ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 31, nº 1 (1981), p. 239-255

<http://www.numdam.org/item?id=AIF_1981__31_1_239_0>

Ann. Inst. Fourier, Grenoble
31, 1 (1981), 239-255

A FACTORIZATION THEOREM IN BANACH LATTICES AND ITS APPLICATION TO LORENTZ SPACES

by Shlomo REISNER

1.

A Köthe function space is a Banach lattice of locally integrable, real-valued functions (more precisely, equivalent classes of functions, modulo equality a.e.) on a σ-finite, complete measure space (Ω, Σ, μ), which satisfy the two conditions

(i) If $|f| \leqslant |g|$ with $f \in L_0(\mu), g \in L$, then $f \in L$ and $\|f\| \leqslant \|g\|$ ($L_0(\mu)$ is the space of all μ-measurable functions).

(ii) For every $A \in \Sigma$ with $\mu(A) < \infty$, the characteristic function of A, χ_A, is in L.

For background on Köthe function spaces and Banach lattices in general we refer to [7], part II. We use standard notation of Banach space theory.

In particular when L is a Köthe function space, L^* is its dual space. L' is the subspace of L^* consisting of functionals φ for which there is $g \in L_0(\mu)$ so that $\varphi(f) = \int_\Omega f(t)\, g(t)\, dt$ for all $f \in L$ (in the sequel we use the same letter for φ and g). The adjoint of a linear operator T is denoted by T^*.

We say that a linear operator $T : E \longrightarrow L$ (resp. $T : L \longrightarrow E$) where E is a Banach space and L is a Banach lattice, is p-convex (resp. q-concave) for $1 \leqslant p, q \leqslant \infty$, if there is $K > 0$ such that for all $x_1, \ldots, x_n \in E$,

$$\Big\|\Big(\sum_i |\mathrm{T}x_i|^p\Big)^{1/p}\Big\| \leqslant \mathrm{K}\Big(\sum_i \|x_i\|^p\Big)^{1/p}$$

$\Big($resp. for all $f_1,\ldots,f_n \in \mathrm{L}$, $\Big(\sum\limits_i \|\mathrm{T}f_i\|^q\Big)^{1/q} \leqslant \mathrm{K}\,\Big\|\Big(\sum\limits_i |f_i|^q\Big)^{1/q}\Big\|\Big)$. We denote $\inf \mathrm{K} = \mathrm{K}^{(p)}(\mathrm{T})$ (resp. $\mathrm{K}_{(q)}(\mathrm{T})$). If the identity I of L is p-convex (resp. q-concave) we say that L is a p-convex (resp. q-concave) lattice, and denote

$$\mathrm{K}^{(p)}(\mathrm{L}) = \mathrm{K}^{(p)}(\mathrm{I}) \ \ (\text{resp. } \mathrm{K}_{(q)}(\mathrm{L}) = \mathrm{K}_{(q)}(\mathrm{I})).$$

The following theorem was proved by Lozanovskii in [9] (for another proof in the discrete case see [5]).

THEOREM. – *Let* L *be a Köthe function space on* (Ω, Σ, μ). *Every* $g \in \mathrm{L}_1(\mu)$ *has, for every* $\epsilon > 0$, *a factorization* $g = g_1 g_2$ *with* $g_1, g_2 \in \mathrm{L}_0(\mu)$ *and* $\|g_2\|_{\mathrm{L}'}\ \|g_1\|_{\mathrm{L}} \leqslant (1+\epsilon)\,\|g\|_{\mathrm{L}_1(\mu)}$.

We interpret this theorem as follows: The multiplication operator $\mathrm{T}_g : \mathrm{L}_\infty(\mu) \longrightarrow \mathrm{L}_1(\mu)$ $(\mathrm{T}_g f = gf)$ has a factorization $\mathrm{T}_g = \mathrm{T}_{g_2} \circ \mathrm{T}_{g_1}$ with

$$\|\mathrm{T}_{g_2} : \mathrm{L} \longrightarrow \mathrm{L}_1(\mu)\|\ \|\mathrm{T}_{g_1} : \mathrm{L}_\infty \longrightarrow \mathrm{L}\| \leqslant (1+\epsilon)\,\|\mathrm{T}_g\|$$

(if X, Y are Köthe function spaces on (Ω, Σ, μ) and T is a linear operator in $\mathrm{L}_0(\mu)$ we denote by $\|\mathrm{T} : \mathrm{X} \longrightarrow \mathrm{Y}\|$ the norm of T as an operator from X into Y).

We show in Section 2 that with this interpretation the factorization theorem has a generalization concerning p-convex q-concave Köthe function spaces. Moreover, this generalization has an inverse which makes it a characterization of p-convex, q-concave Köthe function spaces.

In Section 3 we make use of this characterization to find a necessary and sufficient condition on a non-increasing sequence w or function W in order that the Lorentz sequence space $d(w, p)$ or function space $\mathrm{L}_{\mathrm{W},p}$ be q-concave. For Köthe function spaces L and M and $0 < \theta < 1$ we construct, following [2], the Köthe function space

$$\mathrm{L}^\theta \mathrm{M}^{1-\theta} = \{f \in \mathrm{L}_0(\mu)\ ;\ |f| \leqslant \lambda g^\theta h^{1-\theta} \text{ for some } g \in \mathrm{L},\ h \in \mathrm{M},\ \|g\|_{\mathrm{L}} = \|h\|_{\mathrm{M}} = 1 \text{ and } \lambda \geqslant 0\}$$

with the norm $\|f\|_{\mathrm{L}^\theta \mathrm{M}^{1-\theta}} = \inf\{\lambda\ ;\ \lambda \text{ as above}\}$.

A result that we use in the sequel is the recent result of Pisier [10] which says that if a Köthe function space L has $K^{(p)}(L) = K_{(q)}(L) = 1$ then there is a Köthe function space X with $L = [L_t(\mu)]^\theta X^{1-\theta}$ with θ and t such that

$$\frac{\theta}{t} + \frac{1-\theta}{1} = \frac{1}{p}\,; \ \frac{\theta}{t} + \frac{1-\theta}{\infty} = \frac{1}{q} \tag{1}$$

$\left(\text{i.e. } t = \frac{1}{s'}\,,\ \theta = 1 - \frac{1}{s} = \frac{1}{s'}\right)$.

2.

In this section L is a Köthe function space on a σ-finite measure space (Ω, Σ, μ). We assume that L' is a norming subspace of L^*, i.e., that for all $f \in L$ $\|f\| = \sup\limits_{\|g\|_{L'}=1} \int_\Omega fg\,d\mu$.

Let $g \in L_0(\mu)$, the multiplication operator T_g in $L_0(\mu)$ is defined by $T_g f = gf$.

THEOREM 1. – *Let* $1 \leqslant p < q \leqslant \infty$ *and let* s *be defined by* $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.

L is p-convex and q-concave if and only if there is $K > 0$ so that for all $g \in L_s(\mu)$ the multiplication operator T_g has a factorization as a composition of multiplication operators T_{h_2} and T_{h_1} in the form

$$\begin{tikzcd}
L_q(\mu) \arrow[rr, "T_g"] \arrow[dr, "T_{h_1}"'] & & L_p(\mu) \\
& L \arrow[ur, "T_{h_2}"'] &
\end{tikzcd}$$

with $\|T_{h_2}\|\,\|T_{h_1}\| \leqslant K$. Moreover, if $K^{(p)}(L)$ and $K_{(q)}(L)$ are given, we may choose $K = (1+\epsilon)\,K^{(p)}(L)\,K_{(q)}(L)$ with arbitrarily small $\epsilon > 0$. If, on the other hand, K is given then $K^{(p)}(L)\,K_{(q)}(L) \leqslant K^2$.

Proof. – Necessity. Suppose L is p-convex and q-concave. By the result of Pisier which is quoted in Section 1, L is $K^{(p)}(L)\,K_{(q)}(L)$-isomorphic to $[L_t(\mu)]^\theta X^{1-\theta}$ for an appropriate Köthe function space X on (Ω, Σ, μ) and for θ, t which satisfy (1). (1) implies that $L_p = L_t^\theta L_1^{1-\theta}$, $L_q = L_t^\theta L_\infty^{1-\theta}$, $L_s = L_\infty^\theta L_1^{1-\theta}$ (from here on we write L_p instead of $L_p(\mu)$).

Let $g \in L_s$. $g = g_1^{\frac{1}{1-\theta}}$ (where $g_1 = g^{\frac{1}{1-\theta}}$

and
$$\|g_1\|_{L_1} = \|g\|_{L_s}^{\frac{1}{1-\theta}}.$$

Let $g_1 = g_{1,1} g_{1,2}$ be the factorization of g_1 through X by Lozanovskii's theorem of Section 1. If $h_1 = g_{1,1}^{1,s}$, $h_2 = g_{1,2}^{1/s}$ then clearly $h_2 h_1 = g_1^{1/s} = g$ and also

$$\|T_{h_2} : L \longrightarrow L_p\| \; \|T_{h_1} : L_q \longrightarrow L\| \leqslant K^{(p)}(L)\, K_{(q)}(L)\, \|g\|_{L_s} (1+\epsilon)^{1/s}$$

(see Diagram (2)).

$$\begin{tikzcd}
L_t^\theta L_\infty^{1-\theta} \arrow[rrr, "g = g_1^{1-\theta}"] \arrow[dr, "T_1"'] \arrow[drr, "T_{g_{1,1}^{1-\theta}}" near start] & & & L_t^\theta L_1^{1-\theta} \\
& L_t^\theta \arrow[urr, "T_1" near end] & X^{1-\theta} \arrow[ur, "T_{g_{1,2}^{1-\theta}}"'] &
\end{tikzcd} \tag{2}$$

Sufficiency. – Suppose that every $g \in L_s$ has a factorization $g = h_2 h_1$ with $\|T_{h_2} : L \longrightarrow L_p\| \; \|T_{h_1} : L_q \longrightarrow L\| \leqslant K \|g\|_{L_s}$.

We define a positive homogeneous functional $!\cdot!$ on L by

$$!f! = \sup_{\|g\|_{L_s}=1} \; \inf_{\substack{h_1, h_2 \in L_0 \\ g = h_2 h_1}} \|h_2 f\|_{L_p} \; \|T_{h_1} : L_q \longrightarrow L\|.$$

We denote the lattice semi-norm which is induced by this functional by $|||\cdot|||$.

$$|||f||| = \inf \left\{ \sum_{k=1} !f_k! \; ; \; f_k \geqslant 0 \; ; \; |f| = \sum_{k=1} f_k \right\}.$$

We show that this is in fact a norm and that the formal inclusion map $I:(L,\|\cdot\|)\longrightarrow(L,|||\cdot|||)$ is a lattice isomorphism with $K^{(p)}(I)\,K_{(q)}(I^{-1})\leqslant K$. Clearly by showing this we complete the proof.

a) $K^{(p)}(I)\leqslant K$. Let $\{f_i\}_{i=1}^n\subset L$, then

$$\left|\left|\left|\left(\sum_{i=1}^n |f_i|^p\right)^{1/p}\right|\right|\right| \leqslant \,!\left(\sum_{i=1}^n |f_i|^p\right)^{1/p}!$$

$$= \sup_{\|g\|_{L_s}=1}\ \inf_{\substack{h_1,h_2\in L_0\\ g=h_2h_1}} \left\|h_2\left(\sum_{i=1}^n |f_i|^p\right)^{1/p}\right\|_{L_p} \|T_{h_1}:L_q\longrightarrow L\|.$$

Now, for all $h\in L_0$

$$\left\|h\left(\sum_i |f_i|^p\right)^{1/p}\right\|_{L_p} \leqslant \left(\sum_i \|hf_i\|_{L_p}^p\right)^{1/p}$$

$$=\left(\sum_i \|f_i\|_L^p \left\|h\,\frac{f_i}{\|f_i\|_L}\right\|_{L_p}^p\right)^{1/p} \leqslant \|T_h:L\longrightarrow L_p\|\left(\sum_i \|f_i\|_L^p\right)^{1/p}.$$

Hence, the assumption on L yields

$$\left|\left|\left|\left(\sum_i |f_i|^p\right)^{1/p}\right|\right|\right| \leqslant \left(\sum_i \|f_i\|_L^p\right)^{1/p} \sup_{\|g\|_{L_s}=1}\ \inf_{\substack{h_1,h_2\in L_0\\ g=h_1h_2}} \|T_{h_2}:L\longrightarrow L_p\|$$

$$\|T_{h_1}:L_q\longrightarrow L\| \leqslant K\left(\sum_i \|f_i\|_L^p\right)^{1/p}.$$

b) $K_{(q)}(I^{-1})\leqslant 1$. To show this we show

$$K^{(q')}((I^{-1})^*|_{L'})\leqslant 1,\ \left(\frac{1}{q}+\frac{1}{q'}=1\right).$$

(One can verify that I^{-1} is well defined and bounded, and in particular that $|||\cdot|||$ is a norm, by noting in the course of the following argument that for all $g\in L'$

$$\sup_{|||f|||\leqslant 1}\int_\Omega gf\,d\mu \leqslant \sup_{\|f\|\leqslant 1}\int_\Omega gf\,d\mu,$$

and using the fact that L' is a norming subspace of L^*). (3) implies $K_{(q)}(I^{-1})\leqslant 1$ by [6] (th. 5) and by the fact that L is isometric to a subspace of $(L')^*$ since L' is a norming subspace of L^*. Let $\{g_i\}_{i=1}^m\subset L'$ and $0\leqslant f\in L$. We denote $\varphi=\left(\sum_{i=1}^m |g_i|^{q'}\right)^{1/q'}$.

Let $g_0 \in L_s$ with $\|g_0\|_{L_s} = 1$ be defined by $g_0 = \dfrac{(f\varphi)^{1/s}}{\left(\int_\Omega f\varphi \, d\mu\right)^{1/s}}$, we also define $h_1^0, h_2^0 \in L_0$ by

$$h_1^0 = \begin{cases} \left(\dfrac{g_0^p f^p}{\varphi^{q'}}\right)^{\frac{1}{p+q'}} & ; \varphi \neq 0 \\ 0 & ; \varphi = 0 \end{cases} \qquad h_2^0 = \begin{cases} \left(\dfrac{g_0^{q'} \varphi^{q'}}{f^p}\right)^{\frac{1}{p+q'}} & ; f \neq 0 \\ 0 & ; f = 0 . \end{cases}$$

Then $g_0 = h_2^0 h_1^0$ and using Hölder's inequality we get for r with $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q'} = 1 + \dfrac{1}{s}$

$$\begin{aligned} \varphi(f) &= \int_\Omega f\varphi \, d\mu = \|g_0 f\varphi\|_{L_r} = \|h_2^0 f\|_{L_p} \, \|h_1^0 \varphi\|_{L_{q'}} \\ &= \inf_{h_2 h_1 = g_0} \|h_2 f\|_{L_p} \, \|h_1 \varphi\|_{L_{q'}} \leqslant \sup_{\|g\|_{L_s}=1} \; \inf_{g = h_2 h_1} \|h_2 f\|_{L_p} \, \|h_1 \varphi\|_{L_{q'}} . \end{aligned}$$

Now, like in part a) of the proof we have

$$\begin{aligned} \|h_1 \varphi\|_{L_{q'}} &= \left\| h_1 \Big(\sum_i |g_i|^{q'} \Big)^{1/q'} \right\|_{L_{q'}} \\ &\leqslant \|T_{h_1} : L' \longrightarrow L_{q'}\| \Big(\sum_i \|g_i\|_{L'}^{q'} \Big)^{1/q'} . \end{aligned}$$

It is clear from the assumptions on L that

$$\|T_{h_1} : L' \longrightarrow L_{q'}\| = \|T_{h_1} : L_q \longrightarrow L\|$$

hence $\varphi(f) \leqslant \Big(\sum_i \|g_i\|_{L'}^{q'} \Big)^{1/q'} \; !f!$ and sub-linearity shows

$$\varphi(f) \leqslant \Big(\sum_i \|g_i\|_{L'}^{q'} \Big)^{1/q'} \; |||f||| .$$

Therefore $|||\varphi|||_* = \left|\left|\left| \Big(\sum_i |g_i|^{q'} \Big)^{1/q'} \right|\right|\right|_* \leqslant \Big(\sum_i \|g_i\|_{L'}^{q'} \Big)^{1/q'}$ (where $|||\cdot|||_*$ is the norm dual to $|||\cdot|||$). q.e.d.

If μ is a probability measure and $g \equiv 1$ on Ω, then T_g is the inclusion map $i : L_q(\mu) \longrightarrow L_p(\mu)$. From Theorem 1 it follows that if L is p-convex and q-concave then there is a factorization of i in the form

$$\begin{tikzcd} L_q(\mu) \arrow[rr, "i"] \arrow[dr, "T_h"'] & & L_p(\mu) \\ & L \arrow[ur, "T_{1/h}"'] & \end{tikzcd}$$

That is, for all $f \in L_q(\mu)$

$$\|hf\|_L \leqslant K\,\|f\|_{L_q(\mu)} = K\,\|hf\|_{L_q\left(\frac{d\mu}{h^q}\right)}$$

and for all $g \in L$

$$\|g\|_{L_p\left(\frac{d\mu}{h^p}\right)} = \|g/h\|_{L_p(\mu)} \leqslant K\,\|g\|_L \,.$$

In other words:

COROLLARY 1. – *If* μ *is a finite measure and* L *is p-convex and q-concave, then there exists* $0 < h \in L_0(\mu)$ *such that*

$$L_q\left(\frac{d\mu}{h^q}\right) \subset L \subset L_p\left(\frac{d\mu}{h^p}\right)$$

(set-inclusions with bounded inclusion operators).

3.

In this section we demonstrate an application of the factorization Theorem 1 for the calculation of the convexity exponent of Lorentz sequence and function spaces.

Let $w = (w_i)_{i=1}^{\infty}$ be a positive, non increasing sequence which tends to 0 and satisfies $\sum_{i=1}^{\infty} w_i = \infty$. For $1 \leqslant p < \infty$ the Lorentz sequence space $d(w\,,p)$ is defined by

$$d(w\,,p) = \left\{ \nu = (\nu_i)_{i=1}^{\infty} \in c_0 \;;\; \|\nu\| = \left(\sum_{i=1}^{\infty} \nu_i^{*p}\, w_i\right)^{1/p} < \infty \right\}$$

($\nu^* = (\nu_i^*)_{i=1}$ is the non increasing rearrangement of $|\nu|$). The space $d(w\,,p)$, equipped with the norm $\|\cdot\|$ is a Köthe sequence space which is p-convex with constant 1. It is not r-convex for any $r > p$ since it contains subspaces isomorphic to ℓ_p with the unit basis elements supported on disjoint blocks. $d(w, p)$ is reflexive if and only if $p > 1$. Let W be a positive, continuous, non-increasing function on $(0, \infty)$ which satisfy

$$\lim_{t\to\infty} W(t) = 0,\ \lim_{t\to 0} W(t) = \infty,\ \int_0^\infty W(t)\,dt = \infty,\ \int_0^1 W(t)\,dt = 1.$$

For $1 \leqslant p < \infty$ the Lorentz function space $L_{W,p}(0,\infty)$ introduced in [8] is the space of all functions $f \in L_0(0,\infty)$ which satisfy

$$\|f\| = \left\{ \int_0^\infty f^*(t)^p\, W(t)\,dt \right\}^{1/p} < \infty$$

(f^* is the non-increasing rearrangement of $|f|$). If we assume only those conditions on W which involve the interval $(0, 1]$, and define the norm by integration on $(0,1]$, we get the space $L_{W,p}(0,1]$. In the sequel I denotes $(0,\infty)$ or $(0,1]$. We write $L_{W,p}$ instead of $L_{W,p}(I)$ if we do not specify I exactly or if it is clear from the context which I we deal with. $L_{W,p}$ are Köthe function spaces in which the norm is order continuous, hence $L'_{W,p} = L^*_{W,p} \cdot L_{W,p}$ is p-convex with constant 1; it is not r-convex for any $r > p$ by the same reason as that of $d(w, p)$ (cf. [3]).

An *automorphism* of I on itself is a 1-1 (a.e.) map τ of I on itself such that τ and τ^{-1} are measurable and τ preserves measure.

DEFINITION 1. – *Let* $w = (w_i)_{i=1}^\infty$ *be a positive, non increasing sequence and let* W *be a positive, non increasing function defined in* I *and integrable on finite intervals. For* $p > 0$

a) *We say that* w *is p-regular if*

$$w_n^p \sim \frac{1}{n}\sum_{i=1}^n w_i^p\ ;\quad n \in \mathbf{N}$$

b) *We say that* W *is p-regular if*

$$W(x)^p \sim \frac{1}{x}\int_0^x W(t)^p\,dt\ ;\ x \in I.$$

THEOREM 2. – *For* $1 \leqslant p < \infty$ *let* X *be one of the spaces* $d(w, p)$, $L_{W,p}(0,1)$ *or* $L_{W,p}(0,\infty)$.

a) *For* $p < q < \infty$ *a necessary and sufficient condition for* X *to be q-concave is that the sequence* w *or the function* W *is* $\frac{s}{p}$*-regular, where* s *is defined by* $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.

b) *If* $q(x) = \inf\{q\,;\mathrm{X}$ *is* q*-concave*$\} < \infty$ *then* X *is not* $q(\mathrm{X})$ *concave*.

c) *A necessary and sufficient condition for the existence of* $q < \infty$ *so that* X *is q-concave (i.e. for* X *not to contain* ℓ_∞^n *uniformly) is that* w *or* W *is 1-regular.*

We prove Theorem 2 for function spaces; the proof for sequence spaces is analogous.

LEMMA 1. – *For a positive, non increasing function* W *defined in* I *and* $p > 0$, *the following are equivalent:*

a) W *is p-regular*.

b) $\sup_{x\in \mathrm{I}} \frac{1}{x}\int_0^x \left(\frac{\mathrm{W}(t)}{\mathrm{W}(x-t)}\right)^p dt < \infty$.

If, in addition, $p > 1$ *then a) and b) are equivalent to*

c) $$\sup_{x\in \mathrm{I}} \frac{\left(\frac{1}{x}\int_0^x \mathrm{W}(t)^p\,dt\right)^{1/p}}{\frac{1}{x}\int_0^x \mathrm{W}(t)\,dt} < \infty .$$

The equivalence of a) and b) is very simple and we omit its proof. The equivalence of b) and c) will follow from lemmas 2) and 3) in the sequel.

LEMMA 2. – *For* $0 < p < \infty$ *there is* $\mathrm{K}(p) > 0$ *so that if* f *and* g *are positive, non increasing functions on* $(0,\infty)$ *then for all* $x > 0$

$$\frac{1}{x}\int_0^x \left(\frac{f(t)}{g(x-t)}\right)^p dt > \mathrm{K}(p)\,\frac{\frac{1}{x}\int_0^x f(t)^p\,dt}{\left(\frac{1}{x}\int_0^x g(t)\,dt\right)^p}.$$

Proof. – We put $\overline{g}(x) = \frac{1}{x}\int_0^x g(t)\,dt$.

$$\mu\left\{t\,;\,\frac{1}{g(t)}\leqslant\frac{1}{2\overline{g}(x)}\right\}=\mu\{t\,;g(t)\geqslant 2\overline{g}(x)\}\leqslant\frac{x}{2}$$

(μ-Lebesgue measure).

Therefore, in the interval $\left(0\,,\,\frac{x}{2}\right]$, $\frac{1}{2\overline{g}(x)}<\frac{1}{g(x-t)}$ and we get

$$\frac{1}{x}\int_0^x\left(\frac{f(t)}{g(x-t)}\right)^p dt>\frac{1}{x}\int_0^{x/2}\left(\frac{f(t)}{g(x-t)}\right)^p dt$$

$$>\frac{1}{x}\int_0^{x/2}\left(\frac{f(t)}{2\overline{g}(x)}\right)^p dt>\frac{1}{2x}\int_0^{x}\left(\frac{f(t)}{2\overline{g}(x)}\right)^p dt$$

$$=\frac{1}{2^{p+1}}\,\frac{\frac{1}{x}\int_0^x f(t)^p\,dt}{\left(\frac{1}{x}\int_0^x g(t)\,dt\right)^p}\qquad\text{q.e.d.}$$

LEMMA 3. – *Let* f *be a positive, non increasing function defined in* I *and* $1<p<\infty$. *Suppose for some* $\mathrm{K}>0$

$$\sup_{x\in \mathrm{I}}\frac{\frac{1}{x}\int_0^x f(t)^p\,dt}{\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p}\leqslant \mathrm{K}\,;\tag{4}$$

then there is $\mathrm{N}>0$ *so that*

$$\sup_{x\in\mathrm{I}}\frac{1}{x}\int_0^x\left(\frac{f(t)}{f(x-t)}\right)^p dt\leqslant\mathrm{N}\,.\tag{5}$$

Proof. – We put $\overline{f}(x)=\frac{1}{x}\int_0^x f(t)\,dt$. It is enough to show that for some $c>0$, for all x

$$cf(x)\geqslant\overline{f}(x)\tag{6}$$

since then

$$\frac{1}{x}\int_0^x\left(\frac{f(t)}{f(x-t)}\right)^p dt\leqslant\frac{1}{x}\int_0^x\left(\frac{f(t)}{f(x)}\right)^p dt$$

$$\leqslant\frac{c^p}{x}\int_0^x\left(\frac{f(t)}{\overline{f}(x)}\right)^p dt\leqslant c^p\,\mathrm{K}\,.$$

We prove therefore that (4) implies (6).

Let A be such that $\dfrac{\log \sqrt{A}}{\sqrt{A}} \leqslant \dfrac{1}{2}$ and suppose $\dfrac{f(x_0)}{\bar{f}(x_0)} \leqslant \dfrac{1}{A}$.

Let $x_1 = \sqrt{A}\, x_0$ $\Big($later we show that in the case $I = (0,1]$ we may assume $x_0 \leqslant \dfrac{1}{\sqrt{A}}\Big)$. Let $x_0 \leqslant t \leqslant x_1$, then

$$\bar{f}(t) \geqslant \frac{1}{t} \int_0^{x_0} f(s)\, ds = \frac{x_0}{t}\, \bar{f}(x_0) \geqslant \frac{1}{\sqrt{A}}\, A\, f(x_0) \geqslant \sqrt{A}\, f(t).$$

Hence

$$\begin{aligned}\int_{x_0}^{x_1} f(t)\, dt &\leqslant \frac{1}{\sqrt{A}} \int_{x_0}^{x_1} \Big(\frac{1}{t} \int_0^t f(s)\, ds\Big) dt \\ &\leqslant \frac{1}{\sqrt{A}} \int_{x_0}^{x_1} \frac{1}{t} \int_0^{x_1} f(s)\, ds\, dt \\ &= \frac{1}{\sqrt{A}} \Big(\log \frac{x_1}{x_0}\Big) \int_0^{x_1} f(s)\, ds \\ &= \frac{\log \sqrt{A}}{\sqrt{A}} \int_0^{x_1} f(s)\, ds \leqslant \frac{1}{2} \int_0^{x_1} f(s)\, ds.\end{aligned}$$

We get $\displaystyle\int_0^{x_1} f(t)\, dt \leqslant \int_0^{x_0} f(t)\, dt + \frac{1}{2} \int_0^{x_1} f(t)\, dt$ or

$$\int_0^{x_1} f(t)\, dt \leqslant 2 \int_0^{x_0} f(t)\, dt.$$

From (4) it follows now

$$K^{1/p} \geqslant \frac{\Big(\frac{1}{x_1} \int_0^{x_1} f(t)^p\, dt\Big)^{1/p}}{\frac{1}{x_1} \int_0^{x_1} f(t)\, dt} \geqslant \frac{\Big(\frac{x_0}{x_1}\Big)^{1/p} \Big(\frac{1}{x_0} \int_0^{x_0} f(t)^p\, dt\Big)^{1/p}}{2\, \frac{x_0}{x_1} \Big(\frac{1}{x_0} \int_0^{x_0} f(t)\, dt\Big)} \geqslant \frac{1}{2} \Big(\frac{x_0}{x_1}\Big)^{\frac{1}{p} - 1}$$

(the last inequality by Hölder).

Whence $K^{1/p} \geqslant \dfrac{1}{2} \Big(\dfrac{1}{\sqrt{A}}\Big)^{\frac{1}{p} - 1}$ or $A \leqslant (2K^{1/p})^{\frac{2}{1 - 1/p}}$ which proves the assertion. If $I = (0,1]$, the preceding argument shows that if $c \geqslant (2K^{1/p})^{\frac{2}{1-1/p}}$ and $\dfrac{\log \sqrt{c}}{\sqrt{c}} \leqslant \dfrac{1}{2}$ then for $0 < x < \dfrac{1}{\sqrt{c}}$

holds $\overline{f}(x) \leqslant c\, f(x)$. On the other hand, for $\frac{1}{\sqrt{c}} \leqslant x \leqslant 1$

$$\frac{f(x)}{\overline{f}(x)} \geqslant \frac{f(1)}{\overline{f}\left(\frac{1}{\sqrt{c}}\right)}$$

which completes the proof in this case as well. q.e.d.

LEMMA 4. – *Let* W *be a positive, non increasing continuous fonction defined on* I. *Then* $A = \{p > 0\,;\, W$ *is* p*-regular*$\}$ *is an open interval (if it is not empty).*

Proof. – By Hölder's inequality A is an interval. Suppose that W is p-regular for some $p > 0$. Then there is $0 < c < 1$ such that for all $x \in I$

$$\frac{c}{x} \leqslant \frac{W(x)^p}{\int_0^x W(t)^p\, dt} = \frac{d}{dx}\left(\log \int_0^x W(t)^p\, dt\right). \tag{7}$$

For $0 < x_0 < x_1$ integration yields

$$c \log \frac{x_1}{x_0} \leqslant \log \frac{\int_0^{x_1} W(t)^p\, dt}{\int_0^{x_0} W(t)^p\, dt}$$

whence

$$x_0^{-c} \int_0^{x_0} W(t)^p\, dt \leqslant x_1^{-c} \int_0^{x_1} W(t)^p\, dt. \tag{8}$$

From (7) and (8) we get

$$x_0^{1-c}\, W(x_0)^p \leqslant x_0^{1-c} \frac{1}{x_0} \int_0^{x_0} W(t)^p\, dt \leqslant x_1^{1-c} \frac{1}{x_1} \int_0^{x_1} W(t)^p\, dt$$
$$\leqslant \frac{1}{c}\, x_1^{1-c}\, W(x_1)^p. \tag{9}$$

We choose $\epsilon > 0$ such that $\theta = (1-c)(1+\epsilon) < 1$. Let $p_1 = p(1+\epsilon)$. We claim that W is p_1-regular. Raising the ends of (9) to the power $1+\epsilon$ we get

$$x_0^\theta\, W(x_0)^{p_1} \leqslant K\, x_1^\theta\, W(x_1)^{p_1} \tag{10}$$

for some constant K. Let $x \in I$, in the interval $\left(0, \frac{x}{2}\right)$, $t < x - t$ hence (10) yields

$$\frac{1}{x}\int_0^x\left(\frac{W(t)}{W(x-t)}\right)^{p_1}dt \leqslant \frac{2}{x}\int_0^{x/2}\left(\frac{W(t)}{W(x-t)}\right)^{p_1}dt$$

$$\leqslant \frac{2K}{x}\int_0^{x/2}\left(\frac{x-t}{t}\right)^{\theta}dt \leqslant \frac{2K}{x}\int_0^{x}\left(\frac{x-t}{t}\right)^{\theta}dt < M$$

where M does not depend on x, since $\theta < 1$. We have shown that $\sup\limits_{x\in I}\frac{1}{x}\int_0^x\left(\frac{W(t)}{W(x-t)}\right)^{p_1}dt < \infty$ which, by Lemma 1, is equivalent to p_1-regularity of W. q.e.d.

We omit the proof of the following simple lemma.

LEMMA 5. – *Let* $g \in L_0(I)$. *The multiplication operator* T_g *is a bounded operator from* $L_{W,p}$ *into* L_p *if and only if*

$$\|T_g : L_{W,p} \longrightarrow L_p\|^p = \sup_{x\in I}\frac{\int_0^x g^*(t)^p\,dt}{\int_0^x W(t)\,dt} < \infty .$$

Proof of Theorem 2. – Sufficiency. Suppose W is s/p-regular. By Theorem 1 and p-convexity of $L_{W,p}$, it is enough to show that for some $K > 0$, for every $g \in L_s$, there is $h \in L_0$ such that

$$\left.\begin{array}{ll} & \text{support } g \subset \text{support } h \\ \text{and} & \|T_h : L_{W,p} \longrightarrow L_p\|\ \|T_{g/h} : L_q \longrightarrow L_{W,p}\| \leqslant K\,\|g\|_{L_s}, \end{array}\right\} \quad (11)$$

$\left(\text{we put } \frac{g}{h}(t) = 0 \text{ if } h(t) = 0\right)$.

Since $L_{W,p}$ is rearrangement invariant we may assume g is positive and non-increasing. We take $h = W^{1/p}$. Of course $\|T_h : L_{W,p} \longrightarrow L_p\| = 1$.

We show that for positive non increasing $g \in L_s$

$$\|T_{g/h} : L_q \longrightarrow L_{W,p}\| \leqslant K\,\|g\|_{L_s},$$

i.e. that for all $\varphi \in L_q$ and every automorphism σ of I on itself

$$\left\{\int_I\left(\frac{\varphi(\sigma(t))\,g(\sigma(t))}{W(\sigma(t))^{1/p}}\right)^p W(t)\,dt\right\}^{1/p} \leqslant K\,\|g\|_{L_s}.$$

Since $\left\{ \int_I \left(\frac{\varphi(\sigma(t))\, g(\sigma(t))}{W(\sigma(t))^{1/p}}\right)^p W(t)\, dt \right\}^{1/p}$

$$\leqslant \|\varphi\|_{L_q} \left\{ \int_I \left(\frac{g(\sigma(t))}{W(\sigma(t))^{1/p}}\right)^s W(t)^{s/p}\, dt \right\}^{1/s}$$

it is enough to show that for all σ

$$\left\{ \int_I g(t)^s \left(\frac{W(\sigma(t))}{W(t)}\right)^{s/p} dt \right\}^{1/s} \leqslant K\, \|g\|_{L_s}$$

which is equivalent to the fact that for all non-increasing $0 \leqslant g \in L_1$ and all σ

$$\int_I g(t) \left(\frac{W(\sigma(t))}{W(t)}\right)^{s/p} dt \leqslant K\, \|g\|_{L_1} . \tag{12}$$

In fact, if $g = \frac{1}{x}\, \chi_{(0,x]}$ for some $x \in I$ then, by Lemma 1 and s/p-regularity of W

$$\int_I g(t) \left(\frac{W(\sigma(t))}{W(t)}\right)^{s/p} dt = \frac{1}{x} \int_0^x \left(\frac{W(\sigma(t))}{W(t)}\right)^{s/p} dt$$

$$\leqslant \frac{1}{x} \int_0^x \left(\frac{W(t)}{W(x-t)}\right)^{s/p} dt \leqslant K .$$

Now, for other positive, non-increasing functions $g \in L_1$, (12) follows from the fact that the convex hull of the functions $\frac{1}{x}\, \chi_{(0,x]}$ is dense in L_1-norm in $\{f ;\, \|f\|_{L_1} \leqslant 1 ,\ 0 \leqslant f - \text{non increasing}\}$.

Necessity. – Assume $L_{W,p}$ is q-concave ($q < \infty$). Since it is also p-convex, it is necessary that for every $g \in L_s$ there is $h \in L_0$ such that (11) holds. In particular, from Lemma 5 we conclude (applying (11) to $g = \frac{1}{x^{1/s}}\, \chi_{(0,x]}$) that for every $x \in I$ there is $h \in L_0$ so that

$$\frac{\int_0^x h^*(t)^p\, dt}{\int_0^x W(t)\, dt} \leqslant K^p \quad \text{and} \quad \|T_{g/h} : L_q \longrightarrow L_{W,p}\| \leqslant 1 . \tag{13}$$

We may, of course, assume that h is non increasing and that support $h \subset (0,x]$. For all $\varphi \in L_q$ we have

$$\left\{ \frac{1}{x^{p/s}} \int_0^x \left(\frac{\varphi}{h}\right)^*(t)^p \, \mathrm{W}(t)\, dt \right\}^{1/p} \leqslant \|\varphi\|_{\mathrm{L}_q} . \tag{14}$$

Let $0 < \epsilon < x$. We define W_ϵ to be equal to the constant $\mathrm{W}(\epsilon)$ in $(0, \epsilon]$ and to $\mathrm{W}(t)$ for $t \geqslant \epsilon$. Let the bounded function φ be defined by

$$\varphi(x-t) = \frac{1}{x^{1/q}} \left(\frac{\mathrm{W}_\epsilon(t)}{h(x-t)^p}\right)^{\frac{s}{pq}} ; \quad 0 < t \leqslant x$$

$$\varphi(t) = 0 \qquad ; \quad x < t .$$

Then $\left(\frac{\varphi}{h}\right)^*(t) = \dfrac{\varphi(x-t)}{h(x-t)}$ and by (14)

$$\|\varphi\|_{\mathrm{L}_q} \geqslant \left\{ \frac{1}{x^{p/s}} \int_0^x \frac{1}{x^{p/q}} \, \frac{(\mathrm{W}_\epsilon(t)^{s/pq})^p \mathrm{W}(t)}{(h(x-t)^{\frac{ps}{pq}})^p \, h(x-t)^p} \, dt \right\}^{1/p}$$

$$\geqslant \left\{ \int_0^x \frac{1}{x} \left(\frac{\mathrm{W}_\epsilon(t)}{h(x-t)^p}\right)^{\frac{s}{pq}q} dt \right\}^{1/q} \left\{ \frac{1}{x} \int_0^x \left(\frac{\mathrm{W}_\epsilon(t)}{h(x-t)^p}\right)^{s/p} dt \right\}^{1/s}$$

$$= \|\varphi\|_{\mathrm{L}_q} \left\{ \frac{1}{x} \int_0^x \left(\frac{\mathrm{W}_\epsilon(t)}{h(x-t)^p}\right)^{s/p} dt \right\}^{1/s} .$$

It follows now from Lemma 2 and from (13) that

$$1 \geqslant \frac{1}{x} \int_0^x \left(\frac{\mathrm{W}_\epsilon(t)}{h(x-t)^p}\right)^{s/p} dt \geqslant \mathrm{K}\left(\frac{s}{p}\right) \frac{\frac{1}{x} \int_0^x \mathrm{W}_\epsilon(t)^{s/p} dt}{\left(\frac{1}{x} \int_0^x h(t)^p dt\right)^{s/p}}$$

$$\geqslant \frac{\mathrm{K}\left(\frac{s}{p}\right)}{\mathrm{K}^s} \, \frac{\frac{1}{x} \int_0^x \mathrm{W}_\epsilon(t)^{s/p} dt}{\left(\frac{1}{x} \int_0^x \mathrm{W}(t)\, dt\right)^{s/p}} .$$

Since ϵ is arbitrarily close to 0 it follows that for all $x \in \mathrm{I}$

$$\frac{\left(\frac{1}{x} \int_0^x \mathrm{W}(t)^{s/p} dt\right)^{p/s}}{\frac{1}{x} \int_0^x \mathrm{W}(t)\, dt} \leqslant \frac{\mathrm{K}^p}{\mathrm{K}\left(\frac{s}{p}\right)^{p/s}} .$$

By Lemma 1 this is equivalent to $\frac{s}{p}$-regularity of W. This proves part a) of the theorem. Part b) follows from part a) and Lemma 4.

Part c). If $L_{W,p}$ is q-concave with $q < \infty$ then W is $\frac{s}{p}$-regular with $\frac{s}{p} > 1$. Therefore it is also 1-regular (it is also easy to construct directly subspaces of $L_{W,p}$ which are uniformly isomorphic to ℓ^n_∞, if W is not 1-regular).

On the other hand, if W is 1-regular, then from Lemma 4 it follows that it is r-regular for some $r > 1$. Part a) implies now that $L_{W,p}$ is q-concave for some $q < \infty$. (We remark that this last argument provides an alternative proof for the isomorphic parts of Theorem 3.1 in [4] and Theorem 1 in [1]; i.e. for 1-regularity being a necessary and sufficient condition for $L_{W,p}$ or $d(w, p)$ to be isomorphic to a uniformly convex space when $p > 1$). q.e.d.

This is a part of a Ph. D. thesis prepared under the guidance of Professor P. Saphar. The author wishes to thank Professor Saphar for his quidance and Dr. Y. Benyamini for many valuable discussions.

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Manuscrit reçu le 17 juin 1980.

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