]1, \infty]$ by $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $f \in \mathscr{L}^q(\mathbf{R}, \mu)$ satisfies

$$\int x^n f(x)\, d\mu(x) = 0 \quad \text{for} \quad n = 0, 1, \ldots .$$

It suffices to show that $f = 0$ μ-a.e. Defining

$$F(z) = \int e^{ixz} f(x)\, d\mu(x) \quad \text{for} \quad z \in \Omega = \left\{ z = u + iv \mid |v| < \frac{\alpha}{p} \right\},$$

we see by Hölder's inequality that F is holomorphic in Ω, and $F^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$. This shows that F is identically zero and hence $f = 0$ μ-a.e. □

It is perhaps surprising that there exist determinate measures μ for which $\mathscr{P}$ is not dense in $\mathscr{L}^p(\mathbf{R}, \mu)$ for any $p > 2$. This follows from the next theorem.

THEOREM 7. – *Let $\mu \in \mathscr{M}_1^*$ be indeterminate and* N*-extremal, hence of the form*

$$\mu = \sum_{n=0}^{\infty} a_n \epsilon_{x_n}.$$

The measure

$$\mu' = \mu - a_0 \epsilon_{x_0} = \sum_{n=1}^{\infty} a_n \epsilon_{x_n}$$

is determinate and $\mathscr{P}$ is not dense in $\mathscr{L}^p(\mathbf{R}, \mu')$ for any $p > 2$.

Proof. – It follows from the proof of Theorem 3.4 in [1] that μ' is determinate. For the sake of completeness we include the following proof which seems to be new:

Suppose that μ' is indeterminate and let $h'(x, y)$ be the reproducing kernel associated with $\mu'/\mu'(\mathbf{R})$. The function f defined by

$$f(x) = \begin{cases} h'(x_0, x) & \text{for} \quad x \neq x_0 \\ \dfrac{a_0 - 1}{a_0} & \text{for} \quad x = x_0 \end{cases}$$

defines a non-zero element in $\mathscr{L}^2(\mathbf{R}, \mu)$, and for any $p \in \mathscr{P}$ we have

$$\int p(x) f(x)\, d\mu(x) = a_0 p(x_0) f(x_0) + \int p(x)\, h'(x_0, x)\, d\mu'(x)$$

$$= (a_0 - 1)\, p(x_0) + \mu'(\mathbf{R})\, p(x_0) = 0$$

since $\mu'(\mathbf{R}) = 1 - a_0$. This contradicts the fact that $\mathscr{P}$ is dense in $\mathscr{L}^2(\mathbf{R}, \mu)$.

We next show that $\mathscr{P}$ is not dense in $\mathscr{L}^p(\mathbf{R}, \mu')$ where $p > 2$ is fixed. Let q denote the dual exponent such that $\frac{1}{p} + \frac{1}{q} = 1$. We choose $z \in \mathbf{C} \setminus \mathbf{R}$ and define

$$g(y) = (z - y)\,(x_0 - y)\, h(z, y), \quad y \in \mathbf{R},$$

where h is the reproducing kernel associated with μ. Then $g \in \mathscr{L}^q(\mathbf{R}, \mu)$ and a fortiori $g \in \mathscr{L}^q(\mathbf{R}, \mu')$, as is easily seen by Hölder's inequality, cf. the proof of Theorem 2. For $p \in \mathscr{P}$ we have by (3)

$$0 = \int p(y)\,(z - y)\,(x_0 - y)\, h(z, y)\, d\mu(y) = \int g(y)\, p(y)\, d\mu'(y),$$

and since g is not equal to 0 μ'-a.e. (Theorem 3), it follows that $\mathscr{P}$ is not dense in $\mathscr{L}^p(\mathbf{R}, \mu')$. □

4. On the support of N-extremal measures.

Let μ be an indeterminate N-extremal measure. By Theorem 7 we obtain a determinate measure when all the mass is removed at one point of the support of μ. By removing further the mass at a finite number of points the measure remains determinate. If we only change the mass at a finite number of points of $\operatorname{supp}(\mu)$, we obtain again an indeterminate N-extremal measure. This is an immediate consequence of the fact that the original measure and the new measure have bounded densities with respect to each other.

We shall now see that if we move a finite number of points of $\operatorname{supp}(\mu)$, then the measure remains indeterminate and N-extremal (but associated with a different sequence of moments). This follows from

THEOREM 8. – *Let* $\mu \in \mathscr{M}_1^*$ *be indeterminate and* N-*extremal, hence of the form*

$$\mu = \sum_{n=0}^{\infty} a_n \epsilon_{x_n} .$$

For $y \in \mathbf{R} \setminus \operatorname{supp}(\mu)$ *the measure*

$$\nu = a_0 \epsilon_y + \sum_{n=1}^{\infty} a_n \epsilon_{x_n}$$

is indeterminate and N-*extremal.*

In the proof of Theorem 8 we need

LEMMA 2. – *Let* $\mu \in \mathscr{M}^*$ *be determinate and suppose that* $x_0 \in \mathbf{R}$ *satisfies* $\mu(\{x_0\}) = 0$.

Then $\mathscr{P}$ *is dense in* $\mathscr{L}^2(\mathbf{R}, \mu + a\epsilon_{x_0})$ *for all* $a > 0$.

Proof. – Without loss of generality we may assume that μ is a probability measure, and we may assume that μ has infinite support, the case of finite support being trivial. From [1] p. 60-64 or [8] p. 45 follow that

$$\inf \left\{ \int |p(y)|^2 \, d\mu(y) \mid p \in \mathscr{P}, \ p(x_0) = 1 \right\} = \mu(\{x_0\}) = 0 . \qquad (7)$$

Let f be a continuous function with compact support and let $\epsilon > 0$ be given.

There exists $p \in \mathscr{P}$ such that

$$\|f - p\|_{\mathscr{L}^2(\mu)} < \frac{\epsilon}{2} ,$$

and because of (7) there exists $q \in \mathscr{P}$ satisfying

$$q(x_0) = f(x_0) - p(x_0) , \ \|q\|_{\mathscr{L}^2(\mu)} < \frac{\epsilon}{2} ,$$

hence

$$\|f - p - q\|_{\mathscr{L}^2(\mu + a\epsilon_{x_0})} = \|f - p - q\|_{\mathscr{L}^2(\mu)} < \epsilon . \qquad \square$$

Proof of Theorem 8. – Since $\nu - a_0 \epsilon_y$ is determinate by Theorem 7, we get by Lemma 2 that $\mathscr{P}$ is dense in $\mathscr{L}^2(\mathbf{R}, \nu)$.

We prove that ν is indeterminate by contradiction. If ν was determinate, we get by Lemma 2 that $\mathscr{P}$ is dense in $\mathscr{L}^2$ with respect to

$$\nu + a_0 \epsilon_{x_0} = \mu + a_0 \epsilon_y ,$$

which is indeterminate. (For $\tau_1, \tau_2 \in \mathcal{M}^*$ such that $\tau_1 \leqslant \tau_2$ we have: If τ_1 is indeterminate then τ_2 is indeterminate.) Having seen that $\mu + a_0 \epsilon_y$ is indeterminate and N-extremal we get by Theorem 7 that μ is determinate, which is a contradiction. □

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