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On 2-cycles of $B\,\mathrm{Diff}(S^1)$ which are represented by foliated S^1-bundles over T^2

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ON 2-CYCLES OF B Diff (S^1) WHICH ARE REPRESENTED BY FOLIATED S^1-BUNDLES OVER T^2

by Takashi TSUBOI

1. Introduction.

In this paper, we give several sufficient conditions for a 2-cycle of B Diff (S^1) represented by a foliated S^1-bundle over a 2-torus to be homologous to zero.

Consider $\mathscr{F}\Omega^r_{3,1}$, the group of C^r-foliated cobordism classes of oriented 3-manifolds with oriented codimension one foliations. By a result of Thurston [26, 27] together with the fact that closed oriented 4-manifolds of arbitrary Euler numbers exist, we have

$$\mathscr{F}\Omega^r_{3,1} \cong \Omega_3(B\bar{\Gamma}^r_1)$$

where $\Omega_3(B\bar{\Gamma}^r_1)$ is the 3-dimensional oriented bordism group of Haefliger's classifying space $B\bar{\Gamma}^r_1$ for $\bar{\Gamma}^r_1$ structures. Since $\Omega_i(X) = H_i(X;\mathbf{Z})$, $i = 0$, 1, 2, 3 . for any topological space X ([30]), we have

$$\Omega_3(B\bar{\Gamma}^r_1) \cong H_3(B\bar{\Gamma}^r_1;\mathbf{Z}).$$

We want to know the structure of these groups. In the case when $r = 0$, the classifying space $B\bar{\Gamma}^0_1$ is contractible (Mather [7]). In the case when $r \geqslant 2$, however, these groups are known to be very large. In fact, according to Thurston, the Godbillon-Vey homomorphism

$$GV : H_3(B\bar{\Gamma}^r_1;\mathbf{Z}) \to \mathbf{R} \ (r \geqslant 2)$$

is surjective ([24]). We are naturally interested in the kernel of the Godbillon-Vey homomorphism.

The Godbillon-Vey numbers of the following six types of foliations are zero; moreover, they are foliated cobordant to zero.

1° Bundle foliations of surface bundles over S^1.

2° Foliations of 3-manifolds which are defined by non-vanishing closed 1-forms.

3° The Reeb foliation of S^3 (Mizutani [11], Sergeraert [20]).

4° Foliations constructed by the spinnable structure of 3-manifolds (Fukui [1], Oshikiri [18]).

5° The foliation obtained by a foliated surgery along a transverse closed curve from a foliation cobordant to zero ([1], [18]).

6° Foliations of closed 3-manifolds whose leaves are noncompact, proper and without holonomy except finitely many compact leaves Mizutani-Morita-Tsuboi [12]).

In [28], Wallet proved that the Godbillon-Vey number of a $\mathrm{Diff}_K^\infty(\mathbf{R})_d$-bundle over a 2-torus is zero, and in [4], Herman generalized it for C^r-foliated ($r \geqslant 2$) S^1-bundles ($\mathrm{Diff}_+^r(S^1)_d$-bundles) over a 2-torus. Moreover, the Godbillon-Vey numbers of the following foliations are zero : foliations with finite depth and with abelian holonomy (Nishimori [17]); foliations without holonomy (Morita-Tsuboi [14], see also Mizutani-Tsuboi [13]); foliations which are almost without holonomy (Mizutani-Morita-Tsuboi [12]).

The question whether these foliations are foliated null-cobordant depends on the question whether foliated S^1-bundles over T^2 are foliated null-cobordant. This problem is closely related to the problem whether the corresponding 2-cycles of $B\,\mathrm{Diff}(S^1)_d$ are null-homologous.

A foliated S^1-bundle over a manifold N is a foliation $\mathscr{F}$ of the total space of an S^1-bundle over N such that every leaf of $\mathscr{F}$ is transverse to the fibers. Such a C^r-foliation is determined by the total holonomy homomorphism $\pi_1(N, *) \to \mathrm{Diff}^r(S^1)$ ($\mathrm{Diff}_+^r(S^1)$ if the foliation is transversely orientable), where $*$ is a base point of N (connected) and $\mathrm{Diff}^r(S^1)$ (resp. $\mathrm{Diff}_+^r(S^1)$) denotes the group of (resp. orientation preserving) C^r-diffeomorphisms of S^1.

Let $\mathrm{Diff}_+^r(S^1)_d$ denote the group $\mathrm{Diff}_+^r(S^1)$ equipped with the discrete topology. ($\mathrm{Diff}_+^r(S^1)$ has the natural C^r-topology). Transversely oriented C^r-foliated S^1-bundles are considered to be $\mathrm{Diff}_+^r(S^1)_d$-bundles. The classifying space $B\,\mathrm{Diff}_+^r(S^1)_d$ for $\mathrm{Diff}_+^r(S^1)_d$-bundles is defined and $H_*(B\,\mathrm{Diff}_+^r(S^1)_d)$ is isomorphic to $H_*(\mathrm{Diff}_+^r(S^1))$, the homology of the abstract group $\mathrm{Diff}^r_+(S^1)$. We also consider $\mathrm{Diff}^r_K(\mathbf{R})_d$-bundles and their classifying space $B\,\mathrm{Diff}^r_K(\mathbf{R})_d$, where $\mathrm{Diff}^r_K(\mathbf{R})_d$ denotes the group of C^r-diffeomorphisms of $\mathbf{R}$ with compact support equipped with the discrete topology.

Since a transversely oriented C^r-foliated S^1-bundle over a closed oriented 2-manifold is a transversely oriented foliation of the total space of the S^1-bundle, we have a natural homomorphism

$$s : H_2(\mathrm{Diff}^r_+(S^1);\mathbf{Z}) \to H_3(B\bar{\Gamma}^r_1;\mathbf{Z}).$$

On the other hand, according to Mather [9], there is an isomorphism

$$\sigma : H_2(\mathrm{Diff}^r_K(\mathbf{R});\mathbf{Z}) \to H_3(B\bar{\Gamma}^r_1;\mathbf{Z}).$$

Choosing an embedding i of $\mathbf{R}$ to an open interval of S^1, we have a homomorphism

$$i_* : H_2(\mathrm{Diff}^r_K(\mathbf{R});\mathbf{Z}) \to H_2(\mathrm{Diff}^r_+(S^1);\mathbf{Z})$$

and the following commutative diagram :

$$\begin{tikzcd}
H_2(\mathrm{Diff}^r_K(\mathbf{R});\mathbf{Z}) \arrow[d, "i_*"] \arrow[dr, "\sigma"] & \\
& H_3(B\bar{\Gamma}^r_1;\mathbf{Z}) \\
H_2(\mathrm{Diff}^r_+(S^1);\mathbf{Z}) \arrow[ur, "s"'] &
\end{tikzcd}$$

This diagram implies that s is surjective. A more interesting fact is the following : every element of $H_3(B\bar{\Gamma}^r_1;\mathbf{Z})$ is represented by a C^r-foliated S^1-bundle over Σ_k, where $k \leqslant 2$ and Σ_k denotes a closed oriented 2-manifold of genus k, the reason being that σ is an isomorphism and that there is a fiber preserving embedding of a trivial $\mathbf{R}$-bundle over $\Sigma_k (k \geqslant 2)$ into a trivial S^1-bundle over Σ_2.

Our main theorem concerns $\mathrm{Diff}^\infty_K(\mathbf{R})_d$-bundles over T^2. We do not know whether every $\mathrm{Diff}^r_+(S^1)_d$-bundle over T^2 is homologous to a union of $\mathrm{Diff}^r_+(S^1)_d$-bundles which belong to the image of i_*. We note that, for a 2-cycle of $B\,\mathrm{Diff}^r_+(S^1)_d$ to belong to the image of i_*, it is necessary that its Euler class is zero. In the case of a $\mathrm{Diff}^r_+(S^1)_d$-bundle over T^2, it is known that its Euler class is zero (Wood [29]).

Our plan is as follows.

In § 2, we review the homology of groups. For a transversely oriented C^r-foliated S^1-bundle over T^2, we have a homomorphism

$$\psi : \mathbf{Z}^2 \cong \pi_1(T^2, *) \to \mathrm{Diff}^r_+(S^1).$$

Put $f = \psi(1,0)$ and $g = \psi(0,1)$. Then, since $fg = gf$, $(f, g) - (g, f)$ is a 2-cycle of $B\,\mathrm{Diff}^r_+(S^1)$ and it is homotopic to the image of the classifying map $B\psi$. In this paper, $\{f, g\}$ denotes the class represented by $(f, g) - (g, f)$. We will prove some formulae for $\{f, g\}$

In § 3, we study the structure of commuting diffeomorphisms of the real line and the circle which have fixed points. We sum up results of Sternberg [22], Kopell [5], Takens [23], Sergeraert [20] and Wallet [28], and state Theorem (3.1) which shows that the structure of such commuting diffeomorphisms is fairly simple. This implies that foliated S^1-bundles over T^2 have fairly simple structure. They have been classified up to topological equivalence (Moussu-Roussarie [15]). Theorem (3.1) gives a more precise classification and the background of our main theorem, Theorem (6.1).

In § 4, we give some preliminary theorems necessary in the later sections. There, results of Mather [8, 10] and Sergeraert [20] play an important role. We also mention some of the attempts, which we made at the beginning of our study in this direction, to prove $\{f, g\} = 0$ when f and g belong to a one parameter subgroup generated by a smooth vectorfield. These attempts give some new examples of foliated S^1-bundles which are null-cobordant. Moreover, the proof of Theorem (4.5) inspired us the proof of our main theorem.

After a discussion on the construction of smooth diffeomorphisms in § 5, we state our main theorem, Theorem (6.1), in § 6. Theorem (6.1) says that, under some conditions on the norms of the commuting diffeomorphisms f and g, the class $\{f, g\}$ is zero in $H_2(\mathrm{Diff}^\infty_K(\mathbf{R}); \mathbf{Z})$. In particular, if f and g belong to a one parameter group of transformations generated by a smooth vectorfield on $\mathbf{R}$ with compact support, then $\{f, g\} = 0$. One expects that one might remove the condition on the norms, but we have not been able to do it so far. As a corollary to our main theorem, we can see that, every C^∞-foliated S^1-bundle over T^2 has a C^∞-foliated S^1-bundle which is topologically equivalent to it and which is C^∞-foliated null-cobordant.

In § 7, we show that our main theorem, (6.1), follows from Theorem (6.3) which says that a C^∞-diffeomorphism of $\mathbf{R}$ with compact support can be written as a composition of commutators of C^∞-diffeomorphisms whose supports are contained in that of the original one. Theorem (6.3) is a generalization of a theorem of Sergeraert [20].

We devote §§ 8 and 9 to examining a method of writing a diffeomorphism of [0,1] close to the identity as a composition of commutators. We show that a diffeomorphism sufficiently close to the identity can be written as a composition of commutators of diffeomorphisms which are close to the identity. In § 8, we treat diffeomorphisms of [0,1] whose supports are contained in (1/8, 7/8). There, we use a device of Mather [10] and an implicit function theorem of Sergeraert [19]. In § 9, we treat $\mathrm{Diff}^\infty_\infty([0,1])$ and analyze with care the proof of a theorem of Sergeraert [20] which says $H_1(\mathrm{Diff}^\infty_\infty([0,1]); \mathbf{Z}) = 0$.

In § 10, we prove Theorem (6.3); this will complete the proof of our main theorem, Theorem (6.1).

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2. Lemmas for the group homology.

In this section we prove some lemmas concerning the group homology. First, we recall the definition of the homology group (with integral coefficients) of a group G.

The homology of a group G is the homology of the following complex :

$$\{0\} \xleftarrow{\partial} \mathbf{Z} \xleftarrow{\partial} \mathbf{Z}[G] \xleftarrow{\partial} \mathbf{Z}[G\times G] \xleftarrow{\partial} \mathbf{Z}[G\times G\times G] \xleftarrow{\partial},$$

where $\mathbf{Z}[G^n] = \mathbf{Z}[G\times G\times\cdots\times G]$ is the free abelian group generated by n-tuples $(f_1,\ldots,f_n)$ $(f_i\in G, i=1,\ldots,n)$.

For $(f_1,\ldots,f_n)\in \mathbf{Z}[G^n]$ $(n\geqslant 2)$,

$$\partial(f_1,\ldots,f_n) = (f_2,\ldots,f_n) + \sum_{i=1}^{n-1} (-1)^i(f_1,\ldots,f_if_{i+1},\ldots,f_n) + (-1)^n(f_1,\ldots,f_{n-1}),$$

and for $(f)\in \mathbf{Z}[G]$, $\partial(f) = 0$.

If $f_1f_2 = f_2f_1 (f_1,f_2 \in \mathrm{G})$, $(f_1,f_2) - (f_2,f_1)$ is a 2-cycle of the above complex. Let $\{f_1,f_2\}$ denote the class of $(f_1,f_2) - (f_2,f_1)$. Obviously, $\{f_1,f_2\} = -\{f_2,f_1\}$.

The inner automorphisms act trivially on the homology group of a group ([6]). In particular, we have

LEMMA (2.1). – *Let f, g, h be elements of* G. *Suppose that $fg = gf$. Then $\{h^{-1}fh,h^{-1}gh\} = \{f,g\}$.*

Proof. – For the sake of completeness, we show the equality. By a direct computation using $fg = gf$, we have

$$\begin{aligned}\partial\{(f,h,h^{-1}gh) - (h,h^{-1}fh,h^{-1}gh) + (h,h^{-1}gh,h^{-1}fh)& \\ - (g,h,h^{-1}fh) + (g,f,h) - (f,g,h)\}& \\ = \{(f,g)-(g,f)\} - \{(h^{-1}fh,h^{-1}gh)-(h^{-1}gh,h^{-1}fh)\}&.\end{aligned}$$

Before stating other lemmas we note the following. Since

$$\Omega_2(\mathrm{BG}) \cong \mathrm{H}_2(\mathrm{BG};\mathbf{Z}) \cong \mathrm{H}_2(\mathrm{G};\mathbf{Z})$$

and BG is an Eilenberg-MacLane space, every 2-cycle of BG is represented by a homomorphism $\psi : \pi_1(\Sigma,*) \to \mathrm{G}$, where Σ is a closed oriented 2-manifold. Moreover, this cycle is null homologous if and only if there is an oriented 3-manifold W which bounds Σ and a homomorphism $\tilde{\psi} : \pi_1(\mathrm{W},*) \to \mathrm{G}$ such that the following diagram commutes :

$$\begin{tikzcd} \pi_1(\Sigma,*) \arrow[r, "\psi"] \arrow[d, "\iota_\#"] & \mathrm{G} \\ \pi_1(\mathrm{W},*) \arrow[ur, "\tilde{\psi}"'] & \end{tikzcd}$$

where $* \in \Sigma = \partial\mathrm{W}$ and $\iota : \partial\mathrm{W} \to \mathrm{W}$. We also note that, when $fg = gf$, the class $\{f,g\}$ is represented by the homomorphism $\psi : \mathbf{Z}^2 = \pi_1(\mathrm{T}^2,*) \to \mathrm{G}$ defined by $\psi(1,0) = f$ and $\psi(0,1) = g$.

LEMMA (2.2). – *Let $f, g_1, \ldots, g_n$ be elements of* G. *Suppose that $fg_i = g_if (i=1,\ldots,n)$. Then*

$$\{f,g_1\cdots g_n\} = \sum_{i=1}^{n} \{f,g_i\}$$

and

$$\{g_1 \cdots g_n, f\} = \sum_{i=1}^{n} \{g_i, f\}.$$

Proof. – Consider n disjoint disks $D_1, \ldots, D_n$ in a 2-disk D^2. Put $V = \left(D^2 - \bigcup_{i=1}^{n} \operatorname{Int} D_i\right) \times S^1$. Then $\pi_1(V,(*,0))$ is isomorphic to $(\mathbf{Z} * \overset{n}{\ldots} * \mathbf{Z}) \times \mathbf{Z}$, where $(*,0) \in V$, $\partial D_i \times \{0\}$'s and $\{*\} \times S^1$ correspond to generators of $\pi_1(V,(*,0))$ and

$$[\partial D_1 \times \{0\}] \ldots [\partial D_n \times \{0\}] = [\partial D^2 \times \{0\}].$$

We can define a homomorphism $\psi : \pi_1(V,(*,0)) \to G$ so that

$$\psi[\partial D_i \times \{0\}]) = g_i \quad (i=1,\ldots,n)$$

and

$$\psi([\{*\} \times S^1]) = f.$$

Since the boundary $\partial B\psi$ of the classifying map $B\psi : V \to BG$ represents $\{f, g_1 \ldots g_n\} - \Sigma\{f, g_i\}$, we have proved Lemma (2.2).

It is worthwhile to note $\{f, id\} = 0$. For,

$$(id, f) - (f, id) = \partial(f, id, f).$$

Using Lemma (2.2), we have

LEMMA (2.3). – *Let f, g be commuting elements of G. Then for integers m, n, we have* $\{f^m, g^n\} = mn\{f, g\}$.

LEMMA (2.4). – *Let f, h be elements of G. Suppose that f commutes with hfh^{-1}. Then f commutes with $hfh^{-1}h^{-1}fh$* and $\{f, hfh^{-1}h^{-1}fh\} = 0$.

Proof. – By Lemma (2.1), we have

$$\begin{aligned}\{f, hfh^{-1}\} &= \{h^{-1}fh, h^{-1}(hfh^{-1})h\} \\ &= \{h^{-1}fh, f\} \\ &= -\{f, h^{-1}fh\}.\end{aligned}$$

Therefore we have

$$\begin{aligned} 0 &= \{f,hfh^{-1}\} + \{f,h^{-1}fh\} \\ &= \{f,hfh^{-1}h^{-1}fh\}. \end{aligned}$$

LEMMA (2.5). – *Let* ψ *be a homomorphism from* $\mathbf{Z}^2$ *to* G. *Let* α, β *and* α', β' *be two pairs of oriented generators of* $\mathbf{Z}^2$. *Then*

$$\{\psi(\alpha),\psi(\beta)\} = \{\psi(\alpha'),\psi(\beta')\}.$$

Proof. – Since $\mathbf{Z}^2 \cong \pi_1(\mathrm{T}^2,*)$, this lemma may look trivial from the topological view point. However, for the sake of completeness, we shall prove it. Since the orientation preserving automorphism group of $\mathbf{Z}^2$ is $\mathrm{SL}(2,\mathbf{Z})$, we have

$$\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix} = \begin{pmatrix} p & q\\ r & s\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \qquad \begin{pmatrix} p & q\\ r & s\end{pmatrix} \in \mathrm{SL}(2,\mathbf{Z}).$$

From Lemmas (2.2), (2.3) and the fact that $\{f,f\} = 0$, it follows that

$$\begin{aligned} \{\psi(\alpha'),\psi(\beta')\} &= \{(\psi(\alpha))^p(\psi(\beta))^q,(\psi(\alpha))^r(\psi(\beta))^s\} \\ &= pr\{\psi(\alpha),\psi(\alpha)\} + ps\{\psi(\alpha),\psi(\beta)\} \\ &\qquad + qr\{\psi(\beta),\psi(\alpha)\} + qs\{\psi(\beta),\psi(\beta)\} \\ &= (ps-qr)\{\psi(\alpha),\psi(\beta)\} \\ &= \{\psi(\alpha),\psi(\beta)\}. \end{aligned}$$

LEMMA (2.6). – *Let* f, g *be commuting elements of* G. *Suppose that there exist coprime integers* m, n *such that* $f^m g^n = id$. *Then* $\{f,g\} = 0$.

Proof. – Let r, s be integers such that $ms - nr = 1$. Consider a change of generators as in Lemma (2.5); then we have

$$\{f,g\} = \{f^m g^n, f^r g^s\} = \{id, f^r g^s\} = 0.$$

LEMMA (2.7). – *Let* f, g *be commuting elements of* G. *Suppose that there exist elements* $h_1, \ldots, h_{2k}$ *of* G *such that*

$$g = \prod_{j=1}^{k} [h_{2j-1},h_{2j}]$$

and

$$fh_j = h_j f \quad (j=1,\dots,2k).$$

Then $\{f,g\} = 0$.

Proof. – Let Σ_k denote a closed oriented 2-manifold of genus k, and let D^2 denote a (small) disk on it. Consider $V = S^1 \times (\Sigma_k - \mathrm{Int}(D^2))$; then

$$\pi_1(V,(0,*)) = \langle \alpha,\beta_1,\dots,\beta_{2k} : \alpha\beta_i = \beta_i\alpha \, (i=1,\dots,2k)\rangle$$

$((0,*) \in S^1 \times \partial(\Sigma_k - \mathrm{Int}(D^2)))$, where α is represented by the curve $S^1 \times \{*\}$ and the curve $\{0\} \times \partial(\Sigma_k - \mathrm{Int}(D^2))$ represents a product of commutators $[\beta_1,\beta_2] \dots [\beta_{2k-1},\beta_{2k}]$.

Since f and h_i 's commute, there is a well-defined representation ψ of $\pi_1(V,(0,*))$ in G such that

$$\psi(\alpha) = f \quad \text{and} \quad \psi(\beta_i) = h_i \; (i=1,\dots,2k).$$

Considering the classifying map $B\psi: V \to BG$, we have $\partial B\psi(V) = (f,g) - (g,f)$, that is, $\{f,g\} = 0$.

3. Commuting diffeomorphisms of the real line and the circle.

In this section we prove the following theorem.

THEOREM (3.1). – *Let f, g be elements of* $\mathrm{Diff}^\infty_+(\mathbf{R})$ *which have fixed points;* $\mathrm{Fix}(f) \neq \varnothing$, $\mathrm{Fix}(g) \neq \varnothing$. *Suppose that f and g are commuting with each other;* $fg = gf$.

Then there is a countable family $\{I_i\}$ *of disjoint open intervals satisfying the following conditions.*

(1) $f \,|\, \mathbf{R} - \cup I_i = id_{\mathbf{R}-\cup I_i}$ *and* $g \,|\, \mathbf{R} - \cup I_i = id_{\mathbf{R}-\cup I_i}$

(2) *For* $f \,|\, \bar{I}_i$ *and* $g \,|\, \bar{I}_i$,

either (A) *there exist coprime integers* m_i, n_i *and a smooth diffeomorphism* h_i *of* $\mathbf{R}$, *such that* $\mathrm{Supp}(h_i) = \bar{I}_i$,

$$f \,|\, \bar{I}_i = (h_i|\bar{I}_i)^{m_i} \quad \text{and} \quad g\,|\,\bar{I}_i = (h_i|\bar{I}_i)^{n_i},$$

or (B) *there exist real numbers* s_i, t_i *and a vectorfield* ξ_i, C^1 *on* $\mathbf{R}$ and C^∞ *on* $\mathbf{R} - \partial I_i$, *such that* $\mathrm{Supp}\,(\xi_i) = \bar{I}_i$, $f|I_i$ *and* $g|\bar{I}_i$ *are the time* s_i *map and the time* t_i *map of* $\xi_i|\bar{I}_i$, *respectively.*

Remark. — For commuting diffeomorphisms of S^1 with fixed points, we have a theorem similar to Theorem (3.1). For, such commuting diffeomorphisms lift to those of $\mathbf{R}$ with fixed points.

This theorem clarifies the structure of the centralizer of a diffeomorphism of $\mathbf{R}$ with fixed points, and we can see to what extent our main theorem to be given in § 6 is effective.

To prove Theorem (3.1), we need some lemmas. Let $\mathbf{R}_+$ denote the set of non-negative real numbers.

LEMMA (3.2) (Kopell [5]). — *Let* f *be an element of* $\mathrm{Diff}^r(\mathbf{R}_+)$ $(r \geqslant 2)$ *such that* $\bigcap_{i=1}^{\infty} f^i([0,x)) = \{0\}$ *for any* $x \in \mathbf{R}_+ - \{0\}$. *Let* g *be a* C^1-*diffeomorphism of* $\mathbf{R}_+$ *which commutes with* f. *Suppose that there exists a point* $y \in \mathbf{R}_+ - \{0\}$, *such that* $g(y) = y$. *Then* $g = id_{\mathbf{R}_+}$.

Moreover, if $j_0^r(f) \neq j_0^r(id)$, *then* g *is completely determined by* $j_0^{r_0}(g)$, *where* $r_0 = \min\{s\,;j_0^s(f) \neq j_0^s(id)\}$.

LEMMA (3.3). — *Let* f, g *be commuting* C^r-*diffeomorphisms* $(r \geqslant 2)$ *of* $\mathbf{R}$ *which have fixed points. Suppose that* $f(0) = 0$ *and the germ of* f *at* 0 *is not that of the identity. Then* $g(0) = 0$.

Proof. — Suppose that $g(0) \neq 0$. Since $fg^n(0) = g^n f(0) = g^n(0)$, $g^n(0)$ is a fixed point of f. Put $\inf\{g^n(0)\,;n \in \mathbf{Z}\} = a$ and $\sup\{g^n(0)\,;n \in \mathbf{Z}\} = b$. Since $\mathrm{Fix}(g) \neq \varnothing$, either $a \neq -\infty$ or $b \neq +\infty$. If $a \neq -\infty$, the point a is a fixed point of f. Applying Lemma (3.2) to $g|[a,b)$ and $f|[a,b)$, we have $f|[a,b) = id_{[a,b)}$, which contradicts the assumption. If $b \neq +\infty$, we have $b \in \mathrm{Fix}(f)$. Applying Lemma (3.2) to $g|(a,b]$ and $f|(a,b]$, we have the same contradiction.

LEMMA (3.4) (Sternberg [22], Kopell [5], Takens [23], Sergeraert [20]). — *Let* f *be a* C^∞-*diffeomorphism of* $\mathbf{R}_+$ *such that* $\mathrm{Fix}(f) = \{0\}$. *If* $j_0^\infty(f) \neq j_0^\infty(id)$, *then there is a* C^∞-*vectorfield* ξ *such that* $j_0^\infty(\xi) \neq j_0^\infty(0)$, f *is the time one map of* ξ, *and the centralizer of* f *in* $\mathrm{Diff}^\infty(\mathbf{R}_+)$ *coincides with the one parameter group of transformations generated by* ξ.

If $j_0^\infty(f) = j_0^\infty(id)$, *there is a vectorfield* ξ, C^1 *on* $[0,\infty)$ *and* C^∞ *on* $(0,\infty)$, *such that* $j_0^1(\xi) = j_0^1(0)$, f *is the time one map of* ξ, *and the centralizer of* f *coincides with the set of those elements of the one parameter group of transformations generated by* ξ *which are of class* C^∞

LEMMA (3.5). – *Let* f *be a* C^∞*-diffeomorphism of* $[0,1]$ *such that* $\mathrm{Fix}(f) = \{0,1\}$. *Let* g *be a* C^∞*-diffeomorphism which commutes with* f. *Then*

either (A) *there are coprime integers* m, n *and an element* h *of* $\mathrm{Diff}^\infty([0,1])$ *such that* $f = h^m$ *and* $g = h^n$,

or (B) *there are real numbers* s, t *and a vectorfield* ξ, C^1 *on* $[0,1]$ *and* C^∞ *on* $(0,1)$, *such that* f *and* g *are the time* s *and the time* t *map of* ξ, *respectively.*

In the latter case, if $j_0^\infty(f) \neq j_0^\infty(id)$ (*resp.* $j_1^\infty(f) \neq j_1^\infty(id)$), *then* ξ *is of class* C^∞ *at* 0 (*resp. at* 1).

Moreover, if $j_0^\infty(f) \neq j_0^\infty(id)$ (*resp.* $j_1^\infty(f) \neq j_1^\infty(id)$), j_0^∞ (*resp.* j_1^∞) : $\langle f,g\rangle \to \mathbf{R}[x]$ *is an injective map, where* $\langle f,g\rangle$ *denotes the subgroup of* $\mathrm{Diff}^\infty([0,1])$ *generated by* f *and* g. (See Wallet [28].)

Proof. – Consider $f|[0,1)$ and $g|[0,1)$. By Lemma (3.4), there corresponds a vectorfield ξ_0 to $f|[0,1)$. Since $g|[0,1)$ commutes with $f|[0,1)$, by Lemma (3.4), $g|[0,1)$ is the time t map of ξ_0 for some real number t. If t is a rational number, put $t = n/m$ $(m,n \in \mathbf{Z}, m \neq 0, (m,n) = 1)$. Let h_0 denote the time $1/m$ map of ξ_0. Then $f|[0,1) = h_0^m$ and $g|[0,1) = h_0^n$. If we take integers p, q such that $pn + qm = 1$, we have

$$h_0 = (g|[0,1))^p (f|[0,1))^q .$$

So, if we put $h = g^p f^q (h \in \mathrm{Diff}^\infty([0,1]))$, we have $f = h^m$ and $g = h^n$. Thus, (A) holds if t is rational.

If t is an irrational number, consider $f|(0,1]$ and $g|(0,1]$. By Lemma (3.4), there corresponds a vectorfields ξ_1 to $f|(0,1]$, and $g|(0,1]$ is the time t_1 map of ξ_1 for some real number t_1. We show that $t_1 = t$ and $\xi_0|(0,1) = \xi_1|(0,1)$. Let $F : \mathbf{R} \times [0,1) \to [0,1)$ and $G : \mathbf{R} \times (0,1] \to (0,1]$ be the one parameter subgroup generated by ξ_0 and ξ_1, respectively. For any real number s, take a sequence $\{(m_i,n_i)\}_{i\in\mathbf{N}}$ of pairs of integers such that $m_i + tn_i \to s$ as $i \to \infty$. Then we have

$$F(m_i+tn_i,1/2) = f^{m_i}g^{n_i}(1/2) = G(m_i+t_1m_i,1/2).$$

If we put $s = 0$, we have $F(m_i+tn_i,1/2) \to 1/2$ as $i \to \infty$. Hence we have $m_i + t_1n_i \to 0$, that is, $t = t_1$. For general s, we have

$$\begin{aligned} F(s,1/2) &= \lim_{i\to\infty} F(m_i+tn_i,1/2) \\ &= \lim_{i\to\infty} G(m_i+tn_i,1/2) \\ &= G(s,1/2). \end{aligned}$$

Therefore, we have $\xi_0|(0,1) = \xi_1|(0,1)$.

Thus ξ_0 and ξ_1 define a vectorfield ξ, C^1 on $[0,1]$ and C^∞ on $(0,1)$. It is obvious that f and g are the time one map and the time t map of ξ, respectively. Consequently, (B) holds if t is irrational.

The rest of Lemma (3.5) follows from Lemmas (3.2) and (3.4).

For a diffeomorphism f of $\mathbf{R}$ (or S^1, $[0,1]$), put

$$\mathrm{Fix}^\infty(f) = \{x \in \mathrm{Fix}(f);\ j_x^\infty(f) = j_x^\infty(id)\}.$$

Following Sergeraert [20], we put

$$\mathrm{Diff}^\infty_\infty([0,1]) = \{f \in \mathrm{Diff}^\infty([0,1]);\ j_0^\infty(f) = j_0^\infty(id), \qquad j_1^\infty(f) = j_1^\infty(id)\}.$$

The following lemma is the main step of the proof of Theorem (3.1).

LEMMA (3.6). — *Let* f *be an element of* $\mathrm{Diff}^\infty_\infty([0,1])$ *such that* $\mathrm{Fix}^\infty(f) = \{0,1\}$. *Let* g *be an element of* $\mathrm{Diff}^\infty([0,1])$ *which commutes with* f. *Then*

either (A) *there are comprime integers* m, n *and an element* h *of* $\mathrm{Diff}^\infty_\infty([0,1])$ *such that* $f = h^m$, $g = h^n$,

or (B) *there are real numbers* s, t *and a vectorfield* ξ, C^1 *on* $[0,1]$ *and* C^∞ *on* $(0,1)$, *such that* $j_0^1(\xi) = j_0^1(0)$, $j_1^1(\xi) = j_1^1(0)$ *and* f *and* g *are the time* s *map and the time* t *map of* ξ, *respectively.*

To prove Lemma (3.6), we need another lemma which gives an estimate on the norm of the vectorfield given in Lemma (3.5). For a real valued function h on $[0,1]$, put

$$|h|_i = \max_{x \in [0,1]} |h^{(i)}(x)|, \quad i = 0, 1, \ldots .$$

We may regard a diffeomorphism f of $[0,1]$ as a function on $[0,1]$ and we have $|f|_i = |f - id|_i$, $i \geqslant 2$. On the other hand, vectorfields on $[0,1]$ are naturally considered as functions on $[0,1]$.

LEMMA (3.7). — *Let f be a C^∞-diffeomorphism of $[0,1]$, such that $f(x) < x$ for any $x \in (0,1)$ and $|f - id|_i \leqslant 1/2$, $|f^{-1} - id|_i \leqslant 1/2$ $(i=0,1,2)$. Suppose that there is a C^1-vectorfield ξ on $[0,1]$ whose time one map coincides with f. Then*

$$|\xi|_0 \leqslant 2|f-id|_0 \ \exp\left(|f-id|_2(1+|f^{-1}-id|_1)\right),$$
$$|\xi|_1 \leqslant 2|f-id|_1 + 4|f-id|_0\,|f-id|_2 \ \exp\left(2|f-id|_2(1+|f^{-1}-id|_1)\right).$$

Proof. — Put $\xi(x) = -\mu(x)(x-f(x))$, where $\mu(x)$ is a positive continuous function on $(0,1)$. Then, according to Sergeraert ([20]), if $f'(0) = 1$, $\mu(x)$ is continuously extended to a function on $[0,1)$ with $\mu(0) = 1$. In the same way, it is easy to show that, if $f'(0) < 1$, $\mu(x)$ is again continuous on $[0,1)$ with $\mu(0) = -(\log(f'(0)))/(1-f'(0))$.

Using the equation $\xi(f(x)) = f'(x)\xi(x)$, we have

$$(*) \quad \mu(x) = \mu(0) \lim_{n\to\infty} \frac{f^n(x) - f^{n+1}(x)}{x - f(x)} \prod_{i=0}^{n-1} \frac{1}{f'(f^i(x))} \qquad ([20],\ 2.10).$$

Following Sergeraert [20], we obtain an estimate on $|\mu|_0$ as follows. First we have

$$(**) \quad \prod_{i=0}^{n-1} f'(f^i(x)) = \frac{f^n(x)-f^{n+1}(x)}{x - f(x)} \prod_{i=0}^{n-1} \left\{1 - \varepsilon(f^i(x)) \frac{(f^i(x)-f^{i+1}(x))^2}{f^{i+1}(x)-f^{i+2}(x)}\right\},$$

where $\varepsilon(x) = -\int_0^1 (1-t)f''(x-t(x-f(x)))\,dt$ ([20], 2.4).

Since $|\varepsilon(x)| \leqslant \frac{1}{2}|f|_2$ and $\left|\dfrac{f^i(x)-f^{i+1}(x)}{f^{i+1}(x)-f^{i+2}(x)}\right| \leqslant |f^{-1}|_1$, we have

$$\sum_{i=0}^{\infty}\left|\varepsilon(f^i(x))\frac{f^i(x)-f^{i+1}(x)}{f^{i+1}(x)-f^{i+2}(x)}(f^i(x)-f^{i+1}(x))\right|$$
$$\leqslant \sum_{i=0}^{\infty}\frac{|f|_2}{2}|f^{-1}|_1\,|f^i(x)-f^{i+1}(x)|$$
$$\leqslant \frac{|f|_2\,|f^{-1}|_1}{2}\,x \quad (x\in[0,1))$$
$$\leqslant \frac{|f|_2\,|f^{-1}|_1}{2} \leqslant \frac{3}{8}\,.$$

By (**), we have

$$(***) \quad \left|\frac{f^n(x)-f^{n+1}(x)}{x-f(x)}\right|\exp(-|f|_2\,|f^{-1}|_1) \leqslant \prod_{i=0}^{n-1} f'(f^i(x))$$
$$\leqslant \left|\frac{f^n(x)-f^{n+1}(x)}{x-f(x)}\right|\exp\left(\frac{|f|_2\,|f^{-1}|_1}{2}\right).$$

Thus, by (*), we have

$$\mu(0)\exp\left(-\frac{|f|_2\,|f^{-1}|_1}{2}\right) \leqslant \mu(x) \leqslant \mu(0)\exp(|f|_2\,|f^{-1}|_1)\,.$$

If $1/2 \leqslant f'(0) < 1$, then we have

$$|\mu(0)| = |-(\log(f'(0)))/(1-f'(0))| < 2\,.$$

On the other hand, if $f'(0)=1$, then $\mu(0)=1$.

Therefore, we have

$$\mu(x) \leqslant 2\exp(|f|_2\,|f^{-1}|_1)\,.$$

An estimate on $|\xi|_0$ follows immediately.

$$\begin{aligned}|\xi|_0 &\leqslant |f-id|_0|\mu|_0\\ &\leqslant 2|f-id|_0\exp(|f|_2|f^{-1}|_1)\\ &\leqslant 2|f-id|_0\exp(|f-id|_2(1+|f^{-1}-id|_1))\,.\end{aligned}$$

To estimate $|\xi|_1$, we use another formula of Sergeraert ([20], 2.6, 2.7). Put

$$K_1(x) = \sum_{i=0}^{\infty} \frac{f''(f^i(x))}{f'(f^i(x))} \prod_{j=0}^{i-1} f'(f^j(x)).$$

Then we have $\xi'(x) = \log(f'(0)) + K_1(x)\xi(x)$. Using (***), we have

$$\begin{aligned}|K_1(x)| &\leqslant \sum_{i=0}^{\infty} \left|\frac{f''}{f'}\right|_0 \left|\frac{f^i(x)-f^{i+1}(x)}{x-f(x)}\right| \exp\left(\frac{|f|_2|f^{-1}|_1}{2}\right)\\ &\leqslant \left|\frac{f''}{f'}\right|_0 \exp\left(\frac{|f|_2\,|f^{-1}|_1}{2}\right)\\ &\leqslant 2|f-id|_2 \exp\left(\frac{|f-id|_2(1-|f^{-1}-id|_1)}{2}\right)\end{aligned}$$

An estimate on $|\xi|_1$ is obtained as follows.

$$\begin{aligned}|\xi|_1 &\leqslant \log(1-|f-id|_1)+|K_1|_0|\xi|_0\\ &\leqslant 2|f-id|_1 + 4|f-id|_0\,|f-id|_1 \exp\,(2|f-id|_2(1+|f^{-1}-id|_1)).\end{aligned}$$

Corollary (3.8). — *Let* $\{f_i\}_{i\in\mathbf{N}}$ *be a family of* C^∞*-diffeomorphisms of* $[0,1]$ *such that* $\mathrm{Fix}(f_i) = \{0,1\}$ $(i\in\mathbf{N})$ *and*

$$\lim_{i\to\infty} |f_i-id|_j = 0 \quad (j = 0,1,2).$$

Suppose that there is a family $\{\xi_i\}_{i\in\mathbf{N}}$ *of* C^1*-vectorfields on* $[0,1]$ *such that the time one map of* ξ_i *coincides with* f_i *for each* i. *Then*

$$\lim_{i\to\infty} |\xi_i|_0 = 0 \qquad \textit{and} \qquad \lim_{i\to\infty} |\xi_i|_1 = 0.$$

Proof. — For a diffeomorphism h of $[0,1]$ such that $|h-id|_1 < 1$, we have

$$\begin{aligned}|h^{-1}-id|_0 &= |h-id|_0\\ |h^{-1}-id|_1 &\leqslant |h-id|_1/(1-|h-id|_1)\end{aligned}$$

and

$$|h^{-1}-id|_2 \leqslant |h-id|_2/(1-|h-id|_1)^3.$$

Therefore we have

$$\lim_{i\to\infty} |f_i^{-1} - id|_j = 0 \quad (j=0,1,2).$$

Since, for each i, f_i^{-1} is the time one map of $-\xi_i$ and either f_i or f_i^{-1} satisfies the assumption of Lemma (3.7) for sufficiently large i, Corollary (3.8) follows from the estimate of Lemma (3.7).

Now we are ready to prove Lemma (3.6).

Proof of Lemma (3.6). – By Lemma (3.3), every point of $\mathrm{Fix}(f) - \mathrm{Fix}^\infty(f)$ is fixed also by g. Put $[0,1] - \mathrm{Fix}(f) = \bigcup_j \mathrm{J}_j$, where J_j's are disjoint open intervals. Then, by Lemma (3.5), for each j, $f|\bar{\mathrm{J}}_j$ and $g|\bar{\mathrm{J}}_j$ satisfy the condition (A) or (B) of Lemma (3.5).

For intervals J_j and J_k $(j \neq k)$, there are only finitely many intervals J_ℓ between them. By the injectivity of the jet map at the fixed points between J_j and J_k, $f|\bar{\mathrm{J}}_j$, $g|\bar{\mathrm{J}}_j$ and $f|\bar{\mathrm{J}}_k$, $g|\bar{\mathrm{J}}_k$ satisfy either (A) of Lemma (3.5) with the same m, n or (B) of Lemma (3.5) with the same s, t.

In the former case, we have $f = h^m$, $g = h^n$, where

$$h = g^p f^q \ (p, q \in \mathbf{Z},\ pn + qm = 1), \qquad h \in \mathrm{Diff}^\infty_\infty([0,1]).$$

Therefore (A) of Lemma (3.6) holds.

In the latter case, for each j, take the vectorfield ξ_j which corresponds to $f|\bar{\mathrm{J}}_j$. If $\bar{\mathrm{J}}_j \cap \bar{\mathrm{J}}_k$ is not empty, that is, is a one point set $\{*\}$, by Lemma (3.4), the vectorfields ξ_j and ξ_k which correspond to $f|\bar{\mathrm{J}}_j$ and $f|\bar{\mathrm{J}}_k$ are of class C^∞ at $*$, and have the same ∞-jets at $*$. Hence the vectorfields ξ_j 's on $\bar{\mathrm{J}}_j$'s define a C^∞-vectorfield ξ on $(0,1)$.

Now we extend ξ so that $\xi(0) = 0$, and show that ξ is of class C^1 at 0. In the same way, we can show that ξ is of class C^1 at 1 with $\xi(1) = 0$.

If there are only finitely many intervals J_j in a neighborhood of 0, 0 is an extreme point of some interval J_j. Therefore, by Lemma (3.5), ξ is of class C^1 at 0.

In the case when there are infinitely many intervals J_j in a neighborhood of 0, by the reordering of the suffixes, we may assume that there is a decreasing sequence of positive real numbers $b_i (i \in \mathbf{N})$ such that $J_j = (b_{j+1}, b_j)$ and $\lim_{i \to \infty} b_i = 0$.

Since $j_0^\infty(f) = j_0^\infty(id)$, for integers m, n $(0 \leqslant m \leqslant n)$, we have

$$\sup_{0 \leqslant y \leqslant x} |(f-id)^{(m)}(y)| \leqslant \frac{x^{n-m}}{(n-m)!} \sup_{0 \leqslant y \leqslant x} |(f-id)^{(n)}(y)|$$
$$\leqslant \frac{x^{n-m}}{(n-m)!} |f-id|_n .$$

In particular,

$$\sup_{Y \in J_j} |(f-id)^{(m)}(y)| \leqslant \frac{b_j^{n-m}}{(n-m)!} |f-id|_n .$$

In order to estimate $\sup_{Y \in J_j} |\xi_j(y)|$ and $\sup_{Y \in J_j} |\xi_j'(y)|$, we consider the linear homeomorphism

$$A_j : [0,1] \to [b_{j+1}, b_j]$$

defined by $A_j(x) = (b_j - b_{j+1})x + b_{j+1}$ and the vectorfield $(A_j^{-1})_* \xi_j$ on $[0,1]$. The vectorfield $(A_j^{-1})_* \xi_j$ corresponds to the diffeomorphism $A_j^{-1} f A_j$ of $[0,1]$, and, for $A_j^{-1} f A_j$ we have

$$|A_j^{-1} f A_j - id|_m = (b_j - b_{j+1})^{m-1} \sup_{y \in J_j} |(f-id)^{(m)}(y)|$$
$$\leqslant \frac{(b_j - b_{j+1})^{m-1} b_j^{n-m} |f-id|_n}{(n-m)!} .$$

Put $n = 3$; then, for $m = 1, 2$, we have

$$\lim_{j \to \infty} |A_j^{-1} f A_j - id|_m = 0.$$

Since we have

$$|h-id|_0 \leqslant |h-id|_1$$

for a diffeomorphism h of $[0,1]$, we have

$$\lim_{j \to \infty} |A_j^{-1} f A_j - id|_0 = 0.$$

Therefore, by Corollary (3.8), we have

$$\lim_{j\to\infty} |(A_j^{-1})_*\xi_j|_m = 0 \quad (m=0,1).$$

Since

$$\sup_{y\in J_j} |\xi_j|_m = (b_j-b_{j+1})^{1-m}|(A_j^{-1})_*\xi_j|_m$$

for any integer $m \geqslant 0$, we have

$$\lim_{j\to\infty} \sup_{y\in J_j} |\xi_j|_m = 0 \quad (m=0,1).$$

Therefore, we have

$$\lim_{x\to 0} \sup_{0<y<x} |\xi^{(m)}(y)| = 0 \quad (m=0,1).$$

Consequently, ξ is of class C^1 at 0.

Thus (B) of Lemma (3.6) holds, and we have proved Lemma (3.6).

Using Lemma (3.4) instead of Lemma (3.5) if necessary, we can prove the following lemmas just as we did in Lemma (3.6).

LEMMA (3.9). — *Let f be a C^∞-diffeomorphism of $\mathbf{R}_+ = [0,\infty)$ such that $\mathrm{Fix}^\infty(f) = \{0\}$. Let g be a C^∞-diffeomorphism of $\mathbf{R}_+$ which commutes with f. Then*

either (A) *there are coprime integers m, n and a C^∞-diffeomorphism h of $\mathbf{R}_+$ such that $\mathrm{Fix}^\infty(h) = \{0\}$, $f = h^m$ and $g = h^n$,*

or (B) *there are real numbers s, t and a vectorfield ξ, C^1 on $\mathbf{R}_+$ and C^∞ on $\mathbf{R}_+ - \{0\}$, such that $j_0^1(\xi) = j_0^1(0)$, and f and g are the time s map and the time t map of ξ, respectively.*

LEMMA (3.10). — *Let f be a C^∞-diffeomorphism of $\mathbf{R}$ such that $\mathrm{Fix}(f) \neq \emptyset$ and $\mathrm{Fix}^\infty(f) = \emptyset$. Let g be a C^∞-diffeomorphism of $\mathbf{R}$ which has fixed points and commutes with f. Then*

either (A) *there are coprime integers m, n and a C^∞-diffeomorphism h of $\mathbf{R}$ such that $f = h^m$ and $g = h^n$,*

or (B) *there are real numbers s, t and a C^∞-vectorfield ξ on $\mathbf{R}$ such that f and g are the time s map and the time t map of ξ, respectively.*

Now, we prove Theorem (3.1).

Proof of Theorem (3.1). — First note that, if the germ of f at a point x of $\mathrm{Fix}^\infty(f)$ is not that of the identity, then $x \in \mathrm{Fix}^\infty(g)$. For, by Lemma (3.3), $x \in \mathrm{Fix}(g)$. If $x \in \mathrm{Fix}(g) - \mathrm{Fix}^\infty(g)$, x is an isolated fixed point of g. Since the ∞-jet of f at x coincides with that of the identity, by Lemma (3.2) or (3.4), the germ of f at x coincides with that of the identity; this is a contradiction.

Now, put $\mathbf{R} - (\mathrm{Fix}^\infty(f) \cap \mathrm{Fix}^\infty(g)) = \cup \mathrm{I}_i$, where I_i 's are disjoint open intervals. We show that $\{\mathrm{I}_i\}$ is the desired family. It is obvious that $\{\mathrm{I}_i\}$ satisfies Theorem (3.1) (1). Theorem (3.1) (2) is shown as follows.

Since $\partial \mathrm{I}_i \subset \mathrm{Fix}^\infty(f) \cap \mathrm{Fix}^\infty(g)$, $\bar{\mathrm{I}}_i$ is invariant under f and g. By the choice of I_i, either f or g is not the identity on $\bar{\mathrm{I}}_i$, so we may assume that $f|\mathrm{I}_i \neq id_{\bar{\mathrm{I}}_i}$. Then $\mathrm{Fix}^\infty(f|\bar{\mathrm{I}}_i) = \partial\bar{\mathrm{I}}_i$. For, if $\mathrm{Fix}^\infty(f) \cap \mathrm{I}_i \neq \emptyset$, there is a point x of $\mathrm{Fix}^\infty(f) \cap \mathrm{I}_i$ such that the germ at x of f is not that of the identity. Therefore $x \in \mathrm{Fix}^\infty(g)$; consequently, $x \in \mathrm{Fix}^\infty(f) \cap \mathrm{Fix}^\infty(g)$, which contradicts the definition of $\{\mathrm{I}_i\}$.

If I_i is a bounded interval, applying Lemma (3.6), we have h_i or ξ_i for $f|\bar{\mathrm{I}}_i$ and $g|\bar{\mathrm{I}}_i$. Take the extension of h_i (resp. ξ_i) such that $h_i|_{\mathbf{R}-\mathrm{I}_i} = id_{\mathbf{R}-\mathrm{I}_i}$(resp. $\xi_i|_{\mathbf{R}-\mathrm{I}_i}=0$); then we obtain the desired diffeomorphism (resp. the desired vectorfield).

If I_i is not a bounded interval, I_i is a half line or the whole line. In the case when I_i is the whole line, Theorem (3.1) follows from Lemma (3.10). In the case when I_i is a half line, applying Lemma (3.9), we obtain the desired h_i or ξ_i as in the case of bounded intervals.

We have proved Theorem (3.1).

4. Preliminary theorems.

In this section, first we give certain classes of foliated $S^1-(\mathbf{R}-)$ bundles over a 2-torus which are homologous to zero in the classifying space. To prove the homological triviality of these 2-cycles, we use theorems due to Mather [8,10] and Sergeraert [20] which say that the one dimensional homology group of certain groups of diffeomorphisms are zero.

THEOREM (4.1). — *Let* f, g *be elements of* $\mathrm{Diff}^r_+(S^1)$ (resp. $\mathrm{Diff}^r_K(\mathbf{R})$) $(r \geqslant 3)$. *Suppose that there exist a finite number of disjoint open intervals* $I_1, \ldots, I_p$, $J_1, \ldots, J_q$ *such that* $\mathrm{Supp}\,(f) \subset \bigcup_{i=1}^{p} I_i$ *and* $\mathrm{Supp}(g) \subset \bigcup_{j=1}^{q} J_j$.

Then $f \circ g = g \circ f$ *and* $\{f,g\} = 0$ *in*

$$H_2(\mathrm{Diff}^r_+(S^1)) \quad (\textit{resp. } H_2(\mathrm{Diff}^r_K(\mathbf{R}))).$$

Proof. — By a theorem of Mather [8, 10], we have $H_1(\mathrm{Diff}^r_K(\mathbf{R})) = 0$ $(r \geqslant 3)$. Therefore, g can be written as a product of commutators :

$$g = [h_1,h_2][h_3,h_4] \ldots [h_{2k-1},h_{2k}],$$

where $h_i \in \mathrm{Diff}^r_+(S^1)$ (resp. $\mathrm{Diff}^r_K(\mathbf{R})$) and

$$\mathrm{Supp}\,(h_i) \subset \bigcup_{j=1}^{q} J_j (i=1,\ldots,2k).$$

Since f and h_i 's commute, by Lemma (2.7), we have $\{f,g\} = 0$.

Using a theorem of Sergeraert [20] which says that

$$H_1(\mathrm{Diff}^\infty_\infty([0,1])) = 0,$$

we can prove the following theorem in the same way.

THEOREM (4.2). — *Let* f, g *be elements of* $\mathrm{Diff}^\infty_+(S^1)$ (*resp.* $\mathrm{Diff}^\infty_K(\mathbf{R})$). *Suppose that there exist a finite number of disjoint open intervals* $I_1, \ldots, I_p$, $J_1, \ldots, J_q$ *such that*

$$\mathrm{Supp}\,(f) \subset \bigcup_{i=1}^{p} \bar{I}_i \ (\textit{the closure}) \qquad \textit{and} \qquad \mathrm{Supp}(g) \subset \bigcup_{j=1}^{q} \bar{J}_j \ (\textit{the closure}).$$

Then f *commutes with* g *and*

$$\{f,g\} = 0 \qquad \textit{in} \qquad H_2(\mathrm{Diff}^\infty_+(S^1)) \quad (\textit{resp. } H_2(\mathrm{Diff}^\infty_K(\mathbf{R}))).$$

The following theorem is a corollary to Theorem (4.2). For f, $g \in \mathrm{Diff}^r_+(S^1)$ (or $\mathrm{Diff}^r_K(\mathbf{R})$) and $x \in S^1$ (or $\mathbf{R}$), let $\overline{\mathrm{Orb}}_{\langle f,g \rangle}(x)$ denote the closure of the orbit of x under the action of the subgroup generated by f and g.

THEOREM (4.3). — *Let f, g be commuting orientation preserving C^∞-diffeomorphisms of a circle;* i.e., $f,\ g \in \mathrm{Diff}^\infty_+(S^1)$ *and* $f \circ g = g \circ f$. *Suppose*

(1) $\mathrm{Fix}(f) \neq \varnothing$, $\mathrm{Fix}(g) \neq \varnothing$,

(2) $\mathrm{Int}\,(\overline{\mathrm{Orb}}_{\langle f,g\rangle}(x)) = \varnothing$ *for any* $x \in S^1$

and

(3) $\mathrm{Fix}^\infty(f) \cap \mathrm{Fix}^\infty(g)$ *has only finitely many connected components. Then* $\{f,g\} = 0$ *in* $H_2(\mathrm{Diff}^\infty_+(S^1))$.

Remark. — By Theorem (3.1), the above condition

$$\mathrm{Int}\,(\overline{\mathrm{Orb}}_{\langle f,g\rangle}(x)) = \varnothing$$

implies that the leaf through $x \in S^1(= \pi^{-1}(*))$ of the foliated bundle is a proper leaf.

Proof. — First assume that $\mathrm{Fix}^\infty(f) \cap \mathrm{Fix}^\infty(g)$ is non-empty, and put $S^1 - (\mathrm{Fix}^\infty(f) \cap \mathrm{Fix}^\infty(g)) = \cup I_i$, where I_i 's are disjoint open intervals of S^1. Note that the number of intervals is finite. By Theorem (3.1) and the condition (2) above, for each i, we have coprime integers m_i, n_i and a C^∞-diffeomorphism h_i, such that $f\,|\,\bar{I}_i = (h_i|\bar{I}_i)^{m_i}$ and $g\,|\,\bar{I}_i = (h_i|\bar{I}_i)^{n_i}$. Therefore we have

$$f = \prod_i h_i^{m_i} \qquad \text{and} \qquad g = \prod_j h_j^{n_j}.$$

By Lemmas (2.2) and (2.3), we have

$$\begin{aligned}\{f,g\} &= \left\{\prod_i h_i^{m_i}, \prod_j h_j^{n_j}\right\}\\ &= \sum_{i,j} m_i n_j \{h_i, h_j\}.\end{aligned}$$

If $i \neq j$, we have $\{h_i,h_j\} = 0$ by Theorem (4.2). Since $\{h_i,h_i\} = 0$, we have $\{f,g\} = 0$.

Similarly, if $\mathrm{Fix}^\infty(f) \cap \mathrm{Fix}^\infty(g) = \varnothing$, we have coprime integers m, n and a C^∞-diffeomorphism h such that $f = h^m$, $g = h^n$. Therefore, we have $\{f,g\} = 0$.

Theorems (4.1), (4.2) and Corollary (4.3) are essentially due to the results of Mather [8, 10] and Sergeraert [20] which say the one dimensional homology groups of certain groups of diffeomorphisms are zero. When the foliated $S^1-(\mathbf{R}-)$ bundle has a locally dense leaf, the problem is essentially a problem of the two dimensional homology group of $\mathrm{Diff}^r_+(S^1)$ $(\mathrm{Diff}^r_K(\mathbf{R}))$. Note that, when a 2-torus T^2 bounds a 3-manifold W^3, the induced homomorphism $H_1(T^2) \to H_1(W^3)$ is never injective. Our cobordism is obtained from the non-commutativity of $\pi_1(W^3,*)$.

In the rest of this section, we consider the case when the commuting diffeomorphisms f and g belong to a one parameter subgroup generated by a smooth vectorfield. Then, the foliated bundle may have compact leaves with non-trivial holonomy and locally dense leaves. Under some special conditions, we can prove rather easily the fact that $\{f,g\} = 0$. The proof of them gives us the idea of the proof of our main theorem.

LEMMA (4.4). – *Let ξ be a C^r-vectorfield $(r=1,\ldots,\infty)$ on S^1 (resp. on $\mathbf{R}$ with compact support). Let $f_{(t)}$ denote the time t map of ξ. Let w be a non-zero real number. Suppose that there is an element k of $\mathrm{Diff}^r_+(S^1)$ (resp. $\mathrm{Diff}^r_K(\mathbf{R})$) such that $kf_{(s)}k^{-1} = f_{(ws)}$ for any real number s. Then $\{f_{(1)},f_{(w+(1/w))}\} = 0$ in $H_2(\mathrm{Diff}^r_+(S^1))$ (resp. $H_2(\mathrm{Diff}^r_K(\mathbf{R}))$).*

Proof. – By Lemma (2.4), we have

$$\begin{aligned} 0 &= \{f_{(1)},kf_{(1)}k^{-1}k^{-1}f_{(1)}k\} = \{f_{(1)},f_{(w)}f_{(1/w)}\} \\ &= \{f_{(1)},f_{(w+(1/w))}\}. \end{aligned}$$

We give an application of Lemma (4.4). The linear action of $SL(2,\mathbf{R})$ on 2-plane induces an action on the set of rays through the origin, which is an S^1. Consider the vectorfield ξ of S^1 which corresponds to the element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of the Lie algebra of $SL(2,\mathbf{R})$. Then the time t map $f_{(t)}$ corresponds to $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Since

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 1 & x^2t \\ 0 & 1 \end{pmatrix},$$

by Lemma (4.4), we have $\{f_{(1)},f_{(x^2+(1/x^2))}\} = 0$. Therefore we have $\{f_{(1)},f_{(t)}\} = 0$ for any $t \geqslant 2$. For any t, chose t_1, $t_2 \geqslant 2$ such that

$t = t_1 - t_2$. Then, by Lemmas (2.2) and (2.3), we have

$$\{f_{(1)},f_{(t)}\} = \{f_{(1)},f_{(t_1)}\} - \{f_{(1)},f_{(t_2)}\} = 0.$$

Thus we have $\{f_{(1)},f_{(t)}\} = 0$ for any real number t.

Remark. — Given a vectorfield ξ with singular points and a nonzero real number w, we can construct a homeomorphism k fixing the singular points and satisfying the assymption of Lemma (4.4). However, k is not smooth in general. In fact, using the normal form of Takens [23], we can prove the following : Let ξ be a C^∞-vectorfield on S^1 with singular points. If there is an positive integer r such that $r \leqslant \max \{\ell ; j_p^\ell(\xi) = j_p^\ell(0)\} < \infty$ for every singular point p, then k is of class C^r, but not of class C^{r+1} in general.

The proof of the following theorem inspired us the proof of our main theorem.

THEOREM (4.5). — *Let ξ be a C^r-vectorfield ($r = 3,4,\ldots,\infty$) on $\mathbf{R}$ with compact support. Let $f_{(s)}$ denote the time s map of ξ. Let t be a real quadratic irrational number. Then $\{f_{(1)},f_{(t)}\} = 0$ as an element of $H_2(\mathrm{Diff}^r_K(\mathbf{R}))$.*

Proof. — We may assume that the support of ξ is contained in the open interval $(1/2,1)$. Let η be a C^∞-vectorfield on $\mathbf{R}$ with compact support such that $\eta(x) = -x^2(\partial/\partial x)$ for $x \in [0,1]$. Let k be the time one map of η. Then we have

$$k^m(x) = x/(mx+1) \qquad \text{for} \qquad x \in [0,1]$$

and

$$k^m([1/2,1]) = [1/(m+2),1/(m+1)], \qquad \text{for} \qquad m = 0,1,\ldots.$$

For a real number t satisfying $|t| < 1$, we define a vectorfield ζ on $\mathbf{R}$ with compact support as follows.

$$\zeta = \begin{cases} 0 & \text{on} \quad (-\infty,0] \cup [1,\infty), \\ k^m_*(t^m\xi) & \text{on} \quad [(1/(m+2),1/(m+1)] \quad (m=0,1,\ldots). \end{cases}$$

Then ζ is of class C^r. For, we have

$$\zeta(x) = t^m(1-mx)^2\xi(x/(1-mx))$$

for

$$x \in [1/(m+2), 1/(m+1)].$$

Differentiating ζ, we obtain $j^r_0(\zeta) = j^r_0(0)$.

Let $F_{(s)}$ be the time s map of ζ. Then $F_{(s)}$ commutes with $f_{(u)}$ for any s and u, and we have

$$kF_{(1)}k^{-1} = F_{(1/t)}f_{(-1/t)}$$

and

$$k^{-1}F_{(1)}k = F_{(t)}k^{-1}f_{(1)}k.$$

First we show that $\{f_{(1)}, f_{(t)}\} = 0$ if $t + t^{-1} \in \mathbf{Q}$. Since

$$\{f_{(1)}, f_{(t+(1/t))}\} = 0,$$

we may assume $|t| < 1$ changing t and t^{-1} if necessary. By Lemma (2.4), we have

$$\{F_{(1)}, kF_{(1)}k^{-1}k^{-1}F_{(1)}k\} = 0,$$

that is

$$\{F_{(1)}, F_{(t+(1/t))}k^{-1}f_{(1)}kf_{(-1/t)}\} = 0.$$

By Lemma (2.2), we have

$$\{F_{(1)}, F_{(t+(1/t))}\} + \{F_{(1)}, k^{-1}f_{(1)}k\} + \{F_{(1)}f_{(-1)}, f_{(-1/t)}\} + \{f_{(1)}, f_{(-1/t)}\} = 0.$$

By Theorem (4.1), we have

$$\{F_{(1)}, k^{-1}f_{(1)}k\} = 0 \qquad \text{and} \qquad \{F_{(1)}f_{(-1)}, f_{(-1/t)}\} = 0.$$

By Lemma (2.6) and the assumption that $t + t^{-1} \in \mathbf{Q}$, we have

$$\{F_{(1)}, F_{(t+(1/t))}\} = 0.$$

Therefore, we have

$$\{f_{(1)}, f_{(-1/t)}\} = 0.$$

Using $\{f_{(1)}, f_{(t+(1/t))}\} = 0$, we have

$$\{f_{(1)}, f_{(t)}\} = -\{f_{(1)}, f_{(1/t)}\} = \{f_{(1)}, f_{(-1/t)}\} = 0.$$

Let t be a real quadratic irrational number; i.e.,

$$t = p_1/q_1 + (p_2/q_2)\sqrt{n},$$

where p_1, p_2, q_1, q_2 and n are integers and $q_1 \neq 0$, $q_2 \neq 0$ and $n \geqslant 2$. Using the above result for the vectorfield $(2q_2)^{-1}\xi$ and the number $s = (n+1+2\sqrt{n})/(n-1)$ which satisfies $s + s^{-1} = 2(n+1)/(n-1) \in \mathbf{Q}$, we have

$$[f_{(1/(2q_2))}, f_{((1/(2q_2))((n+1+2\sqrt{n})/(n-1)))}\} = 0.$$

By Lemma (2.6), we have

$$\{f_{(1/(2q_2))}, f_{((1/2q_2))((n+1)/(n-1)))}\} = 0;$$

thus by Lemma (2.2),

$$\{f_{(1/(2q_2))}, f_{((1/q_2)(\sqrt{n}/(n-1)))}\} = 0.$$

By Lemma (2.3), multiplying this by $2p_2q_2(n-1)$, we have

$$\{f_{(1)}, f_{((p_2/q_2)\sqrt{n})}] = 0.$$

On the other hand, by Lemma (2.6), we have

$$\{f_{(1)}, f_{(p_1/q_1)}\} = 0.$$

Thus, by Lemma (2.2), we have

$$\{f_{(1)}, f_{(t)}\} = \{f_{(1)}, f_{(p_1/q_1)}\} + \{f_{(1)}, f_{((p_2/q_2)\sqrt{n})}] = 0.$$

5. Criterion for the smoothness of certain homeomorphisms.

In this section, we prove Lemmas (5.1)-(5.3).

Let f be an element of $\mathrm{Diff}^\infty_K(\mathbf{R})$. Then, $\mathbf{R} - \mathrm{Fix}^\infty(f)$ is a countable union of disjoint open intervals; $\mathbf{R} - \mathrm{Fix}^\infty(f) = \cup \mathrm{I}_i$. Let f_i denote the diffeomorphism of $\mathbf{R}$ defined by

$$f_i | \bar{\mathrm{I}}_i = f | \bar{\mathrm{I}}_i \qquad \text{and} \qquad f_i | \mathbf{R} - \mathrm{I}_i = id_{\mathbf{R}-\mathrm{I}_i}.$$

Then f_i 's are of class C^∞ and $f = \Pi f_i$.

Conversely, let $\{\mathrm{I}_i\}_{i\in\mathbf{N}}$ be a family of disjoint open intervals of $\mathbf{R}$ such that $\cup \mathrm{I}_i$ is bounded. Let $f_i (i \in \mathbf{N})$ be a C^∞-diffeomorphism of $\mathbf{R}$ whose support is contained in $\bar{I}_i$. Then $f = \Pi f_i$ is a homeomorphism of $\mathbf{R}$, but f is not necessarily a diffeomorphism of $\mathbf{R}$. For f to be a diffeomorphism, we need some conditions on the norms of f_i 's.

We introduce some notations which are similar to those used in § 3. For a function λ on $\mathbf{R}$, the norms $|\lambda|_n$ and $||\lambda||_n$ are defined as follows :

$$|\lambda|_n = \sup_{x \in \mathbf{R}} |\lambda^{(n)}(x)| \qquad \text{and} \qquad ||\lambda||_n = \sum_{i=0}^{n} |\lambda|_i .$$

$|\cdot|_n$ and $||\cdot||_n$ are defined for functions on S^1 and intervals in the same way. We define symbols $\underset{(r)}{\leqslant}$ and $\underset{*}{\leqslant}$ as follows. For functions $f(q,r)$ and $g(q,r)$, we mean by $f(q,r) \underset{(r)}{\leqslant} g(q,r)$ that, for every r, there is a constant C_r such that $f(q,r) \leqslant C_r g(q,r)$ for any q. We mean by $f(q,r) \underset{*}{\leqslant} g(q,r)$ that there is a constant C such that $f(q,r) \leqslant Cg(q,r)$ for any q and r.

For a C^r-function λ on $[0,\infty)$, if $j_0^r(\lambda) = j_0^r(0)$, where 0 denotes the zero map, we have

$$(*) \qquad \sup_{0 \leqslant y \leqslant x} |\lambda^{(i)}(y)| \leqslant \frac{x^{r-i}}{(r-i)!} \sup_{0 \leqslant y \leqslant x} |\lambda^{(r)}(y)|$$

for each $i \, (0 \leqslant i \leqslant r)$. Therefore, for any interval $[a,b]$ $(-\infty < a < b < \infty)$ and for any positive integer n, there is a positive real number C such that

$$|f - id|_n \leqslant ||f - id||_n \leqslant C |f - id|_n$$

for any $f \in \mathrm{Diff}^{\infty}_{\infty}([a,b])$.

For $f \in \mathrm{Diff}^{\infty}_{\mathrm{K}}(\mathbf{R})$, take the decomposition $f = \prod_i f_i$. Using the inequality (*) we have the following estimate

$$(**) \qquad |f_i - id|_j \leqslant \frac{\ell_i^{r-j}}{(r-j)!} |f - id|_r$$

for any integers j, $r \, (0 \leqslant j \leqslant r)$, where ℓ_i denotes the length of I_i.

Conversely, we have the following lemmas.

LEMMA (5.1). — *Let* $\{I_i\}_{i\in N}$ *be a family of disjoint open intervals of* $\mathbf{R}$ *such that* $\cup I_i$ *is bounded. Let* $g_i (i\in \mathbf{N})$ *be a* C^r*-function on* $\mathbf{R}$ *whose support is contained in* $\bar{I}_i$. *Suppose that* $\{|g_i|_r\}$ *is bounded. Then* $g = \sum_i g_i$ *is a* C^{r-1}*-function.*

Proof. — Let ℓ_i denote the length of I_i as before. Using (**), we have

$$\sum_i \|g_i\|_{r-1} = \sum_{j=0}^{r-1} \sum_i |g_i|_j \leqslant \sum_{j=0}^{r-1} \sum_i \frac{\ell_i^{r-j}}{(r-j)!} |g_i|_r .$$

Since $\sum_i \ell_i$ is bounded, $\sum_i \ell_i^{r-j}$ is bounded $(0\leqslant j\leqslant r-1)$. Therefore, the assumption that $\{|g_i|_r\}$ is bounded implies that $\sum_i \|g_i\|_{r-1} < \infty$, that is, as m tends to ∞, $\sum_{i=0}^{m} g_i$ converges to g in the $\|.\|_{r-1}$ norm. Thus g is of class C^{r-1}.

LEMMA (5.2). — *Let* $\{I_i\}$ *be a family of disjoint open intervals of* $\mathbf{R}$, *such that* $\cup I_i$ *is bounded. Let* $f_i (i\in \mathbf{N})$ *be a* C^r*-diffeomorphism* $(r\geqslant 2)$ *of* $\mathbf{R}$ *whose support is contained in* $\bar{I}_i$. *Suppose that* $\{|f_i - id|_r\}_{i\in N}$ *is bounded. Then* $f = \prod_i f_i$ *is a* C^{r-1}*-diffeomorphism of* $\mathbf{R}$.

Proof. — Since $f = id + \sum_i (f_i - id)$, by Lemma (5.1), f is a C^{r-1} map. Since, for each i, $\inf_x f_i'(x) > 0$ and

$$|f_i - id|_1 \leqslant \frac{\ell_i^{r-1}}{(r-1)!} |f_i - id|_{r-1},$$

we have $f' \geqslant \inf_i \inf_x f_i'(x) > 0$. Thus f is a diffeomorphism of class C^{r-1}

We shall use these lemmas in the following sections in order to construct some diffeomorphisms. At the end of this section, we give a lemma which we shall use in § 7.

LEMMA (5.3). — *Let* $\mathscr{D}_1^r$ *denote the set of* C^r*-diffeomorphisms of* $\mathbf{R}$ *which are the identity except on an interval of length* 1. *Then, there is a positive* *integer* M_r *such that,*

(1) *if* f_1, $f_2 \in \mathscr{D}^r_1$ *satisfy* $|f_i - id|_r \leqslant 1$ $(i=1,2)$, *then*

$$|f_1 f_2 - id|_r \leqslant M_r |f_1 - id|_r + |f_2 - id|_r$$

and

(2) *if* $f_i \in \mathscr{D}^r_1 (i=1,\dots,N)$ *satisfies* $|f_i - id|_r \leqslant (NM_r)^{-1}$, *then*

$$|\prod_{i=1}^{N} f_i - id|_r \leqslant M_r N \max_{1 \leqslant i \leqslant N} |f_i - id|_r .$$

Proof. – (1) is a consequence of Faa-di-Bruno's formula and the fact that $|f - id|_j \leqslant |f - id|_r$ $(j=0,\dots,r)$. (2) is immediately obtained from (1).

6. Main theorem.

Our main theorem is as follows.

THEOREM (6.1). – *Let* $\{I_i\}_{i \in N}$ *be a family of disjoint open intervals of* $\mathbf{R}$ *such that* $\cup I_i$ *is bounded. Let* f, g *be maps of* $\mathbf{R}$ *onto* $\mathbf{R}$ *satisfying the following conditions.*

(i) $f | \mathbf{R} - \cup I_i = id_{\mathbf{R} - \cup I_i}$ *and* $g | \mathbf{R} - \cup I_i = id_{\mathbf{R} - \cup I_i}$.

(ii) *For* $f | \bar{I}_i$ *and* $g | \bar{I}_i$,

either (*a*) *there exist coprime integers* m_i, n_i *and a* C^∞*-diffeomorphism* h_i *such that* $\mathrm{Supp}(h_i) \subset \bar{I}_i$, $f | \bar{I}_i = (h_i | \bar{I}_i)^{m_i}$ *and* $g | \bar{I}_i = (h_i | \bar{I}_i)^{n_i}$,

or (*b*) *there exist real numbers* s_i, t_i *and a* C^∞*-vectorfield* ξ_i *on* $\mathbf{R}$ *such that* $\mathrm{Supp}(\xi_i) \subset \bar{I}_i$, $f | \bar{I}_i$ *and* $g | \bar{I}_i$ *are the time* s_i *map and the time* t_i *map of* $\xi_i | \bar{I}_i$, *respectively.*

Moreover, (*c*) $\{\max\{|m_i|, |n_i|\} . |h_i - id|_r ; i (\in \mathbf{N})$ *satisfying* $(a)\}$ *and* $\{\max\{|s_i|, |t_i|\} . |\xi_i|_r ; i (\in \mathbf{N})$ *satisfying* $(b)\}$ *are bounded for any non-negative integer* r.

Then, f, g *belong to* $\mathrm{Diff}^\infty_K(\mathbf{R})$, $fg = gf$ *and* $\{f,g\} = 0$ *in* $H_2(\mathrm{Diff}^\infty_K(\mathbf{R}))$.

Remarks. – 1. That f, $g \in \mathrm{Diff}^\infty_K(\mathbf{R})$ follows from Lemmas (5.1)-(5.3) (see also (7.6)). It is obvious that f commutes with g.

2. The condition (ii) (c) is satisfied if the sets $\{(m_i, n_i)\}$ and $\{(s_i, t_i)\}$ are finite. In particular, if f and g belong to a one parameter subgroup of

$\mathrm{Diff}_K^\infty(\mathbf{R})$ generated by a C^∞-vectorfield on $\mathbf{R}$ with compact support, then $\{f,g\} = 0$ in $H_2(\mathrm{Diff}_K^\infty(\mathbf{R}))$.

3. For commuting elements f, g of $\mathrm{Diff}_K^\infty(\mathbf{Ç})$, we have Theorem (3.1). The conclusion of Theorem (3.1) is much weaker than the assumption of Theorem (6.1). However, we do not know whether there exist commuting diffeomorphisms f, g with compact support for which the vectorfield of Theorem (3.1) (B) is not of class C^∞ or the condition (ii) (*c*) of Theorem (6.1) does not hold.

We have a corollary to Theorem (6.1).

COROLLARY (6.2). — *In any topological equivalence class of transversely oriented C^r-foliated ($r \geqslant 2$) S^1-bundles over T^2, there is a C^∞-foliated S^1-bundle which is C^∞-foliated cobordant to zero.*

Proof. — The foliations of foliated S^1-bundles over T^2 are classified up to topological equivalence (Moussu-Roussarie [15]). If the foliation of a foliated S^1-bundle over T^2 has no compact leaves, in its topological equivalence class, there is a foliation defined by a smooth non-vanishing closed 1-form, which is cobordant to zero. If the foliation has a compact leaf, the foliated bundle is defined by commuting diffeomorphisms of S^1 which have fixed points (by changing the fibration if necessary). Then there is a family $\{I_i\}$ of disjoint open intervals with respect to which these commuting diffeomorphisms satisfy (2) (A), (B) of Theorem (3.1). Changing diffeomorphisms h_i and vectorfields ξ_i if necessary, we obtain a C^∞-foliated S^1-bundle over T^2 which is topologically equivalent to the original one and whose total holonomy satisfies (ii) (*a*), (*b*), (*c*) of Theorem (6.1). By a result of [12, § 5], the new foliated S^1-bundle is C^∞-foliated cobordant to a union of foliated S^1-bundles over T^2 whose total holonomies fix an open interval of S^1 and satisfy (ii) (*a*), (*b*), (*c*) of Theorem (6.1). Then, by Theorem (6.1), the 2-cycles corresponding to these foliated S^1-bundles are homologous to zero in $B\,\mathrm{Diff}_+^\infty(S^1)_d$. This proves Corollary (6.2).

In the next section, we show that, to prove Theorem (6.1), we only need the following theorem which is a generalization of a theorem of Sergeraert [20].

THEOREM (6.3). — *Let f be a C^∞-diffeomorphism of $\mathbf{R}$ with compact support. Then there exists a finite collection $\{h_1,\ldots,h_{2k}\}$ of C^∞-*

diffeomorphisms such that

$$\mathrm{Fix}^{\infty}(h_i) \supset \mathrm{Fix}^{\infty}(f) \quad (i=1,\ldots,2k)$$

and

$$f = \prod_{j=1}^{k} [h_{2j-1}, h_{2j}].$$

7. Reduction of the main theorem.

As in the proof of Theorem (4.5), let η be a C^{∞}-vectorfield on $\mathbf{R}$ with compact support such that $\eta(x) = -x^2(\partial/\partial x)$ for $x \in [0,1]$, and let k be the time one map of η.

In this section, we prove that Theorem (6.3) implies Theorem (6.1). We divide the proof into the following three lemmas (Lemmas (7.1), (7.2) and (7.3)). Lemma (2.4) plays an important role in the proof of Lemmas (7.1) and (7.2).

LEMMA (7.1). — *If* $\{f,g\}$ *satisfies the assumption of Theorem* (6.1), *there are commuting elements* G, H *of* $\mathrm{Diff}_K^{\infty}(\mathbf{R})$ *satisfying* $\{f,g\} = -\{G,H\}$ *and the following conditions : There are disjoint open intervals* J_i *such that* $\bigcup_i J_i$ *is bounded, and for each* i, *there is a* C^{∞}*-diffeomorphism* G_i *such that* $\mathrm{Supp}(G_i) \subset \bar{J}_i$,

d) either $G|\bar{J}_i = id_{\bar{J}_i}$ *and* $H|\bar{J}_i = G_i|\bar{J}_i$ *(put* $m_i = 1$ *in this case), or* $G|\bar{J}_i = G_i|\bar{J}_i$ *and* $H|\bar{J}_i = (G_i|\bar{J}_i)^{m_i}$ *for some integer* m_i, *and*

e) for any non-negative integer r, $\{(|m_i|+1)|G_i - id|_r\}$ *is bounded.*

LEMMA (7.2). — *For* $\{G,H\}$ *of Lemma* (7.1), *there are commuting elements* U, V *of* $\mathrm{Diff}_K^{\infty}(\mathbf{R})$ *satisfying* $\{G,H\} = -\{U,V\}$ *and the following conditions : There are disjoint open intervals* K_i *such that* $\bigcup_i K_i$ *is bounded and a finite set* $\{(p_j,q_j)\}_{j=1,\ldots,J}$ *of pairs of coprime integers such that, for each* i, *there is an element* (p_j,q_j) *of this finite set satisfying* $(U|\bar{K}_i)^{p_j}(V|\bar{K}_i)^{q_j} = id_{\bar{K}_i}$.

LEMMA (7.3). — *For* $\{U,V\}$ *of Lemma* (7.2), $\{U,V\} = 0$.

Proof of Lemma (7.1). — Suppose that f and g satisfy the assumption (i) and (ii) of Theorem (6.1). We may assume that the supports of f and g are contained in $(1/2,1)$.

We define a diffeomorphism F with compact support as follows. We have subintervals I_i 's, and for each i, either a C^∞-diffeomorphism h_i and coprime integers m_i, n_i, or a C^∞-vectorfield ξ_i and real numbers s_i, t_i. For each i, we choose a sequence $\{a_{i,j}\}_{j\in \mathbf{N}\cup\{0\}}$ of real numbers $a_{i,j}$, inductively. Put either $a_{i,0} = m_i$ and $a_{i,1} = n_i$ or $a_{i,0} = s_i$ and $a_{i,1} = t_i$. Suppose that we have defined $a_{i,j}$ for $j \leqslant p$. If $a_{i,p} = 0$ $(p \geqslant 1)$, put $a_{i,p+1} = 0$. If $a_{i,p} \neq 0$, we define $a_{i,p+1}$ so that

$$a_{i,p+1} = - a_{i,p-1} + u_{i,p}a_{i,p}$$

for some integer $u_{i,p}$ and $|a_{i,p+1}| \leqslant |a_{i,p}|/2$.

In the case when $a_{i,0} = m_i$ and $a_{i,1} = n_i$, put

$$F_i = \prod_{j=0}^{\infty} k^j h_i^{a_{i,j}} k^{-j}.$$

In the case when $a_{i,0} = s_i$ and $a_{i,1} = t_i$, put

$$\zeta_i = \sum_{j=0}^{\infty} (k^j)_*(a_{i,j}\xi_i),$$

and let F_i be the time one map of ζ_i. In this case, let h_i^t denote the time t map of ξ_i; then we can write

$$F_i = \prod_{j=0}^{\infty} k^j h_i^{a_{i,j}} k^{-j}.$$

Finally put $F = \prod_i F_i$.

By Lemmas (5.1) and (5.2), to show that F is a well-defined C^∞-diffeomorphism, it suffices to show that, for each r, $\{|k^j h_i^{a_{i,j}} k^{-j} - id|_r\}_{i,j}$ and $\{|(k^j)_*(a_{i,j}\xi_i)|_r\}_{i,j}$ are bounded.

LEMMA (7.4). — *If* $\mathrm{Supp}(h) \subset (1/2,1)$ *and* $|h - id|_r < 1$,

$$|k^p h k^{-p} - id|_r \underset{(r)}{\leqslant} |h - id|_r p^{2r}.$$

Proof. – For $x \in (1/(p+2), 1/(p+1))$, put

$$y = x/(1-px) = k^{-p}(x).$$

Then

$$(k^p h k^{-p})'(x) = (1+py)^2(1+ph(y))^{-2}h'(y).$$

Put $L_p(y) = (1+py)(1+ph(y))^{-1}$; then we have

$$(L_p(y)-1)(p^{-1}+h(y)) = y-h(y)$$

and

$$\Sigma\binom{r}{q}(L_p-1)^{(r-q)}(y)(p^{-1}+h)^{(q)}(y) = (id-h)^{(r)}(y).$$

Since we have

$$\sup_{y\in[1/2,1]} |(L_p-1)(y)| \leqslant 2|h-id|_0,$$

if we have $\sup\limits_{y\in[1/2,1]} |(L_p-1)^{(q)}| \underset{(q)}{\leqslant} \|h-id\|_q$ for $q < r$ (as the induction hypothesis), the above formula together with

$$\inf\{(p^{-1}+h)(y); y \in [1/2,1]\} \geqslant 1/2$$

implies that

$$\sup_{y\in[1/2,1]} |(L_p-1)^{(r)}(y)| \underset{(r)}{\leqslant} \|h-id\|_r.$$

On the other hand, we have, for $r \geqslant 1$,

$$\begin{aligned}
&(k^p h k^{-p}-id)^{(r)}(x)\\
&= (d/dx)^{r-1}((L_p(y)^2-1)h'(y)+(h'(y)-1))\\
&= \sum_{q=1}^{r} \sum_{r_1+\cdots+r_{q-1}=r-1} C_{r_1,\ldots,r_{q-1}}(d/dy)^{q-1}((L_p(y)^2-1)h'(y)\\
&\qquad\qquad + (h'(y)-1))(d/dx)^{r_1}(y)\ldots(d/dx)^{r_{q-1}}(y).
\end{aligned}$$

Since $1/(1-px) \in ((p+2)/2, p+1)$, we have

$$(d/dx)^r(y) = r!\,p^{r-1}(1-px)^{-r-1} \underset{(r)}{\leqslant} p^{2r}.$$

Therefore, we have

$$\begin{aligned} |k^p h^k k^{-p} - id)|_r &\underset{(r)}{\leqslant} \sum_{q=1}^{r} |(L_p(y)^2 - 1)h'(y) \to h'(y) - 1|_q p^{2r} \\ &\underset{(r)}{\leqslant} ||h - id||_r p^{2r} \\ &\underset{(r)}{\leqslant} |h - id|_r p^{2r}. \end{aligned}$$

LEMMA (7.5). — *Let* ξ *be a* C^∞*-vectorfield on* $\mathbf{R}$ *such that* $\mathrm{Supp}(\xi) \subset (1/2,1)$. *Then*

$$|(k^p)_*(a\xi)|_r \underset{(r)}{\leqslant} |a| p^{2(r-1)} |\xi|_r \quad (a \in \mathbf{R}).$$

Proof. — Since, for $x \in (1/(p+2), 1/(p+1))$,

$$((k^p)_*(a\xi))(x) = a(1-px)^2 \xi(x/(1-px)),$$

we have

$$((k^p)_*(a\xi))'(x) = a(2(1-px)(-p)\xi(x/(1-px)) + \xi'(x/(1-px)))$$

and

$$\begin{aligned} ((k^p)_*(a\xi))''(x) = a(&2p^2\xi(x/(1-px)) \\ &- 2p(1-px)^{-1}\xi'(x/(1-px)) + (1-px)^{-2}\xi''(x/(1-px))). \end{aligned}$$

For $r \geqslant 2$ and $r \geqslant q \geqslant 1$, there is a homogeneous polynomial $P_{r,q}(z_1, z_2)$ in z_1, z_2 of degree $2(r-1)$, such that

$$((k^p)_*(a\xi))^{(r)}(x) = a \sum_{q=1}^{r} P_{r,q}(p,(1-px)^{-1})\xi^{(q)}(x/(1-px)).$$

This proves Lemma (7.5).

LEMMA (7.6). — *Under the assumption* (ii) (*c*) *of Theorem* (6.1) *for* h_i*'s, we have, for any positive real number* ε *and for every* r, $\max\{|m_i|, |n_i|\}\ |h_i - id|_r \leqslant \varepsilon$ *except a finite number of* h_i*'s.*

Proof. — By (**) of § 5, we have

$$|h_i - id|_r \leqslant \ell_i |h_i - id|_{r+1},$$

where ℓ_i is the length of I_i. Therefore,

$$\max\{|m_i|, |n_i|\}\,|h_i - id|_r \leqslant \ell_i \max\{|m_i|, |n_i|\}\,|h_i - id|_{r+1}.$$

Since except finitly many ℓ_i's, ℓ_i's are sufficiently small, we have proved Lemma (7.6).

Proof of Lemma (7.1) (continued). — By the definition of $a_{i,p}$, we have

$$|a_{i,p}| \leqslant 2^{-p+1}\max\{|a_{i,0}|, |a_{i,1}|\} = 2^{-p+1}\max\{|m_i|, |n_i|\}.$$

By Lemma (7.6), except finitely many i's, we have

$$|h_i - id|_r \leqslant (\max\{|m_i|, |n_i|\}\mathrm{M}_r)^{-1} \leqslant (|a_{i,p}|\mathrm{M}_r)^{-1}.$$

By Lemmas (5.3) and (7.4), we have

$$\begin{aligned} |k^p h_i^{a_{i,p}} k^{-p} - id|_r &\underset{(r)}{\leqslant} |a_{i,p}|p^{2r}|h_i - id|_r \\ &\leqslant 2^{-p+1}p^{2r}\max\{|a_{i,0}|, |a_{i,1}|\}\,|h_i - id|_r. \end{aligned}$$

Therefore, $\{|k^p h_i^{a_{i,p}} k^{-p} - id|_{i,p}$ is bounded.

On the other hand, we have

$$\begin{aligned} |(k^p)_*(a_{i,p}\xi_i)|_r &\underset{(r)}{\leqslant} |a_{i,p}|p^{2(r-1)}|\xi_i|_r \\ &\leqslant 2^{-p+1}p^{2(r-1)}\max\{|a_{i,0}|, |a_{i,1}|\}|\xi_i|_r. \end{aligned}$$

This implies that $\{|(k^p)_*(a_{i,p}\xi_i)|_r\}_{i,p}$ is bounded.

We have proved that F is of class C^∞.

For F and k, we have

$$k\mathrm{F}k^{-1} = \prod_i \prod_{j=1}^{\infty} k^j h_i^{a_{i,j-1}} k^{-j}$$

and

$$k^{-1}\mathrm{F}k = \prod_i \prod_{j=-1}^{\infty} k^j h_i^{a_{i,j+1}} k^{-j}.$$

Put

$$\mathrm{G} = \prod_i \prod_{j=1}^{\infty} k^j h_i^{a_{i,j}} k^{-j}$$

and

$$H = \prod_i \prod_{j=1}^{\infty} k^j h_i^{(a_{i,j-1}+a_{i,j+1})} k^{-j}.$$

Then, by Lemma (2.4), we have

$$\begin{aligned} 0 &= \{F, kFk^{-1}k^{-1}Fk\} \\ &= \{fG, Hgk^{-1}fk\} \\ &= \{f,H\} + \{f,g\} + \{f,k^{-1}fk\} + \{G,H\} + \{G,g\} + \{G,k^{-1}fk\}. \end{aligned}$$

By Theorem (4.1), we have

$$\begin{aligned} \{f,H\} &= 0, & \{f,k^{-1}fk\} &= 0, \\ \{G,g\} &= 0, & \{G,k^{-1}fk\} &= 0. \end{aligned}$$

Therefore, we have

$$\{f,g\} = -\{G,H\}.$$

Since, by construction, G and H satisfy the conditions (*d*), (*e*) of Lemma (7.1), we have proved Lemma (7.1).

Proof of Lemma (7.2). – Again, we may assume that the supports of G and H are contained in (1/2,1). For each i, we have an interval J_i satisfying (*d*) and (*e*) of Lemma (7.1). We choose a sequence $\{b_{i,j}\}$ of integers which plays a similar role as the sequence $\{a_{i,j}\}$ used above.

If $G|\bar{J}_i = id_{\bar{J}_i}$, we put $b_{i,0} = 0$, $b_{i,1} = 1$ and $b_{i,j} = 0$, $j \geqslant 2$.

If $G|\bar{J}_i = G_i|\bar{J}_i$ and $H|\bar{J}_i = (G_i|\bar{J}_i)^{m_i}$, put $b_{i,0} = 1$, $b_{i,1} = m_i$.

In the following definition, if we have $b_{i,p} = 0$, put $b_{i,j} = 0$ for $j \geqslant p+1$.

If $b_{i,j+1}/b_{i,j}$ is an integer, we define $b_{i,j+\ell}$, $2 \leqslant \ell \leqslant 4$ (or $2 \leqslant \ell \leqslant 5$) as follows, making $b_{i,j+4}/b_{i,j+3}$ (or $b_{i,j+5}/b_{i,j+4}$) an integer.

If $|b_{i,j+1}/b_{i,j}| \leqslant 2$, we define $b_{i,j+\ell} = 0$, $\ell \geqslant 2$.

If $b_{i,j+1}/b_{i,j} = 4q$, $q \in \mathbf{Z}$ and $q > 0$, put

$$\begin{aligned} b_{i,j+2} &= (1/2)(b_{i,j+1} - 2b_{i,j}) & &\text{(i.e., } b_{i,j+1} = 2(b_{i,j} + b_{i,j+2})\text{).} \\ b_{i,j+3} &= -2b_{i,j} & &\text{(i.e., } b_{i,j+2} = (1/2)(b_{i,j+1} + b_{i,j+3})\text{),} \end{aligned}$$

and

$$b_{i,j+4} = (1/2)(-b_{i,j+1}+4b_{i,j}) \qquad \text{(i.e., } b_{i,j+3}=-2(b_{i,j+2}+b_{i,j+4})\text{)}.$$

If $b_{i,j+1}/b_{i,j} = 4q$, $q \in \mathbf{Z}$ and $q < 0$, put

$$b_{i,j+2} = (1/2)(-b_{i,j+1}-2b_{i,j}) \qquad \text{(i.e., } b_{i,j+1}=-2(b_{i,j}+b_{i,j+2})\text{)},$$
$$b_{i,j+3} = 2b_{i,j} \qquad \text{(i.e., } b_{i,j+2}=-(1/2)(b_{i,j+1}+b_{i,j+3})\text{)},$$

and

$$b_{i,j+4} = (1/2)(-b_{i,j+1}+4b_{i,j}) \qquad \text{(i.e., } b_{i,j+3}=2(b_{i,j+2}+b_{i,j+4})\text{)}.$$

If $b_{i,j+1}/b_{i,j} = 4q+1$, $q \in \mathbf{Z}$ and $q \geqslant 1$, put

$$b_{i,j+2} = b_{i,j+1} - b_{i,j} \qquad \text{(i.e., } b_{i,j+1}=b_{i,j}+b_{i,j+2}\text{)},$$
$$b_{i,j+3} = (1/2)(-b_{i,j+1}-b_{i,j}) \qquad \text{(i.e., } b_{i,j+2}=2(b_{i,j+1}+b_{i,j+3})\text{)},$$
$$b_{i,j+4} = 2b_{i,j} \qquad \text{(i.e., } b_{i,j+3}=-(1/2)(b_{i,j+2}+b_{i,j+4})\text{)}$$

and

$$b_{i,j+5} = (1/2)(b_{i,j+1}-b_{i,j}) \qquad \text{(i.e., } b_{i,j+4}=-2(b_{i,j+3}+b_{i,j+5})\text{)}.$$

If $b_{i,j+1}/b_{i,j} = -4q-1$, $q \in \mathbf{Z}$ and $q \geqslant 1$, put

$$b_{i,j+2} = -b_{i,j+1} - b_{i,j} \qquad \text{(i.e., } b_{i,j+1}=-(b_{i,j}+b_{i,j+2})\text{)},$$
$$b_{i,j+3} = (1/2)(-b_{i,j+1}+b_{i,j}) \qquad \text{(i.e., } b_{i,j+2}=-2(b_{i,j+1}+b_{i,j+3})\text{)},$$
$$b_{i,j+4} = 2b_{i,j} \qquad \text{(i.e., } b_{i,j+3}=(1/2)(b_{i,j+2}+b_{i,j+4})\text{)},$$

and

$$b_{i,j+5} = (1/2)(b_{i,j+1}+b_{i,j}) \qquad \text{(i.e., } b_{i,j+4}=2b_{i,j+3}+b_{i,j+5})\text{)}.$$

If $b_{i,j+1}/b_{i,j} = 4q+2$, $q \in \mathbf{Z}$ and $q \geqslant 1$, put

$$b_{i,j+2} = (1/2)(b_{i,j+1}-2b_{i,j}) \qquad \text{(i.e., } b_{i,j+1}=2(b_{i,j}+b_{i,j+2})\text{)},$$
$$b_{i,j+3} = -2b_{i,j} \qquad \text{(i.e., } b_{i,j+2}=(1/2)(b_{i,j+1}+b_{i,j+3})\text{)},$$

and

$$b_{i,j+4} = (1/2)(-b_{i,j+1}+6b_{i,j}) \qquad \text{(i.e., } b_{i,j+3}=-(b_{i,j+2}+b_{i,j+4})\text{)}.$$

If $b_{i,j+1}/b_{i,j} = -4q-2$, $q \in \mathbf{Z}$ and $q \geqslant 1$, put

$$b_{i,j+2} = (1/2)(-b_{i,j+1}-2b_{i,j}) \qquad \text{(i.e., } b_{i,j+1}=-2(b_{i,j}+b_{i,j+2})\text{)},$$
$$b_{i,j+3} = 2b_{i,j} \qquad \text{(i.e., } b_{i,j+2}=-(1/2)(b_{i,j+1}+b_{i,j+3})\text{)},$$

and

$$b_{i,j+4} = (1/2)(b_{i,j+1}+6b_{i,j}) \qquad \text{(i.e., } b_{i,j+3}=b_{i,j+2}+b_{i,j+4}).$$

If $b_{i,j+1}/b_{i,j} = 4q + 3$, $q \in \mathbf{Z}$ and $q \geqslant 0$, put

$$\begin{aligned} b_{i,j+2} &= b_{i,j+1} - b_{i,j} && \text{(i.e., } b_{i,j+1}=b_{i,j}+b_{i,j+2}), \\ b_{i,j+3} &= (1/2)(-b_{i,j+1}-b_{i,j}) && \text{(i.e., } b_{i,j+2}=2(b_{i,j+1}+b_{i,j+3})), \\ b_{i,j+4} &= 2b_{i,j} && \text{(i.e., } b_{i,j+3}=-(1/2)(b_{i,j+2}+b_{i,j+4})) \end{aligned}$$

and

$$b_{i,j+5} = (1/2)(b_{i,j+1}-3b_{i,j}) \qquad \text{(i.e., } b_{i,j+4}=-(b_{i,j+3}+b_{i,j+5})).$$

If $b_{i,j+1}/b_{i,j} = -4q - 3$, $q \in \mathbf{Z}$ and $q \geqslant 0$, put

$$\begin{aligned} b_{i,j+2} &= -b_{i,j+1} - b_{i,j} && \text{(i.e., } b_{i,j+1} = -(b_{i,j}+b_{i,j+2})), \\ b_{i,j+3} &= (1/2)(-b_{i,j+1}+b_{i,j}) && \text{(i.e., } b_{i,j+2}=-2(b_{i,j+1}+b_{i,j+3})), \\ b_{i,j+4} &= 2b_{i,j} && \text{(i.e., } b_{i,j+3} = (1/2)(b_{i,j+2}+b_{i,j+4})), \end{aligned}$$

and

$$b_{i,j+5} = (1/2)(-b_{i,j+1}+3b_{i,j}) \qquad \text{(i.e., } b_{i,j+4}=b_{i,j+3}+b_{i,j+5}).$$

Note that, in each case, we have $|b_{i,j+\ell}| \leqslant |b_{i,j+1}|$, for $2 \leqslant \ell \leqslant 4$ (or $2 \leqslant \ell \leqslant 5$) and $|b_{i,j+4}| \leqslant (1/2)|b_{i,j+1}|$ (or $|b_{i,j+5}| \leqslant (1/2)|b_{i,j+1}|$). Therefore, we have

$$|b_{i,j}| \leqslant 2^{(5-j)/4} \max\{|b_{i,0}|, |b_{i,1}|\}$$

Put $\mathrm{S} = \prod_i \prod_{j=0}^{\infty} k^j \mathrm{G}_i^{b_{i,j}} k^{-j}$

Then, using Lemmas (7.4) and (7.6) as in the proof of Lemma (7.1), we see that, for each r, $\{|k^j \mathrm{G}_i^{b_{i,j}} k^{-j} - id|_r\}_{i,j}$ is bounded; consequently, S is of class C^∞

Put

$$\mathrm{U} = \prod_i \prod_{j=1}^{\infty} k^j \mathrm{G}_i^{b_{i,j}} k^{-j}$$

and

$$\mathrm{V} = \prod_i \prod_{j=1}^{\infty} k^j \mathrm{G}_i^{(b_{i,j-1}+b_{i,j+1})} k^{-j}.$$

Then, by Lemma (2.4),

$$\begin{aligned} 0 &= \{S, kSk^{-1}k^{-1}Sk\} \\ &= \{GU, VHk^{-1}Gk\} \\ &= \{G,V\} + \{G,H\} + \{G,k^{-1}Gk\} + \{U,V\} + \{U,H\} + \{U,k^{-1}Gk\}. \end{aligned}$$

As before, by Theorem (4.1), we have

$$\{G,V\} = 0, \qquad \{G,k^{-1}Gk\} = 0, \qquad \{U,H\} = 0$$

and

$$\{U,k^{-1}Gk\} = 0.$$

Therefore, we have

$$\{G,H\} = - \{U,V\}.$$

By the choice of $\{b_{i,j}\}$, for each i and each $j \geqslant 1$,

$$p_{i,j}b_{i,j} + q_{i,j}(b_{i,j-1} + b_{i,j+2}) = 0,$$

where

$$\{(p_{i,j},q_{i,j})\} = \{(0,1), (1,0), (1,1), (1,-1), (2,1), (2,-1), (1,2), (1,-2)\},$$

hence we have Lemma (7.2).

Proof of Lemma (7.3). – Let $M(p_j,q_j)(j=1,\ldots,J)$ denote the set of suffixes i such that

$$(V|\bar{K}_i)^{p_j}(V|\bar{K}_i)^{q_j} = id_{\bar{K}_i}.$$

Choose a pair of integers u_j, v_j such that $u_jp_j + v_jq_j = 1$ for each (p_j,q_j). Define a C^∞-diffeomorphism $W_j\,(j=1,\ldots,J)$ by

$$W_j|\mathbf{R} - \cup K_i = id_{\mathbf{R}-\cup K_i},$$

$$W_j|\bar{K}_i = \begin{cases} id_{\bar{K}_i} & \text{for} \quad i \notin M(p_j,q_j), \\ (U|\bar{K}_i)^{v_j}(V|\bar{K}_i)^{-u_j} & \text{for} \quad i \in M(p_j,q_j). \end{cases}$$

Then we have

$$U = \prod_{j=1}^{J} W_j^{q_j} \qquad \text{and} \qquad V = \prod_{j=1}^{J} W_j^{-p_j}.$$

By Lemma (2.3), we have

$$\{U,V\} = \prod_{i=1}^{J} \prod_{j=1}^{J} - q_i p_j \{W_i, W_j\}$$

By Theorem (6.3), we can write $W_j (j = 1,\ldots,J)$ as a composition of commutators

$$W_j = \prod_{i=1}^{k_j} [h_{j,2i-1}, h_{j,2i}]\,;$$

so that

$$\mathrm{Fix}^\infty(W_j) \subset \mathrm{Fix}^\infty(h_{j,i}) \quad (i = 1,\ldots,2k_j).$$

Therefore, we have $h_{j,i}W_\ell = W_\ell h_{j,i}$ provided $j \neq \ell$. Then, by Lemma (2.7), we have $\{W_i, W_j\} = 0$ if $i \neq j$. Since $\{W_i, W_i\} = 0$, we have proved that $\{U,V\} = 0$.

8. Small commutators of $\mathrm{Diff}_K^\infty(\mathbf{R})$.

This section and the following section are devoted to examining a method of writing a diffeomorphism of $\mathbf{R}$ with compact support as a composition of commutators.

In this section, we prove the following theorem.

THEOREM (8.1). — *Put* $\mathcal{D}_K(n,c) = \{ f \in \mathrm{Diff}^\infty([0,1])\,;$ $\mathrm{Supp}(f) \subset (1/8, 7/8),\ |f - id|_n < c\}$. *Then, for any positive integer* m, *there exist a positive integer* n *and a positive real number* c *such that every element* f *of* $\mathcal{D}_K(n,c)$ *is written as a composition of four commutators;* $f = [h_1,h_2][h_3,h_4][h_5,h_6][h_7,h_8]$, *where each* $h_i (i=1,\ldots,8)$ *belongs to* $\mathrm{Diff}^\infty([0,1])$ *and satisfies*

$$\mathrm{Supp}(h_i) \subset (0,1) \qquad and \qquad |h_i - id|_m \underset{*}{\leqslant} (|f - id|_n)^{1/(m+2)}.$$

First we give some notations and the outline of the proof of Theorem (8.1).

Let μ be a vectorfield on $[0,1]$ such that $\mathrm{Supp}(\mu) \subset (0,1)$ and $\mu(x) = (\partial/\partial x)_x$ for $x \in [1/16, 15/16]$. Let A_t be the time t map of $\beta\mu$ and $A = A_1$, where we shall fix a real number β depending on f later. If β is small, but sufficiently large with respect to $f - id$, Af can be considered as a small perturbation of A. Since A is a translation by β on

[1/16, 15/16], using the device of Mather [10], we can construct a diffeomorphism $\Gamma(f)$ of S^1 associated with Af. If f is sufficiently near the identity, $\Gamma(f)$ is sufficiently near $\Gamma(id)$. Now, we use the following theorem of Herman [3] and Sergeraert [19]. Note that, for any rotation ρ, we can define a Γ so that $\Gamma(id) = \rho$.

THEOREM (8.2) (Herman [3], Sergeraert [19]). — *Let* γ *be a real number which satisfies the Diophantine condition, that is, for* γ *there are positive real numbers* C, ε, *such that* $|p\gamma+q| \geqslant C/|p|^{\varepsilon}$ *for any integers* $p, q(p \neq 0)$. *Let* Φ_γ *be the map defined by*

$$\Phi_\gamma : \mathrm{Diff}^\infty_+(S^1) \times S^1 \to \mathrm{Diff}^\infty_+(S^1)$$
$$(\psi,\lambda) \to R_\lambda \psi^{-1} R_\gamma \psi,$$

where R_λ *denotes the rotation by* λ.

Then there exist a neighborhood V *of* R_γ *in* $\mathrm{Diff}^\infty_+(S^1)$ *in* C^∞*-topology and a weak* $\mathscr{L}$*-morphism* s *of class* C^∞;

$$s : V \to \mathrm{Diff}^\infty_+(S^1) \times S^1$$

such that $\Phi_\gamma s = id_V$ *and* $s(R_\gamma) = (id,0)$.

By Theorem (8.2), we can write the diffeomorphism $\Gamma(f)$ of S^1, which is close to R_γ, in the form $R_\lambda\psi^{-1}R_\gamma\psi$, where ψ is close to the identity. Using ψ, we construct diffeomorphisms close to the identity of [0,1], and, by these diffeomorphisms we can write f as a composition of four commutators.

Now we prove Theorem (8.1). The proof is devided into 7 steps.

Step 1. — In this step, we determine a positive integer n and a positive real number β for our m. First we fix a number γ which satisfies the Diophantine condition and $0 < \gamma < 1$. By Theorem (8.2), we have an inequality

$$\|ds(\varphi,\hat{\varphi})\|_i \underset{(i)}{\leqslant} (1+\|\varphi\|_{i+d_{1,i}})\|\hat{\varphi}\|_0 + \|\hat{\varphi}\|_{i+d_{1,i}}$$

for any i, where $\hat{\varphi}$ is a tangent vector (see [19]). Therefore, if we write $s(\varphi) = (\psi(\varphi), \lambda(\varphi))$, we have

$$\|\psi - id\|_i \underset{(i)}{\leqslant} \|\varphi - R_\gamma\|_{i+d_{1,i}}$$

and

$$|\lambda| \underset{(i)}{\leqslant} \|\varphi - R_\gamma\|_{i+d_{1,i}}.$$

For our m, put n to be $m + d_{1,m}$. Changing n by a greater integer if necessary, we may assume that if $\|\varphi - R_\gamma\|_n$ is sufficiently small, φ belongs to the neighborhood V of R_γ. For this n, put $\alpha = |f - id|_n$ and $\beta = \alpha^{1/(m+2)}$. Thus we have determined A_t.

Step 2. – In this step, we define the diffeomorphism $\Gamma(f)$ of S^1. $\Gamma(f)$ will be defined up to the multiplication of rotations to the left.

Let a be a real number such that $(7/8) + \beta < a < (15/16) - \beta$. If α is *sufficiently small*, Af has no fixed points in $[1/8, 7/8]$. Therefore, for any $x \in [1/16, (1/16) + \beta)$, there exists a unique integer N and $y \in [a, a+\beta)$ such that $y = (Af)^N(x)$. Let $\sigma_a(f)$ denote this map; $y = \sigma_a(f)(x)$.

Define $B : [1/16, (1/16) + \beta) \to S^1$ by $B(x) = (x - (1/16))/\beta \mod 1$ and $B_a : [a, a+\beta) \to S^1$ by $B_a(x) = (x-a)/\beta \mod 1$. Then $B_a\sigma_a(f)B^{-1}$ is a well defined C^∞-diffeomorphism of S^1 (Mather [10]). Put $\Gamma_a(f) = B_a\sigma_a(f)B^{-1}$. Mather proved that, if $\Gamma_a(f)$ is a rotation, Af is conjugate to A ([10]). Note that, for $a' \in ((7/8) + \beta, (15/16) - \beta)$,

$$B_{a'}\sigma_{a'}(f)B^{-1} = R_{(a-a')/\beta}B_a\sigma_a(f)B^{-1}$$

and

$$\Gamma_{a'}(f) = R_{(a-a')/\beta}\Gamma_a(f).$$

Step 3. – An estimate on $\Gamma_a(f)$.

Let N be the minimal integer such that $(1/16) + N > (7/8) + \beta$. Put $a = (1/16) + N\beta$. Since $\Gamma_a(id) = id$, $\Gamma_a(f)$ is close to the identity if f is close to the identity. In fact, we have the following estimate.

Since $13/(16\beta) < N < 7/(8\beta)$, $|f - id|_r \leqslant |f - id|_n = \alpha$ $(r \leqslant n)$, and $\beta = \alpha^{1/(m+2)}$, we have, for *sufficiently small* α,

$$|f - id|_r \leqslant (NM_r)^{-1} \qquad \text{for} \qquad r \leqslant n.$$

Let f_1 denote the diffeomorphism of $\mathbf{R}$ which coincides with f on $[0,1]$ and with the identity on $\mathbf{R} - [0,1]$, and let T denote the translation of $\mathbf{R}$ by β. Then we have

$$|T^{-i}fT^i - id|_r = |f - id|_r \leqslant (NM_r)^{-1}$$

and

$$\begin{aligned}
\sup_{x\in[1/16,(1/16)+\beta]} & |(\sigma_a(f)-\sigma_a(\mathrm{id}))^{(r)}(x)| \\
&= \sup_{x\in[1/16,(1/16)+\beta]} |((\mathrm{A}f)^{\mathrm{N}}-\mathrm{A}^{\mathrm{N}})^{(r)}(x)| \\
&\leqslant \sup_{x\in\mathbf{R}} |((\mathrm{T}f_1)^{\mathrm{N}}-\mathrm{T}^{\mathrm{N}})^{(r)}(x)| = |(\mathrm{T}f_1)^{\mathrm{N}}-\mathrm{T}^{\mathrm{N}}|_r \\
&= |\mathrm{T}^{\mathrm{N}}(\mathrm{T}^{-\mathrm{N}+1}f_1\mathrm{T}^{\mathrm{N}-1})\dots(\mathrm{T}^{-2}f_1\mathrm{T}^2)(\mathrm{T}^{-1}f_1\mathrm{T})f_1-\mathrm{T}^{\mathrm{N}}|_r \\
&= |(\mathrm{T}^{-\mathrm{N}+1}f_1\mathrm{T}^{\mathrm{N}-1})\dots(\mathrm{T}^{-2}f_1\mathrm{T}^2)(\mathrm{T}^{-1}f_1\mathrm{T})f_1-id|_r \\
&\leqslant \mathrm{NM}_r|f-id|_r \\
&\leqslant (7/8)\mathrm{M}_r(\alpha/\beta).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|\Gamma_a(f)-id|_r &= \beta^{r-1} \sup_{x\in[1/16,(1/16)+\beta]} |(\sigma_a(f)-\sigma_a(id))^{(r)}(x)| \\
&\leqslant (7/8)\mathrm{M}_r(\beta^{r-2}\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\|\Gamma_a(f)-id\|_n &\leqslant \sum_{r=0}^{n} (7/8)\mathrm{M}_r(\beta^{r-2}\alpha) \\
&\leqslant (7/8)(n+1)\mathrm{M}_n(\alpha/\beta^2) = (7/8)(n+1)\mathrm{M}_n\alpha^{m/(m+2)}
\end{aligned}$$

Step 4. – In this step, we apply Theorem (8.2).

If $\Gamma_a(f)$ is sufficiently close to the identity, $\Gamma_{\alpha+\beta(1-\gamma)}(f)$ is sufficiently close to R_γ. In fact, we have

$$\|\Gamma_{a+\beta(1-\gamma)}(f)-\mathrm{R}_\gamma\|_n \leqslant (7/8)(n+1)\mathrm{M}_n\alpha^{m/(m+2)}$$

If α is *sufficiently small*, by Theorem (8.2) and the choice of n in Step 1, we can find a number λ and a diffeomorphism ψ of S^1 such that

$$\Gamma_{a+\beta(1-\gamma)}(f) = \mathrm{R}_\lambda\psi^{-1}\mathrm{R}_\gamma\psi,$$

where

$$|\lambda| \underset{*}{\leqslant} \|\Gamma_{a+\beta(1-\gamma)}(f)-\mathrm{R}_\gamma\|_n \underset{*}{\leqslant} \alpha^{m/(m+2)}$$

and

$$\|\psi-id\|_m \underset{*}{\leqslant} \|\Gamma_{a+\beta(1-\gamma)}(f)-R_\gamma\|_n \underset{*}{\leqslant} \alpha^{m/(m+2)}$$

Step 5. — In this step, using ψ, we define diffeomorphisms $\tilde{k}_0$, $\tilde{k}_1$ of $[0,1]$ (see [10]).

Let $\pi : \mathbf{R} \to \mathbf{R}/\mathbf{Z} = S^1$ denote the canonical projection. Let ν be a C^∞-function on $\mathbf{R}/\mathbf{Z}$ such that

$$\begin{aligned} &\nu(x) \in [0,1], && x \in \mathbf{R}/\mathbf{Z}, \\ &\nu(x) = 0 \quad \text{on} && [3/8, 5/8] \mod 1 \end{aligned}$$

and

$$\nu(x) = 1 \quad \text{on} \quad [-1/8, 1/8] \mod 1 .$$

Define a diffeomorphism k_0 of $S^1 = \mathbf{R}/\mathbf{Z}$ by

$$k_0 = \pi(\nu.(\widetilde{\psi - id})) + id ,$$

where $\widetilde{\psi - id} : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ is a lift of $\psi - id$ which is close to zero. Then, since ψ is *sufficiently close* to the identity, k_0 is a well-defined diffeomorphism of S^1. Put $k_1 = \psi k_0^{-1}$. It is clear that $\psi = k_1 k_0$, k_0 is the identity on $[3/8, 5/8]$ mod 1 and k_1 is the identity on $[-1/8, 1/8]$ mod 1.

By the Leibnitz formula, we have

$$\|k_0 - id\|_m \underset{*}{\leqslant} \|\psi - id\|_m \underset{*}{\leqslant} \alpha^{m/(m+2)}$$

and

$$\|k_1 - id\|_m \underset{*}{\leqslant} \|\psi - id\|_m \underset{*}{\leqslant} \alpha^{m/(m+2)} .$$

Let $\tilde{k}_0$, $\tilde{k}_1$ be the diffeomorphisms of $[0,1]$ which are defined by

$$\tilde{k}_0(x) = \begin{cases} B_{a+(\beta/2)} R_{1/2} k_0 R_{1/2}^{-1} B_{a+(\beta/2)}^{-1}(x), & x \in [a+(\beta/2), a+(3/2)\beta] \\ x & x \in [0,1] - [a+(\beta/2), a+(3/2)\beta], \end{cases}$$

$$\tilde{k}_1(x) = \begin{cases} B_{a+2\beta} k_1 B_{a+2\beta}^{-1}(x), & x \in [a+2\beta, a+3\beta] \\ x & x \in [0,1] - [a+2\beta, a+3\beta]. \end{cases}$$

$\tilde{k}_0$ and $\tilde{k}_1$ are well-defined and we have

$$\Gamma_{a+3\beta}(\tilde{k}_1 \tilde{k}_0) = \psi$$

and

$$\Gamma_{a+5\beta}(A_3 \tilde{k}_0^{-1} A_{-3} \tilde{k}_1^{-1}) = \psi^{-1} .$$

The norms of $\tilde{k}_0$ and $\tilde{k}_1$ are estimated as follows.

$$\begin{aligned}|\tilde{k}_0 - id|_m &= \beta^{1-m}|k_0 - id|_m \\ &\leqslant \beta^{1-m}\|k_0 - id\|_m \\ &\underset{*}{\leqslant} \beta^{1-m}\alpha^{m/(m+2)} = \alpha^{1/(m+2)}.\end{aligned}$$

In the same way, we have

$$|\tilde{k}_1 - id|_m \underset{*}{\leqslant} \alpha^{1/(m+2)}.$$

Step 6. – In this step, we show that $\Gamma_{a+(8+\lambda)\beta}$ of the composition of f and some A_t, $\tilde{k}_0^{\pm 1}$, $\tilde{k}_1^{\pm 1}$ is the identity.

Since $\Gamma_{a+\beta(1-\gamma)}(f) = R_\lambda \psi^{-1} R_\gamma \psi$, we have

$$\Gamma_{a+\beta(1-\gamma)+\beta(1+\lambda)}(f) = \psi^{-1} R_\gamma \psi.$$

Since $\Gamma_{a+3\beta}(\tilde{k}_1\tilde{k}_0) = \psi$, we have

$$\Gamma_{a+(2+\lambda-\gamma)\beta+3\beta}(A_{(2+\lambda-\gamma)\beta}\tilde{k}_1\tilde{k}_0 A_{-(2+\lambda-\gamma)\beta} f) = R_\gamma \psi.$$

Therefore, we have

$$\Gamma_{a+(5+\lambda)\beta}(A_{(2+\lambda-\gamma)\beta}\tilde{k}_1\tilde{k}_0 A_{-(2+\lambda-\gamma)\beta} f) = \psi.$$

Using, $\Gamma_{a+5\beta}(A_{3\beta}\tilde{k}_0^{-1}A_{-3\beta}\tilde{k}_1^{-1}) = \psi^{-1}$, we obtain

$$\Gamma_{a+(5+\lambda)\beta+3\beta}(g) = id,$$

where

$$g = A_{(3+\lambda)\beta}A_{3\beta}\tilde{k}_0^{-1}A_{-3\beta}\tilde{k}_1^{-1}A_{-(3+\lambda)\beta}\, A_{(2+\lambda-\gamma)\beta}\tilde{k}_1\tilde{k}_0 A_{-(2+\lambda-\gamma)\beta} f.$$

Step 7. – Completion of the proof.

By a result of Mather [10], there exists a diffeomorphism H of $[0,1]$ such that $Ag = HAH^{-1}$ Therefore, we have

$$\begin{aligned}f = [A_{(2+\lambda-\gamma)\beta}, \tilde{k}_0^{-1}\tilde{k}_1^{-1}][\tilde{k}_0^{-1}\tilde{k}_1^{-1}, A_{(3+\lambda)\beta}]& \\ [A_{(3+\lambda)\beta}\tilde{k}_0^{-1}A_{-(3+\lambda)\beta}, A_{3\beta}][A^{-1}, H]&.\end{aligned}$$

For a diffeomorphism g such that $\Gamma_{15/16}(g)$ is a rotation, H is defined so that

$$H(x) = (Ag)^q A^{-q}(x) \qquad \text{for} \qquad x \in [1/8, 15/16],$$

where q is a positive integer satisfying $A^{-q}(x) \in [1/16, 1/8]$,

H is the identity on $[0, 1/8]$,
and

$H = A_t$ on $[15/16, 1]$,
where t is a real number such that $\Gamma_{15/16}(g) - \Gamma_{15/16}(id) = R_t$.

It is easy to check that $H(x)$ is well-defined and $HA = AgH$.

In our case, by the definition of g, using the same argument as in Step 3, we obtain

$$\sup_{x \in [1/8, 7/8]} |(H - id)^{(m)}(x)| \leqslant (7/8) M_m(\alpha/\beta).$$

On the other hand, $H|[(7/8)+10\beta, 1]$ coincides with

$$A_{\lambda\beta}|[(7/8)+10\beta, 1];$$

therefore

$$\sup_{x \in [(7/8)+10\beta, 1]} |(H - id)^{(m)}(x)| \underset{*}{\leqslant} \lambda\beta \underset{*}{\leqslant} \alpha^{(m+1)/(m+2)}$$

Finally, $H|[7/8, (7/8)+10\beta]$ is a composition of $(Ag)^{11}$, H restricted to $[1/8, 7/8]$ and A^{-11};

$$H|[7/8, (7/8)+10\beta] = ((Ag)^{11}(H|[1/8, 7/8])A^{-11})|[7/8, (7/8)+10\beta]$$

Using the estimates on $|\tilde{k}_0 - id|_m$, $|\tilde{k}_1 - id|_m$ and

$$\sup_{x \in [1/8, 7/8]} |(H - id)^{(m)}(x)|,$$

by Lemma (5.3), we have

$$\sup_{x \in [7/8, (7/8)+10\beta]} |(H - id)^{(m)}(x)| \underset{*}{\leqslant} \alpha^{1/(m+2)}$$

Consequently, $|H - id|_m \underset{*}{\leqslant} \alpha^{1/(m+2)}$

Since

$$|A_t - id|_m \underset{*}{\leqslant} \beta = \alpha^{1/(m+2)} \quad (|t| \leqslant 5),$$

$$|\tilde{k}_0 - id|_m \underset{*}{\leqslant} \alpha^{1/(m+2)}, \qquad |\tilde{k}_1 - id|_m \underset{*}{\leqslant} \alpha^{1/(m+2)}$$

and

$$|\mathrm{H}-id|_m \underset{*}{\leqslant} \alpha^{1/(m+2)},$$

we have written f as a composition of four commutators so that the $|\,.\,|_m$-norm of every element appearing in the commutators is not greater than $\alpha^{1/(m+2)}$ up to a constant.

We fix the real number c of the statement of Theorem (8.1) so that c is small enough for the requirements of Steps 2, 3, 4 and 5. Thus we have proved Theorem (8.1).

9. Small commutators of $\mathrm{Diff}^\infty_\infty([0,1])$.

In this section we prove the following theorem.

THEOREM (9.1). – *Put $\mathscr{D}(n,c) = \{f \in \mathrm{Diff}^\infty_\infty([0,1]);\ |f-id|_n < c\}$. For any positive integer m, there exist a positive integer n and a positive real number c, such that, for any element f of $\mathscr{D}(n,c)$, there are elements $\bar{g}$, $\bar{h}$ of $\mathrm{Diff}^\infty_\infty([0,1])$ such that*

(1) $f = \bar{g}\bar{h}\bar{g}^{-1}\bar{h}^{-1}$ *on* $[0,1/8]$,

(2) $\bar{g}|[5/6,1] = id_{[5/6,1]}$, $\bar{h}|[5/6,1] = id_{[5/6,1]}$, *and*

(3) $|\bar{g}-id|_m \underset{*}{\leqslant} (|f-id|_n)^{1/3}$, $|\bar{h}-id|_m \underset{*}{\leqslant} (|f-id|_n)^{1/3}$.

Sergeraert proved the following proposition ([20]).

PROPOSITION (9.2). – *Suppose that, for an element f of $\mathrm{Diff}^\infty_\infty([0,1])$, there exist elements g, h of $\mathrm{Diff}^\infty([0,1])$ such that*

(1) $f = ghg^{-1}h^{-1}$ *on* $[0,1/6]$,

(2) $g|[5/6,1] = id_{[5/6,1]}$, $h|[5/6,1] = id_{[5/6,1]}$,

(3) $j^\infty_0(h) = j^\infty_0(id)$,

and

(4) $j^m_0(g) = j^m_0(id)$, $j^{m+1}_0(g) \neq j^{m+1}_0(id)$ *for some* $m \geqslant 1$.

Then, for any positive real number ε, there exist elements $\bar{g}$ and $\bar{h}$ of $\mathrm{Diff}^\infty_\infty([0,1])$ which are ε-$\|\,.\,\|_m$-close to g and h, respectively, and satisfy (1) and (2) of Theorem (9.1).

By Proposition (9.2), to prove Theorem (9.1), it suffices to show that there are elements g, h of $\mathrm{Diff}^\infty([0,1])$ which satisfy the conditions (1)-(4) of Proposition (9.2) and admit the estimate (3) of Theorem (9.1).

We obtain such g, h as follows.

Let ξ be a C^∞-vectorfield on $[0,1]$ such that $\xi(x) = \beta x^{m+1}(\partial/\partial x)$ on $[0,1/6]$ and $\xi = 0$ on $[5/6,1]$, where β is a positive real number. Let $G : \mathbf{R} \times [0,1] \to \mathbf{R}$ denote the one parameter group of transformations generated by ξ, and let $g(x) = G(1,x)$.

Since the ∞-jets at 0 of g^{-1} and $g^{-1}f$ coincide, by a theorem of Takens [23], there is a local diffeomorphism h between neighborhoods of 0 such that $hg^{-1}h^{-1} = g^{-1}f$ annd ${}^\infty_0(h) = j^\infty_0(id)$. Thus we have $f = ghg^{-1}h^{-1}$ in a neighborhood of 0. We have a more explicit definition of h ([20]). Assume that β is sufficiently large with respect to f. Then we have a function $\tau : [0,1/6] \to \mathbf{R}$ defined by $G(\tau(x),x) = f(x)$. Sergeraert [20] showed that $\tau(x)$ is of class C^∞ and $j^\infty_0(\tau) = j^\infty_0(0)$. Put

$$T(x) = \sum_{i=0}^{\infty} \tau((g^{-1}f)^i(x));$$

then $T(x)$ is a well-defined C^∞ function of $[0,1/6]$ ([20]). Then our h is defined by

$$h^{-1}(x) = G(T(x),x).$$

Therefore, to prove Theorem (9.1), it suffices to choose n, c and β so that g and h satisfy the estimate (3) of Theorem (9.1).

Now put $n = 3(m+1)$, $\alpha = |f-id|_n$ and $\beta = \alpha^{1/3}$. Then we have

$$|g-id|_m \underset{*}{\leqslant} \beta = \alpha^{1/3}.$$

We will estimate the derivatives of τ and T and show that

$$|h-id|_m \underset{*}{\leqslant} \alpha^{1/3}.$$

Estimate on τ. By the definition of τ, if α is *sufficiently small*, we have

$$\tau(x) = -\beta^{-1}m^{-1}(f(x))^{-m} + \beta^{-1}m^{-1}x^{-m}, \qquad x \in [0,1/6].$$

Put $P(x) = -m^{-1}x^{-m}$. Then, for $r \geqslant 0$,

$$\tau^{(r)}(x) = \beta^{-1}((P^{(r)} \circ f)(x) - P^{(r)}(x)) + \beta^{-1}(P^{(r)} \circ f)(x)((f'(x))^r - 1)$$
$$+ \beta^{-1} \sum_{q=1}^{r-1} \sum_{r_1+\cdots+r_q=r} C_{r_1,\ldots,r_q}(P^{(q)} \circ f)(x) f^{(r_1)}(x) \ldots f^{(r_q)}(x).$$

Using $|(f-id)^{(i)}(x)| \leqslant ((n-i)!)^{-1} x^{n-i} |f-id|_n \quad (0 \leqslant i \leqslant n)$,

$$(f-id)^{(i)}(x) = f^{(i)}(x) \quad (i \geqslant 2)$$

and

$$P^{(q)}(x) = (-m)^{-1}(-m)(-m-1) \ldots (-m-q+1)x^{-m-q},$$

if α is *sufficiently small*, we have, for $0 \leqslant r \leqslant n$,

$$|(P^{(r)} \circ f)(x) - P^{(r)}(x)| \leqslant \sup_{t \in [0,1]} |P^{(r+1)}((f(x)-x)t+x)| \,.\, |f(x)-x|$$
$$\underset{(m,r)}{\leqslant} \left(x - \frac{x^n}{n!}|f-id|_n\right)^{-m-r-1} \frac{x^n}{n!}|f-id|_n$$
$$\underset{(m,n,r)}{\leqslant} x^{n-m-r-1}|f-id|_n,$$

$$|(P^{(r)} \circ f)(x)((f'(x))^r - 1)| \underset{(m,r)}{\leqslant} \left(x - \frac{x^n}{n!}|f-id|_n\right)^{-m-r} \frac{x^{n-1}}{(n-1)!}|f-id|_n$$
$$\underset{(n,m,r)}{\leqslant} x^{n-m-r-1}|f-id|_n$$

and

$$\left|\sum_{q=1}^{r-1} \sum_{r_1+\cdots+r_q=r} C_{r_1,\cdots,r_q}(P^{(q)} \circ f)(x) f^{(r_1)}(x) \ldots f^{(r_q)}(x)\right|$$
$$\underset{(m,r)}{\leqslant} \sum_{q=1}^{r-1} \sum_{r_1+\cdots+r_q=r} \left(x - \frac{x^n}{n!}|f-id|_n\right)^{-m-q} \prod_{r_i \geqslant 2} \frac{x^{n-r_i}}{(n-r_i)!}|f-id|_n$$
$$\underset{(n,m,r)}{\leqslant} \sum_{q=1}^{r-1} x^{-m-q} \sum_{r_1+\cdots+r_q=r} \prod_{r_i \geqslant 2} x^{n-r_i}|f-id|_n$$
$$\underset{(n,m,r)}{\leqslant} x^{n-m-r-1}\,|f-id|_n,$$

where the last inequality holds because

$$-m-q+\sum_{r_1+\cdots+r_q=r,\,r_i\geqslant 2}(n-r_i)$$
$$=(n-m-r-1)+(n-1)\left(q-\left(r-\sum_{r_i\geqslant 2}r_i\right)-1\right)$$
$$\leqslant n-m-r-1.$$

Therefore, we have, for $1\leqslant r\leqslant m+1$,

$$|\tau^{(r)}(x)|\underset{(m,n,r)}{\leqslant}\beta^{-1}x^{n-m-r-1}|f-id|_n=\alpha^{2/3}x^{2(m+1)-r}.$$

Estimate on T.

Let χ be the C^∞-vectorfield of $[0,1/6]$ whose time one map coincides with $g^{-1}f$. Put

$$\mu_q(x)=(\chi(x))^{q-1}\chi^{(q)}(x)$$

and

$$\varphi(x)=(g^{-1}f)''(x)/(g^{-1}f)'(x)=(\log((g^{-1}f)'))'(x).$$

Then we have the following equalities due to Sergeraert ([20], p. 270 and Lemma (3.6)).

$$(*)\quad T^{(q)}(x)(\chi(x))^q+\sum_{\ell=1}^{q-1}T^{(\ell)}(x)\sum_{\alpha_1+\cdots+(q-1)\alpha_{q-1}=q-\ell}E_{\alpha_1,\ldots,\alpha_{q-1}}[\chi'(x)]^{\alpha_1}\ldots[\chi^{(q-1)}(x)]^{\alpha_{q-1}}[\chi(x)]^{q-\Sigma\alpha_j}$$
$$=\sum_{i\geqslant 0}\sum_{\ell=1}^{q}\tau^{(\ell)}((g^{-1}f)^i(x))\sum_{\alpha_1+\cdots+(q-1)\alpha_{q-1}=q-\ell}E_{\alpha_1,\ldots,\alpha_{q-1}}[\chi'((g^{-1}f)^i(x))]^{\alpha_1}\ldots[\chi^{(q-1)}((g^{-1}f)^i(x))]^{\alpha_{q-1}}[\chi((g^{-1}f)^i(x))]^{q-\Sigma\alpha_j},$$

for $q\geqslant 1$.

$$(**)\quad \mu_r(x)+\sum_{\alpha_1+\cdots+(r-1)\alpha_{r-1}=r}C_{\alpha_1,\ldots,\alpha_{r-1}}[\mu_1(x)]^{\alpha_1}\ldots[\mu_{r-1}(x)]^{\alpha_{r-1}}$$
$$=\sum_{i\geqslant 0}\chi((g^{-1}f)^i(x))\sum_{q=0}^{r-1}\varphi^{(q)}((g^{-1}f)^i(x))[\chi((g^{-1}f)^i(x))]^q\sum_{\alpha_1+\cdots+(r-1)\alpha_{r-1}=r-q-1}D_{q,\alpha_1,\ldots,\alpha_{r-1}}[\mu_1((g^{-1}f)^i(x))]^{\alpha_1}\ldots[\mu_{r-1}((g^{-1}f)^i(x))]^{\alpha_{r-1}},$$

for $r\geqslant 0$.

On the other hand, if $\alpha=|f-id|_n$ is *sufficiently small*, we have

$|\tau(x)| < 1/2$ and, for $x \in [0,1/6]$, $i \geqslant 0$, we have

$$(g^{-1}f)^i(x) = \mathrm{G}\left(\sum_{j=1}^{i-1}\sum_{j=0}^{i-1}\tau((g^{-1}f)^j(x))-i,x\right)$$
$$< \mathrm{G}(-i/2,x) = x(1+2^{-1}\beta mix^m)^{-1/m}.$$

Here we use the equality $\mathrm{G}(t,x) = x(1-t\beta mx^m)^{-1/m}$ for

$$x \in [0,1/6],\ t \leqslant 0.$$

We note the following inequality

$$(***)\quad \sum_{i=0}^{\infty}((g^{-1}f)^i(x))^k < \int_0^\infty x^k(1+2^{-1}m\beta x^m t)^{-k/m}\,dt + x^k$$
$$= \left.\frac{x^k(1+2^{-1}m\beta x^m t)^{-(k/m)+1}}{(-(k/m)+1)(2^{-1}\beta mx^m)}\right|_0^\infty + x^k$$
$$= 2((k/m)-1)\beta^{-1}m^{-1}x^{k-m} + x^k$$

for $k > m$.

The inequalities (*) and (***) imply that an estimate on $\mathrm{T}^{(q)}$ follows from those on $\mathrm{T}^{(r)}$ and $\chi^{(r)}\,(r<q)$. On the other hand, the inequalities (**) and (***) imply that estimates on $\chi^{(r)}$ and μ_r follow from those on $\chi^{(\ell)}$, μ_ℓ and $\varphi^{(\ell)}(\ell<r)$.

Estimate on φ. — Since

$$|g^{-1}f-id|_{m+1} \underset{(m)}{\leqslant} |f-id|_{m+1} + |g-id|_{m+1} \underset{*}{\leqslant} \beta,$$

we have

$$|((g^{-1}f)')^{(i)}(x)| \leqslant \frac{x^{m-i}}{(m-i)!}|g^{-1}f-id|_{m+1} \underset{(m)}{\leqslant} x^{m-i}\beta,$$

for $m \geqslant i \geqslant 1$. Therefore, we have

$$|\varphi^{(r)}(x)| = \left|\sum_{q=1}^{r+1}\sum_{r_1+\cdots+r_q=r+1}\mathrm{C}_{r_1,\ldots,r_q}\log^{(q)}((g^{-1}f)'(x))\right.$$
$$\left.\cdot((g^{-1}f)')^{(r_1)}(x)\ldots((g^{-1}f)')^{(r_q)}(x)\right|$$
$$\underset{(r)}{\leqslant}\sum_{q=1}^{r+1}\beta^q x^{qm-\Sigma r_i}$$
$$\underset{(r)}{\leqslant}\beta x^{m-r-1},$$

for $0 \leqslant r \leqslant m-1$.

Estimates on μ_r *and* $\chi^{(r)}$. – For $\mu_1(x) = \chi'(x)$, we have ([20], 2.11)

$$\mu_1(x) = \sum_{i=0}^{\infty} \varphi((g^{-1}f)^i(x))\chi((g^{-1}f)^i(x)).$$

Since $|\chi(x)| \underset{*}{\leqslant} \beta x^{m+1}$ and $|\varphi(x)| \underset{*}{\leqslant} \beta x^{m-1}$, we have, by (***),

$$|\mu_1(x)| \underset{*}{\leqslant} \sum_{i=0}^{\infty} \beta^2((g^{-1}f)^i(x))^{2m}$$

$$\leqslant \beta^2(2\beta^{-1}m^{-1}x^m + x^{2m}) \underset{(m)}{\leqslant} \beta x^m.$$

Inductively, if we assume that $|\mu_\ell(x)| \underset{(\ell)}{\leqslant} \beta^\ell x^{m\ell}$ for $\ell < r(\leqslant m)$, using (**), we have

$$\begin{aligned}
|\mu_r(x)| \underset{(r)}{\leqslant} & \sum_{\alpha_1 + \cdots + (r-1)\alpha_{r-1} = r} (\beta x^m)^{\alpha_1} \cdots ((\beta x^m)^{r-1})^{\alpha_{r-1}} \\
& + \sum_{i \geqslant 0} (\beta (g^{-1}f)^i(x))^{m+1} \sum_{q=0}^{r-1} \beta((g^{-1}f)^i(x))^{m-q-1} \\
& \cdot (\beta((g^{-1}f)^i(x))^{m+1})^q \sum_{\alpha_1 + \cdots + (r-1)\alpha_{r-1} = r-q-1} (\beta x^m)^{\alpha_1} \ldots ((\beta x^m)^{r-1})^{\alpha_{r-1}} \\
\underset{*}{\leqslant} & (\beta x^m)^r + \sum_{i=0}^{\infty} \beta^{r+1}((g^{-1}f)^i(x))^{m(r+1)} \\
\underset{(m,r)}{\leqslant} & (\beta x^m)^r .
\end{aligned}$$

Therefore, we have

$$|\mu_r(x)| \underset{(r)}{\leqslant} \beta^r x^{mr} \qquad \text{for} \qquad 1 \leqslant r \leqslant m.$$

Since we have $\beta x^{m+1} \underset{*}{\leqslant} \chi(x)$,

$$|\chi^{(r)}(x)| \underset{*}{\leqslant} \beta^r x^{mr}/(\beta x^{m+1})^{r-1} = \beta x^{m-r+1} \qquad \text{for} \qquad 1 \leqslant r \leqslant m.$$

Estimate on T *(continued).* – For $T'(x)$, we have

$$T'(x)\chi(x) = \sum_{i=0}^{\infty} \tau'((g^{-1}f)^i(x))\chi((g^{-1}f)^i(x)).$$

By the estimate on τ and (***),

$$|\mathrm{T}'(x)\chi(x)| \underset{(m,n)}{\leqslant} \Sigma\alpha^{2/3}((g^{-1}f)^i(x))^{n-m-2}.\beta((g^{-1}f)^i(x))^{m+1}$$
$$= \alpha\Sigma((g^{-1}f)^i(x))^{n-1} \underset{*}{\leqslant} \alpha\beta^{-1}x^{n-m-1}$$

Therefore, $|\mathrm{T}'(x)| \underset{(m)}{\leqslant} \alpha^{1/3}x^{m+1}$ Inductively, if we assume that

$$|\mathrm{T}^{(\ell)}(x)| \underset{(\ell,m)}{\leqslant} \alpha^{1/3}x^{m+2-\ell} \qquad \text{for} \quad 1 \leqslant \ell < q(\leqslant m+1),$$

we have, by (*) and (***),

$$|\mathrm{T}^{(q)}(x)(\chi(x))^q| \underset{(m,q)}{\leqslant} \sum_{\ell=1}^{q-1} \alpha^{1/3}x^{m+2-\ell}$$
$$\sum_{\alpha_1+\cdots+(q-1)\alpha_{q-1}=q-\ell} (\beta x^{m-1+1})^{\alpha_1} \ldots (\beta x^{m-(q-1)+1})^{\alpha_{q-1}}(\beta x^{m+1})^{q-\Sigma\alpha_j}$$
$$+ \sum_{i=0}^{\infty} \sum_{\ell=1}^{q} \alpha^{2/3}((g^{-1}f)^i(x))^{2(m+1)-\ell}$$
$$\sum_{\alpha_1+\cdots+(q-1)\alpha_{q-1}=q-\ell} [\beta((g^{-1}f)^i(x))^{m-1+1}]^{\alpha_1} \ldots$$
$$[\beta((g^{-1}f)^i(x))^{m-(q-1)+1}]^{\alpha_{q-1}}[\beta((g^{-1}f)^i(x))^{m+1}]^{q-\Sigma\alpha_j}$$
$$\underset{(q)}{\leqslant} \alpha^{1/3}\beta^q x^{m+2+mq} + \alpha^{2/3}\beta^q \sum_{i=0}^{\infty} ((g^{-1}f)^i(x))^{2m+2+mq}$$
$$\underset{*}{\leqslant} \alpha^{1/3}\beta^q x^{m+2+mq}.$$

By $\beta x^{m+1} \underset{*}{\leqslant} \chi(x)$ $(x \in [0,1/6])$, we have $|\mathrm{T}^{(q)}(x)| \underset{(q,m)}{\leqslant} \alpha^{1/3}x^{m+2-q}$.

Therefore, we have

$$|\mathrm{T}^{(q)}(x)| \underset{(q,m)}{\leqslant} \alpha^{1/3}x^{m+2-q} \qquad \text{for} \qquad 1 \leqslant q \leqslant m+1.$$

Since $|\mathrm{T}(x)| \underset{*}{\leqslant} \Sigma\,\alpha^{2/3}((g^{-1}f)^i(x))^{2(m+1)} \underset{(m)}{\leqslant} \alpha^{1/3}x^{m+2}$, the above estimate is also valid for $q = 0$.

Estimate on h. – Since $h^{-1}(x) - x = \mathrm{G}(\mathrm{T}(x),x) - \mathrm{G}(0,x)$, we have

$$|h^{-1}(x)-x| \underset{*}{\leqslant} \beta x^{m+1}|\mathrm{T}(x)| \underset{(m)}{\leqslant} \alpha^{2/3}x^{2m+3}$$

if α is *sufficiently small.* For $r \geqslant 1$, we have

$$(h^{-1}-id)^{(r)}(x) = (\xi \circ h^{-1}.T')^{(r-1)}(x) + ((\xi \circ h^{-1}-\xi)/\xi)^{(r-1)}(x).$$

Assume that $|(h^{-1}-id)^{(\ell)}(x)| \underset{(m,\ell)}{\leqslant} \alpha^{2/3}x^{2m+3-\ell}$ for $0 \leqslant \ell < r(\leqslant m+1)$. Then by the formula

$$(\xi \circ h^{-1}T')^{(r-1)}(x)$$
$$= \sum_{\ell=0}^{r-1} \sum_{r_1+\cdots+r_\ell+j=r} C_{r_1,\ldots,r_\ell,j}\xi^{(\ell)}(h^{-1}(x)).(h^{-1})^{(r_1)}(x)\ldots(h^{-1})^{(r_\ell)}(x).T^{(j)}(x),$$

we have

$$|(\xi \circ h^{-1}.T')^{(r-1)}(x)| \underset{(r,m)}{\leqslant} \Sigma\Sigma\beta x^{m+1-\ell}\alpha^{1/3}x^{m+2-j} \underset{*}{\leqslant} \alpha^{2/3}x^{2m+3-r}$$

On the other hand, by the equalities $\xi(x) = \beta x^{m+1}(x \in [0,1/6])$,

$$((\xi \circ h^{-1}-\xi)/\xi)^{(r-1)}(x) = \sum_{q=0}^{r-1}\binom{r-1}{q}(1/\xi)^{(r-q-1)}(x)(\xi \circ h^{-1}-\xi)^{(q)}(x),$$

and

$$(\xi \circ h^{-1}-\xi)^{(q)}(x) = \xi^{(q)}(h^{-1}(x))((h^{-1})'(x))^q - \xi^{(q)}(x)$$
$$+ \sum_{\ell=1}^{q-1} \sum_{q_1+\cdots+q_\ell=q} C_{q_1,\ldots,q_\ell}\xi^{(\ell)}(h^{-1}(x))(h^{-1})^{(q_1)}(x)\ldots(h^{-1})^{(q_\ell)}(x),$$

we have

$$|((\xi \circ h^{-1}-\xi)/\xi)^{(r-1)}(x)| \underset{(m,r)}{\leqslant} \beta^{-1}x^{-m-r+q}\Bigg(\beta\alpha^{2/3}x^{3m+3-q}$$
$$+ \sum_{\ell=1}^{q-1} \sum_{q_1+\cdots+q_\ell=q} \beta x^{m+1-\ell} \prod_{q_i \geqslant 2} \alpha^{2/3}x^{2m+3-q_i}\Bigg)$$
$$\underset{*}{\leqslant} \beta^{-1}x^{-m-r+q}\Bigg(\beta\alpha^{2/3}x^{3m+3-q}$$
$$+ \sum_{\ell=1}^{q-1} \sum_{q_1+\cdots+q_\ell=q} \beta\alpha^{2/3}x^{3m+3-q+(2m+2)\left(\ell-\left(q-\sum_{q_i\geqslant 2} q_i\right)-1\right)}\Bigg)$$
$$\underset{*}{\leqslant} \beta^{-1}x^{-m-r+q}(\beta\alpha^{2/3}x^{3m+3-q}) = \alpha^{2/3}x^{2m+3-r}$$

Therefore, we have

$$|(h^{-1}-id)^{(r)}(x)| \underset{(m,r)}{\leqslant} \alpha^{2/3}x^{2m+3-r} \quad \text{for} \quad 0 \leqslant r \leqslant m+1.$$

This implies that

$$|(h-id)^{(r)}(x)| \underset{(r,m)}{\leqslant} \alpha^{2/3}x^{2m+3-r}, \qquad x \in [0,1/6].$$

We can define $h|[1/6,1]$ so that h satisfies (2) of Proposition (9.2) and $|h-id|_m \underset{*}{\leqslant} \alpha^{1/3}$. Take c so small that all the estimates are valid; then we complete the proof of Theorem (9.1).

As a corollary to Theorems (8.1) and (9.1), we obtain the following theorem.

THEOREM (9.3). — *For any positive integer m, there exist a positive integer n and a positive real number c, such that, for any element f of* $\mathrm{D}(n,c)$, *there are twelve elements $h_1, \ldots, h_{12}$ of* $\mathrm{Diff}^\infty_\infty([0,1])$ *such that*

$$f = \prod_{i=1}^{6} [h_{2i-1}, h_{2i}]$$

and $$|h_j - id|_m \underset{*}{\leqslant} (|f-id|_n)^{1/(3(m+2))} \qquad (j = 1,\ldots,12).$$

Proof. — For m, we have n_0 and c_0 given by Theorem (8.1). Then, for n_0, we take $n = 3(n_0+1)$ of Theorem (9.1). By Theorem (9.1), if $|f-id|_n$ is sufficiently small, we have $\bar{g}_0$, $\bar{h}_0$ and $\bar{g}_1$, $\bar{h}_1$ such that

$$|\bar{g}_i - id|_{n_0} \underset{*}{\leqslant} (|f-id|_n)^{1/3}, \qquad |\bar{h}_i - id|_{n_0} \underset{*}{\leqslant} (|f-id|_n)^{1/3}$$

(i=1,2) and

$$\mathrm{Supp}\,(f[\bar{g}_0,\bar{h}_0]^{-1}[\bar{g}_1,\bar{h}_1]^{-1}) \subset (1/8, 7/8).$$

Since $|f[\bar{g}_0,\bar{h}_0]^{-1}[\bar{g}_1,\bar{h}_1]^{-1} - id|_{n_0} \underset{*}{\leqslant} (|f-id|_n)^{1/3}$, we can choose c so that

$$|f[\bar{g}_0,\bar{h}_0]^{-1}[\bar{g}_1,\bar{h}_1]^{-1} - id|_{n_0} < c_0.$$

Then by Theorem (8.1), there are $h_i (i=1,\ldots,8)$ such that

$$f[\bar{g}_0,\bar{h}_0]^{-1}[\bar{g}_1,\bar{h}_1]^{-1} = [h_1,h_2][h_3,h_4][h_5,h_6][h_7,h_8]$$

and

$$\begin{aligned} |h_i - id|_m &\underset{*}{\leqslant} (|f[\bar{g}_0,\bar{h}_0]^{-1}[\bar{g}_1,\bar{h}_1]^{-1} - id|_{n_0})^{1/(m+2)} \\ &\underset{*}{\leqslant} (|f-id|_n)^{1/(3(m+2))}. \end{aligned}$$

10. Completion of the proof of the main theorem.

We are now in a position of completing the proof of our main Theorem (6.1). To this end, as we saw in § 7, we need only prove Theorem (6.3).

Proof of Theorem (6.3). – Put $\mathbf{R} - \mathrm{Fix}^\infty(f) = \cup \mathrm{I}_i$, where $\{\mathrm{I}_i\}_{i\in\mathrm{N}}$ is a family of disjoint open intervals. Let ℓ_i denote the length of I_i. We may assume that, for any i, $\ell_i \leqslant 1$. We may also assume that $\{\ell_i\}_{i\in\mathrm{N}}$ is a decreasing sequence; $\ell_i \geqslant \ell_j$ provided that $i \leqslant j$.

Consider the linear homeomorphism A_i which maps $[0,1]$ onto $\bar{\mathrm{I}}_i$. Then we have, for each i and for any integer n, $r\,(0 \leqslant n \leqslant r)$,

$$\sup_{0\leqslant x\leqslant 1} |(\mathrm{A}_i^{-1}(f|\bar{\mathrm{I}}_i)\mathrm{A}_i - id_{[0,1]})^{(n)}(x)| = \ell_i^{n-1} \sup_{x\in \bar{\mathrm{I}}_i} |(f|\bar{\mathrm{I}}_i - id_{\bar{\mathrm{I}}_i})^{(n)}(x)|$$
$$\leqslant ((r-n)\,!)^{-1}\,\ell_i^{r-1}|f-id|_r .$$

Here we used the inequality (**) of § 5.

For any positive integer m, by Theorem (9.3), we have a positive integer n_m and a positive real number c_m such that if $\mathrm{A}_i^{-1}(f|\bar{\mathrm{I}}_i)\mathrm{A}_i$ belongs to $\mathscr{D}(n_m,c_m)$, then there are twelve elements $h_{i,1}, \ldots, h_{i,12}$ of $\mathrm{Diff}^\infty_\infty([0,1])$ satisfying

$$\mathrm{A}_i^{-1}(f|\bar{\mathrm{I}}_i)\mathrm{A}_i = \prod_{j=1}^{6} [h_{i,2j-1}, h_{i,2j}]$$

and

$$|h_{i,j} - id|_m \leqslant \mathrm{C}_m(|\mathrm{A}_i^{-1}(f|\bar{\mathrm{I}}_i)\mathrm{A}_i - id|_{n_m})^{1/(3(m+2))} .$$

Conjugating by A_i, we have $\mathrm{A}_i h_{i,j} \mathrm{A}_i^{-1} \in \mathrm{Diff}^\infty_\infty(\bar{\mathrm{I}}_i)$ $(i=1,\ldots,12)$,

$$f|\bar{\mathrm{I}}_i = \prod_{j=1}^{6} [\mathrm{A}_i h_{i,2j-1}\mathrm{A}_i^{-1}, \mathrm{A}_i h_{i,2j}\mathrm{A}_i^{-1}]$$

and

$$\sup_{x\in\bar{\mathrm{I}}_i} |(\mathrm{A}_i h_{i,j}\mathrm{A}_i^{-1} - id_{\bar{\mathrm{I}}_i})^{(m)}(x)| \leqslant \ell_i^{1-m}\mathrm{C}_m(|\mathrm{A}_i^{-1}(f|\bar{\mathrm{I}}_i)\mathrm{A}_i - id_{[0,1]}|_{n_m})^{1/(3(m+2))} .$$

Take a sequence $\{r_m\}_{m\in\mathrm{N}}$ of positive integers such that

$$r_m \geqslant \max\{n_m, 3(m+2)(m+1)+1\}$$

and

$$(r_m - 1)/(m+2) \geqslant ((r_{m-1} - 1)/(m+1)) + 1 \quad (m \geqslant 2).$$

For each m, there is a positive integer N_m such that, for any $i \geqslant N_m$,

$$((r_m - n_m)!)^{-1} \ell_i^{r_m - 1} |f - id|_{r_m} \leqslant c_m \text{ holds.}$$

We take a sequence $\{N'_m\}$ of positive integers satisfying the following conditions.

(1) $N'_1 = N_1$, $N'_m \geqslant N_m$

and

(2) $$C_m \ell_i^{(r_m - 1)/(3(m+2))} (|f - id|_{r_m})^{1/(3(m+2))} \leqslant C_{m-1} \ell_i^{(r_{m-1} - 1)/(3(m+1))} (|f - id|_{r_{m-1}})^{1/(3(m+1))}$$

holds for any $i \geqslant N'_m$.

Such a sequence exists because of the inequality

$$(r_m - 1)/(m+2) \geqslant ((r_{m-1} - 1)/(m+1)) + 1 \quad (m \geqslant 2).$$

Now we write f as a composition of commutators satisfying the condition of Theorem (6.3).

For $i = 1, \ldots, N'_1 - 1$, applying a theorem of Sergeraert [20], we write $f|\bar{I}_i$ as a composition of commutators of $\mathrm{Diff}^\infty_\infty(\bar{I}_i)$;

$$f|\bar{I}_i = \prod_{j=1}^{k_i} [h_{i,2j-1}, h_{i,2j}] \quad (h_{i,j} \in \mathrm{Diff}^\infty_\infty(\bar{I}_i)).$$

For each m, for each $i = N'_m, \ldots, N'_{m+1} - 1$, since $N'_m \geqslant N_m$, we have

$$|A_i^{-1}(f|\bar{I}_i)A_i - id_{[0,1]}|_{n_m} \leqslant c_m,$$

that is, $A_i^{-1}(f|\bar{I}_i)A_i \in \mathscr{D}(n_m, c_m)$. Therefore, using Theorem (9.3) as before, we have

$$A_i h_{i,j} A_i^{-1} \in \mathrm{Diff}^\infty_\infty(\bar{I}_i) \quad (j = 1, \ldots, 12),$$

$$f|\bar{I}_i = \prod_{j=1}^{6} [A_i h_{i,2j-1} A_i^{-1}, A_i h_{i,2j} A_i^{-1}]$$

and

$$\begin{aligned}\sup_{x\in \bar{I}_i} |(A_i h_{i,j} A_i^{-1} - id)^{(m)}(x)| &\leqslant C_m \ell_i^{1-m}(|A_i^{-1}(f\,|\,\bar{I}_i)A_i - id_{[0,1]}|_{n_m})^{1/(3(m+2))}\\ &\leqslant C_m \ell_i^{1-m+((r_m-1)/(3(m+2)))}(|f - id|_{r_m})^{1/(3(m+2))}.\end{aligned}$$

Since $\ell_i \leqslant 1$, by the choice of r_m, we have

$$\ell_i^{1-m+((r_m-1)/(3(m+2)))} \leqslant 1.$$

Define a diffeomorphism h_j $(j=1,\ldots,12)$ by

$$\begin{aligned} &h_j|\mathbf{R} - \cup I_i = id_{\mathbf{R}-\cup I_i},\\ &h_j|\bar{I}_i = id_{\bar{I}_i} \quad \text{for} \quad i = 1,\ldots,N'_1 - 1,\\ &h_j|\bar{I}_i = A_i h_{i,j} A_i^{-1} \quad \text{for} \quad i \geqslant N'_1. \end{aligned}$$

If $N'_m \leqslant i \leqslant N'_{m+1-1}$ $(m \geqslant 1)$ and $m' \leqslant m$, by the choice of N'_m, we have

$$\begin{aligned} &\sup_{x\in \bar{I}_i} |(A_i h_{i,j} A_i^{-1} - id_{\bar{I}_i})^{(m')}(x)|\\ &\leqslant ((m-m')!)^{-1}\ell_i^{m-m'} \sup_{x\in \bar{I}_i} |(A_i h_{i,j} A_i^{-1} - id_{\bar{I}_i})^{(m)}(x)|\\ &\leqslant ((m-m')!)^{-1}\ell_i^{m-m'} C_m \ell_i^{1-m+((r_m-1)/(3(m+2)))}(|f - id|_{r_m})^{1/(3(m+2))}\\ &\leqslant C_{m'} \ell_i^{1-m'+((r_{m'}-1)/(3(m'+2)))}(|f - id|_{r_{m'}})^{1/(3(m'+2))}. \end{aligned}$$

Therefore, $\left\{\sup_{x\in \bar{I}_i} |(A_i h_{i,j} A_i^{-1} - id_{\bar{I}_i})^{(m)}(x)|\,;\ i \geqslant N'_m\right\}$ is bounded for each m. Since

$$\left\{\sup_{x\in \bar{I}_i} |(A_i h_{i,j} A_i^{-1} - id_{\bar{I}_i})^{(m)}(x)|;\ 1 \leqslant i \leqslant N'_m - 1\right\}$$

is of course bounded,

$$\left\{\sup_{x\in \bar{I}_i} |(A_i h_{i,j} A_i^{-1} - id_{\bar{I}_i})^{(m)}(x)|\,;\ i \in \mathbf{N}\right\} \text{ is bounded.}$$

Therefore, by Lemma (5.2), h_j is a C^{m-1}-diffeomorphism. Since m is arbitrary, h_j is of class C^∞.

For $i = 1,\ldots,N'_1 - 1$, let $\bar{h}_{i,j}$ $(j=1,\ldots,2k_i)$ be the diffeomorphism defined by

$$\bar{h}_{i,j}|\bar{I}_i = h_{i,j} \quad \text{and} \quad \bar{h}_{i,j}|\mathbf{R} - I_i = id_{\mathbf{R}-I_i}.$$

By construction, we have

$$f = \prod_{i=1}^{N_1'-1} \prod_{j=1}^{k_i} [\bar{h}_{i,2j-1}, \bar{h}_{i,2j}] \prod_{j=1}^{6} [h_{2j-1}, h_{2j}]$$

and

$$\begin{aligned} &\mathrm{Fix}^{\infty}(\bar{h}_{i,j}) \supset \mathrm{Fix}^{\infty}(f) \quad (i=1,\dots,N_1'-1\,;\ j=1,\dots,2k_i), \\ &\mathrm{Fix}^{\infty}(h_j) \supset \mathrm{Fix}^{\infty}(f) \quad (j=1,\dots,12). \end{aligned}$$

We have proved Theorem (6.3).

(*Added in proof*: The author heard that recently G. H. Davis obtained independently our Theorem (6.1) in the case of Remark (2). He also proved a theorem similar to our Theorem (9.3).)

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