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C^1 - MINIMAL SUBSETS OF THE CIRCLE

by Dusa McDUFF

1. Introduction.

In this note we give a partial answer to the following question which was raised by M. Herman. For which Cantor subsets K of the circle T does there exist a C^1-diffeomorphism of T having minimal set K ? (For short, such sets will be called C^1-minimal sets.) Recall that any homeomorphism f of T which has no periodic points has a unique minimal set, which is either the whole circle, in which case the homeomorphism is conjugate to an irrational rotation, or is a Cantor set. Denjoy showed in [1] that the latter case cannot occur if f is C^1 and its first derivative has bounded variation. He also constructed examples of C^1-diffeomorphisms f which have minimal sets which are Cantor sets and so are not conjugate to rotations. Since the group of homeomorphisms of T acts transitively on the collection of Cantor subsets of T, every Cantor set is the minimal set of some homeomorphism of T. However, not every Cantor set is C^1-minimal. For instance, we will see that the usual ternary Cantor set, obtained by removing the interval $(1/2, 1)$ from $T = \mathbf{R}/\mathbf{Z}$ and then the middle third of $[0, 1/2]$, and so on, is not C^1-minimal.

Given any positive numbers ℓ_n, $n \in \mathbf{Z}$, with $\sum_{n=-\infty}^{\infty} \ell_n \leqslant 1$ and such that $\ell_n/\ell_{n+1} \longrightarrow 1$ as $|n| \longrightarrow \infty$, one can construct a Cantor set K, and a C^1-diffeomorphism f with minimal set K, such that the complement $\complement K$ of K is the union of connected components $I_n = f^n(I_0)$, $n \in \mathbf{Z}$, of lengths $\ell_n = \ell(I_n)$. (See [1]

18-20, and § 2 below. Note that the derivative of f is identically equal to 1 on K, so that this construction is rather special. Other examples are given in [1] 29-30 and [2] X.3.) If one rearranges these lengths ℓ_n into a decreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$, then it is easy to see that $\lim_{i\to\infty} \lambda_i/\lambda_{i+1}$ is also equal to 1. Therefore, it seems reasonable to ask the following question.

Suppose that K *is any* C^1*-minimal set, and let* $\lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$ *be the lengths of the components of its complement, arranged in decreasing order. Then must* $\lim_{i\to\infty} \lambda_i/\lambda_{i+1} = 1$?

I do not know the answer. However, as a special case of the results in § 4 we will see that the set of ratios $\{\lambda_i/\lambda_{i+1} : i \geqslant 1\}$ is bounded, and has 1 as a non-trivial limit point. Thus there must be a subsequence consisting of ratios $\lambda_i/\lambda_{i+1} > 1$ which converge to 1. It follows that the ternary Cantor set, which has λ_i/λ_{i+1} equal to 1 or 3 for all i, is not C^1-minimal. See Corollary 4.3 and the note immediately following.

We will prove the following localization result in § 3 : if K is C^1-minimal then, given any open set $U \subseteq T$ such that $U \cap K \neq \emptyset$, there is an open subset $V \subseteq U$ such that $V \cap K$ is non-empty and C^1-minimal. One concludes that:

THEOREM 1.1. – *Suppose that* K *is* C^1*-minimal and that* U *is an open subset of* T *with* $U \cap K \neq \emptyset$. *Let* $\lambda_1^U \geqslant \lambda_2^U \geqslant \ldots$ *be the lengths of the components of* $\complement K$ *which are contained in* U, *arranged in decreasing order. Then the set* $\{\lambda_i^U/\lambda_{i+1}^U : i \geqslant 1\}$ *is bounded and has 1 as a non-trivial limit point.*

Sharper restrictions on the $\lambda_i^U/\lambda_{i+1}^U$ may be obtained by using Proposition 4.2 of § 4 rather than its corollary.

None of the conditions discussed so far is C^1-invariant. For example, it is not hard to see that if K_0 is the ternary Cantor set, one can find a C^1-diffeomorphism g such that gK_0 satisfies the conclusion of Theorem 1.1. (All that is necessary is that g take the components of equal length in $\complement K_0$ to components of slightly differing lengths.) However, because the derivative of a C^1-diffeomorphism varies very little on sufficiently small sets, one can often formulate C^1-invariant conditions by "localization". For example, it is easy to check that the following condition is C^1-invariant:

For every open subset $U \subseteq T$ *with* $U \cap K \neq \emptyset$, $\lim_{i \to \infty} \lambda_i^U / \lambda_{i+1}^U = 1$.

This condition is not sufficient for K to be C^1-minimal since it does not take into account the homogeneity conditions discussed below and in § 5. (Even if it did, it would be unlikely to be sufficient.) However, it is satisfied by all the C^1-minimal sets which I know of, and so it may be a necessary condition.

So far, we have only looked at conditions on the lengths of the components of the complement of a C^1-minimal set. Clearly, the way in which these components are placed around the circle is also crucial. In particular, C^1-minimal sets have the following homogeneity properties:

(H_1): *Given neighbourhoods* U, V *of two "interior" points* $x, y \in K$, *there are smaller neighbourhoods* U', V' *of* x, y *and a* C^1*-diffeomorphism* $g_{y,x} : U' \longrightarrow V'$ *which maps* $U' \cap K$ *onto* $V' \cap K$.

(H_2): *Given neighbourhoods* U, V *of the closures* $\overline{I}, \overline{J}$ *of two components of* $\complement K$, *there are smaller neighbourhoods* U', V' *of* $\overline{I}, \overline{J}$ *and a* C^1*-diffeomorphism* $g_{J,I} : U' \longrightarrow V'$ *which maps* $U' \cap K$ *onto* $V' \cap K$.

(An "interior" point of K is one which is not contained in the closure $\overline{I}$ of any complementary component.)

In fact, one can choose the $g_{y,x}$ and $g_{J,I}$ to be suitable powers of f, where f is a C^1-diffeomorphism with minimal set K. This follows easily from the fact that f is semi-conjugate to an irrational rotation, see § 2.

Conditions (H) imply, for instance, that if K has positive Lebesgue measure so does any non-empty subset of the form $U \cap K$. Also, the lengths of the components of $\complement K$ which are contained in the open set U must tend to zero at the "same" rate for different U, in a sense which is made precise in Proposition 5.2 of § 5. For example, there is no C^1-minimal set K with $\{\lambda_i^U\} = \{c/i^2\}$ and $\{\lambda_i^V\} = \{c'/i^3\}$ for some $U, V \subseteq T$.

Since the ternary Cantor set satisfies (H_1) and (H_2) but is not C^1-minimal, these properties alone are not sufficient for C^1-minimality. However, we will prove in § 3 by a cutting and pasting argument, that if K is C^1-homogeneous (that is, satisfies

(H_1) and (H_2)) and is locally C^1-minimal, then it is C^1-minimal, as long as the local diffeomorphisms $g_{y,x}$ and $g_{J,I}$ which provide the homogeneity are "compatible" with the diffeomorphisms of T whose minimal sets are $U \cap K$, (see Proposition 3.4).

The methods used in this note are completely elementary. In order to make the paper self-contained, I will begin by recalling Denjoy's description of the structure of homeomorphisms whose minimal set is a Cantor set. Sections § 3, § 4 and § 5 are mutually independent and may be read in any order.

I wish to thank M. Herman for raising the problem and discussing it with me, and J. Milnor for some helpful suggestions.

2. Homeomorphisms whose minimal set is a Cantor set.

This section is a review of well-known facts. Proofs may be found in [1] and [2] II.7, X.3.

If f is a homeomorphism of T whose minimal set is a Cantor set K, then f is semi-conjugate to an irrational rotation R_α. (This number α is called the rotation number of f.) This means that there is a continuous monotone map φ of degree 1 such that the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "\varphi"'] & T \arrow[d, "\varphi"] \\ T \arrow[r, "R_\alpha"] & T. \end{tikzcd}$$

In particular, f has no fixed or periodic points. The map φ is uniquely determined by f up to composition on the left by a rotation. Observe that $\varphi K = T$. (For φK is a closed subset of T which is invariant under R_α.) In fact, φ maps each component I of $\complement K$ to a single point, so that $\varphi(\complement K)$ is a countable R_α-invariant set. Moreover φ is $1-1$ on the "interior" $\{K - \cup \bar{I} : I \subset \complement K\}$ of K. Note that the countable set $D = \varphi(\complement K)$ is uniquely determined by f up to being rotated. One can show that its isometry class, together with α, determines the C^0-conjugacy class of f [3].

Conversely, starting from any countable, R_α-invariant subset $D \subset T$, one may construct f, φ and K as above, with $\varphi(\complement K) = D$.

To do this, one chooses disjoint, closed intervals $\bar{I}_d \subset T$, for $d \in D$, which have the same ordering as the points in D and are dense in T. Then there is a continuous map $\varphi : T \longrightarrow T$, such that $\varphi^{-1}(d) = \bar{I}_d$ for all d, and which is $1-1$ on $\varphi^{-1}(T - D)$. The restriction of f to $K = T - \cup I_d$ is then determined. (Here I_d is the interior of $\bar{I}_d$.) Since R_α has minimal set T, it is easy to see that f has the unique minimal set K. Also, it is not hard to prove that f may be chosen to be C^1, with derivative $Df \equiv 1$ on K, provided that the sum of the lengths $\ell(I_d)$ of the intervals I_d is $\leqslant 1$, and that the ratios $\ell(I_d)/\ell(I_{d+\alpha})$, $d \in D$, may be arranged into a sequence which converges to 1. Note that when $\Sigma\, \ell(I_d) < 1$, there are many ways of placing the intervals I_d in T. However only one yields a C^1-diffeomorphism f. For, if $Df \equiv 1$ on K, then f must preserve the restriction $m|K$ of Lebesgue measure to K. Hence $\varphi_*(m|K)$ is R_α-invariant, and so must be a multiple of m. It is easy to check that this happens for a unique (up to rotation) choice of the I_d.

3. Cutting and pasting C^1-minimal sets.

In this section we describe some easy ways of making new C^1-minimal sets out of old ones. In particular, we will show that every C^1-minimal set is locally C^1-minimal and C^1-homogeneous, and will discuss the converse.

PROPOSITION 3.1. – *Let* K *be minimal for the* C^1*-diffeomorphism* f, *and let* A *be any open arc of the form* $(x, f^k x)$, *where* $x \in \complement K$ *and* $k \neq 0$. *Then* $A \cap K$ *is* C^1*-minimal.*

Note. – We will always consider $T = \mathbf{R}/\mathbf{Z}$ to be oriented in the obvious way, and will denote by (a, b) the open arc with first endpoint $a \in T$ and second endpoint $b \in T$. In particular, (a, b) cannot equal T. Its length is the fractional part $(b - a)$ of $b - a$.

Proof of (3.1). – By § 2, f is semi-conjugate to a rotation R_α. Thus there is $\varphi : T \longrightarrow T$ such that $R_\alpha \circ \varphi = \varphi \circ f$. We may choose φ so that $\varphi(x) = 0$. Then $\varphi(f^k x) = k\alpha$ modulo $\mathbf{Z}$, and so $\varphi(A)$ has length $(k\alpha)$. Let $\hat{T}$ be the circle of length $(k\alpha)$

which is obtained from T by collapsing $T - \varphi(A)$ to a single point, and let $\pi : T \longrightarrow \hat{T}$ be the projection. Then $\hat{\varphi} = \pi \circ \varphi$ maps T onto $\hat{T}$ and is $1-1$ on the "interior" points of $A \cap K$. Now choose m so that $(m\alpha)/(k\alpha)$ is < 1 and irrational. (It suffices to choose m so that $m > |k|$ and $0 < (m\alpha) < (k\alpha)$. For, if $(m\alpha)/(k\alpha)$ were rational it would have to equal m/k.) Then the translation τ of $\hat{T}$ by $(m\alpha)$ has no periodic points. Moreover the countable set $\hat{D} = \pi D = \hat{\varphi}(\complement K) \subset \hat{T}$ is invariant under τ. Indeed, if $\hat{T}$ is identified with the arc $[0, (k\alpha)) \subset T$ in the obvious way, then τ is translation by $(m\alpha)$ on $[0, (k\alpha) - (m\alpha))$, and is translation by $(m\alpha) - (k\alpha)$ on $[(k\alpha) - (m\alpha), (k\alpha))$. Hence, τ may be lifted to a C^1-diffeomorphism h of T, such that $\hat{\varphi} \circ h = \tau \circ \hat{\varphi}$. In fact, if $I = (a, b)$ is the component of $\complement K$ which contains x, we may put $h = f^m$ on the arc $[b, f^{k-m}(a)]$ and $h = f^{m-k}$ on the arc $[f^{k-m}(b), f^k(a)]$, and extend over the rest of T by any C^1-diffeomorphisms $f^{k-m}(\overline{I}) \longrightarrow [f^k(a), b]$ and $[f^k(a), b] \longrightarrow f^m(\overline{I})$ which coincide with f^m or f^{m-k}, as required, near the ends of these intervals. Thus we have constructed a C^1-diffeomorphism h which is semi-conjugate to τ. Since τ has no periodic points, its minimal set is $\hat{T}$. Because $\hat{\varphi}$ maps the "interior" points of $A \cap K$ injectively onto the dense subset $\hat{T} - \hat{D}$ of $\hat{T}$, it follows easily that the minimal set of h is $A \cap K$. □

As a corollary we see that C^1-minimal sets are "locally C^1-minimal".

COROLLARY 3.2. – *If* K *is* C^1*-minimal, any* $x \in K$ *is contained in an arbitrarily small open arc* A *such that* $A \cap K$ *is also* C^1*-minimal.*

Note. – The different possible choices for m in (3.1) give rise to different diffeomorphisms h with minimal set $A \cap K$. However, the restriction of any such h to $A \cap K$ has the form $h_1^{\ell} h_2^{j}$, for some $\ell \neq 0$ and $0 \leqslant j < n$, where h_1 and h_2 are fixed diffeomorphisms such that h_1 has minimal set $A \cap K$ and $h_2^n = \mathrm{id}$ on $A \cap K$. To see this, observe first that the restriction of h to $A \cap K$ is completely determined by its rotation number $(m\alpha)/(k\alpha)$. Therefore, the set of such h corresponds to the irrational elements of the group $\mathcal{R}$ consisting of all ratios $(m\alpha)/(k\alpha) \bmod \mathbf{Z}$, where

$0 \leqslant (m\alpha) < (k\alpha)$. Suppose that $(k\alpha) = k\alpha + k'$, and put n equal to the greatest common factor of k and k'. Then there are unique integers a and a' such that $ak' - a'k = n$ and $0 < a\alpha + a' < k\alpha + k'$. Set $\beta = (a\alpha + a')\,(k\alpha + k')^{-1} = (a\alpha)/(k\alpha)$. Then it is easy to check that $\mathcal{R}$ consists of the numbers $\ell\beta + j/n$, where $\ell \in \mathbf{Z}$ and $0 \leqslant j < n$. Now let h_1 and h_2 be the diffeomorphisms corresponding to β and $1/n$ respectively. Then $h_2^n = \mathrm{id}$ on $A \cap K$, and if h corresponds to $m = \ell\beta + j/n$, we clearly have $h = h_1^\ell h_2^j$ on $A \cap K$.

A similar remark can be made about the diffeomorphisms h constructed in (3.3).

The next result shows how one can piece together C^1-minimal sets.

PROPOSITION 3.3. – *Suppose that* $A \cap K$ *is minimal for the* C^1*-diffeomorphism* f, *and that* K *has a covering by disjoint open arcs* $A_i = (x_i, y_i)$, $1 \leqslant i \leqslant n$, *which satisfy the following conditions:*

(i) *for each* i, *there is a* C^1*-diffeomorphism* g_i *of* A_i *into* A *such that* $g_i(A_i \cap K) = g_i(A_i) \cap K$; *and*

(ii) *the components of* $\complement\,(A \cap K)$ *which contain the points* $g_i(x_i)$, $g_i(y_i)$, $1 \leqslant i \leqslant n$, *are all in the same* f*-orbit.*

Then K *is* C^1*-minimal.*

Proof. – For simplicity we will first assume that $n = 2$. Let $\varphi : T \longrightarrow T$ be the semi-conjugating map of f to R_α. Then by (ii) the points $\varphi(g_i(x_i))$ and $\varphi(g_i(y_i))$, where $i = 1, 2$, are all in the same R_α-orbit. Therefore, if μ_i denotes the length of the arc $\varphi(g_i A_i)$, for $i = 1, 2$, we have $\mu_1 + \mu_2 = p + q\alpha$ for some integers p and q.

Let $\hat{T}$ be a circle of length $\mu = \mu_1 + \mu_2$, which we will consider to be the union of a copy, $[0, \mu_1]$, of $\varphi(g_1 A_1)$ with a copy, $[\mu_1, \mu]$, of $\varphi(g_2 A_2)$. Then there is a monotone map $\hat{\varphi} : T \longrightarrow \hat{T}$ such that $\hat{\varphi}(A_1 \cap K) \subseteq [0, \mu_1]$ and $\hat{\varphi}(A_2 \cap K) \subseteq [\mu_1, \mu]$, which is given by $\varphi \circ g_i$ on $A_i \cap K$, for each i. Set $\hat{D} = \hat{\varphi}(\complement K)$. Then it is easy to see that $\hat{D}$ is invariant under translation by α. Choose m so that $(m\alpha)/\mu$ is irrational. (Since $\mu \equiv (q\alpha) \bmod \mathbf{Z}$, this can be done as in (3.1).) Then the map $\tau : \hat{T} \longrightarrow \hat{T}$ which translates $\hat{T}$ by $(m\alpha)$ has no periodic points. Also $\tau(\hat{D}) = \hat{D}$. Therefore, in order to show that $K = (A_1 \cap K) \cup (A_2 \cap K)$ is C^1-minimal, it suffices to construct a C^1-diffeomorphism h of T which lifts τ, in the sense that $\tau \circ \hat{\varphi} = \hat{\varphi} \circ h$. However, it is easy to check that such a map h can be constructed from f, g_1 and g_2 as in (3.1). This completes the proof when $n = 2$. The proof for $n > 2$ is similar. □

Observe that condition (ii) of (3.3) is automatically satisfied if f acts transitively on the components of $\complement\,(A \cap K)$.

Note. – Let K be a Cantor set which is minimal for some C^1 f. If K has many C^1-symmetries, that is C^1-diffeomorphisms g of T which restrict to non-trivial homeomorphisms of K, one can use (3.1) and (3.3) to construct other C^1-diffeomorphisms with minimal set K as follows. Suppose, for example, that g is a symmetry of K which fixes a point $x \in \complement K$ and takes the arc $A = (x, f(x))$ to an arc $B = (x, f^k(x))$ which contains A. Then there is a C^1-diffeomorphism $\hat{g}$ of T which takes K onto

$$K \cap (f^k(x), f(x)) = K \cap (\complement\,(B - A)).$$

By (3.1) $\hat{g}K$ is minimal for some C^1 h. Therefore K is minimal for $\hat{g}^{-1}h\hat{g}$. One can construct examples where the rotation number of $\hat{g}^{-1}h\hat{g}$, which has the form $(m\alpha)/(1 - (k-1)\alpha)$, is not a rational multiple of the rotation number α of f. Hence $(\hat{g}^{-1}h\hat{g})^n$, $n \neq 0$, is not equal on K to the conjugate of any power of f.

Finally, let us consider the question of whether every homogeneous and locally C^1-minimal Cantor set K is C^1-minimal. More precisely:

Let K *be a Cantor set which satisfies* (H_1) *and* (H_2) *in* § 1, *and also satisfies*

(L): *any* $x \in K$ *is contained in an arbitrarily small open arc* A *such that* $A \cap K$ *is* C^1*-minimal.*

Then must K *be* C^1*-minimal ?*

Note that, by Proposition 3.1, we may replace condition (L) by:

(L′): *there is an open arc* A *such that the set* $A \cap K$ *is non-empty and minimal for a* C^1*-diffeomorphism* f.

It follows easily from (3.3) that if K satisfies (H_1), (H_2) and (L′) and if, in addition, the diffeomorphism f of (L′) acts transitively on the components of $\complement\,(A \cap K)$, then K is C^1-minimal. It seems unlikely, however, that these three conditions are sufficient in general for C^1-minimality. We will prove the following weaker statement which assumes some compatibility between the $g_{y,x}$ and $g_{J,I}$ of conditions (H) and the f of (L′).

PROPOSITION 3.4. – *A Cantor set* K *is* C^1*-minimal if and only if it satisfies* (H_1), (H_2) *and* (L′), *as well as:*

(HL): *the* $g_{y,x}$ *and* $g_{J,I}$ *of conditions* (H) *may be chosen so that the local diffeomorphisms* $\hat{g}_2^{-1}\hat{g}_1$, *where each* $\hat{g}_i$ *has the form* $g_{y,x}$ *or* $g_{J,I}$, *respect one of the orbits* $\mathcal{O} = \{f^k I : k \in \mathbf{Z}\}$ *of* f *on the complement of* $A \cap K$. *Thus we require that* $\hat{g}_2^{-1}\hat{g}_1(I') \in \mathcal{O}$ *whenever* $I' \in \mathcal{O}$ *is entirely contained in the domain of* $\hat{g}_2^{-1}\hat{g}_1$.

Proof. – It is clear that any C^1-minimal set K satisfies all these conditions. For we may choose the arc A in (L′) so that $A \cap K = K$, and then choose the $g_{y,x}$ and $g_{J,I}$ to be powers of f. To prove the converse, it suffices to construct a covering of K by disjoint arcs $A_1, \ldots, A_n$, which satisfies the conditions of (3.3). This may be done in the following way. First choose open arcs $B_i \subseteq A$ and local diffeomorphisms $\hat{g}_i$, of the form $g_{y,x}$ or $g_{J,I}$, so that the arcs $\hat{g}_1 B_1, \ldots, \hat{g}_n B_n$ cover T. By (H_2) we may suppose that every component J of $\complement K$ is entirely contained in at least one of the $\hat{g}_i B_i$. Notice that, by (HL), if $J \subset \hat{g}_i B_i \cap \hat{g}_j B_j$, then $\hat{g}_i^{-1} J \in \mathcal{O}$ if and only if $\hat{g}_j^{-1} J \in \mathcal{O}$. It follows easily that there is a covering of K by disjoint arcs $A_1 \subset \hat{g}_1 B_1, \ldots, A_n \subset \hat{g}_n B_n$ where each A_i has endpoints in $\hat{g}_i \mathcal{O}$. Therefore, setting $g_i = \hat{g}_i^{-1}$ for all i, the conditions of (3.3) are satisfied. □

4. The lengths of the complementary intervals.

Suppose that K is a Cantor set in T and let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$ be the lengths of its complementary intervals, as in § 1. Further, let $J_j = [\alpha_j, \beta_j]$, $j \geqslant 1$, be disjoint, possibly degenerate, closed subintervals of $(0,1]$, which are arranged in decreasing order and which contain the λ_i. Thus $\{\lambda_i : i \geqslant 1\} \subseteq J_1 \cup J_2 \cup \ldots$, and $\alpha_{j+1} \leqslant \beta_{j+1} < \alpha_j$ for all j. We will show that if K is a C^1-minimal set the "gap" ratios α_j/β_{j+1} cannot be too large relative to the "interval" ratios β_j/α_j. As a first step, we show:

LEMMA 4.1. – *If* K *is* C^1*-minimal, the gap ratios* α_j/β_{j+1} *are bounded.*

Proof. – Clearly, it suffices to show that the ratios λ_i/λ_{i+1} are bounded. So, suppose that K is minimal for the C^1-diffeomorphism f, and choose $c > 0$ so that $Df(x) \geqslant c$ for all $x \in T$. Then $\ell(fI) \geqslant c\ell(I)$ for all components I of $\complement K$. It follows easily that $\lambda_i/\lambda_{i+1} \leqslant 1/c$ for all $i \geqslant 1$. For, because $\lim_{k\to\infty} \ell(f^k I) = 0$ for all I, there is, for any i, a component $I' = f^k I$ such that $\ell(I') \geqslant \lambda_i$ and $\ell(fI') \leqslant \lambda_{i+1}$. Hence $\lambda_i/\lambda_{i+1} \leqslant \ell(I')/\ell(fI') \leqslant 1/c$, as claimed. □

PROPOSITION 4.2. – *Suppose that the* λ_i, α_j *and* β_j *above satisfy the following condition:*

(*) *for each* $N > 0$ *there is* $\eta = \eta(N) > 0$ *such that* $\alpha_{j+n-1}/\beta_{j+n} \geqslant (1+\eta)\,\beta_j/\alpha_j$ *for* $-N \leqslant n \leqslant N$ *and all* $j > N$.

Then K *is not* C^1*-minimal.*

In particular, suppose that $\sigma_1 > \sigma_2 > \ldots > 0$ is the set obtained from the λ_i by deleting repetitions, and that we choose $\alpha_j = \beta_j = \sigma_j$ for all j. Then each J_j is a single point $\{\sigma_j\}$, and

$$\{\lambda_i : i \geqslant 1\} \subseteq J_1 \cup J_2 \cup \ldots .$$

Also, the interval ratios β_j/α_j are all equal to 1, while the gap ratios α_j/β_{j+1} run over the set of all ratios λ_i/λ_{i+1} which are > 1. Therefore, Lemma 4.1, together with the case $N = 1$ of Proposition 4.2, implies that:

COROLLARY 4.3. – *If* K *is* C^1*-minimal, then the ratios* λ_i/λ_{i+1} *are bounded and have* 1 *as a non-trivial limit point.*

Note. – This corollary implies in particular that a Cantor set K_0, whose complement consists of intervals of lengths σ^k, $k \in \mathbf{Z}$, for some $0 < \sigma < 1$, cannot be C^1-minimal. This may be proved more easily by observing that any C^1-diffeomorphism f such that $f(K_0) = K_0$ is equal on K_0 to the restriction of some PL homeomorphism of T. Since any PL homeomorphism of T either has periodic points or is conjugate to a rotation (see [2] VI.4, 5), K_0 cannot be minimal for f.

For a given set of λ_i's one can improve on (4.3) by choosing the intervals J_j more carefully. Here is an example.

COROLLARY 4.4. – *Let* μ, σ *be any two positive numbers. Then there is no* C^1*-minimal set* K *such that*

$$\{\lambda_i : i \geqslant 1\} \subseteq \{\mu^k, \sigma^k : k \in \mathbf{Z}\}.$$

Proof. – If $\mu^k = \sigma^\ell$ for some $k, \ell \in \mathbf{Z}$, this reduces to (4.3). Therefore, we may assume that $\mu^k \neq \sigma^\ell$ for any k, ℓ. Then 1 is a limit point of the ratios μ^k/σ^ℓ so that (4.3) does not apply. For convenience, let us assume that $\mu < \sigma < 1$. Then $\{\lambda_i : i \geqslant 1\} \subseteq \{\mu^k, \sigma^k : k \geqslant 1\}$.

Let the J_j consist of the following intervals, arranged in decreasing order:

(a) intervals $[\mu^k, \sigma^\ell]$ with $k, \ell \geqslant 1$ and $\sigma^\ell/\mu^k < \sigma^{-\frac{1}{4}}$,

(b) intervals $[\sigma^\ell, \mu^k]$ with $k, \ell \geqslant 1$ and $\mu^k/\sigma^\ell < \sigma^{-\frac{1}{4}}$,

(c) the points $\{\sigma^\ell\}$, $\{\mu^k\}$, $k, \ell \geqslant 1$, which are not contained in intervals of types (a) or (b).

Then the gap ratios are $> \sigma^{-\frac{1}{4}}$ and the interval ratios are $< \sigma^{-\frac{1}{4}}$. Moreover, even though both the gap ratios and the interval ratios approach $\sigma^{-\frac{1}{4}}$ arbitrarily closely, condition $(*)$ of (4.2) is satisfied. To prove this we must show that for each N the ratios $(\alpha_{j+n-1}/\beta_{j+n})/(\beta_j/\alpha_j)$ are bounded away from 1 for $|n| \leqslant N$ and all $j > N$. Consider the case $r = \alpha_{j+n-1}/\beta_{j+n} = \mu^{k'}/\sigma^{\ell'}$ and $s = \beta_j/\alpha_j = \sigma^\ell/\mu^k$. Then $\beta_{j+n} = \sigma^{\ell'}$ and $\beta_j = \sigma^\ell$. Using the fact that $|n| \leqslant N$ and that each power $\sigma, \sigma^2, \sigma^3, \ldots$ belongs to a different J_j, it is not hard to see that $|\ell' - \ell| \leqslant N$. Therefore we have $r > \sigma^{-\frac{1}{4}}$, $s < \sigma^{-\frac{1}{4}}$ while $rs = \mu^{k'-k}/\sigma^{\ell'-\ell}$ is bounded away from $\sigma^{-\frac{1}{2}}$. The desired conclusion follows easily. □

The final result in this section is a version of Proposition 4.2 localized at an orbit in $\complement$ K.

PROPOSITION 4.5. – *Let* K *be minimal for the homeomorphism* f *and suppose that there is a component* I_0 *of* $\complement$ K *such that the set* $\{\lambda_i : i \geq 1\}$ *of lengths of the components* $f^n I_0$, $n \in \mathbf{Z}$, *together with appropriate* α_j, β_j, *satisfies condition* (*) *of (4.2). Then* f *is not* C^1.

We will now begin the proof of Proposition 4.2. Throughout the following discussion we consider a fixed Cantor set K together with a fixed choice of intervals $J_j = [\alpha_j, \beta_j]$.

DEFINITION 4.6. – *The depth* $d(I)$ *of a component* I *of* $\complement$ K *is the integer* j *such that* $\ell(I) \in J_j$.

Note 4.7. – $\ell(I) \geq \ell(I')$ implies that $d(I) \leq d(I')$. Conversely, if $d(I) \leq d(I') = j$ then $\ell(I) \geq \alpha_j$ while $\ell(I') \leq \beta_j$. Thus $\ell(I')/\ell(I) \leq \beta_j/\alpha_j$.

The following lemma shows the importance of condition (*).

LEMMA 4.8. – *Suppose that* K *is minimal for the* C^1*-diffeomorphism* f *and that* (*) *is satisfied. Then* K *may be covered by disjoint open arcs* $A_1, \ldots, A_r$ *in such a way that, for any pair* I, I′ *of components of* $\complement$ K *which are both contained in the same* A_i, $d(I) \leq d(I') \Longrightarrow d(fI) \leq d(fI')$.

Proof. – The idea is the following. Choose the covering $A_1, \ldots, A_r$ of K so that the derivative Df of f varies very little on each A_i. Then for any pair I, I′ of components of $\complement$ K which are both contained in A_i the difference $\ell(fI)/\ell(I) - \ell(fI')/\ell(I')$ is small. However, if $d(I) \leq d(I')$ while $d(fI) > d(fI')$, then by (4.7) $\ell(I')/\ell(I) \leq \beta_j/\alpha_j$ while $\ell(fI')/\ell(fI) \geq \alpha_{j+n-1}/\beta_{j+n}$ for some n. We will see that $|n|$ is bounded. (Its bound depends on Df.) If (*) holds, the gap ratios $\alpha_{j+n-1}/\beta_{j+n}$ are definitely bigger than the interval ratios β_j/α_j. We will see that this implies that the difference $\ell(f\mathrm{I})/\ell(\mathrm{I}) - \ell(f\mathrm{I}')/\ell(\mathrm{I}')$ is quite large, and so derive a contradiction.

Here is the proof in detail. By (*) with $\mathrm{N} = 1$, there is $\eta' > 0$ such that $\alpha_j/\beta_{j+1} \geqslant (1+\eta')\,\beta_j/\alpha_j \geqslant 1 + \eta'$ for all $j \geqslant 1$. Therefore, by Lemma 4.1, there is L such that $\beta_j/\alpha_j \leqslant \alpha_j/\beta_{j+1} \leqslant \mathrm{L}$ for all j. Hence

$$(1+\eta')^{\mathrm{N}} \leqslant \alpha_j/\beta_{j+\mathrm{N}} \leqslant \mathrm{L}^{2\mathrm{N}} \tag{I}$$

for all j. In particular, one can choose N so that

$$\alpha_j/\beta_{j+\mathrm{N}} > \sup\{\mathrm{D}f(x)\,,\, \mathrm{D}f^{-1}(x) : x \in \mathrm{T}\} \quad \text{for all } j\,.$$

Then, if $\ell(\mathrm{I}) \in \mathrm{J}_j$, both $\ell(f\mathrm{I})$ and $\ell(f^{-1}\mathrm{I})$ are $> \beta_{j+\mathrm{N}}$. It follows that

$$|d(f\mathrm{I}) - d(\mathrm{I})| \leqslant \mathrm{N} \quad \text{for all I}\,. \tag{II}$$

Let $\delta < \eta(\mathrm{N})/\mathrm{L}^{2\mathrm{N}}$, and cover K by open subsets $\mathrm{W}_1, \ldots, \mathrm{W}_s$ of T so that $|\mathrm{D}f(x) - \mathrm{D}f(y)| < \delta/2$ for all $x, y \in \mathrm{W}_i$, $1 \leqslant i \leqslant s$. Then, it is easy to see that, if I, I' are any two components of $\complement\,\mathrm{K}$ which are contained in the same W_i, we have

$$|\ell(f\mathrm{I}')/\ell(\mathrm{I}') - \ell(f\mathrm{I})/\ell(\mathrm{I})| < \delta\,. \tag{III}$$

Since the covering $\mathrm{A}_1, \ldots, \mathrm{A}_r$ of K can be chosen to exclude any finite set of components of K, it will suffice to show that, if $\mathrm{N} \leqslant d(\mathrm{I}) \leqslant d(\mathrm{I}')$ while $d(f\mathrm{I}) > d(f\mathrm{I}')$, then (III) does not hold.

So, suppose that $\mathrm{N} \leqslant j = d(\mathrm{I}) \leqslant d(\mathrm{I}')$ and that

$$j + n = d(f\mathrm{I}) > d(f\mathrm{I}')\,.$$

Then $\ell(\mathrm{I}') \leqslant \beta_j$ and $\ell(f\mathrm{I}') \geqslant \alpha_{j+n-1}$, and so

$$\ell(f\mathrm{I}')/\ell(\mathrm{I}') - \ell(f\mathrm{I})/\ell(\mathrm{I}) \geqslant \alpha_{j+n-1}/\beta_j - \beta_{j+n}/\alpha_j$$
$$= (\alpha_{j+n-1}/\beta_{j+n} - \beta_j/\alpha_j)\,.\ \beta_{j+n}/\beta_j\,.$$

By (II), $|n| \leqslant \mathrm{N}$. Therefore, because $j \geqslant \mathrm{N}$, we may apply (*) and (I) to get

$$\ell(f\mathrm{I}')/\ell(\mathrm{I}') - \ell(f\mathrm{I})/\ell(\mathrm{I}) \geqslant \eta(\mathrm{N})\,\beta_{j+n}/\alpha_j \geqslant \eta(\mathrm{N})/\mathrm{L}^{2\mathrm{N}} > \delta\,.$$

Thus (III) does not hold. □

LEMMA 4.9. – *Let* K, f *be as in Lemma 4.8, and let* $\mathrm{L}_0 = \sup\{\beta_j/\alpha_j : j \geqslant 1\}$. *Then, for all* $\epsilon > 0$, *there is an open* *subset* U *of* T *with* $U \cap K \neq \emptyset$ *such that, for all components* I' *of* $\complement K$ *which are contained in* U *and all* $k \geqslant 0$, $\ell(f^k I') < L_0 \epsilon$.

Proof. – Let $A_1, \ldots, A_r$ be the covering of K constructed in Lemma 4.8. By making ϵ smaller if necessary, we may suppose that any component of $\complement K$ with length $< L_0 \epsilon$ is entirely contained in some A_i. Let $\mathfrak{I}$ be the set of all components I of $\complement K$ such that $\ell(f^k I) < \epsilon$ for all $k \geqslant 0$. Then the components in $\mathfrak{I}$ accumulate on K so that there is a connected open set U such that

(a) $U \cap K \neq \emptyset$, and

(b) one of the components $I_0 \subset U$ of maximal length belongs to $\mathfrak{I}$.

We will now show by induction on k that the following statements hold for all $k \geqslant 0$.

(P_k): $f^k(U) \cap K \subset$ *some* A_i;

(Q_k): *for all* $I' \subset U$, $d(f^k I') \geqslant d(f^k I_0)$.

Note that (Q_0) holds because, by (b), $\ell(I') \leqslant \ell(I_0)$ for all $I' \subset U$.

Proof that $(Q_k) \Longrightarrow (P_k)$.

Recall that, by hypothesis on ϵ, every component of $\complement K$ of length $< L_0 \epsilon$ lies in some arc A_i. Note also, that by (4.7), $d(I_1) \geqslant d(I_2)$ implies that $\ell(I_1) \leqslant L_0 . \ell(I_2)$. Hence (Q_k) implies that, for all $I' \subset U$, we have $\ell(f^k I') \leqslant L_0 . \ell(f^k I_0)$. But $\ell(f^k I_0) < \epsilon$, since $I_0 \in \mathfrak{I}$ by (b). Therefore, every interval $f^k I'$ must lie in some arc A_i. But the A_i are disjoint, and both U and the A_i are connected. It follows that all the components $f^k I'$, for $I' \subset U$, must lie in the same A_i. Thus $f^k(U) \cap K \subset A_i$, as required. □

Proof that $(P_k), (Q_k) \Longrightarrow (Q_{k+1})$.

This follows immediately from Lemma 4.8.

Thus (Q_k) holds for all k. Since $\ell(f^k I_0) < \epsilon$ for all $k \geqslant 0$, it follows that $\ell(f^k I') < L_0 \epsilon$ for all $k \geqslant 0$. This completes the proof of Lemma 4.9. □

It is now easy to prove Proposition 4.2.

Proof of Proposition 4.2.

It suffices to show that Lemma 4.9 cannot be true. Suppose to the contrary that Lemma 4.9 holds for some K, f. Choose $\epsilon > 0$ so that there is a component, I say, of $\complement$ K with length $> L_0 \epsilon$, and let U be as in the lemma. Then the components $f^{-k}(I)$ for $k \geqslant 0$ accumulate on K so that $f^{-k}(I) \subset U$ for some $k \geqslant 0$. Put $I' = f^{-k}(I)$. Then $\ell(f^k I') = \ell(I) > L_0 \epsilon$, a contradiction. □

We will finish this section with:

Proof of Proposition 4.5.

Let μ be the sum of the lengths of the components of $\complement$ K which are not in the orbit $f^n(I_0)$, $n \in \mathbf{Z}$, of I_0, and let $\hat{T}$ be the circle of length $1 - \mu$ obtained from T by collapsing all these components to points. Let $\pi : T \longrightarrow \hat{T}$ be the quotient map, so that $\ell(\pi I) = \ell(I)$ for all $I = f^n(I_0)$. Clearly there is a homeomorphism $\hat{f}$ of $\hat{T}$ so that the diagram

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "\pi"'] & T \arrow[d, "\pi"] \\ \hat{T} \arrow[r, "\hat{f}"] & \hat{T} \end{tikzcd}$$

commutes. Then $\hat{f}$ has minimal set $\hat{K} = \pi K$. By hypothesis the lengths λ_i of the components of $\complement \hat{K}$ together with appropriate α_j, β_j satisfy $(*)$. The claim is that f cannot be C^1.

Suppose to the contrary that f is C^1. Then $\hat{f}$ need not be C^1, and so we cannot immediately apply (4.2). However, it is not hard to check that Lemma 4.8 still holds for $\hat{K}$, $\hat{f}$. For, in the proof of this lemma, the differentiability of f was used only to construct the covering W_t for which (III) holds, and such a covering can be found for $\hat{f}$ too. Similarly, Lemma 4.1 is also true for $\hat{K}$. Since the rest of the proof of (4.2) was based only on (4.1) and (4.8), and did not mention the differentiability of f, one may derive a contradiction as before. □

5. A consequence of homogeneity.

In this section we give precise form to the statement made in § 1 that, if K is C^1-minimal, the sequences $\{\lambda_i^U\}$ tend to 0 at approximately the same rate.

DEFINITION 5.1. – *Let* $\{\mu_i\}$ *and* $\{\mu_i'\}$ *be two sequences of positive numbers which tend to* 0, *with decreasing rearrangements* $\{\mu_{\pi(i)}\}$ *and* $\{\mu'_{\rho(i)}\}$. *Then we will say that* $\{\mu_i\} \leqslant \{\mu_i'\}$, *if there are integers* k *and* $L > 0$ *such that* $\mu_{k\pi(i)} \leqslant L\mu'_{\rho(i)}$ *for all* $i \geqslant 1$. *Further, if* $\{\mu_i\} \leqslant \{\mu_i'\}$ *and* $\{\mu_i'\} \leqslant \{\mu_i\}$, *we will say that the two sequences are equivalent, and will write* $\{\mu_i\} \sim \{\mu_i'\}$.

It is easy to check that the relation $\sim$ is an equivalence relation. Note that $\{\mu_i\} \sim \{\mu_{ki+m}\}$ if $\{\mu_i\}$ is decreasing. On the other hand $\{1/i^2\} \not\sim \{1/i^3\}$. The main result of this section is:

PROPOSITION 5.2. – *If* K *is* C^1-*minimal, then* $\{\lambda_i^U\} \sim \{\lambda_i^V\}$, *where* U *and* V *are any open subsets of* T *such that* $U \cap K \neq \phi \neq V \cap K$.

Before proving this, it will be convenient to prove the following lemma.

LEMMA 5.3. – *If* $\{\mu_i\}$ *and* $\{\mu_i'\}$ *are two positive sequences which tend to* 0, *and are such that* $(1/L) . \mu_i \leqslant \mu_i' \leqslant L\mu_i$ *for all* $i \geqslant 1$, *then* $\{\mu_i\} \sim \{\mu_i'\}$.

Proof. – We may suppose that $\{\mu_i\}$ is decreasing. Let π be a permutation of $\mathbf{N}$ such that $\{\mu'_{\pi(i)}\}$ is decreasing. It will suffice to show that $(1/L) . \mu_i \leqslant \mu'_{\pi(i)} \leqslant L\mu_i$ for all i.

This may be seen as follows. Since $\mu_i' \leqslant L\mu_i \leqslant L\mu_n$ for all $i \geqslant n$, there are at most $n-1$ of the μ_i' which are $> L\mu_n$. Hence, $\mu'_{\pi(n)}$, which is the nth largest of the μ_i', must be $\leqslant L\mu_n$. Similarly, there are at least n of the μ_i' which are $\geqslant (1/L) . \mu_n$. Hence the nth largest of the μ_i' must also be $\geqslant (1/L) . \mu_n$. □

Proof of Proposition 5.2.

By the homogeneity of K, it suffices to prove that $\{\lambda_i^U\} \sim \{\lambda_i^V\}$ in the following two cases: (i) $V = gU$ for some C^1-diffeomorphism g such that $gK = K$, and (ii) $V \subseteq U$.

Case (i). – The proof in this case follows immediately from Lemma 5.3 and the fact that, if $I \subseteq U$ and

$$L = \sup \{Dg(x), Dg^{-1}(x) : x \in T\}$$

then $(1/L) . \ell(I) \leqslant \ell(gI) \leqslant L\, \ell(I)$.

Case (ii). – Note first that $\{\lambda_i^{V'}\}$ is a subsequence of $\{\lambda_i^U\}$ whenever $V' \subseteq U$. Hence $\{\lambda_i^V\} \leqslant \{\lambda_i^U\}$, and it will clearly suffice to show that $\{\lambda_i^U\} \leqslant \{\lambda_i^{V_0}\}$, where V_0 is any connected open subset of V.

Let $g_1, \ldots, g_m$ be C^1-diffeomorphisms which leave K invariant and are such that $U \subset g_1 V_0 \cup \ldots \cup g_m V_0$. (The g_i may be taken to be iterates of f, where f is the diffeomorphism with minimal set K.) Because V_0 is connected, there are only finitely many components I of $\complement K$ which lie in U but not in any of the sets $g_j V_0$. Since the equivalence class of $\{\lambda_i^U\}$ is not changed by the deletion of a finite number of its terms, we may ignore these I. Then $\{\lambda_i^U\}$ is a subsequence of the disjoint union $\{\lambda_i^{g_1 V_0}\} \cup \ldots \cup \{\lambda_i^{g_m V_0}\}$. But $\{\lambda_i^{g_j V_0}\} \sim \{\lambda_i^{V_0}\}$ by case (i), and it is easy to check that, for any sequence $\{\mu_i\}$, we have $\{\mu_i\} \sim \{\mu_i\} \cup \ldots \cup \{\mu_i\}$. Hence $\{\lambda_i^U\} \leqslant \{\lambda_i^{V_0}\}$, as required. □

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