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BMO AND COMMUTATORS OF MARTINGALE TRANSFORMS

by Svante JANSON [1]

0. Introduction.

The connection between BMO and commutators of singular integrals on $\mathbf{R}^n$ was found by Coifman, Rochberg and Weiss [1]. Their result has been further developed by Uchiyama [5] and myself [4]. This paper shows that these results hold also for the martingale transforms studied in [3].

1. The transform.

We state the basic definitions and properties of our transforms. More details are given in [3].

We assume that $(\Omega, \mathcal{F}, \mu)$ is a probability space and that $\{\mathcal{F}_n\}_{n=0}^{\infty}$ is an increasing sequence of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_n$ is generated by d^n disjoint atoms of probability d^{-n}. d is here and in the sequel a fixed integer. Thus, an atom Q of $\mathcal{F}_n$ is the union of d atoms of $\mathcal{F}_{n+1}$ which will be denoted $Q^1 \ldots Q^d$.

For f an integrable function, we define $f_n = E(f \mid \mathcal{F}_n)$. On any atom of $\mathcal{F}_n$, f_n is constant and f_{n+1} assumes d values. Hence, still studying one atom only, $f_{n+1} - f_n$ may be regarded as a vector in C^d, which will be called the local difference of f on the atom.

(*) Research has partly been done during boring lessons at Arméns stabs- och sambandsskola (Army School of Staff Work and Communications) in Uppsala.

It is easily seen that every local difference actually belongs to the $d-1$ dimensional space $V = \{(x_i)_{i=1}^d ; \Sigma x_i = 0\}$.

Let A be a linear operator in V.

We define, whenever possible, Tf to be the function whose local differences are obtained from those of f by the operator A. (Also $Tf_0 = 0$).

We will need the fact that T is a bounded operator on L^p, $1 < p < \infty$.

We will represent A by a $d \times d$ matrix. This represents an extension of A to an operator of C^d into C^d and may be chosen in many ways, but we will use the unique choice $(a_{ij})_{i,j}$ such that $\Sigma_j a_{ij} = \Sigma_i a_{ij} = 0$. Note that the identity mapping in V is represented by $\tilde{I} = (\delta_{ij} - 1/d)_{i,j=1}^d$, and the corresponding transform is $Tf = f - Ef$. C will denote various positive constants.

2. The commutator.

For any integrable function f on Ω, we define C_f to be the commutator of multiplication by f and the operator T above, i.e. $C_f g = f Tg - T(fg)$. If $f \in L^q$, it is obvious that C_f is a continuous linear operator from L^p to L^r, $1 < r < p < \infty$ and $1/r = 1/p + 1/q$.

The following theorem is less trivial. Here

$$\mathrm{BMO} = \{f ; \sup_{n,\omega} E(|f - f_n| \mid \mathcal{F}_n) < \infty\}.$$

We also have $\sup E(|f - f_n|^p \mid \mathcal{F}_n) < \infty$ for $f \in \mathrm{BMO}$ and any $p < \infty$.

THEOREM 1. – *If* $f \in \mathrm{BMO}$, *then* C_f *is a bounded linear operator in* L^p, $1 < p < \infty$.

Proof. – This is a simple adaptation of the proof of [4], Lemma 11, but for completeness, we give the main steps. We define $g^* = \sup_n E(|g| \mid \mathcal{F}_n)$ and $g^\# = \sup_n E(|g - g_n| \mid \mathcal{F}_n)$. Choose q and r such that $1 < q < qr < p$. Assume that $\omega \in Q$, an atom of $\mathcal{F}_n$, and $g \in L^p$. Let $g_1 = g \cdot X_Q$, $g_2 = g - g_1$ and $a = f_n(\omega)$.

$C_f g = C_{f-a} g = (f-a) Tg - T(f-a) g_1 - T(f-a) g_2$. We treat three terms separately.

$$\begin{aligned}
E(|(f-a)\,Tg\,|\,|Q) &\leqslant E(|f-a|^{q'}|Q)^{1/q'}\,E(|Tg|^q\,|Q)^{1/q} \\
&\leqslant C((Tg^q)^*(\omega))^{1/q}
\end{aligned}$$

$$\begin{aligned}
E(|T(f-a)\,g_1\,|\,|Q) &\leqslant d^{-n/r}\,\|T(f-a)\,g_1\|_r \leqslant C d^{-n/r}\,\|(f-a)g_1\|_r \\
&\leqslant C((g^{rq})^*(\omega))^{1/rq}\,,
\end{aligned}$$

and $T(f-a)\,g_2$ is constant on Q. Hence

$$E(|C_f g - (C_f g)_n\,|\,|Q) \leqslant C((Tg^q)^*(\omega))^{1/q} + C((g^{rq})^*(\omega))^{1/rq}\,,$$

and since the right hand side is independent of Q,

$$(C_f g)^\# \leqslant C((Tg^q)^*)^{1/q} + C((g^{rq})^*)^{1/rq} \in L^p\,.$$

Now $C_f g \in L^p$ follows as in the real-variable case [2].

In order to prove the converse, we obviously have to exclude some cases, e.g. when T is the identity. The proper requirement turns out to be the following.

We define A to be degenerate if there exists i_0 such that $a_{i_0 j} = a_{j i_0} = -a_{i_0 i_0}/(d-1)$ for every $j \neq i_0$, otherwise A is non-degenerate.

Equivalently A is degenerate if and only if it is a multiple of Υ plus a matrix having all entries in one row and in the corresponding column equal to zero.

Remark. – This property is weaker than the property required for the characterization of H^1 and BMO by a different method in [3] (viz. that A has no real eigenvector). In the important special case $a_{ij} = \alpha_{i-j}$ (where $\alpha_{-k} = \alpha_{d-k}$), A is non-degenerate unless it is a multiple of the identity.

THEOREM 2. –

a) *Assume that* A *is non-degenerate. If* C_f *is bounded on any* L^p, *then* $f \in$ BMO.

b) *If* A *is degenerate, this fails for every* L^p, $1 < p < \infty$.

Proof. – Assume that C_f is bounded on L^p. We choose an atom Q of $\mathcal{F}_n (n \geqslant 1)$.

Choose $j, k \neq i$ and define g to be $\chi_{Q^j} - \chi_{Q^k}$. All local differences but one of g are zero and we find that $Tg = a_{ij} - a_{ik}$

on Q^i. Since fg is zero on Q^i, $T(fg)$ is constant there. Thus there is a constant a such that

$$\begin{aligned} |a_{ij} - a_{ik}|\, E(|f - a|\,|Q^i) &\leqslant E(|(a_{ij} - a_{ik})(f - a)|^p\,|Q^i)^{1/p} \\ &= E(|C_f g|^p\,|Q^i)^{1/p} \leqslant |Q^i|^{-1/p}\,\|C_f g\|_p \\ &\leqslant C\,|Q^i|^{-1/p}\,\|g\|_p = C\,. \end{aligned}$$

Consequently $E(|f - a|\,|Q^i) \leqslant C$ unless the $d - 1$ values of a_{ij}, $j \neq i$, all are equal. In that case their common value must be $-a_{ii}/(d-1)$.

Now we note that the transpose operator C'_f is bounded on $L^{p'}$. However $C'_f = -C^t_f$, where C^t_f is the commutator of f and the operator T^t obtained as above from the transpose matrix A^t. Hence we also have $E(|f - a|\,|Q^i) \leqslant C$ unless $a^t_{ij} = -a^t_{ii}/(d-1)$. This together with the preliminary result above shows that $E(|f - a|\,|Q^i) \leqslant C$ unless A is degenerate. Since every atom, except Ω itself, is of the form Q^i for some Q and i, this completes the proof of part a).

For the converse, let us assume that A is degenerate ; $A = \lambda\tilde{I} + A'$ where $a'_{i_0 j} = a'_{j i_0} = 0$, $j = 1 \ldots d$. Hence $Tg = \lambda(g - Eg) + T'g$. We choose a positive integer N. $\Omega, \Omega^{i_0}, (\Omega^{i_0})^{i_0} \ldots$ is a sequence of atoms of $\mathcal{F}_0$, $\mathcal{F}_1, \ldots$ respectively. Let $Q \in \mathcal{F}_N$ be the $(N+1)$: *th* of these, and define $f = X_Q$. Then it is easy to see, for any g, that $T(f(g - g_N)) = fT(g - g_N)$, $T'(fg_N) = 0$ and $fT'g_N = 0$. Consequently

$$\begin{aligned} C_f g &= \lambda f(g - Eg) + fT'g - \lambda(fg - E(fg)) - T'(fg) \\ &= \lambda(-fEg + E(fg)) \end{aligned}$$

and $\|C_f g\|_p \leqslant |\lambda|\,(\|f\|_p + \|f\|_{p'})\,\|g\|_p$. Hence

$$\|C_f\| \leqslant |\lambda|\,(\|f\|_p + \|f\|_{p'})\,.$$

If the result of part a) were to hold, we would by the closed graph theorem have $\|f\|_{BMO} \leqslant C\,\|C_f\| \leqslant C(\|f\|_p + \|f\|_{p'})$, but we see that this is impossible by letting $N \longrightarrow \infty$.

3. Various extensions.

On $\mathbf{R}^n$, Uchiyama [5] showed that the commutator is compact if and only if the function belongs to the subspace CMO of BMO. This holds in our case too.

DEFINITION. – $\mathrm{CMO} = \{f \in \mathrm{BMO}\ ;\ f_n \longrightarrow f$ *in* BMO *as* $n \longrightarrow \infty\} = \{f\ ;\ \sup \mathrm{E}(|f - f_n| \,|\, \mathcal{F}_n) \longrightarrow 0,\ n \longrightarrow \infty\}$. CMO *is the closure in* BMO *of the functions that are* $\mathcal{F}_n$*-measurable for some* n.

THEOREM 3. – *Assume that* A *is non-degenerate. Then* C_f *is a compact operator of* L^p *into itself if and only if* $f \in \mathrm{CMO}$ $(1 < p < \infty)$.

Proof. – If $f \in \mathrm{CMO}$, $\|\mathrm{C}_{f_n} - \mathrm{C}_f\| = \|\mathrm{C}_{f_n - f}\| \leqslant \mathrm{C}\,\|f_n - f\|_{\mathrm{BMO}}$ by Theorem 1. Hence $\mathrm{C}_{f_n} \longrightarrow \mathrm{C}_f$, and since the range of C_{f_n} is finite-dimensional, C_f is compact.

Conversely, if $f \notin \mathrm{CMO}$, there exists an infinite sequence Q_n of atoms such that $\mathrm{E}(|f - \mathrm{E}(f \,|\, \mathrm{Q}_n)| \,|\, \mathrm{Q}_n) \geqslant \mathrm{C}$ for some positive C. Thus, by the proof of Theorem 2, there exist functions g_n such that g_n and $\mathrm{C}_f g_n$ are supported on Q_n, $\|g_n\| \leqslant 1$ and $\|\mathrm{C}_f g_n\| \geqslant \mathrm{C}$. There is no convergent subsequence of $\{\mathrm{C}_f g_n\}$, and hence C_f is not compact.

Quantitative estimates of the rate of convergence of f_n to f correspond to the commutator mapping one space into another.

THEOREM 4. – *Assume that* A *is non-degenerate. Let* $1 < p < \infty$ *and let* φ *be a positive increasing convex function on* R^+ *such that* $\varphi(0) = 0$, $\varphi(2t) \leqslant \mathrm{C}\varphi(t)$ *and* $t^{-1/p}\varphi^{-1}(t)$ *is decreasing. Then* $\mathrm{E}(|f - f_n| \,|\, \mathcal{F}_n) \leqslant \mathrm{C}d^{-n/p}\varphi^{-1}(d^n)$ *if and only if* C_f *maps* L^p *into the Orlicz space* $\mathrm{L}\varphi$.

The proof is similar to the one for $\mathbf{R}^n$ given in [4].

We may also study more general operators. Let us assume that we begin with one linear operator A_Q in V for every atom Q. Define the operator T as before, now applying A_Q to the local difference on Q.

It is clear, by the same proof as above, that Theorem 1 (and one direction of Theorems 3 and 4) holds if the operators A_Q are uniformly bounded. Also, if the operators are uniformly non-degenerate, all theorems above hold.

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