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LITTLEWOOD-PALEY DECOMPOSITIONS AND FOURIER MULTIPLIERS WITH SINGULARITIES ON CERTAIN SETS

by P. SJÖGREN and P. SJÖLIN

Introduction.

The well-known Hörmander-Mihlin multiplier theorem in $\mathbf{R}$ says that any bounded function $m(x)$ such that $|x|\, m'(x)$ is bounded belongs to the space M_p of Fourier multipliers for L^p, $1 < p < \infty$. We shall generalize this result. A closed null set $E \subset \mathbf{R}$ will be said to have property $HM(p)$ if any bounded function m such that $d_E m'$ is bounded belongs to M_p. Here d_E denotes the distance to E. We shall prove that property $HM(p)$ is equivalent to the Littlewood-Paley decomposition property for L^p with respect to the complementary intervals of E. There are also equivalent properties of E related to the Marcinkiewicz multiplier theorem.

As is well known, the Littlewood-Paley decomposition, and thus also property $HM(p)$, hold for $1 < p < \infty$ when E is a lacunary sequence tending to 0. We prove that these properties are preserved if we, roughly speaking, add to such an E uniformly lacunary sequences, one converging to each point of E. Sets obtained by iteration of this procedure are called lacunary, and they are shown to have the two properties. Further, we give a simple necessary condition for the properties, saying that any bounded part of E should not contain too many points. And finally, Cantor sets of type $\{\Sigma \epsilon_j \ell_j \,;\, \epsilon_j = 0, 1\}$ are shown never to have the properties for $p \neq 2$.

The precise formulations of these one-dimensional results are given in Section 1. And Section 2 deals with the two-dimensional case, which is more complicated. Then E will be a set of directions. We compare the following three properties of E: firstly, the Littlewood-Paley decomposition property with respect to the complementary sectors of E, secondly a Hörmander-Mihlin property for homogeneous multipliers with singularities on rays in the E directions, and, thirdly, the boundedness on L^p of the maximal function with respect to rectangles in the E directions. Improving earlier results of J.-O. Strömberg and A. Cordoba - R. Fefferman, A. Nagel - E.M. Stein - S. Wainger [5] have shown that the first and third properties hold for lacunary sequences of directions. Extending the definition of lacunary sets described above to sets of directions, we prove that such sets have all three properties (see Corollary 2.4). We finally give some necessary conditions.

As for notations, C is a generic constant, not always the same, and $f \sim g$ means $1/C \leqslant f/g \leqslant C$. The definition of the Fourier transform we use is $\hat{f}(\xi) = \int e^{-i\xi\cdot x} f(x)\, dx$.

1. One-dimensional results.

Let $E \subset \mathbf{R}$ be a closed null set and I_k, $k = 1, 2, \ldots,$ the complementary intervals of E, i.e., the components of $\mathbf{R}\setminus E$. We denote by χ_k the characteristic function of I_k. Call S_k the operator given by $(S_k f)\hat{} = \chi_k \hat{f}$.

DEFINITION. – *Let* $1 < p < \infty$. E *is said to have property* LP(p) *(Littlewood-Paley) if there is a constant* C *such that for all* $f \in L^p$ $C^{-1}\|f\|_p \leqslant \|(\Sigma\, |S_k f|^2)^{1/2}\|_p \leqslant C\|f\|_p$.

The smallest such constant is called the LP(p) constant of E. Further, E is said to have property HM(p) (Hörmander-Mihlin) if any function $m \in C^1(\mathbf{R}\setminus E)$ such that $m(x)$ and $d_E(x)\, m'(x)$ are bounded is in M_p. And E is said to have property Mar(p) (Marcinkiewicz) if any bounded function m locally of bounded variation in $\mathbf{R}\setminus E$ such that $\sup_k \int_{I_k} |dm| < \infty$ is in M_p.

If E has property HM(p), it follows from the closed graph theorem that there is an associated constant C such that the M_p norm of m is bounded by $C(\sup |m| + \sup |d_E \,.\, m'|)$. A similar remark applies to property Mar(p). Notice that the three properties defined, and the associated constants, are invariant under translation and dilation.

THEOREM 1.1. – *If* $1 < p < \infty$ *and* $E \subset \mathbf{R}$ *is a closed null set, then properties* LP(p), HM(p), *and* Mar(p) *are equivalent, and so are the associated constants.*

Proof. – If E has property HM(p) or Mar(p), it follows that $\Sigma \pm \chi_k \in M_p$, uniformly for all sign combinations. Averaging as usual by means of Rademacher functions, one obtains $\|(\Sigma |S_k f|^2)^{1/2}\|_p \leqslant C\|f\|_p$. By a duality argument, the converse inequality follows, cf. [7, p. 105]. Thus E has property LP(p).

To prove that LP(p) implies HM(p), assume

$$\sup |m(x)| < \infty, \quad \sup |d_E(x)\, m'(x)| < \infty. \tag{1.1}$$

Select a function $\varphi \in C^\infty(\mathbf{R}\backslash E)$ which equals 1 in the leftmost third and 0 in the rightmost third of each bounded I_k, and satisfies the same inequalities (1.1) as m. On unbounded intervals I_k, let $\varphi = 1$. Then φm also satisfies (1.1). Let $m_k^1 = \chi_k \varphi m$. It follows that m_k^1 is a translate of an ordinary Hörmander-Mihlin multiplier in $\mathbf{R}$, with bounds uniform in k. By D.S. Kurtz and R.L. Wheeden [4], the m_k^1 are uniformly bounded Fourier multipliers on weighted $L^2(\mathbf{R})$, with any weight in Muckenhoupt's class A_2. But then these multipliers define a bounded operator on $L^p(\ell^2)$ for $1 < p < \infty$, as proved by J.L. Rubio de Francia [6]. This means that if $(f_k)_1^\infty$ are functions in $L^p(\mathbf{R})$ with $(\Sigma |f_k|^2)^{1/2} \in L^p$ and $\hat{F}_k^1 = m_k^1 \hat{f}_k$, then

$$\|(\Sigma |F_k^1|^2)^{1/2}\|_p \leqslant C\|(\Sigma |f_k|^2)^{1/2}\|_p .$$

The same thing holds for m_k^2 and F_k^2, defined by replacing φ by $1 - \varphi$. Letting $F_k = F_k^1 + F_k^2$, so that $\hat{F}_k = \chi_k m \hat{f}_k$, we thus have

$$\|(\Sigma |F_k|^2)^{1/2}\|_p \leqslant C\|(\Sigma |f_k|^2)^{1/2}\|_p . \tag{1.2}$$

Now take $f \in L^p$ and $f_k = S_k f$, so that $F_k = S_k F$, where $\hat{F} = m\hat{f}$. If E has property LP(p), (1.2) says that $\|F\|_p \leqslant C\|f\|_p$ and property HM(p) follows.

Finally, to prove that LP(p) implies Mar(p), we proceed as in [7, p. 111-112] (see also the last part of the proof of our Theorem 2.1). Theorem 1.1 is proved.

Let p' be the exponent dual to p. Since $M_{p'} = M_p$, clearly the three properties of Theorem 1.1 are also equivalent to LP(p'), HM(p'), and Mar(p'). Notice that the three properties are hereditary to subsets, with smaller or equivalent constants. They are also hereditary to certain larger sets, as we shall now see.

DEFINITION. – *If* E *and* E$'$ *are closed null sets in* $\mathbf{R}$, *we call* E$'$ *a successor of* E *if there exists a constant* $c > 0$, *called the successor constant, such that* $x, y \in E'$ *and* $x \neq y$ *implies* $|x - y| \geqslant c d_E(x)$.

A sequence $(x_j)_1^\infty$ or $(x_j)_{-\infty}^{+\infty}$ converging to x is called lacunary if $x_j \neq x$ for all j and there exists $\theta > 1$ so that $(x_j - x)/(x_{j+1} - x) > \theta$ for all j. Then the above definition implies that if I_k is a bounded complementary interval of E, then $E' \cap I_k$ is contained in the union of two lacunary sequences converging to the endpoints of I_k, and analogously for an unbounded I_k.

We define lacunary sets of order n inductively as follows. A lacunary set of order 0 is a one-point set, and a lacunary set of order $n \geqslant 1$ is a successor of a lacunary set of order $n - 1$. Thus a double exponential sequence like $\{2^i + 2^j : i, j \in \mathbf{Z}\} \cup \{0\}$ is a lacunary set of order 2.

THEOREM 1.2. – *If* E *has property* LP(p), *then so does any successor of* E. *A lacunary set of finite order has property* LP(p) *for* $1 < p < \infty$.

Proof. – The second statement is a consequence of the first one. Assume E$'$ is a successor of E. Let $]a_k, b_k[$ be the complementary intervals of E$'$ and χ_k' their characteristic functions. Take non-negative functions $\psi_k \in C^\infty(\mathbf{R} \setminus E)$ such that

(i) $\psi_k = 1$ on $[a_k, b_k]$

(ii) $\sup\limits_k \sup\limits_x (\psi_k(x) + d_E(x) \, |\psi_k'(x)|) < \infty$

(iii) $\operatorname{supp} \psi_k \subset [a_k - d_E(a_k)/2 \, , \, b_k + d_E(b_k)/2]$.

Notice that $d_E(a_k)$ and $d_E(b_k)$ may be 0. Then the ψ_k have bounded overlap, so $\Sigma \pm \psi_k$ is uniformly in M_p if E has property HM(p). Let $\hat{G}_k = \psi_k \hat{f}$ and $\hat{F}_k = \chi'_k \hat{f}$ for $f \in L^p$. Averaging, we have $\|(\Sigma |G_k|^2)^{1/2}\|_p \leqslant C\|f\|_p$. Using Hilbert transforms, we get $\|(\Sigma |F_k|^2)^{1/2}\|_p \leqslant C\|(\Sigma |G_k|^2)^{1/2}\|_p$, and property LP($p$) for E' follows, by duality.

Notice that the LP(p) constant of E' can be estimated in terms of that of E and the successor constant.

Remark. – Theorem 1.2 implies that the following strong Hörmander-Mihlin-Marcinkiewicz property is equivalent to those of Theorem 1: Let m be bounded and locally of bounded variation in $R \backslash E$ and such that $\sup_I \int_I |dm(x)| < \infty$, where the sup is taken over all intervals I with $|I| = \operatorname{dist}(I, E)$. Then $m \in M_p$. This is easily proved by means of property Mar(p) for a successor of E.

Next, we give a simple necessary condition.

THEOREM 1.3. – *Let* E *have property* LP(p) *for some* $p > 2$. *Then there exists a constant* C *such that if* I *is a bounded interval and* $0 < d < |I|$, *then* $E \cap I$ *contains at most* $C(|I|/d)^{2/p}$ *points all of which have mutual distances at least* d.

Proof. – By translation and dilation, we may assume $I = [0, 1]$. Take f so that $\hat{f} \in C_0^\infty$ and $\hat{f} = 1$ in $[0, 2]$. Let $x_1, \ldots, x_n$ be points of $E \cap I$ of mutual distances at least d. Then the set $D = \{x_1, x_1 + d, x_2, x_2 + d, \ldots, x_n + d\}$ is easily seen to be a successor of E with constant $c = 1$. Thus D has properties LP(p) and LP(p') with constant independent of d. Denoting by S_J the operator $\widehat{S_J f} = \chi_J \hat{f}$ for any interval J, we get

$$\left\| \left(\sum_1^n |S_{[x_j, x_j + d]} f|^2 \right)^{1/2} \right\|_{p'} \leqslant C\|f\|_{p'} .$$

Hence, $n^{1/2} \left\| \dfrac{\sin d\xi/2}{\xi} \right\|_{p'} \leqslant C\|f\|_{p'}$, which implies $n \leqslant C d^{-2/p}$. The proof is complete.

From Theorem 1.3 we get the well-known result that no sequence of type $(n^\alpha)_{n=1}^\infty$ has property LP(p) for $p \neq 2$, $\alpha \neq 0$.

Consider now Cantor sets of type $E = \left\{ \sum_1^\infty \epsilon_j \ell_j \,;\, \epsilon_j = 0 \text{ or } 1 \right\}$, where ℓ_j, $j = 1, 2, \ldots$, are positive numbers satisfying $\ell_{j+1} < \ell_j/2$. For $\ell_j = 2.3^{-j}$, we get the classical Cantor set. Such sets will satisfy the necessary condition of Theorem 1.3, if the ℓ_j are small enough, but clearly they are not lacunary of finite order.

THEOREM 1.4. – *A Cantor set* E *of the above type has property* LP(p) *for no* $p \neq 2$.

To prepare for the proof, fix $p \in \,]1, 2[$ and let

$$m_p = \pi^{-1} \int_0^\pi |\cos x|^p \, dx .$$

By Hölder's inequality, $m_p < m_2^{p/2} = 2^{-p/2}$ with strict inequality, so we can take s_p with $m_p < s_p < 2^{-p/2}$.

It is easy to prove that

$$\int h(x) \, |\cos Qx|^p dx \longrightarrow m_p \int h(x) \, dx \,, \ Q \longrightarrow \infty \,,$$

for an integrable h. We need a uniform iterated version of a special case of this.

LEMMA 1.5. – *There exist numbers* $(A_j)_1^\infty$ *in* $]1, \infty[$ *such that if* $(Q_j)_0^\infty$ *are positive and* $Q_j/Q_{j-1} \geqslant A_j$ *for* $j = 1, 2, \ldots$, *then for any natural* N

$$\int \left| \frac{\sin Q_0 x}{x} \prod_1^N \cos Q_j x \right|^p dx \leqslant s_p^N \int \left| \frac{\sin Q_0 x}{x} \right|^p dx .$$

Proof. – We can clearly assume $Q_0 = 1$. Let for $N = 0, 1, \ldots$

$$h_N(x) = \left| \frac{\sin x}{x} \prod_1^N \cos Q_j x \right|^p .$$

Assuming $A_1, \ldots, A_{N-1}$ constructed, we must find A_N. Take $B > 0$ so that

$$\int_{|x| > B} h_0 \, dx \leqslant (s_p - m_p) \, s_p^{N-1} \int h_0 dx/2 \,, \tag{1.3}$$

and observe that $h_N \leqslant h_0$. For any Q_N, we have

$$\int_{-B}^{B} h_N(x) \, dx \leqslant \Sigma \int_{k\pi/Q_N}^{(k+1)\pi/Q_N} h_{N-1}(x) \, |\cos Q_N x|^p \, dx \,,$$

where the sum is taken over those k for which the interval $J_k = [k\pi/Q_N, (k+1)\pi/Q_N]$ intersects $[-B, B]$. Thus, at most $2BQ_N/\pi + 2$ values of k occur. For each k, take $x_k \in J_k$ so that $h_{N-1}(x_k)$ equals the mean value of h_{N-1} in J_k. Then

$$\begin{aligned}\int_{J_k} & h_{N-1}(x)\,|\cos Q_N x|^p\,dx \\ &\leqslant \int_{J_k} h_{N-1}(x_k)\,|\cos Q_N x|^p\,dx + \frac{\pi}{Q_N}\int_{J_k} \sup|h'_{N-1}|\,|\cos Q_N x|^p\,dx \\ &= m_p \int_{J_k} h_{N-1}(x)\,dx + \frac{\pi^2}{Q_N^2}\sup|h'_{N-1}|\,m_p\,.\end{aligned}$$

Summing in k, we obtain

$$\int_{-B}^{B} h_N(x)\,dx \leqslant m_p \int h_{N-1} + \frac{C_N}{Q_N}\sup|h'_{N-1}|\,. \tag{1.4}$$

Since the Q_j are increasing, it is easy to see that $\sup|h'_{N-1}| \leqslant C'_N Q_{N-1}$. So if Q_{N-1}/Q_N is sufficiently small, the last term of (1.4) will be dominated by $(s_p - m_p)\,s_p^{N-1}\int h_0\,dx/2$. From this and (1.3)-(1.4) the lemma follows, by the induction assumption.

Proof of Theorem 1.4. – Given an integer $N > 0$, select a finite subsequence $\ell_{n_0}, \ell_{n_1}, \ldots, \ell_{n_N}$ such that $n_0 = 1$ and $\ell_{n_j}/\ell_{n_{j+1}} > A_{N-j}$ for $j = 0, \ldots, N-1$. Writing $Q_j = \ell_{n_{N-j}}/2$, we thus have $Q_j/Q_{j-1} > A_j$ as in the lemma. Clearly, all points $\sum_0^N 2\epsilon_j Q_j$, $\epsilon_j = 0$ or 1, are in E, so if E has property LP(p), these points form a set E_N with LP(p) constant bounded uniformly in N. Define f_N so that $\hat{f}_N$ is the sum over $\epsilon_1, \ldots, \epsilon_N$ of the characteristic functions of the intervals

$$\left[\sum_{j=1}^N 2\epsilon_j Q_j\,,\ \sum_{j=1}^N 2\epsilon_j Q_j + 2Q_0\right],$$

so that

$$\hat{f}_N = \chi_{[0,2Q_0]} * \sum_{\epsilon_1,\ldots,\epsilon_N} \delta_{\sum_{j=1}^N 2\epsilon_j Q_j}\,.$$

One finds

$$\begin{aligned} f_N(x) &= (2\pi)^{-1}\,\frac{e^{2iQ_0x}-1}{x}\sum_{\epsilon_1,\ldots,\epsilon_N} e^{2i\sum_{j=1}^N \epsilon_j Q_j x} \\ &= (2\pi)^{-1}\,\frac{e^{2iQ_0x}-1}{x}\prod_{j=1}^N (1+e^{2iQ_jx}),\end{aligned}$$

and thus

$$|f_N(x)| = \pi^{-1} 2^N \left| \frac{\sin Q_0 x}{x} \prod_{j=1}^{N} \cos Q_j x \right| .$$

But the Littlewood-Paley sum of f corresponding to E_N is $\pi^{-1} 2^{N/2} \left| \frac{\sin Q_0 x}{x} \right|$. Property $LP(p)$ implies

$$\pi^{-p} 2^{Np/2} \int \left| \frac{\sin Q_0 x}{x} \right|^p dx$$

$$\leqslant C \pi^{-p} 2^{Np} \int \left| \frac{\sin Q_0 x}{x} \prod_{j=1}^{N} \cos Q_j x \right|^p dx .$$

But this is false for large N when $p < 2$, by Lemma 1.5. The case $p > 2$ follows, so the theorem is proved.

2. Two-dimensional results.

Let E denote a closed subset of S^1 with measure 0. We then have $S^1 \backslash E = \bigcup_{k=1}^{\infty} I_k$, where I_k are the open component intervals of $S^1 \backslash E$. Let $D_k = \{x \in \mathbf{R}^2 ; x' \in I_k\}$, where $x' = x/|x|$, and $E_0 = \{\theta \in \mathbf{R} ; (\cos\theta , \sin\theta) \in E\}$.

We shall now define properties $LP(p)$, $HM(p)$, and $Max(p)$, $1 < p < \infty$, for a set E of this type. Define operators S_k by setting $(S_k f)\hat{} = \chi_{D_k} \hat{f}$, where χ_{D_k} denotes the characteristic function of D_k. Then E is said to have property $LP(p)$ if

$$\|(\Sigma |S_k f|^2)^{1/2}\|_p \sim \|f\|_p , \ f \in L^p(\mathbf{R}^2) .$$

We let (r, θ) denote polar coordinates in $\mathbf{R}^2$ and shall consider functions $m \in L^\infty(\mathbf{R}^2)$ with the following property:

$$\begin{gathered} m(x) = m_0(\theta) , \ m_0 \in C^2(\mathbf{R}\backslash E_0) , \ m_0 \ \text{has period } 2\pi , \\ |m_0^{(k)}(\theta)| \leqslant C d_{E_0}(\theta)^{-k} , \ k = 0, 1, 2 . \end{gathered} \tag{2.1}$$

The set E is said to have property $HM(p)$ if every function m satisfying (2.1) is a Fourier multiplier for $L^p(\mathbf{R}^2)$. For $\alpha \in S^1$ we set

$$M_\alpha f(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x + t\alpha)| \, dt , \ x \in \mathbf{R}^2 , \ f \in C_0^\infty(\mathbf{R}^2) ,$$

and $M_E f = \sup_{\alpha \in E} M_\alpha f$, $f \in C_0^\infty(\mathbf{R}^2)$. We say that E has property Max(p) if M_E can be extended to a bounded linear operator on $L^p(\mathbf{R}^2)$. This is equivalent to L^p boundedness of the maximal function operator defined with respect to all rectangles in the E directions.

In this section, we study the relations between the above three properties and prove that lacunary sets of finite order have all the properties for $1 < p < \infty$.

Observe first that HM(p) implies LP(p). This follows from the fact that if $m = \Sigma \pm \chi_{D_k}$, then m satisfies condition (2.1). The next theorem is a partial converse of this observation.

THEOREM 2.1. – *Assume* $2 < p < \infty$ *and* $1 < r < (p/2)'$. *If* E *has properties* Max(r) *and* LP(p), *then* E *has property* HM(p).

Proof. – We set $I_k = \{(\cos\theta, \sin\theta) ; a_k < \theta < b_k\}$. Without loss of generality, we may assume that $0 < \theta_k = b_k - a_k \leqslant \pi/2$. Set $e_k = (\cos a_k, \sin a_k)$, $f_k = (\cos b_k, \sin b_k)$ and let the coordinates (ξ_k, η_k) of a point $x \in D_k$ be defined by $x = \xi_k e_k + \eta_k f_k$. Choose $\varphi \in C_0^\infty(\mathbf{R})$ such that $\varphi(t) = 1$, $1 \leqslant t \leqslant 2$, and $\varphi(t) = 0$ if $t \leqslant 2/3$ or $t \geqslant 3$. Then set $\varphi_i(t) = \varphi(2^{-i}t)$, $i \in \mathbf{Z}$, and $\varphi_{kij}(x) = \varphi_i(\xi_k)\varphi_j(\eta_k)$. Let R_{kij} denote the parallelogram

$$\{x ; 2^i \leqslant \xi_k \leqslant 2^{i+1}, 2^j \leqslant \eta_k \leqslant 2^{j+1}\},$$

and define the operators S_{kij} and S'_{kij} by the formulas

$$(S_{kij} f)\hat{} = \chi_{R_{kij}} \hat{f}$$

and $(S'_{kij} f)\hat{} = \varphi_{kij} \hat{f}$. We shall prove that

$$\Big\| \Big(\sum_{k,i,j} |S_{kij} f|^2 \Big)^{1/2} \Big\|_p \sim \|f\|_p . \tag{2.2}$$

To do this, we shall use the operators T_t, P_{k,t_2} and Q_{k,t_3} defined in the following way, where $(r_k)_{-\infty}^{\infty}$ is an enumeration of the Rademacher functions:

$$T_t f(x) = \sum_{k,i,j} r_k(t_1)\, r_i(t_2)\, r_j(t_3)\, S'_{kij} f(x),$$

$$(P_{k,t_2} f)\hat{}(x) = \Big(\sum_i r_i(t_2)\, \varphi_i(\xi_k) \Big) \hat{f}(x)$$

and $\quad (Q_{k,t_3}f)\hat{\ }(x) = \Big(\sum_j r_j(t_3)\,\varphi_j(\eta_k)\Big)\hat{f}(x)\,.$

Here $x \in \mathbf{R}^2$ and $t = (t_1, t_2, t_3) \in [0,1]^3$. We then have

$$T_t f = \sum_k r_k(t_1)\, P_{k,t_2}\, Q_{k,t_3} f.$$

With $q = (p/2)'$, property LP(p) implies

$$\begin{aligned} \|T_t f\|_p^2 &\leqslant C\,\|(\Sigma\,|P_{k,t_2} Q_{k,t_3} f|^2)^{1/2}\|_p^2 \\ &= \sup_{\|\psi\|_q = 1} C \int (\Sigma\,|P_{k,t_2} Q_{k,t_3} f\,|^2)\,\psi\,dx \\ &= \sup_{\psi} C\,\Sigma \int |P_{k,t_2}\, Q_{k,t_3}\, S_k f|^2\,\psi\,dx\,. \end{aligned}$$

Introducing the notation e'_k and f'_k for the vectors

$$(\cos(a_k + \pi/2),\ \sin(a_k + \pi/2))$$

and $(\cos(b_k - \pi/2),\ \sin(b_k - \pi/2))$, we easily see that

$$\xi_k = f'_k \cdot x/\sin\theta_k\,.$$

It follows that $(P_{k,t_2} f)\hat{\ }(x) = p_0(f'_k \cdot x)\,\hat{f}(x)$, where

$$p_0(u) = \sum_i r_i(t_2)\,\varphi_i(u/\sin\theta_k)\,,$$

and hence $|p_0(u)| \leqslant C$ and $|p'_0(u)| \leqslant C\,\dfrac{1}{|u|}$. We then choose $s = q/r$ and set $A_\alpha \psi = (M_\alpha\,|\psi|^s)^{1/s}$, $\alpha \in S^1$. Then the restriction of $A_{f'_k}\psi$ to almost every line parallel to f'_k will belong to the class A_2 of weight functions (see [1]). Using the above estimates for p_0 and p'_0 and a similar result for the operator Q_{k,t_3} we therefore obtain

$$\begin{aligned} \|T_t f\|_p^2 &\leqslant \sup_{\psi} C\,\Sigma \int |Q_{k,t_3} S_k f|^2\, A_{f'_k}(\psi)\,dx \\ &\leqslant \sup_{\psi} C\,\Sigma \int |S_k f|^2\, A_{e'_k} A_{f'_k}(\psi)\,dx \\ &\leqslant \sup_{\psi} C \int (\Sigma\,|S_k f|^2)\,(M^2_{E_1}(\psi^s))^{1/s}\,dx \\ &\leqslant \sup_{\psi} C\,\|(\Sigma\,|S_k f|^2)^{1/2}\|_p^2 \Big(\int [M^2_{E_1}(\psi^s)]^{q/s}\,dx\Big)^{1/q} \\ &\leqslant \sup_{\psi} C\,\|f\|_p^2\,\|\psi\|_q = C\,\|f\|_p^2\,, \end{aligned}$$

where $E_1 = \{\alpha \in S^1 ; \alpha \cdot \beta = 0$ for some $\beta \in E\}$. Here we have used property LP(p) for E and also the assumption that M_E and thus also M_{E_1} are bounded on L^r. We have proved that

$$\|T_t f\|_p \leqslant C \|f\|_p \tag{2.3}$$

and it follows that

$$\Big\|\Big(\sum_{k,i,j} |S'_{kij} f|^2\Big)^{1/2}\Big\|_p \leqslant C \|f\|_p . \tag{2.4}$$

From duality, it also follows that (2.3) and (2.4) hold with p replaced by p'.

Now let V_k, $k = 1, 2, 3, \ldots$, be half-planes and assume that the boundary of each V_k is parallel to a vector in E. Define the operator H_k by $(H_k g)\hat{} = \chi_{V_k} \hat{g}$. We then claim that

$$\|(\Sigma |H_k g_k|^2)^{1/2}\|_p \leqslant C \|(\Sigma |g_k|^2)^{1/2}\|_p . \tag{2.5}$$

This is easily proved in the following way (cf. A. Cordoba, R. Fefferman [2]):

$$\begin{aligned}
\|(\Sigma |H_k g_k|^2)^{1/2}\|_p^2 &= \|\Sigma |H_k g_k|^2\|_{p/2} = \sup_{\|\psi\|_q = 1} \int (\Sigma |H_k g_k|^2)\, \psi \, dx \\
&\leqslant \sup_\psi \Sigma \int |g_k|^2 \, (M_{E_1}(\psi^s))^{1/s} dx \\
&\leqslant \sup_\psi \|(\Sigma |g_k|^2)^{1/2}\|_p^2 \;\; \|(M_{E_1}(\psi^s))^{1/s}\|_q \leqslant C \|(\Sigma |g_k|^2)^{1/2}\|_p .
\end{aligned}$$

From duality we then conclude that (2.5) holds also with p replaced by p'. A combination of (2.4) and (2.5) and the analogous inequalities with p' then yields $\|(\Sigma |S_{kij} f|^2)^{1/2}\|_p \leqslant C \|f\|_p$ and $\|(\Sigma |S_{kij} f|^2)^{1/2}\|_{p'} \leqslant C \|f\|_{p'}$, and (2.2) follows.

We shall now use (2.2) to prove that E has property HM(p). Let m and m_0 satisfy (2.1) and assume that $\hat{F} = m\hat{f}$, where $f \in L^p(\mathbf{R}^2)$. Setting $n(\xi, \eta) = m(\xi e_k + \eta f_k)$, $\xi > 0$, $\eta > 0$, we have

$$n(\xi, \eta) = \int_{2^i}^{\xi} \int_{2^j}^{\eta} \frac{\partial^2 n}{\partial t_1 \partial t_2} (t_1, t_2)\, dt_1 dt_2 + \int_{2^i}^{\xi} \frac{\partial n}{\partial t_1} (t_1, 2^j)\, dt_1$$

$$+ \int_{2^j}^{\eta} \frac{\partial n}{\partial t_2} (2^i, t_2)\, dt_2 + n(2^i, 2^j)$$

$$= \int_{2^i}^{2^{i+1}} \int_{2^j}^{2^{j+1}} \frac{\partial^2 n}{\partial t_1 \partial t_2} (t_1, t_2) \chi_{[t_1, 2^{i+1}]}(\xi) \chi_{[t_2, 2^{j+1}]}(\eta) dt_1 dt_2$$

$$+ \int_{2^i}^{2^{i+1}} \frac{\partial n}{\partial t_1} (t_1, 2^j) \chi_{[t_1, 2^{i+1}]}(\xi)\, dt_1$$

$$+ \int_{2^j}^{2^{j+1}} \frac{\partial n}{\partial t_2} (2^i, t_2) \chi_{[t_2, 2^{j+1}]}(\eta)\, dt_2 + n(2^i, 2^j),$$

$$2^i \leqslant \xi \leqslant 2^{i+1},\ 2^j \leqslant \eta \leqslant 2^{j+1}.$$

Setting $\Delta_i = (2^i, 2^{i+1})$ and $\Delta_{ij} = \Delta_i \times \Delta_j$ and observing that $m(x) = n(\xi_k, \eta_k)$ for $x \in \mathrm{D}_k$, we conclude that

$$\mathrm{S}_{kij}\mathrm{F} = \iint_{\Delta_{ij}} \frac{\partial^2 n}{\partial t_1 \partial t_2} (t_1, t_2)\, \mathrm{S}_t \mathrm{S}_{kij} f\, dt_1 dt_2$$

$$+ \int_{\Delta_i} \frac{\partial n}{\partial t_1} (t_1, 2^j)\, \mathrm{S}^1_{t_1} \mathrm{S}_{kij} f\, dt_1 + \int_{\Delta_j} \frac{\partial n}{\partial t_2} (2^i, t_2)\, \mathrm{S}^2_{t_2} \mathrm{S}_{kij} f\, dt_2$$

$$+ n(2^i, 2^j)\, \mathrm{S}_{kij} f,$$

where

$$(\mathrm{S}_t f)\hat{}\,(x) = \chi_{[t_1, 2^{i+1}]}(\xi_k)\, \chi_{[t_2, 2^{j+1}]}(\eta_k) \hat{f}(x),$$

$$(\mathrm{S}^1_{t_1} f)\hat{}\,(x) = \chi_{[t_1, 2^{i+1}]}(\xi_k) \hat{f}(x)$$

and

$$(\mathrm{S}^2_{t_2} f)\hat{}\,(x) = \chi_{[t_2, 2^{j+1}]}(\eta_k) \hat{f}(x).$$

We have $n(\xi, \eta) = m(\xi e_k + \eta f_k) = m_0(\theta)$, and it is easy to see that the relation between θ and (ξ, η) is given by

$$\theta = a_k + \arctan \frac{\eta \sin \theta_k}{\xi + \eta \cos \theta_k}.$$

A computation using this formula and the estimates (2.1) of the derivatives of m_0 then shows that

$$\left| \frac{\partial n}{\partial t_1} (t_1, t_2) \right| \leqslant \mathrm{C} \frac{1}{t_1},$$

$$\left| \frac{\partial n}{\partial t_2} (t_1, t_2) \right| \leqslant \mathrm{C} \frac{1}{t_2}$$

and

$$\left| \frac{\partial^2 n}{\partial t_1 \partial t_2} (t_1, t_2) \right| \leqslant \mathrm{C} \frac{1}{t_1 t_2}.$$

Invoking the Cauchy-Schwarz inequality, we then see that

$$|S_{kij}F|^2 \leqslant C \iint_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2}(t_1, t_2) \right| |S_t S_{kij} f|^2 \, dt_1 dt_2$$
$$+ C \int_{\Delta_i} \left| \frac{\partial n}{\partial t_1}(t_1, 2^j) \right| |S^1_{t_1} S_{kij} f|^2 \, dt_1$$
$$+ C \int_{\Delta_j} \left| \frac{\partial n}{\partial t_2}(2^i, t_2) \right| |S^2_{t_2} S_{kij} f|^2 \, dt_2 + C\, |S_{kij} f|^2 .$$

Now (2.2) yields

$$\|F\|_p^2 \leqslant C \left\| \Big(\sum_{k,i,j} |S_{kij}F|^2 \Big)^{1/2} \right\|_p^2$$
$$\leqslant C \left(\int \left[\sum_{k,i,j} \int_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2}(t) \right| |S_t S_{kij} f(x)|^2 \, dt \right]^{p/2} dx \right)^{2/p}$$
$$+ C \left(\int \left[\sum_{k,i,j} \int_{\Delta_i} \left| \frac{\partial n}{\partial t_1}(t_1, 2^j) \right| |S^1_{t_1} S_{kij} f(x)|^2 \, dt_1 \right]^{p/2} dx \right)^{2/p}$$
$$+ C \left(\int \left[\sum_{k,i,j} \int_{\Delta_j} \left| \frac{\partial n}{\partial t_2}(2^i, t_2) \right| |S^2_{t_2} S_{kij} f(x)|^2 \, dt_2 \right]^{p/2} dx \right)^{2/p}$$
$$+ C \left(\int \Big(\sum_{k,i,j} |S_{kij} f(x)|^2 \Big)^{p/2} dx \right)^{2/p} .$$

We shall only show how to estimate the first term on the right-hand side. The estimates for the other terms are similar. The first term on the right-hand side equals

$$C \sup_{\|\psi\|_q = 1} \int \left[\sum \int_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2}(t) \right| |S_t S_{kij} f(x)|^2 \, dt \right] \psi(x)\, dx$$
$$= C \sup_{\psi} \sum \int_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2}(t) \right| \left[\int |S_t S_{kij} f(x)|^2 \psi(x)\, dx \right] dt$$
$$\leqslant C \sup_{\psi} \sum \int_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2}(t) \right| \left[\int |S_{kij} f(x)|^2 (M^2_{E_1}(\psi^s))^{1/s} dx \right] dt$$
$$= C \sup_{\psi} \int \Big(\sum |S_{kij} f(x)|^2 \Big) (M^2_{E_1}(\psi^s))^{1/s} dx$$
$$\leqslant C \left\| \Big(\sum |S_{kij} f|^2 \Big)^{1/2} \right\|_p^2 \leqslant C \|f\|_p^2 ,$$

where we have invoked (2.2) once more.

It follows that $\|F\|_p \leqslant C\,\|f\|_p$ and hence m is a multiplier for L^p. We conclude that E has property HM(p), and the proof of the theorem is complete.

COROLLARY 2.2. – *Assume* $1 < p \leqslant 2$. *If* E *has properties* Max(p) *and* LP(p), *then* E *has property* HM(p).

Proof. – It is sufficient to prove that Max(p) and LP(p') imply HM(p'), and this follows from Theorem 2.1 since $p < (p'/2)'$.

We define a successor of a set $E \subset S^1$ in the same way as for subsets of $\mathbf{R}$, and we also define lacunary sets of order n, $n = 0, 1, 2, \ldots$, analogously.

THEOREM 2.3. – *Assume* E′ *is a successor of a set* $E \subset S^1$ *and that* E *has properties* Max(p) *and* HM(p), *where* $1 < p \leqslant 2$. *Then* E′ *has properties* Max(p) *and* HM(p).

Proof. – We shall first prove that E′ has property Max(p). Let e_k, f_k, a_k, b_k have the same meaning as in the proof of Theorem 2.1. We may assume $E' \backslash E = \{e_{kj}, f_{kj} : k, j = 1, 2, \ldots\}$, where $e_{kj} = (\cos a_{kj}, \sin a_{kj})$ and $(a_{kj})_{j=1}^{\infty}$ is a lacunary sequence tending to a_k and contained in $]a_k, (a_k + b_k)/2]$, and analogously for f_{kj}. Letting $F = \{e_{kj}\}$, we shall prove that M_F is bounded on L^p. The set $\{f_{kj}\}$ can be treated in a similar way.

Our proof is a modification of that of A. Nagel, E.M. Stein and S. Wainger [5]. First, we prove assertions I and II below.

I. If $p \leqslant r \leqslant 2$ and

$$\Big\| \Big(\sum_{k,j} |M_{e_{kj}} g_{kj}|^2 \Big)^{1/2} \Big\|_r \leqslant C \, \Big\| \Big(\sum |g_{kj}|^2 \Big)^{1/2} \Big\|_r , \tag{2.6}$$

then

$$\|M_F f\|_r \leqslant C \, \|f\|_r . \tag{2.7}$$

II. If (2.7) holds for some r with $1 < r \leqslant 2$, then

$$\Big\| \Big(\sum_{k,j} |M_{e_{kj}} g_{kj}|^2 \Big)^{1/2} \Big\|_q \leqslant C \, \Big\| \Big(\sum |g_{kj}|^2 \Big)^{1/2} \Big\|_q \tag{2.8}$$

for all q satisfying $\frac{1}{2} \leqslant \frac{1}{q} < \frac{1}{2} \left(1 + \frac{1}{r}\right)$.

Assertion II can be proved in the same way as in [5, Lemma 3]. We shall now prove I and first set

$$\mathrm{N}_{hkj} f(x) = \frac{1}{h} \int_{-\infty}^{\infty} \psi(t/h) f(x - te_{kj})\, dt\ ,\ x \in \mathbf{R}^2\ ,$$

where $\psi \in C_0^\infty(\mathbf{R})$, ψ is positive and $\psi(t) = 1$ for $|t| \leq 1$. Also set $m = \hat{\psi}$ and $\delta_{kj} = a_{kj} - a_k$. Let $\phi_1 \in C_0^\infty(\mathbf{R}^2)$ and assume that $\phi_1(x) = 1$ for $|x| \leq 1$. Set $\phi_2 = 1 - \phi_1$ and

$$g_1(x) = m(x_1 + x_2)\, \phi_1(x).$$

Also let $\omega \in C^\infty(\mathbf{R}^2 \backslash \{0\})$ be homogeneous of degree zero and assume that $\omega(x) = 1$, $|x_1 + x_2| < c|x|$ and

$$\omega(x) = 0\,,\ |x_1 + x_2| > 2c|x|\,,$$

where c is a small positive constant. Set

$$g_2(x) = m(x_1 + x_2)\, \phi_2(x)\ (1 - \omega(x))\,.$$

Let $\mathrm{R}_k : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ denote a rotation of angle $-a_k$. We then have

$$\begin{aligned}
(\mathrm{N}_{hkj} f)\hat{}(\xi) &= m(he_{kj} \cdot \xi)\, \hat{f}(\xi) \\
&\equiv m(he_{kj} \cdot \xi)\, \phi_1(h \cos \delta_{kj} (\mathrm{R}_k \xi)_1\,,\ h \sin \delta_{kj} (\mathrm{R}_k \xi)_2)\, \hat{f}(\xi) \\
&\quad + m(he_{kj} \cdot \xi)\, \phi_2(h \cos \delta_{kj} (\mathrm{R}_k \xi)_1\,,\ h \sin \delta_{kj} (\mathrm{R}_k \xi)_2) \\
&\qquad (1 - \omega(h \cos \delta_{kj} (\mathrm{R}_k \xi)_1\,,\ h \sin \delta_{kj} (\mathrm{R}_k \xi)_2))\, \hat{f}(\xi) \\
&\quad + m(he_{kj} \cdot \xi)\, \phi_2(h \cos \delta_{kj} (\mathrm{R}_k \xi)_1\,,\ h \sin \delta_{kj} (\mathrm{R}_k \xi)_2) \\
&\quad \cdot\ \omega(h \cos \delta_{kj} (\mathrm{R}_k \xi)_1\,,\ h \sin \delta_{kj} (\mathrm{R}_k \xi)_2)\, \hat{f}(\xi) \\
&\equiv (\mathrm{A}_{hkj} f)\hat{}(\xi) + (\mathrm{B}_{hkj} f)\hat{}(\xi) + (\mathrm{C}_{hkj} f)\hat{}(\xi)\,.
\end{aligned}$$

Now $e_{kj} = (\cos(a_k + \delta_{kj}),\ \sin(a_k + \delta_{kj}))$ and so

$$e_{kj} \cdot \xi = \cos \delta_{kj} (\mathrm{R}_k \xi)_1 + \sin \delta_{kj} (\mathrm{R}_k \xi)_2\ .$$

Hence,

$$(\mathrm{A}_{hkj} f)\hat{}(\xi) = g_1(h \cos \delta_{kj} (\mathrm{R}_k \xi)_1\,,\ h \sin \delta_{kj} (\mathrm{R}_k \xi)_2)\, \hat{f}(\xi)$$

and

$$(\mathrm{B}_{hkj} f)\hat{}(\xi) = g_2(h \cos \delta_{kj} (\mathrm{R}_k \xi)_1\,,\ h \sin \delta_{kj} (\mathrm{R}_k \xi)_2)\, \hat{f}(\xi)\ .$$

We set $\mathrm{A}^* f = \sup\limits_{h,k,j} |\mathrm{A}_{hkj} f|$ and $\mathrm{B}^* f = \sup\limits_{h,k,j} |\mathrm{B}_{hkj} f|$. From the fact that g_1 and g_2 belong to the Schwartz class $\mathcal{S}$, we conclude that $\mathrm{A}^* f + \mathrm{B}^* f \leq C\, \mathrm{M_E}\, \mathrm{M_{E_1}} f$. We have assumed that E has property Max(p), and it follows by interpolation that E also has property Max(r). Hence,

$$\|\mathrm{A}^* f\|_r + \|\mathrm{B}^* f\|_r \leq C\, \|f\|_r\ . \tag{2.9}$$

Setting

$$(\mathrm{D}_{hkj}f)^{\wedge}(\xi) = m(he_{kj}\cdot\xi)\,\omega(h\cos\delta_{kj}(\mathrm{R}_k\xi)_1\,,\ h\sin\delta_{kj}(\mathrm{R}_k\xi)_2)\,\hat{f}(\xi)\,,$$

we have $\mathrm{C}^*f \leqslant \mathrm{C}\,\mathrm{M}_{\mathrm{E}}\mathrm{M}_{\mathrm{E}_1}\mathrm{D}^*f$, since $\phi_2 = 1 - \phi_1$ and $\phi_1 \in \mathcal{S}$. Here C^* and D^* are defined in the same way as A^* and B^*. It follows that

$$\|\mathrm{C}^*f\|_r \leqslant \mathrm{C}\,\|\mathrm{D}^*f\|_r\,. \tag{2.10}$$

Define the operator K_{kj} by setting

$$(\mathrm{K}_{kj}f)^{\wedge}(\xi) = \omega(\cos\delta_{kj}(\mathrm{R}_k\xi)_1\,,\ \sin\delta_{kj}(\mathrm{R}_k\xi)_2)\,\hat{f}(\xi)\,.$$

Then $\mathrm{D}_{hkj}f = \mathrm{N}_{hkj}\mathrm{K}_{kj}f$, and it follows from (2.6) that

$$\begin{aligned}\|\mathrm{D}^*f\|_r &\leqslant \left\|\Big(\sum_{k,j}\sup_h |\mathrm{D}_{hkj}f|^2\Big)^{1/2}\right\|_r\\ &\leqslant \mathrm{C}\left\|\Big(\sum_{k,j}|\mathrm{M}_{e_{kj}}\mathrm{K}_{kj}f|^2\Big)^{1/2}\right\|_r \leqslant \mathrm{C}\left\|\Big(\sum_{k,j}|\mathrm{K}_{kj}f|^2\Big)^{1/2}\right\|_r.\end{aligned} \tag{2.11}$$

We have $\Big(\sum\limits_{k,j} \pm \mathrm{K}_{kj}f\Big)^{\wedge}(\xi) = m(\xi)\,\hat{f}(\xi)$, where

$$m(\xi) = \sum_{k,j} \pm\,\omega(\cos\delta_{kj}(\mathrm{R}_k\xi)_1\,,\ \sin\delta_{kj}(\mathrm{R}_k\xi)_2)\,.$$

Let E_1' denote the set E rotated an angle $\pi/2$ and E_1'' the set E rotated an angle $-\pi/2$. A computation then shows that $m = m' + m''$, where m' satisfies (2.1) for E_1' and m'' satisfies (2.1) for E_1''. Since E and thus also E_1' and E_1'' have properties HM(p) and HM(r), we conclude that

$$\left\|\sum_{k,j} \pm\,\mathrm{K}_{kj}f\right\|_r \leqslant \mathrm{C}\,\|f\|_r\,.$$

It follows that

$$\left\|\Big(\sum |\mathrm{K}_{kj}f|^2\Big)^{1/2}\right\|_r \leqslant \mathrm{C}\,\|f\|_r\,, \tag{2.12}$$

and a combination of (2.9) – (2.12) shows that $\|\mathrm{N}^*f\|_r \leqslant \mathrm{C}\,\|f\|_r$, where $\mathrm{N}^*f = \sup\limits_{h,k,j}|\mathrm{N}_{hkj}f|$. It follows that M_{F} is bounded on L^r, and hence assertion I is proved.

A repeated application of I and II now shows that M_{F} is bounded on L^p, and hence E' has property Max(p).

It remains to prove that E' has property HM(p). First, let V_k, $k = 1, 2, 3, \ldots,$ be half-planes and assume that the boundary of each V_k is parallel to a vector in E'. Define the operator H_k by $(H_k g)\hat{} = \chi_{V_k} \hat{g}$. It then follows that

$$\|(\Sigma |H_k g_k|^2)^{1/2}\|_r \leqslant C \|(\Sigma |g_k|^2)^{1/2}\|_r, \ p \leqslant r \leqslant p'. \quad (2.13)$$

This can be proved in the same way as (2.5) if we observe that $p < (p'/2)'$ and that E' has property $\mathrm{Max}(p)$. We shall now show that E' has property $\mathrm{LP}(p)$. Write $e_{k0} = f_{k1}$ and let $D_{kj}^{(1)}$ denote the sector between the vectors $e_{k,j-1}$ and e_{kj} and $D_{kj}^{(2)}$ the sector between f_{kj} and $f_{k,j+1}$. Then $D_k = \left(\bigcup_{j=1}^{\infty} D_{kj}^{(1)}\right) \cup \left(\bigcup_{j=1}^{\infty} D_{kj}^{(2)}\right)$, except for a set of measure zero.

Let $\omega_{kj}^{(i)} \in C^\infty(\mathbf{R}^2 \setminus \{0\})$ be homogeneous of degree zero and satisfy $\omega_{kj}^{(i)}(x) = 1$ for $x \in D_{kj}^{(i)}$, where $i = 1, 2$ and $k, j = 1, 2, 3, \ldots$ From the lacunarity of the sequences $(e_{kj})_{j=1}^\infty$ and $(f_{kj})_{j=1}^\infty$, it follows that we can choose the $\omega_{kj}^{(i)}$ so that if we set $m = \sum_{i,k,j} \pm \omega_{kj}^{(i)}$, then m will satisfy condition (2.1) for the set E. Since E has property $\mathrm{HM}(p)$ it follows that m is a Fourier multiplier for $L^r(\mathbf{R}^2)$ for $p \leqslant r \leqslant p'$. Thus, if $(T_{kj}^{(i)} f)\hat{} = \omega_{kj}^{(i)} \hat{f}$, we have

$$\Big\| \sum_{i,k,j} \pm T_{kj}^{(i)} f \Big\|_r \leqslant C \|f\|_r, \ p \leqslant r \leqslant p'.$$

Hence

$$\Big\| \Big(\sum_{i,k,j} |T_{kj}^{(i)} f|^2 \Big)^{1/2} \Big\|_r \leqslant C \|f\|_r, \ p \leqslant r \leqslant p'.$$

An application of (2.13) yields

$$\Big\| \Big(\sum_{i,k,j} |S_{kj}^{(i)} f|^2 \Big)^{1/2} \Big\|_r \leqslant C \|f\|_r, \ p \leqslant r \leqslant p',$$

where $(S_{kj}^{(i)} f)\hat{} = \chi_{D_{kj}^{(i)}} \hat{f}$. It follows that E' has property $\mathrm{LP}(p)$, and using Corollary 2.2, we conclude that E' has property $\mathrm{HM}(p)$.

The proof of the theorem is complete.

A repeated application of Theorem 2.3 gives the following corollary.

COROLLARY 2.4. – *Lacunary sets of finite order have properties* $\mathrm{Max}(p)$, $\mathrm{HM}(p)$ *and* $\mathrm{LP}(p)$ *for* $1 < p < \infty$.

The fact that lacunary sets of order 1 have the properties $\mathrm{Max}(p)$ and $\mathrm{LP}(p)$ for $1 < p < \infty$ was proved in [5].

One- and two-dimensional sets are related as follows. If $E \subset S^1$, we let $E^* = \{rx : r \geqslant 0, x \in E\}$ be the corresponding union of rays.

PROPOSITION 2.5. – *Let* $E \subset S^1$ *have property* LP(p). *Then the intersection of* E^* *with any line not passing through the origin is a one-dimensional set with property* LP(p).

Proof. – Keeping our notation, we see that $\Sigma \pm \chi_{D_k} \in M_p(\mathbf{R}^2)$, uniformly for all sign combinations. In view of M. Jodeit's note [3], this implies that the restriction of $\Sigma \pm \chi_{D_k}$ to any line not containing 0 is in $M_p(\mathbf{R})$, uniformly. The conclusion follows.

COROLLARY 2.6. – *If* $E \subset S^1$ *has property* LP(p), $p > 2$, *then any arc* $I \subset S^1$ *contains at most* $C(|I|/d)^{2/p}$ *points of mutual distances at least* d. *Here* $0 < d < |I|$ *and* $C = C(E)$.

Proof. – This follows if we intersect E^* with the lines $x_1 = \pm 1$, $x_2 = \pm 1$, say, and apply Proposition 2.5 and Theorem 1.3.

From Theorem 1.4 we obtain examples of sets $E \subset S^1$ homeomorphic to the Cantor set not having property LP(p), $p \neq 2$. Simply choose E so that the intersection of E^* with some line is a Cantor set of the type studied in Theorem 1.4.

As to the maximal property, there is a simple necessary condition like that of Corollary 2.6.

PROPOSITION 2.7. – *If* $E \subset S^1$ *has property* Max(p), $1 < p < \infty$, *then* E *contains at most* Cd^{1-p} *points of mutual distances at least* d *for* $0 < d < 2\pi$, *where* $C = C(E)$.

Proof. – Assume E contains points $x_1, \ldots, x_n$ with $|x_i - x_j| \geqslant d$, $i \neq j$. (It is irrelevant whether we consider Euclidean distance in $\mathbf{R}^2$ or arc length in S^1). Let f be the characteristic function of the unit disc. Consider the rectangles with directions in some x_j, centered at 0, and having width 2 and length $10/d$. They will cover a set of area at least n/d on which $M_E f \geqslant Cd$. The maximal property now implies $n \leqslant Cd^{1-p}$.

Notice that this result applies to Cantor sets in S^1 of constant ratio $q < 1/2$ (i.e. $\ell_{j+1}/\ell_j = q$ in the definition in Section 1), and shows that such sets do not have property Max(p) for

$$p < 1 + \log 2/\log q^{-1}.$$

And Corollary 2.6 implies that they do not have property LP(p) for $p > 2 \log q^{-1}/\log 2$.

BIBLIOGRAPHY

[1] A. CORDOBA and C. FEFFERMAN, A weighted norm inequality for singular integrals, *Studia Math.*, 57 (1976), 97-101.

[2] A. CORDOBA and R. FEFFERMAN, On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier analysis, *Proc. Natl. Acad. Sci. USA*, 74 (1977), 423-425.

[3] M. JODEIT Jr, A note on Fourier multipliers, *Proc. Amer. Math. Soc.*, 27 (1971), 423-424.

[4] D.S. KURTZ and R.L. WHEEDEN, Results on weighted norm inequalities for multipliers, *Trans. Amer. Math. Soc.*, 255 (1979), 343-362.

[5] A. NAGEL, E.M. STEIN and S. WAINGER, Differentiation in lacunary directions, *Proc. Natl. Acad. Sci. USA*, 75 (1978), 1060-1062.

[6] J.L. RUBIO de FRANCIA, Vector valued inequalities for operators in L^p spaces, *Bull. London Math. Soc.*, 12 (1980), 211-215.

[7] E.M. STEIN, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, 1970.

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